

**Computer Vision**  
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**Lecture - 42**  
**Range Image Processing – Part II**

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**Parametric curve (2D)**

- Parametric curve:  $X: I \subset \mathbb{R} \rightarrow \mathbb{R}^2$   
 $X(t) = (u(t), v(t))$       $|X'| = \sqrt{(u'(t))^2 + (v'(t))^2}$
- Tangent at a point:  $T(t) = (u'(t), v'(t))$
- Curvature at a point:  $k(t) = \frac{|X' \times X''|}{|X'|^3} \Rightarrow \frac{\begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \\ \frac{\partial^2 u}{\partial t^2} & \frac{\partial^2 v}{\partial t^2} \end{vmatrix}}{\sqrt{(u'(t))^2 + (v'(t))^2}^3}$

We are discussing about Range Image Analysis, in the last lecture we discussed different mechanisms for sensing range images. In this lecture we will be discussing different concepts of differential geometry which will be useful in processing range data. So, let me first explain how parametric curves in 2D they are represented mathematically. Here you consider a particular parametric parameter which is which is a subset of for the real space, the whose values know lie in an interval of real space. And you are mapping these parameter value to a coordinate of a of a two dimensional space.

So, that is why it is a mapping  $\mathbb{R} \rightarrow \mathbb{R}^2$  and in parametric form we can represent this parameter say  $t$  which is a variable, which takes values from an interval of real space. For example, from interval between 0 to 1 the parameter values may vary and for each parameter value you get a co ordinate within that interval. You get a two dimensional co ordinate which is denoted here as  $X(t)$  as a point and its x co-ordinate as  $u(t)$ . It is a function of that parameter and  $v(t)$  that is a y co-ordinate which is also a function of this parameter.

So, this is a parametric description a very simple way to describe a curve as a set of points and it is a continuous curve with the parameter varies continuously and this functions are also continuous over t. So, this curve also a continuous curve can be represented in this fashion. Now when we compute a tangent because of the continuity you come compute the derivatives at those points, if it is continuous of R at least of order one it could be continuous of you in order two also or higher orders. So, at least when it is it is continuous curve of order one then you can compute the tangent and which means mathematically you can very easily get this tangent information by the tangents at those locations also denoted by the corresponding derivatives of those parametric curves, which is giving you the tangent information. Which is giving the vector actually it is giving the directions of the vector with respect to that point and you can also compute the curvature at that point by performing this computations. What you need to do you need to compute the first derivative and second derivative of the parametric curve and then perform this operations.

You can see that numerator is nothing but the determinant of this quantity derived from first derivative and second derivative of the parametric curves , which is shown here as a

determinant value . That is in this case you can see that this 
$$\begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \\ \frac{\partial^2 u}{\partial t^2} & \frac{\partial^2 v}{\partial t^2} \end{vmatrix}$$
 that is that is

giving you the corresponding value. So, given a particular value of t you can get a scalar value here, all the scalar values are obtained and then you perform the determinant of this matrix. And this is the magnitude of the tangent vector.

So, if your tangent vector is  $(u'(t), v'(t))$  that is shown here. So, this magnitude should be  $(u'(t)^2 + v'(t)^2)$  .It is the magnitude of  $X'$  and when you are performing when you are raising it to the power three cube of this amount  $((\sqrt{u'(t)^2 + v'(t)^2})^3)$  . So, this is the quantity you are measuring. So, this is how the curvature could be computed at any parameter value

$$(|X'|)^3 = (\sqrt{u'(t)^2 + v'(t)^2})^3$$

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**Parametric curve (2D)**

- Parametric curve:  $X: I \subset \mathbb{R} \rightarrow \mathbb{R}^2$   

$$X(t) = (u(t), v(t))$$
- Tangent at a point:  $T(t) = (u'(t), v'(t))$
- Curvature at a point:  $k(t) = \frac{|X' \times X''|}{|X'|^3}$   
 Determinant in 2D:  $\begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \\ \frac{\partial^2 u}{\partial t^2} & \frac{\partial^2 v}{\partial t^2} \end{vmatrix}$

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**Parametric curve (3D)**

- Parametric curve:  $X: I \subset \mathbb{R} \rightarrow \mathbb{R}^3$   

$$X(t) = (u(t), v(t), w(t))^T$$
- Tangent at a point:  $T(t) = (u'(t), v'(t), w'(t))^T$
- Curvature at a point:  $k(t) = \frac{|X' \times X''|}{|X'|^3}$   
 Cross Product of 3-vectors

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So, that is just simply explaining that part. Now, let me consider the parametric representation curve presentation is in three dimension. So, like two dimension you can see all this notations are simply extended in this particular context, where the parametric curve representation is a mapping from a one dimensional real space to a three dimensional real space  $\mathbb{R} \rightarrow \mathbb{R}^3$ .

So, again the parameter values varies within an interval of real space  $I$  and you have an additional co-ordinate that in the z which has been shown as a function of w(t) here, in

addition to  $u$  and  $v$  functions for corresponding  $x$  and  $y$  coordinates. Similarly, the tangent which is computed in 2D by taking the derivatives of respective functions that also you do it here. And for curvatures also you see that the expression remains the same only its interpretation is a bit different. You have the cross product of the vectors  $X'$  and  $X''$ .

$$k(t) = \frac{|X' \times X''|}{|X'|^3}$$

So, this is cross product of three vectors you take the double derivatives which means in

this case you are taking  $\begin{bmatrix} u'(t) \\ v'(t) \\ w'(t) \end{bmatrix}$ . So, that is the vector notation, so I am using column

vector notations in this lecture slide everything is shown as row vector, but throughout I have used column vector notation  $(u'(t), v'(t), w'(t))^T$ . So, you should consider you know in some cases I will be using column vector notations for doing the exercises. So, let us consider this is this vector this is what is  $X'(t)$  and  $X''(t)$  which is given as the double derivative with of this function.

$$X'(t) = \begin{bmatrix} u'(t) \\ v'(t) \\ w'(t) \end{bmatrix} \quad X''(t) = \begin{bmatrix} u''(t) \\ v''(t) \\ w''(t) \end{bmatrix}$$

So, given a value of  $t$  you can get these three vectors and then you perform the cross product that is the  $|X'|^3$  interpretation. And similarly  $X'$  cube magnitude we have shown in the two dimensional case you have to extend that concept of these vectors. So, take the magnitude of these vectors and raise it to the power 3 that would give you the denominator that is how this curvature is completed.

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$X(t) = (u(t), v(t), w(t))$   
 $T(t) = (u'(t), v'(t), w'(t))$   
**Parametric curve (3D)**  
 $k(t) = \frac{|X' \times X''|}{|X'|^3}$

So, now we will be discussing about the parametric curve which is lying on a surface. So, in the previous case we have considered any parametric curve ah, but with respect to surface let us discuss as such if I look it in the curve in an independent fashion our previous analysis or previous representations all are valid. We represent it as a parametric curve representation of three functions of parameters for x y z coordinate given as  $u(t), v(t), w(t)$ .

Similarly its tangents are given by their first derivatives and curvatures are computed at every point by computing the cross product of these two terms  $X'$ ,  $X''$  taking the cross product and the magnitude of the cross product. So, that I missed in the previous you know slide.

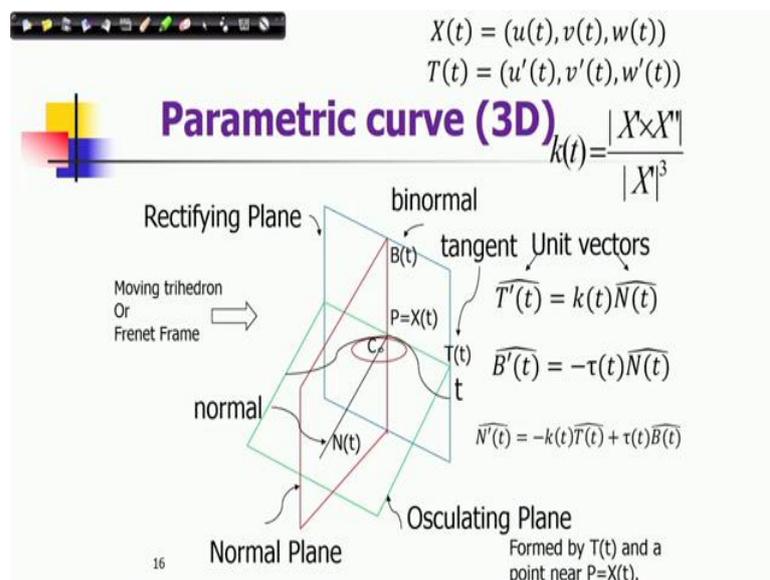
So, actually it is after completing cross product you have to take the magnitude on the cross product and then in the denominator you compute the tangent by taking the first derivative and take its magnitude and raise it to power three. So, if I divide these two terms I will get the curvature, so that is what we discussed.

So now, let us see with respect to surface what other kinds of concepts are there. Consider a point at a particular point of a curve and consider the tangent directions. So, this is the tangent directions at that point say which is given by  $T(t)$  in our notation and consider very near about points.

So, this is a point which is very close infinite infinitesimally closed point on the curve. So, this tangent and this curve this point will form a plane and this is this plane ok. So, it is a kind of plane where the tangent is lying also the curve is lying. So, it is a tangential plane kind of thing and which is called actually osculating plane. We will see the concepts names nomenclatures later on first let me define this concepts.

First so in that plane where the tangent is lying and also curve is laying at that particular point you can define the normal which is lying on that plane. So, normal which is perpendicular to this tangent and which is lying on that plane.

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So, this is the normal of the curve, so this point is p and this is the tangent what I mentioned and this is the normal. You can also consider another plane which is perpendicular to the tangent plane, actually name of the tangent plane is osculating plane as I mentioned and this is normal as I have shown here. And this you know this red colored plane is considered as a perpendicular plane of the osculating plane and this plane is called normal plane we will come to that. And so the curvature at that point is shown here by this quantity that it is basically radius of circle is giving you inverse of the curvature.

So, centre of that circle which forms at that point due to curvature has been shown and it should lie on the normal. And this plane is a normal plane as I was mentioning that normal plane is the perpendicular plane to the osculating plane. So, you have these two

planes now you can have another plane also which is perpendicular to both of these planes and that is called Rectifying plane. So, these are the three planes which are defined with respect to a point P by considering its direction of tangent and also the osculating plane and so normal of the curve at that point.

So, the bi normal is a direction which is perpendicular to normal and tangent of the curve, so that is binormal. So, binormal lies in the rectifying plane and also in the normal plane, so it is an intersection of normal and rectifying plane. So, you can see as if at every point you have defined a co-ordinate system, that means you have three axes which are perpendicular and it is a local co-ordinate system.

So, this is called the moving trihedron or Frenet frame. So, this particular configuration and the relationships between the tangent and normal's in the differential geometric operations those can be expressed. Here all are unit vectors and if you take the derivative of tangent you will get the curvature, you will get the normal vector, unit normal vector which is magnitude which was magnitude not unit normal vector. It is a vector along the normal of the curve with the magnitude's curvature. Similarly if I take the derivative of the bi normal's.

And then the vector what you get once again it is as along the normal of the curve, but its magnitude will be modified by a quantity called  $-\tau$  which is called torsion of this curve. And the derivative of normal is related with the directions of tangent and binormal in a linear form from you can see that this is lying in the plane of rectifying plane particularly this direction.

So, it is some directions in the rectifying plane in a so that is expressed. So, these are the three I should say very fundamental relationships between the change of the directions of tangent binormal and normal with respect to its curvature and torsions of the curve.

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**Parametric surfaces**

- Parametric surface:  $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$X(u,v) = (x(u,v), y(u,v), z(u,v))$$

**Surface Normal:**

$$\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|}$$

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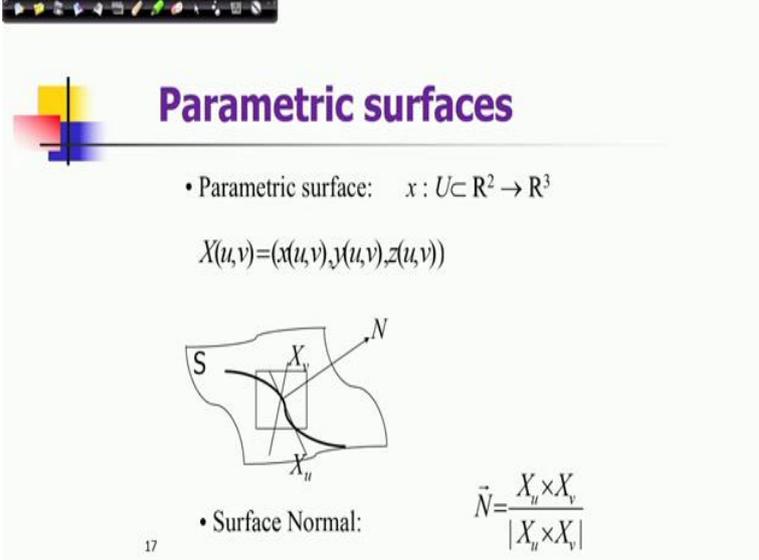
So, we discussed about the representation of parametric curve and also we have discussed some concepts several concepts on normal osculating plane, normal plane, rectifying plane, normal, binormal curvatures and torsion of the curve. So, let us now consider a representation of surfaces, because range data as you understand mostly it is a surface data and of course a curve would lie on a surface and also you can try to treat the surface in totality around your point. So, for parametric surface representation once again here your required two parameters.

So, it is surface is a two dimensional entity. So, it is a mapping from a two dimensional real space to a three dimensional real space ( $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ ). Once again the parameters may vary with in certain finite ranges. So, you can consider mapping from a subset of real space to the three dimensional real space. surface co ordinate is represented by the x y z coordinate and each one is a function of these two parameters u and v.

So, let us consider a curve which is lying on the surface and a at a particular point I can get a directions we have shown here. Say suppose this is a point p and if I considered a curve which is varying over u only, whereas v is kept constant. This is how a curve could be defined when your one parameter is varying, other is constant, then it gets a single parametric description of points and then it becomes a curve on the surface. And you consider another curve on the same point passing through that same point, where it varies over v and u is constant. So, for that you get another say tangent directions.

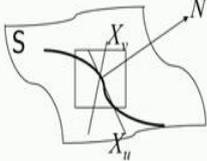
So now, the normal at the surface point, so actually  $X_u$  and  $X_v$  these are all tangents and there is a plane touching that surface at that point, in that plane all this tangents are lying. So, that is called tangent plane and surface and the normal to that tangent plane is the normal at that point which is called Surface Normal.

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**Parametric surfaces**

- Parametric surface:  $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$X(u,v) = (x(u,v), y(u,v), z(u,v))$$


- Surface Normal:  $\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|}$

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So, if I denote it by the  $\vec{N}$  in that case and you can see that how surface normal could be computed using this two tangents, because it is perpendicular to these vectors  $X_u$  and  $X_v$ . So, you can take simply cross product of them then you can get the directions of the normal and as a vector if you normalize it you get the unit vector along this normal. So, that is how normal at this point computed and that would be giving you the unit normal

vector.  $\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|}$

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**Parametric surfaces**

- Parametric surface:  $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$   
 $X(u,v) = (x(u,v), y(u,v), z(u,v))$
- Surface Normal:  $\vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|}$
- Parameterized curve on the surface:  $\beta(t) = (u(t), v(t))$   
 $\beta(t) \equiv (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$
- Tangent vector to the curve  $\beta(t)$  :  $\vec{t} = u'(t)X_u + v'(t)X_v$

So, these are the things we discussed and then how we can represent a curve on the surface in the parameterized forms, I have mentioned that if you make one of the parameter constant and the other parameter varying you get this curve. But in general what we can consider even the parameter  $u$  and  $v$  is also described as a function of a single parameter.

So, let us consider a consider another variable  $t$ , which is again mapping the values from an interval of real space to the two dimensional real space which were actually coordinate which are the values given by the combinations of  $u$  and  $v$ . So, this curve  $\beta(t)$  equals  $(u(t), v(t))$  that itself will give you a parametric.

Say curve on the surface because using these values of  $u$  and  $v$  for a particular  $t$  you can get a curve on the surface. That is how you can represent any general curve on the surface you need not keep one of the parameter  $u$  as constant, the other varying, you can vary both the parameters. But there is a relationship in them where there is a functional relationship in that variation and that functional relationship is denoted by this variable function  $\beta(t)$  and which is denoting that parametric curve.

So, if I expand them we can see that corresponding to  $\beta(t)$  we get a curve on a surface whose  $x$   $y$   $z$  coordinates are given in this fashion, so this is how it is going. So, tangent vector to this curve  $\beta(t)$  it is given in this form when will this representation, it is a

familiar differential operations you first know derivative get derivative of the function  $X$  with respect to  $u$  partial derivative and then you get the derivative of  $u$  with respect  $t$ . Similarly you get derivative of  $X$  with respect to  $v$  and then again you know you get the derivative with respect to  $v(t)$ .

So, each one will give you a vector, so you can see  $X_u$  is also vector  $X_v$  is also vector where as  $u'(t)$  and  $v'(t)$ . You will get a single scalar value because  $u(t)$  is a function, so you will get a function. So, it is a weighted combination of these two vectors that would give you the tangent. So, what I meant here that if I perform the differentiation of this function and take some value  $t_0$  that is a scalar quantity.

So, but  $X_u$  is a vector and  $X_v$  is a vector when you are performing partial derivative you are getting  $(x_u(u, v), y_u(u, v), z_u(u, v))$  for  $X_u$ , so in this way you are getting vectors. So, you are getting the resultant of these two tangent, two vector  $X_u$  and  $X_v$  that is giving up the tangent over that parametric curve.

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**First Fundamental Form**

First fundamental form: The bilinear form that associates two vectors in the tangent plane in the form of dot product.

$$I(\vec{u}, \vec{v}) = \vec{u} \cdot \vec{v}$$

w.r.t.  $\beta(t)$

$$I(\vec{t}, \vec{t}) = \vec{t} \cdot \vec{t} = (u'X_u + v'X_v) \cdot (u'X_u + v'X_v)$$

$$= (X_u \cdot X_u)u'^2 + 2(X_u \cdot X_v)u'v' + (X_v \cdot X_v)v'^2$$

$$= Eu'^2 + 2Fv' + Gv'^2$$

Magnitude of the tangent vector

$$\begin{cases} E = X_u \cdot X_u \\ F = X_u \cdot X_v \\ G = X_v \cdot X_v \end{cases}$$

*Handwritten notes:*  $X(u, v) = (x(u, v), y(u, v), z(u, v))$

So, these are some of the key concepts of differential geometry characterizing the surfaces related to the gradient and curvatures. So, the first fundamental form it relates the change of magnitudes or it relates to the gradient directions at the surface point. And magnitude of that tangent or changes in that surface point in the gradients.

So, this is called this is a bilinear form that associates two vectors in the tangent plane in the form of dot product. So, the definition of the first fundamental form is that in the tangent plane, if I consider two vectors  $u$  and  $v$  and if you take the dot product that would give you the corresponding First Fundamental Form. So, with respect to a curve parametric curve  $\beta(t)$ , then this could be expressed as the dot product of  $t.t$  that is that is know tangential directions or this is the tangent actually this  $t$  is a parameter, but this is tangent.

So, you take the dot product of these two tangents as we have discussed how tangents could be computed, so we are expanding these two tangents. So, you get this particular quantity, so this quantity is related to the first fundamental form. So, where you note that this parameters of  $E$ ,  $F$  and  $G$  which characterizes the first fundamental form, they are expressed as dot products of the corresponding partial derivatives of the functions in this way. I mean partial derivatives of the surface point with respect to  $u$ .

$$\begin{cases} E = x_u \cdot x_u \\ F = x_u \cdot x_v \\ G = x_v \cdot x_v \end{cases}$$

So, this is just to expand. you have say  $X(u,v)$  and this you are representing as say  $X(u,v) = (x(u,v), y(u,v), z(u,v))$ . So, if I take the partial derivative of all these functions and at a given  $u$  and  $v$  you get a vector. So, that is how  $\vec{X}_u$  is a vector so this is a vector this is also. So, you take the magnitude of that vector that is  $E$  similarly you compute  $\vec{X}_v$ . So, you are computing  $\vec{X}_u$  as a vector and similarly you compute  $\vec{X}_v$  as a vector from here and you know you take the dot products and you get this solutions. So, the interpretation of first fundamental form that it is the magnitude of the tangent vector.

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**Second Fundamental Form**

$$X(u,v) = (x(u,v), y(u,v), z(u,v)) \quad \vec{N} = \frac{X_u \times X_v}{|X_u \times X_v|}$$

$$II(\vec{u}, \vec{v}) = \vec{u} \cdot d\vec{N}(\vec{v})$$

$$\vec{t} \cdot \vec{N} = 0$$

$$\frac{d\vec{t}}{dv} \cdot \vec{N} + \vec{t} \cdot d\vec{N}(v) = 0$$

When they are unit vectors  $k\hat{n} \cdot \vec{N} + \vec{t} \cdot d\vec{N}(v) = 0$

$$II(\vec{t}, \vec{t}) = -k \cdot \cos(\varphi)$$

Angle between curve normal and surface normal

For normal section,  $\varphi = \text{zero} \rightarrow II(\vec{t}, \vec{t}) = -k_t$  ← Normal Curvature

The other concept which is used in characterizing surface points and the local surface geometry that is second fundamental form. And in the second fundamental form it is related to the curvatures let us see the definition.

So, as we have already discussed we can see that we have defined the normal as a normal to the tangential plane and which could be derived by taking the tangents at a points along two curves which would be  $X_u$  and  $X_v$ . That means, where varying only u and taking v constant and varying only v taking u constant and you can compute normal. So, in this particular diagram it is shown that if I have a curve where it is varying along v and if I consider with the variations how the normal is changing it is directions.

So, that itself that change of normal that is expressed by a vector  $dN(v)$  that that is shown here and let us continue. So, the definition of second fundamental form is that it is a dot product of two vectors u and v not dot product. So, it is a function of two tangential directions u and v at this point, which is given in this term u vector  $dN(v)$  vector. So, as you vary v so this term is  $dN(v)$ , so it is a tangent it is it is basically dot product of tangent with respect to d and b.

So, tangent is another curve u this is not the same direction. So, we already know  $t \cdot N = 0$ . So, if I take u and v both as t then actually we get this relationships. So, since tangent is perpendicular to normal. So,  $t \cdot N = 0$  and then we take the derivative with

respect to  $v$ . So, you get  $\frac{dt}{dv} \vec{N} + \vec{t} \cdot \overrightarrow{dN(v)} = 0$  so this is the derivative. So, just from there we are deriving with respect to  $v$ .

And so we have already seen that if I take the derivative of tangent, you will get the actually the normal of the curve at that point a vector along the normal of the curve whose magnitude would be the curvature at that point. So, you get this quantity here, so this is coming from this and this is the actually second fundamental form. So, from there I can derive that second fundamental form could be expressed as  $k$  it is a curvature and this is the angle. So, this is cosine of that plane because this is dot product of two vectors normal and these are all unit vectors. So, this is the angle between these two that would give you the corresponding  $\cos(\varphi)$

$$\frac{dt}{dv} \vec{N} + \vec{t} \cdot \overrightarrow{dN(v)} = 0$$

$$k \hat{n} \cdot \vec{N} + \vec{t} \cdot \overrightarrow{dN(v)} = 0$$

$$II(\vec{t}, \vec{t}) = -k \cos(\varphi)$$

So, this is how the second fundamental form is interpreted and you should note that their unit vector, so angle between curve normal and surface normal that is the interpretation. And if you have a normal section, that means when your curve is lying on the normal plane itself then this angle becomes  $0$ . So, it is something like this that you have a normal section. So, your curve is lying like this in a it is intersection here normal plane.

So, your normal section so this is a tangential plane. So, the direction of normal and surface normal both are either  $0^\circ$  or  $180^\circ$ , whatever depending upon topology of that point. But the dot product would be  $0$  and sorry the dot product would give you either  $1$  or  $-1$ . So, in this way you can get the corresponding know surface curve and this is called normal curvature. So, with this let me stop here for this lecture, we will continue our discussion on differential geometry as and it is in the context of analyzing range image and it is applications for characterizing surfaces in our subsequent lectures.

Thank you very much for your attention.