Second Level Algorithms

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Lecture 26

Welcome to the 26-th lecture of the second-level algorithms course. In the last lecture, we started proving the correctness of the push-relabel algorithm. We have seen that the push-relabel algorithm maintains invariants throughout the run of the algorithm. So, when the push-relabel algorithm, if at all terminates, it terminates with maximum flow—maximum s-t flow—but the question is whether it terminates at all or not, so that we will resolve in this lecture. Okay, so let's begin. We have proved that the push-relabel algorithm maintains the three invariants which imply that the optimality condition that is, T being disconnected from S in the residual graph. is maintained throughout the run of the algorithm. Hence, if the push-relabel algorithm terminates then it outputs a maximum s-t flow. The question is why it should terminate and in how many iterations can we bound the number of iterations. We will show that the while loop in the push-relabel algorithm iterates $O(n^3)$ times. In particular, we will prove the following theorem.

The number of reliable operations is $O(n^2)$ and the number of push operations is $O(n^3)$. So, this in particular implies that the number of iterations of the while loop of the push reliable algorithm is $O(n^3)$. So, to prove this we will prove this important lemma first which says that every vertex v which has an excess flow there must be a s to v path in the graph input graph not residual graph where every edge carries some positive amount of flow so for every vertex v not equal to s and t has a positive excess in a preflow f there exists a path from S to V in G where every edge of the path carries some positive amount of flow. equivalently there is a path from V to S in the residual graph G_f.

Let us first prove this lemma and then we will see how using this lemma we can bound first the number of reliable operations and thereby bound the number of push operations. So, proof of the lemma. So, the main point in the proof is to consider this set of vertices A is the set of vertices $v \in V$ such that there is a path from s to v in G where every edge of the path carries a positive amount of flow ok.

So, this is B and this is A. For every vertex, there is a path from S such that every edge of that path carries some positive amount of flow. So, now the main point is to consider the following quantity. Summation $v \in V \setminus A$: flow out of v minus flow into v. So, let us consider this particular quantity, and we will calculate this quantity in two ways, thereby proving the lemma.

The first thing to observe is that for every vertex, flow out is less than or equal to flow in because f is a preflow, right? For every vertex other than the source vertex S, flow out is at most flow in, and therefore, flow out of V is less than or equal to flow into V. So, flow out of v minus flow into v is less than or equal to 0, and for every vertex $v \in V \setminus A$, and capital $V \setminus A$ does not contain s because s belongs to A. Recall what is the definition of A? It is the set of vertices which are reachable from s using only those edges which carry a positive amount of flow.

So, S is reachable from S. So, S belongs to A. So, now you see I have a sum where every term is less than or equal to 0. So, this sum, let us call it S. So, S is less than or equal to 0 because S is a free flow and S belongs to A. Now, consider this sum from edgewise. See, there are four kinds of edges in the graph. One kind is both endpoints are in A.

Another kind is where both endpoints are in B. Another kind is this: Kind 1 is where both endpoints are in A. Kind 2 is where both endpoints are in B-A. The third type is edges starting from A. and go to $V \setminus A$. These are the third type. The fourth type is edges going from $V \setminus A$ to A. Note that edges of the first type do not contribute to the sum 2S. Edges of type 1 do not contribute to S, okay. Now, consider edges of type 2. Edges of type contribute twice: once as a flow out of one vertex and another as a flow into one vertex with opposite signs, hence they cancel.

So, the net contribution of the edges of type 2 to 2S is 0. Now, let us consider the contribution of the third edges. But we claim that the flow in every edge of type 3 must be 0 because A is the set of vertices reachable from S using only those edges which carry a positive amount of flow. So, let us consider an edge X to Y of type 3.

Since X belongs to A, there is an S to X path using only those edges which carry a positive amount of flow. Now, Y does not belong to A. So, if the XY edge carries flow, then Y should also be in A, contradicting the fact that Y does not belong to A. Thus, edges of type 3 must carry 0 flow. To see this, consider an edge X, Y. where X belongs to A and Y belongs to $V \setminus A$. If the edge carries a positive amount of flow, then Y should also belong to A, contradicting our assumption is that y belongs to $V \setminus A$, ok. So, the contribution of edges of type 3 is, ah, type 3 2s is 0. Now, let us consider the fourth type of edge. The fourth type of edge leaves $V \setminus A$. So, if this edge is from, say, x_1 to y_1, then it contributes negatively to the sum S because it is an outgoing flow at x_1.

Is it negative or positive? What is the sum? No, it is an outflow. So, it contributes to the outflow of x_1. On the other hand, it does not contribute to the inflow of y_1. So, it contributes positively.

The edges of type 4 contribute non-negatively, whatever their value is. That is what their contribution is, and the flow value is non-negative. That is all we know—non-negatively to S. S is, from edgewise, the value of S should be greater than or equal to 0 because there are four types of edges. The first three types do not contribute anything, or they contribute 0, and the fourth kind of edge contributes non-negatively. So, S should be greater than or equal to 0.

Let us call it equation 2. Inequality, and this we call it inequality 1. So, you have seen before s is less than or equal to 0. Now, we are seeing s is greater than or equal to 0. So, from 1 and 2, we conclude

that s is equal to 0. But now let us revisit the definition of s. Each term is less than or equal to 0, and the whole sum is 0. So, each term must be 0. However, s is a sum of non-negative or non-positive terms.

Hence, every term of S must be 0. That implies no vertex of $V \setminus A$ has any excess flow; equivalently, every vertex having some positive excess must belong to A.

Every vertex having any positive axis is reachable. From S using only those edges that carry some positive flow. This is the statement of the lemma, and hence we conclude the proof here. So, in the next lecture, we will see how using this lemma we can bound the number of field-level operations and the number of push operations, OK?

So, let us stop here. Thank you very much.