Approximation Algorithm

Prof. Palash Dey

Department of Computer Science and Engineering

Indian Institute of Technology, Kharagpur

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Lecture 33

Lecture 33 : Derandomization using Method of Conditional Expectation

Welcome, in the last lecture we have seen how a simple randomized algorithm can give a half factor approximation for MAX-SAT and MAX-CUT problem. So, in today's lecture we will see a powerful technique for de-randomizing those randomized algorithms and which is quite successful. in derandomizing various randomized algorithms this method is called method of conditional expectation. So, today's topic is method of conditional expectation. using method of conditional expectation. So, often it is desirable to have a deterministic algorithm for some problem and derandomization is a umbrella term used to use the idea of a randomized algorithm to design a deterministic algorithm.

So, it is not formally defined again. So, let us see how we de randomize the randomized algorithm for MAX-SAT. So, let us recall what was the randomized algorithm for MAX-SAT, we simply set each Boolean variable to true or false with equal probability. And let us recall how does the analysis work, we had alg a random variable which is the sum of the indicator random variables $Z_1 + \ldots + Z_m$ where Z_j is a indicator random variable for the event that j-th clause is satisfied by the random assignment.

And then what we showed here is that expectation of ALG is greater than equal to $\frac{m}{2}$. and m is in turn greater than equal to opt. So, this is greater than equal to $\frac{opt}{2}$. Now, what the measure of conditional expectation does? It writes expectation of ALG in terms of conditional expectations. So, let us recall.

So, we had variables of the input formula, we had n variables suppose they are $x_1, x_2, \ldots, x_n$ and what we write we set each of the variable to true and false with equal probability. So, I want to write expectation of ALG as two conditional expectations. So, ALG you observe that it is a function of how I set these variables $x_1, x_2, \ldots, x_n$. So, this expectation of ALG can be written as expectation of ALG given x_1 is set to true times probability that x_1 is set to true plus expectation of ALG given x_1 is set to probability that x_1 is set to false. Now, we set each variable to true and false with equal probability that means, probability that x_1 is set to true is half and probability that x_1 is set to false is also half.

So, this half of expectation of ALG by given x_1 set to true plus half times expectation of ALG x_1 is set to false ok. So, observe that ALG is a convex combination of these two expectation of ALG is a convex combination of these two conditional expectations is a convex combination of expectation of ALG given x_1 is set to true and expectation of ALG given x_1 is set to false. Now, a convex combination this value is at most the maximum of each term. So, let us write first what do you mean by convex combination? Convex combination of say n numbers $a_1,\ldots,a_n$ is $\alpha_1 a_1+\alpha_2 a_2+...+\alpha_n a_n$, where each α_i are non-negative and they sum to 1 ok.

So, here you see the αs are half and half we have two terms α_1 is half α_2 is half both is both are greater than equal to 0 and they sum to 1 that is the convex combination. So, here is an observation which you prove it as a homework that $\alpha_1 a_1+\alpha_2 a_2+...+\alpha_n a_n$ this is less than equal to $max\{a_1,\ldots,a_n\}$ also $\alpha_1 a_1+\alpha_2 a_2+...+\alpha_n a_n$ this is greater than equal to $min\{a_1,\ldots,a_n\}$ ok. So, proof you do as a homework ok. So, now, we use this first one. So, from first one what we can conclude if I apply this first one to expectation of alg this is a convex combination of expectation of ALG given x_1 is set to true and expectation of ALG given x_1 is set to false.

So, we can write expectation of ALG is less than equal to expectation of ALG given x_1 set to true, max of this max of expectation of ALG given x_1 set to true and expectation of ALG given x_1 set to false. So, here is a simple algorithm in the first iteration I will compute these two quantities. So, compute expectation of ALG given x_1 set to true. and expectation of ALG given $x_1$set to false, set x_1 to true. if expectation of alg given x 1 set to true is greater than equal to expectation of ALG given x_1 set to false.

ok otherwise set x_1 to false. So, in the first iteration I compute these two conditional expectations and whichever setting of x_1 gives the more conditional expectation I set x_1 accordingly. This way in general at the beginning of i-th iteration we have set $x_1,\ldots,x_{i-1}$ and in this iteration I will set x_i. So, again I use conditional expectation of ALG.

we have already set x_1 to some values whichever gives higher conditional expectation to x_{i-1}. This again I can write it as half times expectation of given $x_1,\ldots,x_i$ these are already set in the earlier iterations and x_i I said to true plus half times expectation of ALG given $x_1,\dots,x_{i-1}$ x_i set to false. And again I see that this conditional expectation expectation of ALG given $x_1,\dots,x_{i-1}$ this again a convex combination of these two conditional expectations I compute them and set x_i accordingly. So, after n iterations we have set all n variables ok. And what we have ensured is that we have ensured that expectation of ALG which we know this is greater than equal to $\frac{opt}{2}$ this is less than equal to expectation of ALG given x_1 whatever we set this is again less than equal to expectation of ALG given x_1, x_2.

and so on less than equal to expectation of ALG given $x_1,\dots,x_n$ ok which is actually let us call it $ALG^{'}$. $ALG^{'}$ is the number of clauses satisfied by the deterministic algorithm . So, this finishes the description of this de-randomized algorithm except we have not explained how I can compute this conditional expectations. So, that I will explain now and that is very important this framework you can use in any general randomized algorithms. If you can compute this conditional expectations efficiently deterministically then everything this whole algorithm is a deterministic algorithm.

So, next I discuss computing expectation of ALG given $x_1,\dots,x_i$ some setup some setting of this variables $x_1,\dots,x_i$ to true and false. How do I compute this? So, let us recall we have written $ALG = Z_1 + Z_2 + \ldots + Z_m$. So, expectation of ALG given $x_1,\dots,x_i$ is $E[Z_1|x_1,\dots,x_n] + \ldots + E[Z_m|x_1,\dots,x_i]$. So, I have simply taken conditional expectation on both side conditioned upon $x_1,\dots,x_i$. Now, again I apply method of linearity of expectation which works for conditional expectations also.

So, this is $E[Z_1|x_1,\dots,x_n] + \ldots + E[Z_m|x_1,\dots,x_i]$. So, all I need to explain how I can compute conditional expectation of Z_j given $x_1,\dots,x_i$ for any arbitrary $j \in [m]$. So, $j \in [m]$. So, how do I compute expectation of Z_j given $x_1,\dots,x_i$. Again recall that Z_j is a indicator random variable expectation of an indicator random variable may be conditional expectation is the conditional probability of that event.

This is probability that C_j is satisfied given $x_1,\dots,x_i$. That means, I am given a setup of $x_1,\dots,x_i$ and the rest of the variables $x_{i+1},\dots,x_n$ each of them is set to true or false with equal probability. So, what is this probability this is easy to compute. So, let if the assignment of $x_1,\dots,x_i$ satisfies C_j irrespective of how $x_{i+1},\dots,x_n$ are set, then probability of C_j is satisfied given $x_1,\dots,x_i$ is 1. Otherwise let us assume that the setup of $x_1,\dots,x_i$ does not already satisfy C_j.

Otherwise let us assume that the assignment of $x_1,\ldots,x_i$ does not already satisfy C_j. So, again ask what is the number of variables appearing in C_j among $x_{i+1},\ldots,x_n$. Let $l_j^{'}$ be the number of literals in C_j except $x_1,\ldots,x_i$ then it could be 0 also. So, then probability that C j is satisfied given $x_1,\ldots,x_i$ you see there are $l_j^{'}$ many variables literals among $x_{i+1},\ldots,x_n$ or their negations that are there in C_j and only one setup make $C_j$not satisfied. So, this probability is $1-(\frac{1}{2})^{l_j^{'}}$ ok.

So, in particular if $l_j^{'}$ is 0, so this probability is 0. So, we can compute for every C_j this probability and hence we can compute this expectation of Z_j's, hence we can compute expectation of ALG given $x_1,\ldots,x_i$ efficiently, hence we can compute expectation of ALG given $x_1,\ldots,x_i$ in deterministic polynomial time. And that is the only thing we require once we can compute this conditional expectations I can compare these various conditional expectations and set $x_1,\ldots,x_n$ accordingly and hence the number of clauses satisfies is at least $\frac{opt}{2}$. So, you see how the same idea we used conditional expectation method to de-randomize it to get the same approximation guarantee. So, you take it as a homework use method of conditional expectation to derandomize the half factor randomized approximation algorithms for weighted MAX-SAT and weighted max cut ok.

The same technique you can use to de randomize them only thing you need to show is that how you can compute the conditional expectations for these problems. And again if you analyze these terms write ALG as sum of individual indicator random variables and and it and analyze those conditional expectations of those individual indicator random variables and again similarly you can derandomize them without losing any approximation factor there will be half factor approximation algorithm. So, it turns out that many of the algorithms of this course can be de-randomized using this method of conditional expectations. So, whenever it is the case we will explicitly mention it and you can take it as a homework for to apply method of conditional expectation to de-randomize those randomized algorithms ok. So, let us stop here.