Approximation Algorithm

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Lecture 55

Lecture 55 : Approximation Algorithm for Multicut Cont.

Welcome. So, in the last class we have seen a randomized rounding based algorithm for multicut problem and at the end we stated the theorem that our algorithm is a $4\ln(k+1)$ factor approximation algorithm. So, today we will see the proof of that theorem along with the proof of the lemma that was used crucially in our algorithm. So, let us start. Multi cut problem.

So, the theorem that we prove the approximation ratio of our algorithm is at most $4\ln(k+1)$ proof. Let B_i be the set of vertices in the ball $B(s_i, r)$ chosen by the algorithm at iteration i ok. So, if it happens that in iteration i on s_i and t_i are already disconnected then we define B_i to be 0. If s_i and t_i are already disconnected in the beginning of iteration i then we define B_i to be the empty set and let F_i which is the boundary edges of B_i be the set of edges included in the solution F at iteration or i ok.

So, we have then so, if B_i is empty set F_i is also empty. So, if B_i is empty set then F_i is also the empty set. Then we have $F = \cup_{i=1}^{k} F_i$ ok. So, let V_i be the total volume of the edges when or removed sorry in iteration i. Recall in iteration i we are removing the boundary edges or we are removing all the vertices in the ball B_i along with the edges incident on it and let their total volume be V_i.

So, then we have then you see V_i is then greater than equal to we have $V(s_i, r)$ what was the $V(s_i, r)$ you recall we look at the ball of radius r around s_i all the edges whose both end points are in this ball they are volumes plus all these edges whose exactly one end point are there that part whatever is there within the ball of radius r that volume was also added in $V(s_i, r)$ plus $\frac{V^*}{k}$. So, if I remove $\frac{V^*}{k}$ then this is the volume of all edges that are in the ball of radius r B_i which are contained both end points are contained in B_i plus the partial edges the part of the edges whose which is contained in the ball. On other hand, V_i

takes the volume entire edge of this partial edge. For example, then this part belongs to V_i, but does not belong to $V(s_i, r)$. So, then V_i is greater than equal to $V(s_i, r)$ - $\frac{V^*}{k}$.

 So, this is fine and by the choice of r you recall by the choice of r from the lemma. cost of f i which are the cost of boundary edges of this ball is less than equal to $2\ln(k+1)$ $V(s_i, r)$ that is by the choice of r. Now, $V(s_i, r)$ is less than equal to V_i + $\frac{V^*}{k}$ ok. Now, what is then cost of F? This is $\sum c_e$ which is $\sum\sum c_e$, you see because in every iteration we are removing the vertices in the ball of radius r around s_i along with it incident edges, all these edges e in f can appear in at most 1 F_i c_e. So, this is the cost of F_i.

 So, this is less than equal to $2\ln(k+1)\sum\left(V_i+\frac{V^*}{k}\right)$. The first term is the sum of the volumes of all edges which is V^* as we have argued before this is less than equal to or equal to $2\ln(k+1)$. The first term V^* and the second term is also V^* which is then $4\ln(k+1)$ which concludes the proof $\ln(k+1)$ times opt because v star is LP-opt. So, let us write this is LP-opt which is less than equal to opt as usual $4\ln(k+1)$ times opt.

 So, this is a $4\ln(k+1)$ factor approximation algorithm. So, the only thing that we need to prove is the lemma. So, let us recall the lemma. for any s_i, $i\in[k]$ one can find in polynomial time. a radius r less than half such that cost of boundary edges of the ball of radius r around is less than equal to $2\ln(k+1)$ $V(s_i, r)$ which has an extra $\frac{V^*}{k}$ term also.

 So, this is the lemma that we need to prove. So, for simplicity let us write because this is a bit cumbersome notation to work with $C(r)$ is C of cost of boundary edges of the ball of radius r around s_i and $V(r)$ is $V(s_i, r)$. So, our proof uses a method called probabilistic method. What we will show that if we pick r uniformly randomly from the interval 0 to half, then the expected value of this cost $\frac{C(r)}{V(r)}$ is at most $2\ln(k+1)$. So, if this expected value is ah expected value of $\frac{C(r)}{V(r)}$ is at most $2\ln(k+1)$ that means, that there exists an r for which $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$. And then we will see how to compute such an r in polynomial time.

 So, we choose r uniformly randomly from $\left[0,\frac{1}{2}\right]$ it should be strictly less than $\frac{1}{2}$. We will show that the expected value of $\frac{C(r)}{V(r)}$ is at most $2\ln(k+1)$, this will show that there exists an r in $\left[0,\frac{1}{2}\right]$ such that $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$ ok. So, it So, $\frac{C(r)}{V(r)}$ are continuous random variable. So, we need to integrate over this range r in $\left[0,\frac{1}{2}\right]$.

So, let us see whether these functions are differentiable or continuous or what. So, for any value of r in $\left[0,\frac{1}{2}\right]$. such that $V(r)$ is differentiable, we have $\frac{d(V(r))}{d(C(r))}$. This follows from the simple fact that volume of a pipe the differential of the volume of pipe with respect to the length r is the cross sectional area ok. So, you see that this derivatives will change only where this $C(r)$ will change.

So, we observe that $V(r)$ is not differentiable exactly for the values of r such that the ball of radius r changes this is a set and that set changes that means, some vertex gets added if r increases that is there exists a vertex $v\in V$ such that distance of v and s_i is exactly r. So, at those r they are not differentiable this $V(r)$ are not differentiable otherwise it is differentiable. not only that $V(r)$ may not be even continuous or it is it continuous at every year need not be again check that at those are exactly at those are where this ball changes at those are $V(r)$ may not even be continuous $V(r)$ may not be may not even be continuous at r such that. there exists a vertex $v\in V$ such that distance from s_i to v is r.

and where is continuous and differentiable at every other r is both continuous and differentiable at other r ok. So, now, we so, this $V(r)$ function is piecewise continuous and differentiable and these functions we know how to integrate. So, we sort the vertices in $B\left(s_i,\frac{1}{2}\right)$ based on their distances from s_i that is suppose the vertices are $v_0,v_1,\ldots,v_{l-1}$ with distances from being r_0 which is 0.

let us define r_l to be half. So, the expected value of $\frac{C(r)}{V(r)}$ is The probability density function is 1 by the length of the interval which is $\frac{1}{2}\sum\int \square \frac{C(r)}{V(r)}$ dr ok. Now, what is the relationship between $C(r)$ and $V(r)$ that is what we have here that $C(r)=\frac{d(V(r))}{dr}$. So, that we put there $C(r)=\frac{d(V(r))}{dr}$.

So, we get $\frac{d(V(r))}{dr}$. Now, notice that $V(r)$ is a non-decreasing function since $V(r)$ is non-decreasing is expectation of $\frac{C(r)}{V(r)}$.

Now, this can be bounded by less than equal to $2\ln(k+1)$.

So, you take this as a homework you just compute what is $V(0)$ and then use elementary calculus to bound this quantity as $2\ln(k+1)$. So, this is $2\ln(k+1)$. So, this shows that there exist and this shows that expectation of $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$, which in particular shows that there exists an r in $\left[0,\frac{1}{2}\right]$ $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$.

Now, you observe this equality $\frac{C(r)}{V(r)}$ as r increases.

So, observe that for $r\in[r_j,r_{j+1}]$ minus $C(r)$ which is the cross sectional area of the pipes remain constant. while $V(r)$ is non decreasing. So, if there exists an r in the interval $[r_j,r_{j+1}]$ such that $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$, then this holds for r_{j+1} also.

Therefore, there exists an $j\in[0,l-1]$ such that this is less than equal to $2\ln(k+1)$, which in turn implies that $\frac{C(r_{j+1})}{V(r_{j+1})}$ is less than equal to $2\ln(k+1)$.

So, our algorithm it is enough to check for this l values and l is at most n the number of vertices. So, we can so, this theorem show proves that there exist at least one such values among this l values which such that $\frac{C(r)}{V(r)}$ is less than equal to $2\ln(k+1)$. Hence, we can check in polynomial time check and find a value of r in polynomial time such that $\frac{C(r)}{V(r)}$ is at most $2\ln(k+1)$ which proves the lemma ok. So, let us stop here. Thank you.