Second Level Algorithms

Prof. Palash Dey

Department of Computer Science and Engineering

Indian Institute of Technology, Kharagpur

Week – 04

Lecture 16

Welcome to the 16th lecture of the second-level algorithms course. In the last lecture, we have seen that the Ford-Fulkerson method works by studying the execution of that algorithm in a toy example, in particular, in the example where the greedy algorithm fails, and then we wrote an important lemma which will lead us to the proof of the Ford-Fulkerson method, the proof of correctness of the Ford-Fulkerson method. So, let us begin. So, the lemma basically says that given

a flow network G equal to (V, E), and capacity values and a flow, if the following are equivalent. The first one is f is a maximum s-t flow. The second one is G_f does not have any s to t path, and the third one is the value of f is the same

as the capacity of the s-t cut $(U, V \setminus U)$, where U is the set of vertices reachable from s in G_f. So, clearly s belongs to U because s is reachable from s, and t does not belong to U because if there is no s to t path in G_f. So, we will show that these three conditions are equivalent. Let us first prove that 1 implies 2.

That means, assuming that f is a maximum st-flow, we need to show that there is no s-to-t path in G_f. So, we will prove the contrapositive of this statement. That is, not 2 implies not 1. So, not 2 means there exists an s-t path in G_f. So, assume that

there exists an s-to-t path in G_f. Let us call this path P. Then we can augment the flow f using the path P, which, informally speaking, is the same as sending as much flow as you can along that path. to obtain another flow of value more than the value of f. Hence, f is not a maximum flow of G. So, we have proved 1 implies 2.

Now, let us prove 2 implies 3. That means we assume that there is no s-t path in G_f, and we need to show that the flow value of f is the same as the capacity of a u-v cut. So, we claim that for every edge u-v in E with small u belonging to capital U (where capital U is the set of vertices reachable from s) and $v \in V \setminus U$, for every such edge u-v, the flow value f(u, v) must be the same as the capacity of the edge. Indeed, this must be the case; otherwise, If the flow value is strictly less than the capacity of the edge, then there will be a forward edge $uv \in G_f$. However, this implies that V is also reachable from s in G_f, right? So, pictorially, this is how it looks: this is V, this is U, this is s. Capital U is the set of vertices reachable from s in G_f.

Now, if the flow value of this edge is strictly less than the capacity of this edge, then the U to V edge is also present as a forward edge in G_f. Now, U is reachable from S, which means there exists a path from s to u, and because there is an edge from u to v, that means V is also reachable from S. But that contradicts our assumption that v belongs to $V \setminus U$. This contradicts our assumption that $v \in V \setminus U$. So, all the edges from U to $V \setminus U$ the flow values should be the same as the capacity. We also claim that for every edge in E with $x \in V \setminus U$ and $y \in U$, the flow value of such an edge must be 0. Pictorially, if this is V This is U; here is the source node, here is the sink node, this is the residual graph, and this is the edge. So, y is reachable from S, and x is not reachable from S. Now, such edges must carry no flow. This must be the case; otherwise, there is a backward edge $yx \in G_f$.

If there is a positive flow from x to y edge, then by the definition of the residual graph, there will be a back edge from y to x. But again, this leads to a contradiction because y belongs to u, which means there is an s to y path. In G_f, and y to x is there in G_f. So, x must be in u, which contradicts our assumption that x belongs to $V \setminus U$. However, then since y belongs to u, and y comma x, this edge belongs to G_f, y also must—sorry, x also must belong to u. This contradicts our assumption that x belongs to $V \setminus U$. So, all the edges from the u side to the $V \setminus U$ side must carry the full amount of flow, and all the edges from the v side to the u side must carry no flow. Next, what we will show is that the flow value f, which is the sum of the flow values leaving s, must be the same as the sum of the flow values leaving u. In general, for any cut $(U, V \setminus U)$, the value of the flow f is the sum of the flow values leaving u minus the sum of the flow values entering u. But here, because all the incoming edges to u carry 0 units of flow, the sum of the flow values entering u is 0. So, this is so for this particular choice of u. Which is the set of vertices reachable from s, the value of s is actually the same as the sum of the flow values leaving u. So, to prove this, again consider the sum s equal to for every vertex $u \in U$, we sum up the total flow in U minus the total flow out of u. Now, let us see the value of this sum. So, the value of this if I look at from the vertex side, for every $h \in U$ except s, this sum is 0.

So, this is the value of f since the sum is 0 for every vertex u in capital U except the source vertex.

On the other hand, if I look at it from each side, this is the sum of the flows leaving you minus the sum of the flows entering you. Considering the contribution of every edge in S, S equals the total flow leaving U minus the total flow entering U. But as we have already seen, all the edges from $V \setminus U$ to U carry 0 units of flow.

So, the total flow entering U is 0, and the total flow leaving U—we have already argued that all the edges going from U to $V \setminus U$ carry the full amount of flow. So, this is the capacity of $(U, V \setminus U)$. So, what we got here is, on the one hand, S is the value of F; on the other hand, S is the capacity of the S-T cut $(U, V \setminus U)$. So, this proves 2 implies 3. Now, we will show 3 implies 1.We assume that the value of f is the capacity of the s-t cut $(U, V \setminus U)$. We need to show that f is a maximum flow. So, let us take any maximum flow and show that the value of that maximum flow is the value of f. Let f' be any maximum S-T flow.

So, again repeating this analysis, that means considering the sum s, but instead of using f, using f'. What we get is that the value of f' is the capacity of $(U, V \setminus U)$ minus because for f', we cannot say that the total flow entering u is 0, because to argue this, we needed that the flow f_u is the set of vertices reachable in G_f. So, for any other flow f' for this set of vertices u, we cannot claim that this is 0.

So, we have—let us have this term—minus the total flow f' entering u. But we have assumed that the capacity of $(U, V \setminus U)$ is the value of f. So, what we have is the value of f prime is the value of f minus the total flow f' entering u, which is a non-negative number. So, this implies that the value of f' is less than or equal to the value of f. But f' is a maximum s-t flow. So, this implies that the value of f' is the same as the value of f.

So, this proves the lemma and thereby proves the correctness of the Ford-Fulkerson algorithm, except for one point: that it always terminates when all the capacities are rational numbers. So, let us stop here. Thank you.