Algorithmic Game Theory
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Lecture - 10
MSNEs of Matrix Games

Ok. So, in the last lecture, we have seen that in two-player zero-sum game, there exists a mixed strategy in Nash equilibrium. We have proved it. It is constructive we can write the two linear programs and solve it. So, this not only shows the existence of Nash equilibrium, it also shows that we can compute them in polynomial time. And we have also shown that the utility of both the players are exactly their security values.

So, in this lecture also, we will continue our investigation of two-player zero-sum game.

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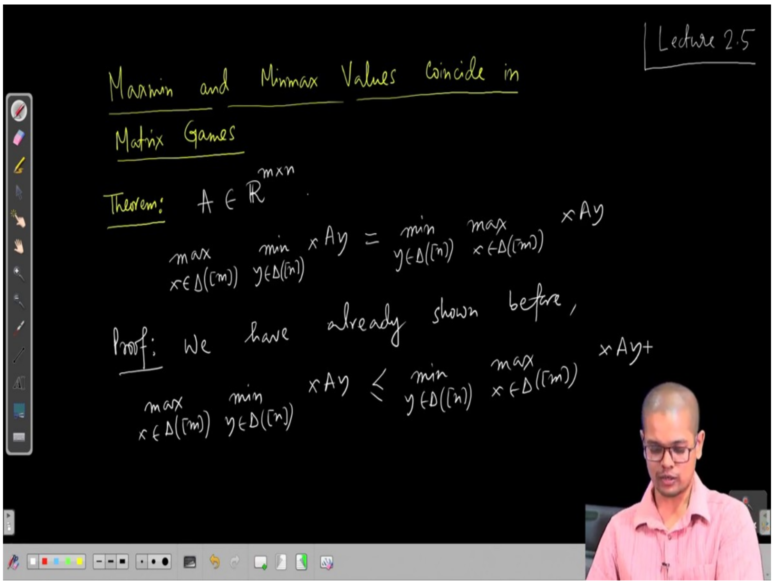

So, we begin with showing an important result which says that the maxmin value and minmax value coincides, maxmin and minmax values coincide in matrix games. So, what is the statement? So, theorem, again as usual I have a real matrix $A \in R^{m \times n}$, then the maximum value; that means, if I take max, max is always over the rows, and min should be always over the columns; $max_{x \in \Delta([m])} min_{y \in \Delta([n])} x\, A\, y$ is same as if I take the min first and then max these two are same, ok.

Proof: So, we have already shown that maxmin value is less than equal to minmax value. So, we have already shown before, we have already shown before that max, min, maxmin value is less than equal to min, max. This we have already shown. So, you only need to show the other inequality.

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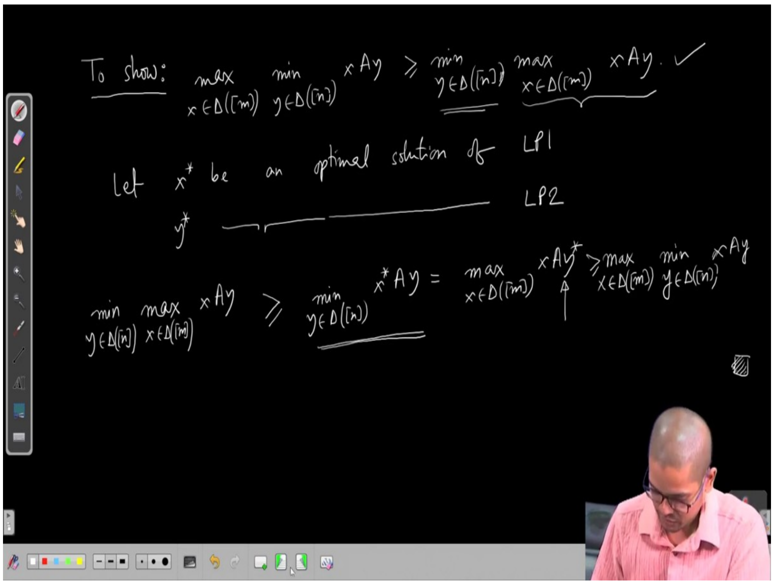

So, to show what? That $max_{x\in\Delta([m])} min_{y\in\Delta([n])} x\,A\,y$ is greater than equal to $min_{y\in\Delta([n])} max_{x\in\Delta([m])} x\,A\,y$ ok. So, towards that let as usual let x^* be an optimal solution of LP 1 and y^* be an optimal solution of LP 2, ok. So, what we have? We have already shown that $min_{y\in\Delta([n])} x^*\,A\,y$ is equal to $max_{x\in\Delta([m])} x\,A\,y^*$.

Now, you see that this term minimum over y in $\Delta([n])$ sort of matches with this part. So, what we do is that, we replace x^* with this maximum thing and what we get is that if we replace x^* with a maximum then we get this; $min_{y\in\Delta([n])} max_{x\in\Delta([m])} x\,A\,y$. Simply because we are replacing a particular x value, which is x^* with the maximizer.

Similarly, we replace y^* with the minimizer. So, this is greater than equal to $max_{x\in\Delta([m])} min_{y\in\Delta([n])} x\,A\,y$. So, this finishes the proof. So, this shows this equal inequality and we have already shown the other inequality and so, this finishes the proof. Our next important result for two-person zero-sum game is this.

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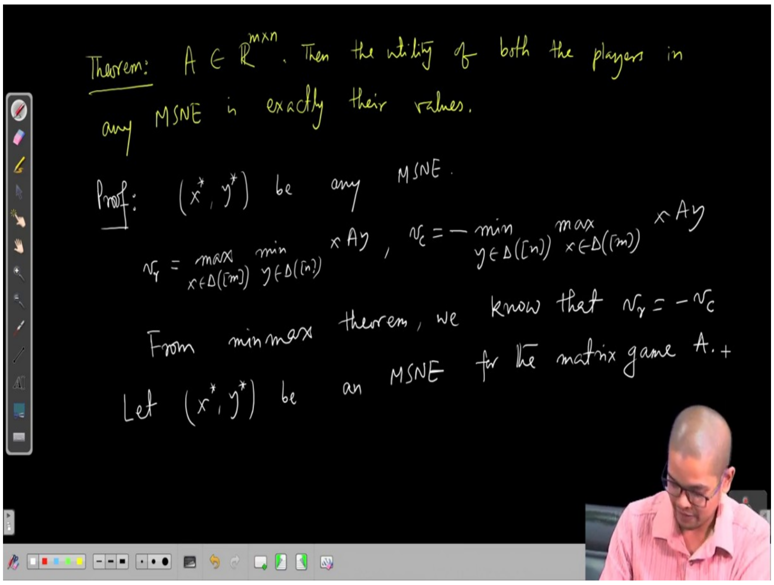

So, again as usual let $A \in R^{m \times n}$. So, before we have, we picked particular mixed strategy Nash equilibrium x^* and y^* in particular, where x^* is a solution for LP 1 and y star is a solution for LP 2. And we showed that for that particular mixed strategy Nash equilibrium both the players enjoy their security level. What we show now is a more general theorem.

So, then the utility of both the players in any MSNE, not necessarily a solution for LP 1 and LP 2 then, the utility of both the players in any MSNE is exactly their values. Proof: So, in this sense, this is this general this works for any mixed strategy Nash equilibrium. So, to begin with let x star comma y star be any mixed strategy Nash equilibrium, ok.

And what is let us write what is the security levels of row player. Let us write it as v_r. So, $v_r = max_{x \in \Delta([m])} min_{y \in \Delta([n])} x\,A\,y$. This is the security level of the row player and the security level of the column player v_c is minimum, there should be a minus before it, $v_c = -min_{y \in \Delta([n])} max_{x \in \Delta([m])} x\,A\,y$.

Why minus simply because the utility matrix of the column player is -A, ok. And from minmax theorem, we know that $v_r = -v_c$ because maxmin value is equal to minmax value, ok. So, let (x^*, y^*) be an MSNE, ok. Let us this be an MSNE for the matrix game A. So, let us see what we have.

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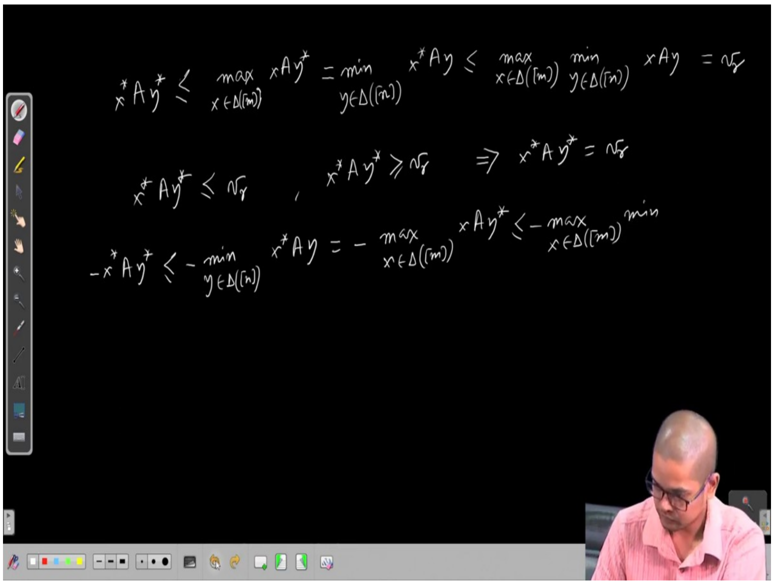

So, what is $x^* Ay^*$? This is the utility that the row player gets. This is less than equal to, instead of, if we replace x^* with $max\, x\, A\, y^*$ because we are reply we are maximizing over x, I get $x^* Ay^*$ less than equal to this.

Now, let us apply minmax theorem, this is equal to $min_{y\in\Delta([m])} x^* A\, y$ and this is less than equal to max. If we max, instead of x^* if we maximize over all x, so max $x\in\Delta([m])$, sorry y should be in over $min_{y\in\Delta([n])} x\, A\, y$. This is nothing but v_r. So, what we get is $x^* Ay^* \leq v_r$.

And, so the utility of row player is at most v_r and of course, we know that $x^* Ay^* \geq v_r$ because if the utility of row player in a MSNE is strictly less than v_r, then player row player can deviate to a strategy which guarantees its value. So, so it will not be an MSNE. So, from these two we can conclude that $x^* Ay^* = v_r$, ok.

Similarly, for the column player you can write $-x^* Ay^*$. This is the utility of the column player. This is less than equal to minus minimum. If I vary y, instead of taking y^* if I vary y and take the minimum, so $min_{y\in\Delta([n])} x^* A\, y$, now applying minmax theorem. This is $-max_{x\in\Delta([m])} x\, A\, y^*$, this is less than equal to instead of taking y^* we will minimize it. And because of the minus sign we get less than equal to, so $min_{x\in\Delta([m])}$

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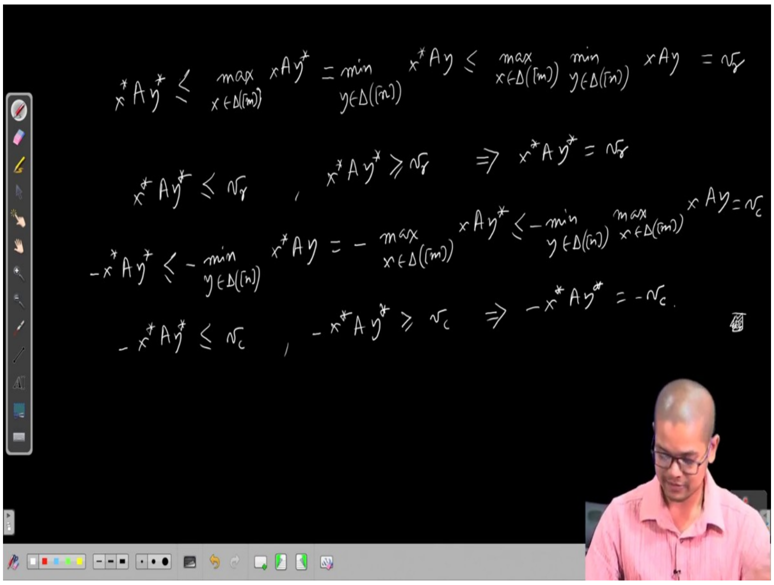

We minimize this over $min_{y\in\Delta([n])} max_{x\in[m]} x\,A\,y$ which is nothing but v_c. So, what we get is the utility of column player $-x^*\,Ay^*$ is less than equal to v_c. But of course, the utility of column player must be at least v_c because it is a MSNE. $x^*\,Ay^*$r must be greater than equal to v_c. So, from these two we can conclude that $-x^*\,Ay^* = -v_c$.

So, this concludes the proof. So, let us do some; let us do some examples. Let us do take some concrete examples of two-player zero-sum game and let us see these concepts.

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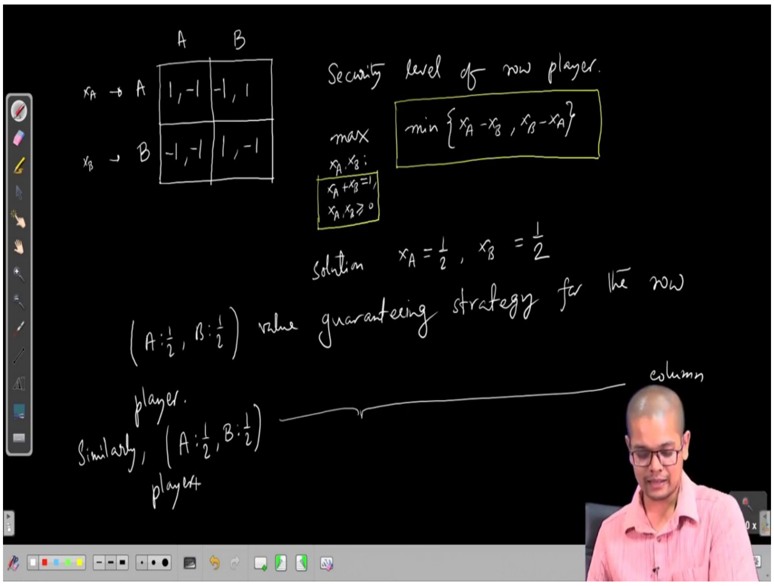

So, let us take the say the matching pennies example A B, A B. Now, if there is a match then row player wins and gets a utility of 1, if there is unmatched mismatch then row player loses. And if row player loses column player wins and vice versa. So, let us find out what is the security value, security level of row player. So, this is max over all mixed strategies.

So, the suppose this is x_A and this is x_B, so overall (x_A, x_B), such that $x_A+x_B=1$ and both are greater than equal to 0. That is what we mean by iterating over all possible mixed strategies of the row player. And for this row mixed strategy what is the utility that column player. So, if column player plays A, the utility is x, utility of row player when column player plays A is x_A-x_B. And when column player plays B, it is x_B-x_A.

So, this particular term minimum of x_A-x_B we want to maximize over this conditions that summation $x_A+x_B=1$ and $x_A, x_B \geq 0$. So, in this case, what we get is, what we the only solution is $x_A=\frac{1}{2}$ and $x_B=\frac{1}{2}$.

So, this shows that; so, this A with $\frac{1}{2}$ and B with probability $\frac{1}{2}$, this is a security or write this way value guaranteeing strategy for the row player. Similarly, symmetrically also you can see that for column player also playing A with probability $\frac{1}{2}$ and playing B with probability $\frac{1}{2}$ is a value guaranteeing strategy for the column player.

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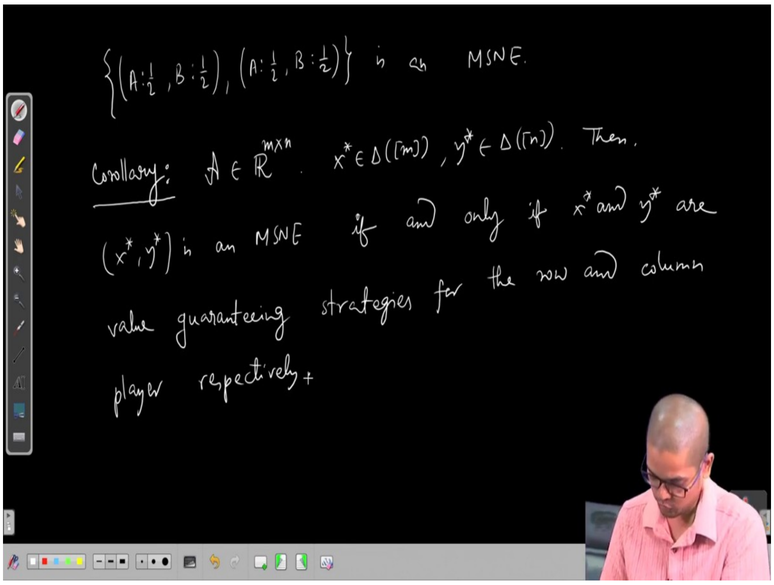

And hence these two strategies form a mixed strategy Nash equilibrium, A with probability half and B with probability half. This is a mixed strategy Nash equilibrium. And to prove uniqueness you know this is an observation or corollary from what we have got. So, let $A \in R^{m \times n}$.

Now, and x^* is a mixed strategy for row player and y^* is a mixed strategy of column player, then the following holds. Then (x^*, y^*) is an MSNE if and only if, if and only if x^* and y^* are value guaranteeing strategy, strategies for the row and column player, respectively. And we can give very interesting or nice cases arising out of it.

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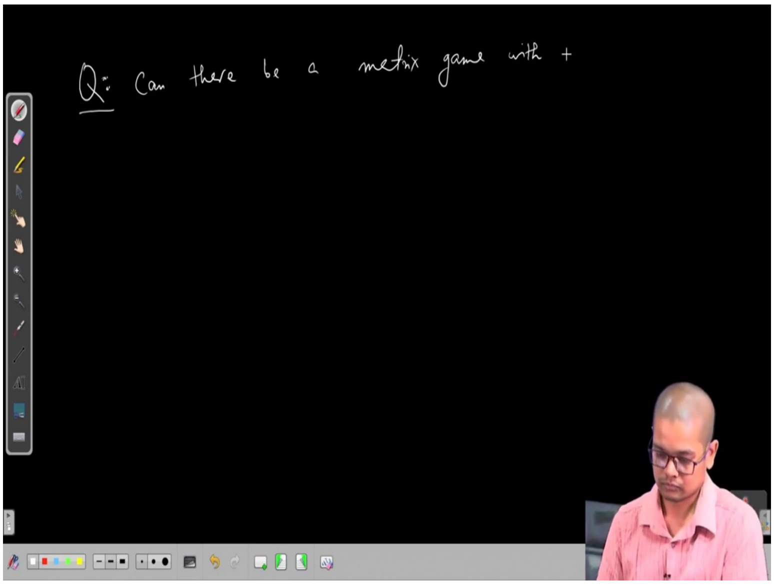

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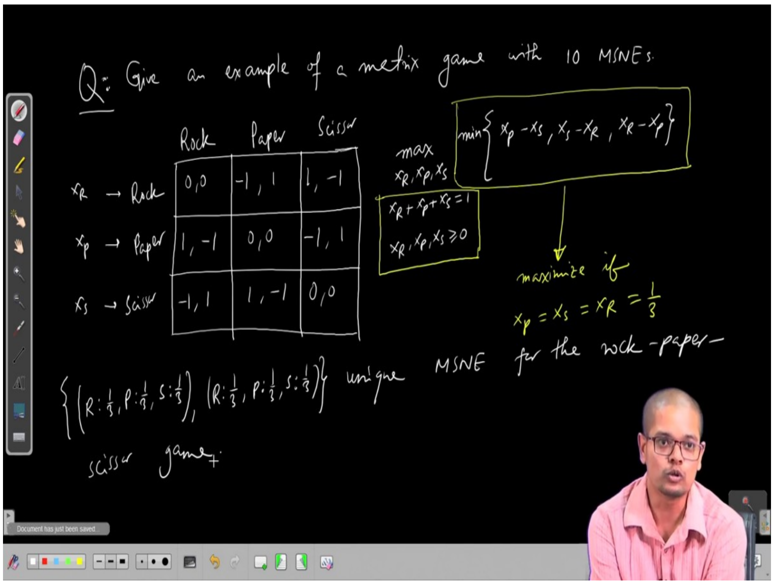

For example so, there is a question. Can there be a matrix game or give an example of a matrix game with I do not know, with say 7 or with 10, with exactly 10 MSNEs. So, how to proceed it?

So, one way to is that you know you can try to ensure that there are exactly say two value maximizing strategies, mixed strategies for column player and 5 for row players, or maybe 10 for 10 value maximizing strategies for row player and 1 for column player and so on. So, this way you can combine.

So, similarly you can; so, last class we also seen the example of a rock-paper-scissor. So, let us discuss that in last week; so, rock, paper, scissor. Diagonals are 0 0. Between rock and paper, rock loses. Between rock and scissor, rock wins. And between paper and scissor, paper loses, this -1, 1, 1 , -1, -1 ,1 this will be symmetrically opposite so, here also.

So, let us compute the value maximizing strategy, value guaranteeing strategy for the row player. So, suppose the individual probabilities are say x_R, x_P, x_S. So, what is the value? Of the row player max x_R, x_P, x_S such that $x_R + x_P + x_S = 1$, and x_R, x_P, x_S this is all, these are all greater than equal to 0.

And when you take the minimum of, if the column player plays rock, then the utility of the row player is $x_P - x_S$. If the column player plays paper, then the utility of row player is $x_S - x_R$. And if the column player plays scissor then the utility of row player is $x_R - x_P$.

So, again we want to minimize, sorry we want to maximize this minimum of these 3 quantities subject to these conditions, that this sum must be 1, and so it will be minimized if. So, this minimum will be maximized if all 3 are same, $x_P = x_S = x_R$. In that case, it will be 0, if these 3 are not same, then then this minimum will be strictly negative. And so, the only solution that we have is these are equal to one-third.

And so, we can see that there is a unique strategy for the row player to achieve its value. Similarly, there is a unique strategy for column player to achieve its value, and hence this rock with probability one-third, paper with probability one-third, and scissor with probability one-third. This is for row player this mixed strategy. Same with column player, rock with probability one-third, paper with probability one-third, and scissor with probability one-third.

This is a unique mixed strategy Nash equilibria for the rock-paper-scissor game. Why unique if it is not unique then there will exist more than 1 value maximizing strategy for either row player or column player which is not true. So, this concludes the proof, ok. So, in the next class, we will see some interesting applications of this matrix games in terms of Yao's lemma and proving lower bounds for algorithmic results, ok.

Thank you.