Advanced Graph Theory
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Lecture – 05
Trees and Distance

Lecture 5; Trees and Distance preface recap of previous lecture.

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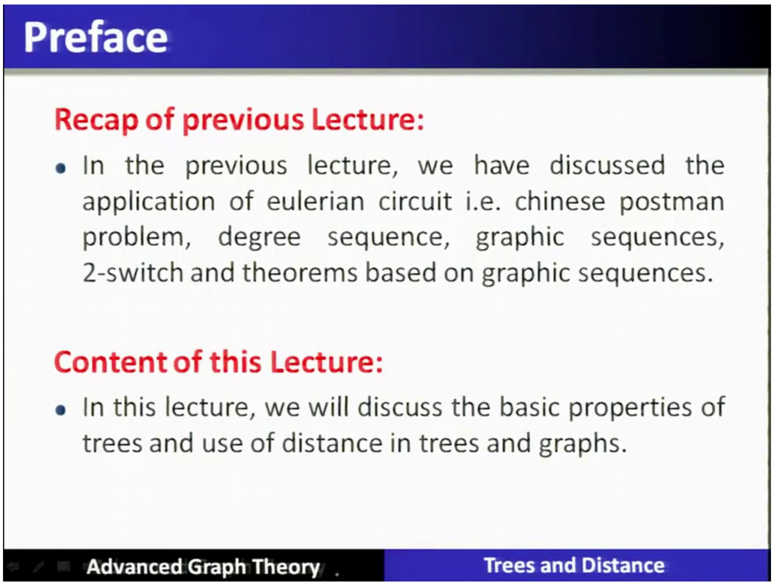

In previous lecture, we have discussed the application of Eulerian circuit that is the Chinese postman problem, and its algorithm, we have seen. We have also seen the degree sequence the graphic sequences and a 2 switch operations. And the theorems based on the graphic sequence we have also proved in the last lecture, content of this lecture in, this lecture we will discuss the basic properties of a trees and the use of distance in the tree and the graph. So, basic properties of a tree mean we are going to basically introduce a theorem which will characterize a tree, so let us start.

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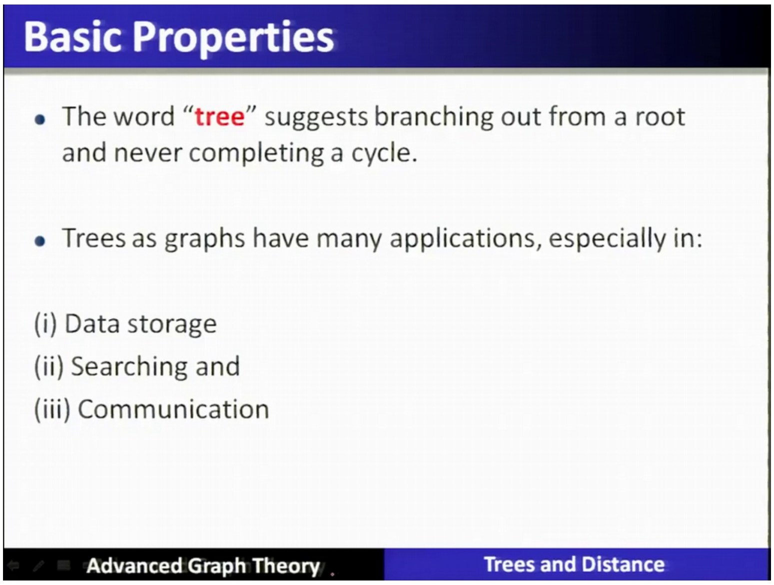

Looking at the basic properties of a tree the word tree suggests branching out from the root and never completing a cycle. So, the trees as the graph have many applications especially in the data storage applications, searching of a data and in the communication network these trees are going to be very, very useful.

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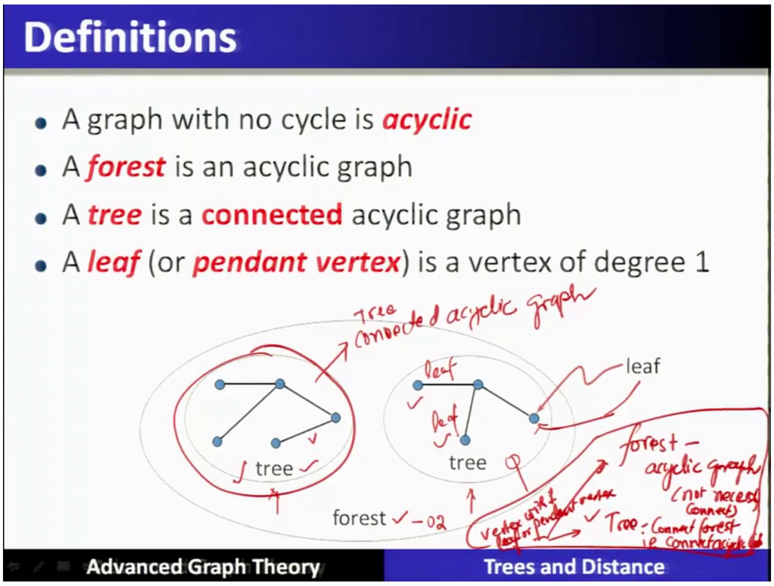

Definition a graph with no cycle is a cyclic, a forest is an acyclic graphic a tree is a connected acyclic graph you can also say a tree is a connected forest why because forest is also an acyclic graph. So, a tree is a connected forest or you can also say tree is a connected acyclic graph or the tree is a graph which is not having any cycle and also it is connected, then one more element is introduced in the tree for our discussion that is called a leaf or a pendent vertex.

Leaf or a pendent vertex is a vertex of degree 1 let us see the example 1 here. So, as far as forest is concerned forest is an acyclic graph. So, here we have given two different acyclic graphs and that constitutes a forest. Now, when we talk about the tree, so tree is a connected acyclic graph, so this particular component which is a tree where you can see that it is connected; that means, all the nodes are connected via an edge and there is no cycle also. So, it is connected acyclic, so tree is a connected acyclic graph.

Now, within this particular tree a node which is called a leaf node has the vertex degree of 1. So, here this is also a leaf node this is also a leaf node, so we have seen three definitions one is the forest the property is that it is basically an acyclic graph that is all which is not necessarily connected now tree is a connected forest so; that means, you can also say it is a connected acyclic graph, similarly in a tree or in a forest the vertex with a degree of 1 is called a leaf node or a leaf vertex or it is called a pendent vertex. So, all these things we have seen here as part of the definitions in the background a spanning sub graph.

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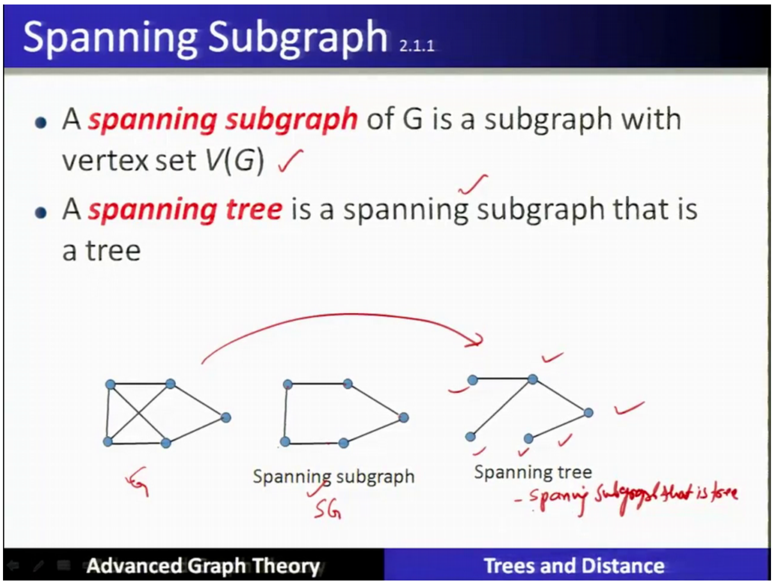

A spanning sub graph of a graph is a sub graph with vertex set VG; that means, a sub graph which includes all the vertices of a graph is called spanning sub graph the spanning tree is a spanning sub graph that is a tree. So, take this particular example this is the graph G, and if we include all the vertices and basically some subset of the edges then it is called basically a spanning sub graph..

Now if you take the subset of this particular graph which includes all the vertices of a graph, but only those particular edges of a graph which will form a tree that which is called a spanning tree, the spanning tree is a spanning sub graph that is a tree that is called a spanning tree.

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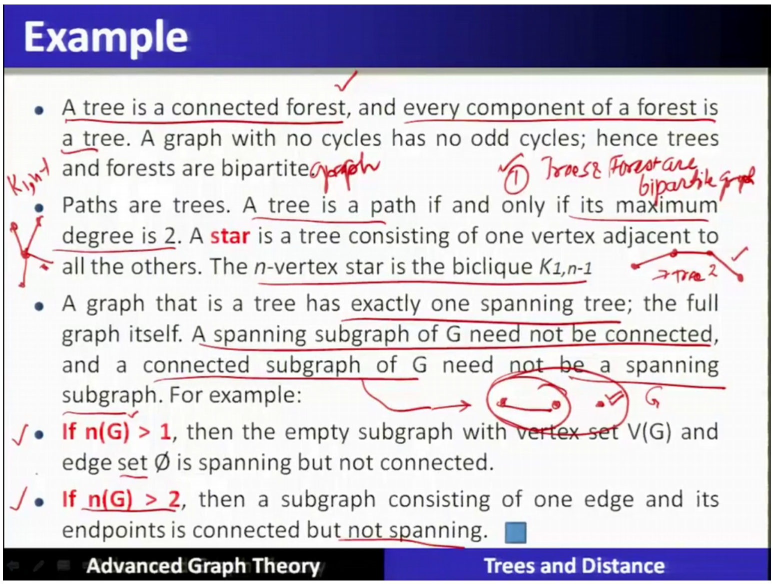

Now, tree is a connected forest this particular definition I have already given you and every component of a forest is a tree that also have stated. So, a graph with no cycles has an odd cycle has no odd cycle, hence tree is a forest or bipartite graph. So, bipartite graph are characterized by the property that they will not have any odd cycle since the graph since the tree do not have any cycle hence it does not have the odd cycle, hence the trees are and the forest are the bipartite graph.

Paths are the trees a tree is a path if an only if its maximum degree is 1. So, you can see this is the path, so this becomes a tree why because it is a connected acyclic graph this tree is also a path if and only if the maximum degree is 2. So, if you include a vertex then it is not a path, but it is a tree. So, this is not a path, path since that why because here the degree of this node is more than 2 hence it is not a path. So, therefore, a tree is a path if and only if its maximum degree is 2 that we have already checked and seen.

Or star is a tree, start is a tree consisting of 1 vertex adjacent to all other going away from that particular going out from that particular vertex is called a star, any start is represented in the form of a biclique that is k 1 comma n minus 1 n minus 1, other vertices now the graph that is a tree has is (Refer Time: 08:33) 1 spanning tree, this is important the full graph itself the spanning sub graph of G need not be connected, and a connected sub graph of G need not be spanning sub graph for example, if the number of nodes in a graph is greater than 1; that means, let us assume it is having the 2 nodes, then empty sub graph, then empty sub graph with a vertex at VG, and except 5 there is no edge here in this example is a spanning sub graph is a spanning, but not connected.

So, a spanning sub graph not necessarily be connected, the other condition says that and a connected sub graph of G need not be spanning sub graph for example, let us assume the order of the graph is greater than 2. So, greater than 2 let us say that it has 3 different vertices the order of the graph is now 3 and 1 and 2 of them is having an edge. So, this is the graph, so this is the connected sub graph of a particular graph G it need not be a spanning sub graph..

So, this is portion is connected, but this portion is left out, so this particular portion is called a connected sub graph of the entire graph G, but it need not have to be or a spanning sub graph because this node is left out, hence it is not a spanning sub graph. So, therefore, when we talk about a spanning sub graph then we have to mention whether it is connected or not, if it is a spanning sub graph which is connected then it will form a tree a spanning acyclic connected sub graph is a tree.

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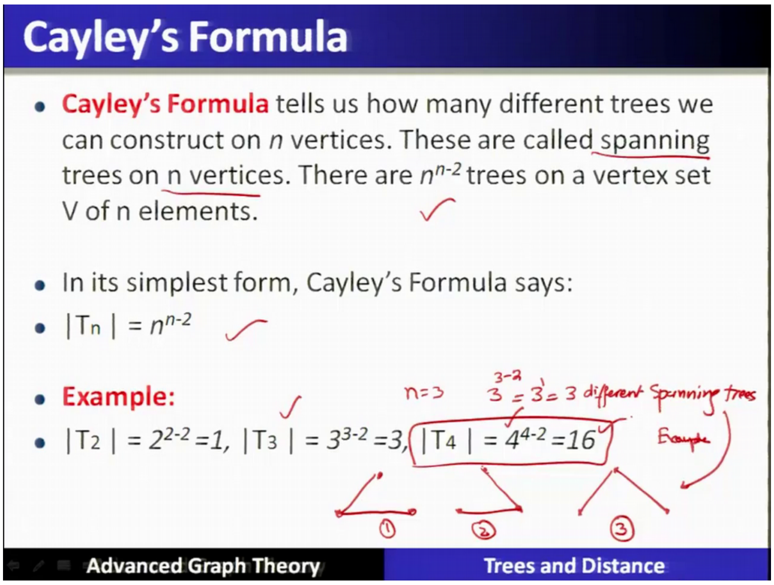

So, cayleys formula gives you about how many different possible trees you can construct out of a nodes. So, if there are n vertices then you can construct the trees including all the vertices spanning all the vertices hence those trees are called spanning trees. So, with n vertices you can construct how many trees n raise power n minus 2 trees on the vertex set of n. So, this particular formula is called cayleys formula let us take the example, let us take the example when it has three different vertices.

If it has three different vertices, you can form three different spanning trees in this way, if this 2 edges are connected, these 2 edges these 2 vertices are connected by an edge or these 2 vertices are connected by an edge, this is 1, 2 and 3. So, by cayleys formula if we say n is equal to 3 then 3 raise to power 3 minus 2 that becomes 3 raise power 1 that is 3 different trees and this is an example which has illustrated. Now, when you have four different vertices given, then there are 16 possible trees according to the cayleys formula.

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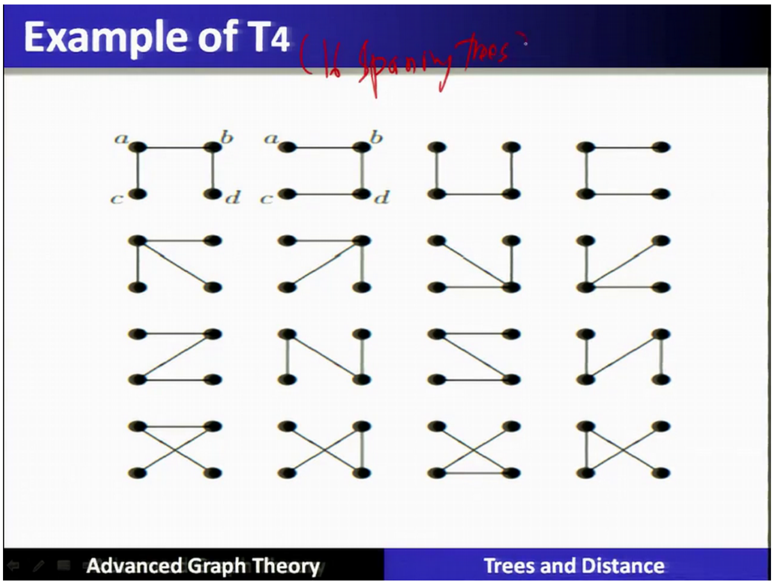

We can see all these 16 different possibilities, which are stated here then there is lemma.

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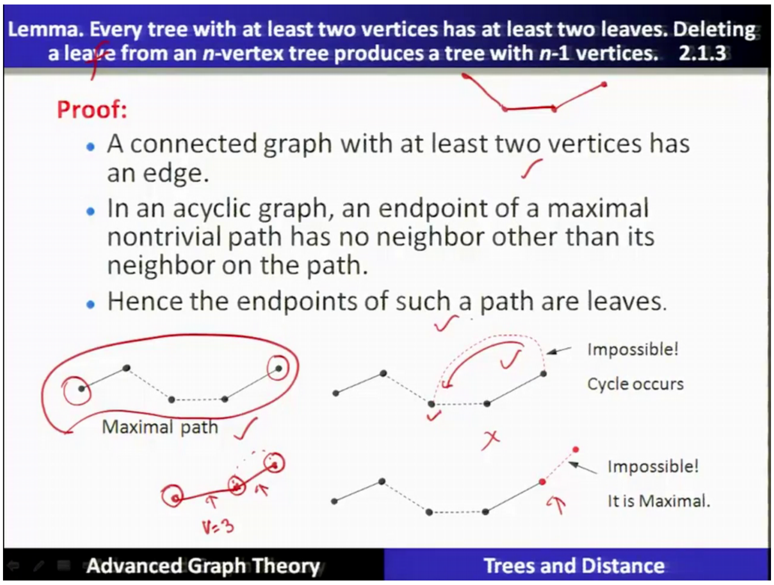

Which is stated as every tree with at least 2 vertices, has at least 2 leaves then another portion says that deleting a leaf from an n vertex tree produces a tree with n minus 1 vertices let us see the proof of it. Now a connected graph with at least 2 vertices has an edge, now in an acyclic graph an end point of a maximal non trivial path has no neighbors other than its neighbors on the path..

So, the maximal path if we consider, then in that particular path these 2 vertices are basically the leaf nodes and hence at least hence a cyclic in an acyclic graph that is a tree with at least 2 vertices has at least 2 leaves. For example if, if let us say assume that at least 2 vertices let us assume that the vertices are total number of vertices is 3 and 2 are the leaf nodes and there is 1 more node available. So, if you place an edge how many edges will be there to form a tree or acyclic graph is required only 2 edges..

Now if we place further more edges like this then the nodes at the end they are called basically the pendent vertices or a leaf node, will have the degree more than 1 and there will be a back edge, and this back edge will be connecting some node in the maximal path for example, this node and thus it will form a cycle and which is contradicting that it is basically a tree which is an acyclic graph, so that is not possible.

Furthermore when we talk about a maximal path; that means, 2 nodes which are farthest apart this particular maximal path cannot be further extended. Why because this is the maximal that is why it is it cannot be contained any other maximum path and that is why it is maximal cannot be extended further.

If let us say it is having an edge then it is go and join the particular already existing vertex in the maximal path and it will form a cycle, which we have seen it is not possible hence, the end points of such a path are basically the leaf nodes. Now we have to see the other part of the proof that deleting a leaf from an n vertex tree will produce a tree with n minus 1 vertex.

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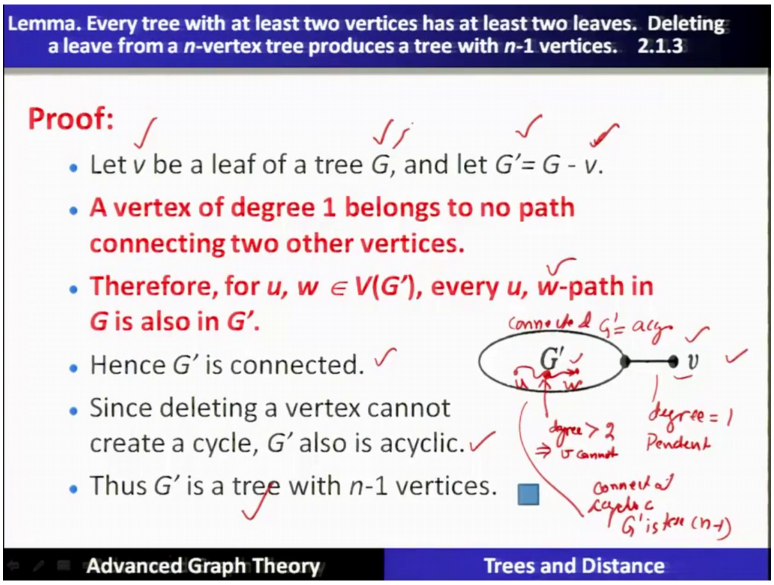

Now, let v be a node of a graph which is basically a leaf of a graph G and this particular leaf node means it is has, it has the degree is equal to 1 why because there is a pendent vertex property of a pendent vertex so; that means, there is 1 edge which will connect to some node in the remaining part of the graph let us call it as G prime. So, we can define G prime as G minus this particular vertex, if we remove this will be the G prime..

So, a vertex of degree 1 belongs to no path connecting the 2 vertices. So, if there are 2 vertices let us say u and w within G prime. So, it cannot come in between why because it is a pendant vertex its degree is 1 so any other vertex between u and v must have the degree more than 2.

Hence v cannot be the node within any connected said pair or vertices within G prime, hence all the pair of vertices they have a path connecting path within it therefore, G prime is still connected a graph. Since deleting a vertex cannot create a cycle therefore, G prime is an acyclic why because the original graph G is a tree which is an acyclic graph. So, if you remove a vertex out of a G the remaining graph will also be acyclic. So, it is a connected it is acyclic..

So, therefore, G prime is a tree, but here the number of nodes is n minus 1 in contrast to G which is having n nodes. So, hence thy statement is proved, so we started with a with a graph we started with a with a tree which is a graph. So, every tree has at least 2 leaves that we assume in the previous theorem, and if you take out of these at least 2 1 leaf let us say v and if we remove it from the graph the resulting graph is G prime. So, G prime is also a tree with n minus 1 vertices that we have already proved, so this particular lemma.

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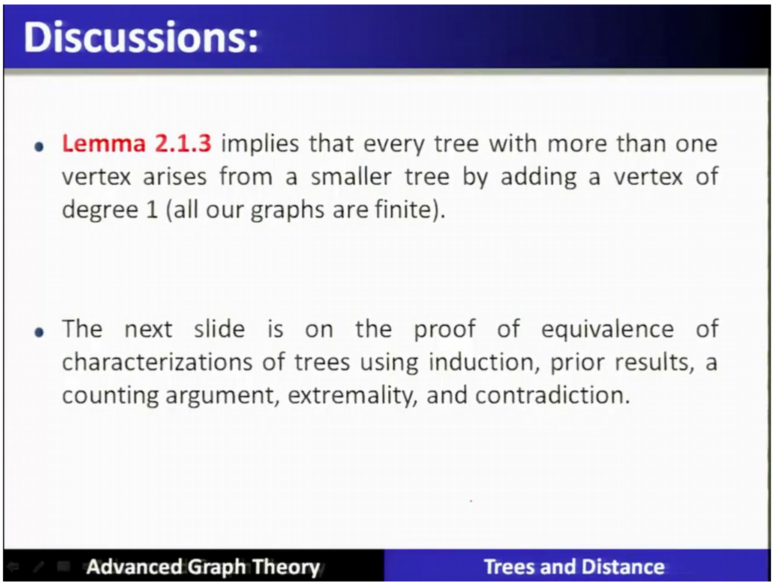

Will imply that implies that every tree with more than 1 vertex arises from a smaller tree by adding a vertex of degree 1. So, this is how we can expand or we can construct a tree from a single node vertex. In the next slide we will be seeing the equivalence and that basic equivalence of trees that is the characterization of the trees, and you will be using different methods of the proof that is called induction prior results also is used then counting argument extremality and contradiction all these methods of different groups we are going to see here when we look upon the characterization of the trees.

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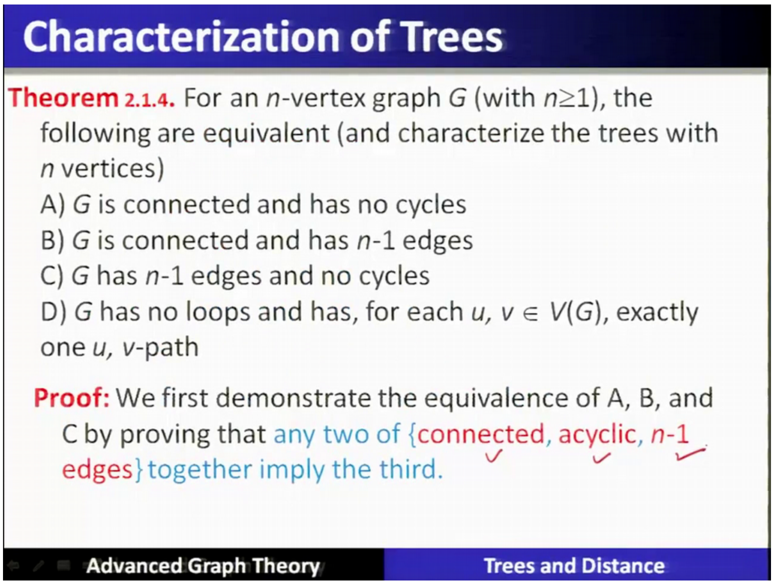

Characterization of the trees there is a theorem which states that for n vertex graph, where n is greater than or equal to 1 the following are equivalent and also will characterize the tree of n vertices. So, following are equivalent and they will characterize the tree in the sense when we say that G is connected and has no cycles this will define as the tree and that is also equivalent as another definition of a tree that G is connected and has an minus 1 edges this also will characterize the tree.

G has n minus 1 edges and no cycle will also characterize the tree, G has no loops and has for each u v pair within the vertex set of G exactly 1 u v path exist then it is also characterize a tree. So, all these four different statements or a properties they will characterize the tree and they are all equivalent that we are going to prove now. So, proof the first we will demonstrate the equivalence of A, B and C by proving that any 2 of these 3, A, B, C, properties they will basically represent the properties three different properties connected acyclic n minus 1 edges..

So, out of these three if you pick any 2 in any of these properties then it will imply the third one, and that is how the equivalence of A, B, C will be proved let us see the first theorem.

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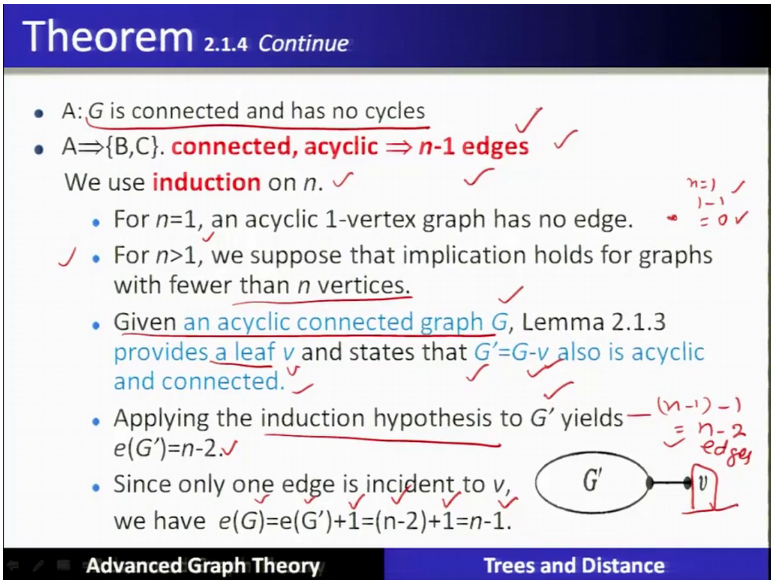

So, this particular theorem has four different sub theorems or four different theorems, which will comprise of A this bigger theorem. So, the first theorem says that A means if A graph G is connected and has no cycles, then this will be equivalent to the properties B and C. So, B and C also as connected end n minus 1 edges n minus 1 edges and no cycles. So, basically n minus 1 edge was not there in A so; that means, connected end acyclic will basically implied that it has n minus 1 edges.

Let us use the induction on n for n is equal to 1 acyclic 1 vertex graph, how many edges are there no edge, hence this particular statement that is having n minus 1 edge is basically satisfied why because, 1 minus 1 that becomes 0 and how many edges are there, there are no edge when n is equal to 1. So, this single vertex node is connected acyclic and now it will be having n minus 1 edge.

Now, if n is greater than 1, and we suppose that implication holds for the graph with a fewer then n vertices. Now given an acyclic connected graph G that is the statement that we are given, if we apply using lemma 2.1.3, here we, we are provided with a leaf node why because every connected acyclic graph G that is a tree will have at least 2 nodes with a degree 1. So, let us assume a leaf node with a in this particular graph and also that particular lemma will state that if we remove it then the resulting graph G prime is also acyclic and connected.

Now, by applying the induction hypothesis which assumes that with the fewer than n vertices this particular condition holds. So, basically for G prime this condition holds means G prime has n minus 1 n is n minus 1 minus 1 that is n minus 2 edges it has. So, that is what we have written over here now since v is a vertex which was there in the original graph. So, let us count how many edges will be there in our original graph, having knowing that G prime is the graph which has n minus 2 edges when we plug in and the degree of this particular leaf or a pendant is 1 that we know.

If we plug in, so the total number of edges in the original graph comes out to be n minus 1, hence this particular part of the theorem we have proved by induction now the second property the second part of the theorem says that B is the condition that if it connected n minus 1. So, we are going to show that it is having no cycles that is equivalent to no cycle, so let us assume that the graph G.

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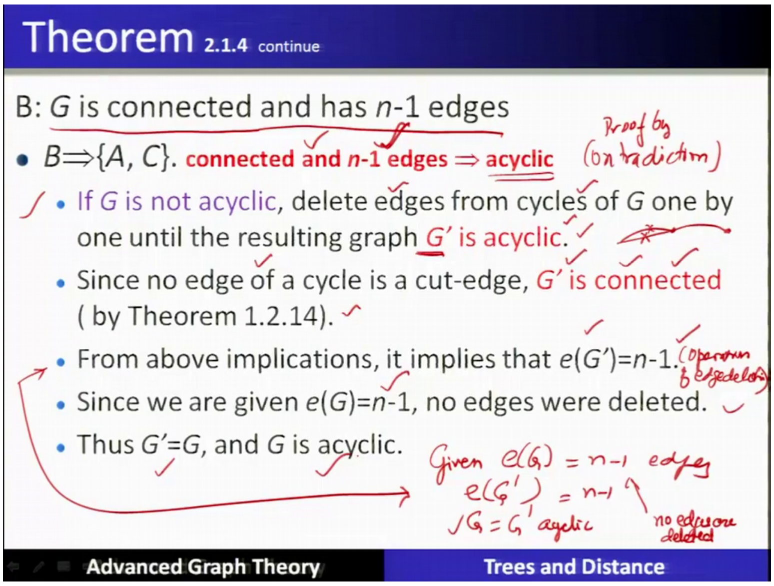

Which is given to us is connected and has n minus 1 edges, and we have to prove that this will imply that it is acyclic. Now, let us prove it by the contradiction suppose if this particular graph is not acyclic, suppose this graph is not acyclic then we can delete the edges from this particular cycle. So, we can delete some of the edges either this edge or the other edge. So, we can delete the edges from the cycle of G 1 by 1 until the resulting graph that is called G prime becomes acyclic so; that means, if a graph which is not acyclic we can convert it to a cyclic graph by removing some of the edges deleting some of the edges.

Now, you know that no edge on the cycle is an cut edge that particular characterization we have seen in the earlier slides therefore, G prime is connected by the theorem why because it is not removing the it is not deleting the cut edges, but it is deleting the edges out of the cycles hence by theorem 1.2.1 4, that we have seen earlier lectures will state that removing the edge from the cycle will not disconnect the graph.

Why because they are not the cut edge, so from the other implication it implies that the number of edges which are present in G prime, why because G prime is connected and also it is acyclic we have made it acyclic it was also connected. So, it has n minus 1 edges because there was we have done through this operation. Now we are already given that the graph G has n minus 1 edges and we have basically obtained..

So, that was given to us here and, but by this particular step we have achieved a graph G prime and that is by deleting the edges and the total number of edges which are there in G prime is also n minus 1. So, how many edges we have deleted we have not deleted any edges.

If no edges are deleted; that means, both the graphs are same since we assume that our graph is basically the G prime is acyclic why because we have obtained acyclic graph by as deletion since no edge is deleted. So, G prime is acyclic hence G is also acyclic hence we have done the proof, now third part which says that G has n minus 1 edges.

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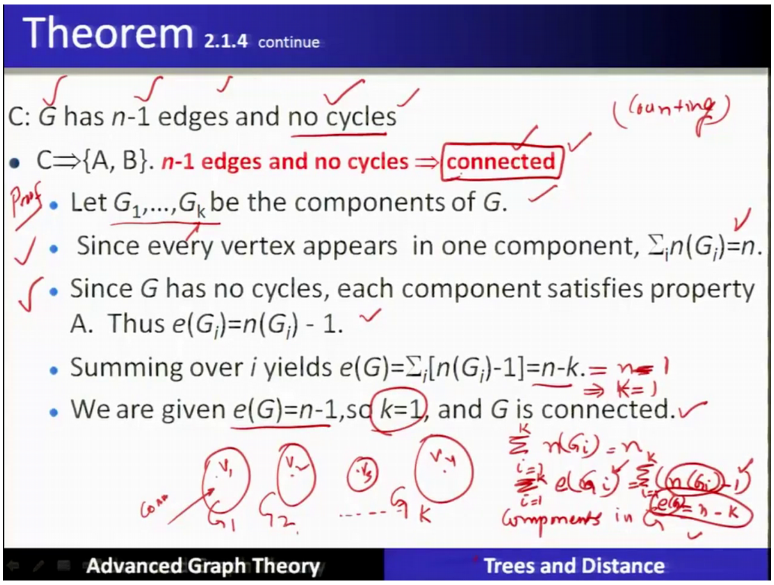

And no cycles; that means, we are going to prove A, B, A, B has this particular condition that it is also necessarily connected, but has n minus 1 edges and no cycles; that means, it will be connected.

Now, let us see the proof of it, so the proof here we are using the counting. So, let us see that G graph which is given to us has k different components some are small some are big. So, 1, 2 and so on up to k different components, in graph G these components we can write down G1, G2 and so on up to G k. Now since every vertex appears in 1 of these for example, if it is V 1 V 1 cannot appear in any other component. So, here V 2 will appear, here V 3 may appear, here V 4 may appear and so on.

So, every vertex appears in 1 of the component, so let us count how many vertices will be there total. So, we will count the number of edges in the component G i and we sum over all i from 1 to k. So, the total number of vertices in this way becomes n for an n vertex graph.

Now, another thing which we have to see is that since it is given that he has no cycles. So, each component will satisfy the property what is that property that every component is connected. So, it will be having how many different edges, so every component will have this n minus 1 different edges. So, let us count these number of edges also 1 to k, so that becomes the summation of you know that n e of G either number of edges in a particular component is n of G i minus 1, and this particular summation will go from i is equal to 1 to k. So, here this particular factor will introduce k and this factor is already known as n so the total number of edges here will be n minus k.

Now, we are given that this particular graph has n minus 1 edges and we have counted that it becomes n minus k. So, n minus k when you make equal to n minus 1 this implies that k is equal to 1. So, the number of component here we have assumed it is k if it is 1 then the graph is connected, hence by if a graph has n minus 1 edges and no cycles we have concluded that this particular graph must be connected.

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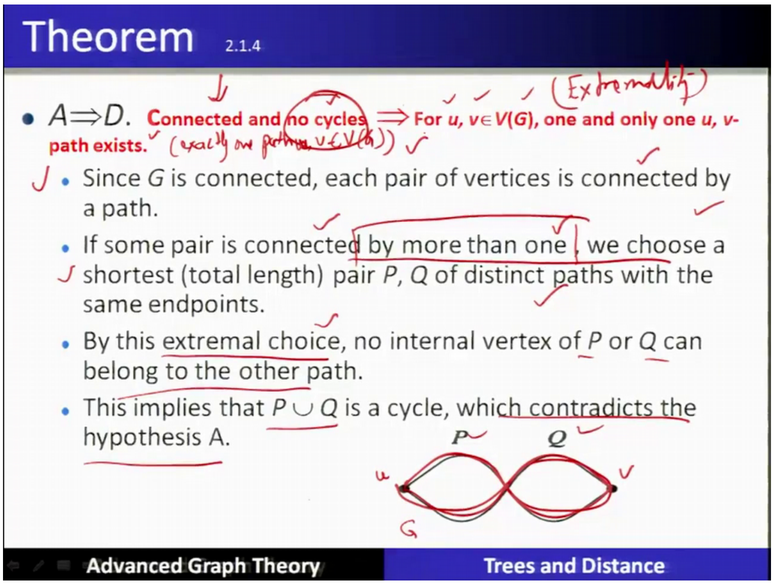

Now the next part of this particular theorem says that if the graph which is given is connected and no cycle, this will imply that for any pair of vertices of a graph there is only 1 and 1 path exist; that means, there is exactly 1 path for every u v, which is there in the vertex that exists that been took away, now since graph G is connected that is given. So, each pair of the vertices is connected by a path this is the definition of a connectivity which we have seen earlier. So, for example, if this particular graph is connected and for any 2 vertices let us say u and v, they are basically having a connection through A path.

So, if some pair is connected by more than 1 such path, so we choose the shortest that is the total length shortest path or pair let us say P and Q there are 2 shortest path exist of the distinct path with the same end points. So, with this extremal choice, extremal in the sense all other path we have ignored only the shortest 1 we have considered and incidentally let us say there are more than 1 shortest path. So, we have now seen the extremal conditions, so we are looking the extremality condition here in the proof. So, by this extremal choice no internal vertex of P or Q can belong to the other path..

So, this implies that P union Q will form a cycle and which contradicts the hypothesis A, which basically assumes that the graph is connected, but it does not have any cycle. Hence this particular assumption that it is connected by more than 1 such paths this also is contradicted hence, it is connected by exactly 1 path the last part of this particular theorem.

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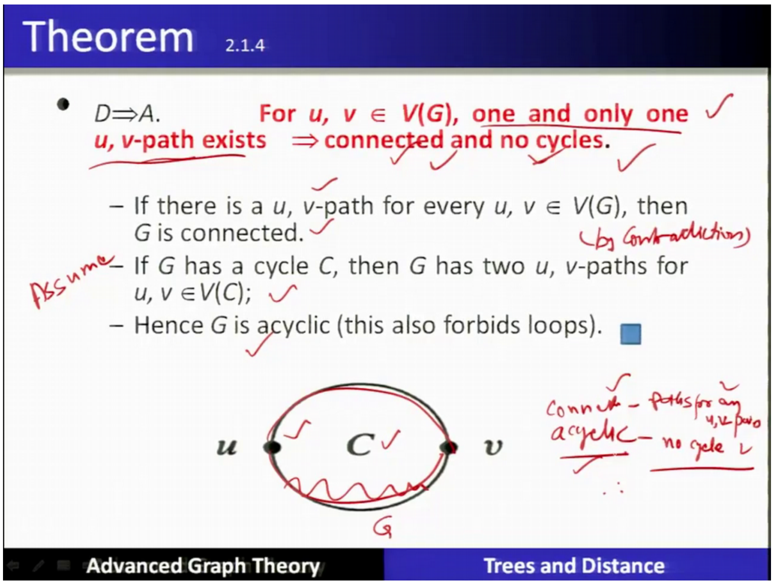

Says that D, the statement of D says that for any pair of vertices in a tree there is exactly 1 u v path exist and this will imply that that particular tree is connected and having no cycles let us see the proof.

Now, if there is a exactly 1 u v path, for every u v then basically u and v is connected let us assume that it contains a cycle, and we are going to prove by contradiction let us assume there is a cycle in G here then G has 2 u v paths, this is 1 path this is another path between u and v and that will go around this particular cycle will prove 2 different paths, hence the G is acyclic why because the statement of this particular theorem opens with a condition it has only 1 u v path now with the 2 u v path. If a cycle exist then it will be having 2 u v path, and hence this particular condition will be violated hence by contradiction it has only 1 path if it is 1 path then it is acyclic..

It is connected also why because this path will provide the connectivity and it will forbid or it will ignore the condition that it will be having a cycle. So, it is a connected graph why because the paths exist, between any 2 pair of vertices, and there is no possibility of a cycle this will conclude that both our conditions are basically satisfied. So, this is equivalent to saying that if let us say there is exactly 1 path exist between u and v. So, basically then the graph will become a connected and also the graph will become acyclic. So, there are some corollaries based on this particular theorem let us see.

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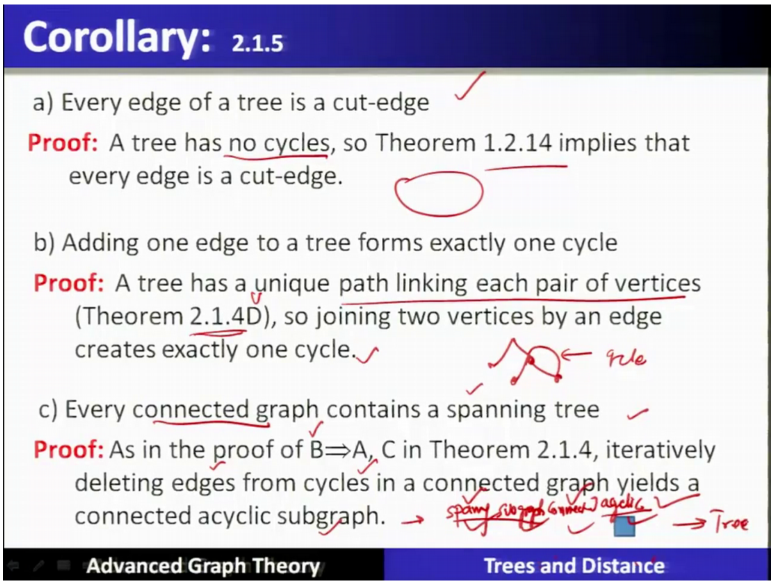

Those corollaries quickly, now every edge of a tree is a cut edge this is quite obvious in the proof and no now the tree has no cycles because it is an acyclic. So, the theorem says that that every edge of a tree is a cut edge why because there is no cycle. So, all the edges of a tree will become will not lie on the cycles if they are not lying on the cycle then every edge is a cut edge.

The other corollary which says that adding 1 edge to a tree will form exactly 1 cycle. So, tree has the unique path linking between every pair of vertices in the previous theorem which we have seen through the decondition. So, joining any 2 vertices by an edge will create exactly 1 cycle. So, here if you join any 2 vertices by an edge it will create a cycle, and that will be exactly 1 cycle which will be created.

Third part of a proof says that every connected graph contains a spanning tree. So, as in the in the proof of the property B, which we have seen in theorem 2.1.4, that we considered a graph and it relatively we were deleting the edges till the graph becomes acyclic. So, by deletion the edges the sub graph which we obtain is a, is a spanning sub graph which is connected and which is acyclic..

So, the connection was already given the condition so; that means, when we delete all the edges from the cycle to make it acyclic. So, it becomes an acyclic graph when we delete the edges; that means, we are not removing not deleting the vertices by spanning sub graph it is. So, spanning sub graph which is connected acyclic which we obtain is nothing, but a tree.

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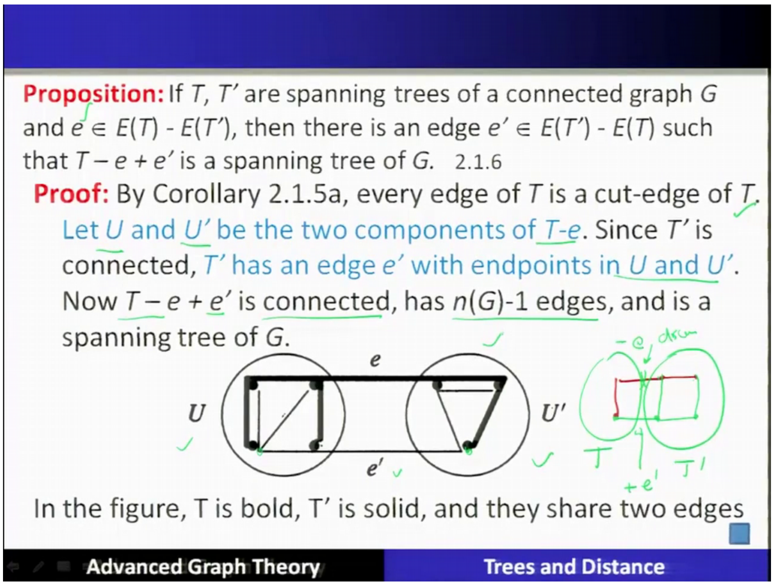

Preposition if T and T prime are the spanning trees of a connected graph G, and there exist an edge which is present in T with, but not present in T prime then there is an edge T prime which is present in T prime, but not in T such that T minus e plus e prime is a spanning tree of G. So, what it says that if there exist 2 different spanning trees of a connected graph, then if we remove an edge from a particular graph from a particular spanning tree T then that particular tree will become disconnected why because every edge is a cut edge that we have seen in the previous theorem.

So, if the, the graph become disconnected, but another is having tree T prime will have an edge e prime basically which connects these 2 disconnected components after removing e, hence after adding that particular e prime the total graph that is the total tree

T minus e plus e prime will become a spanning tree of G. So, let us take the example before we go ahead about proof.

So, there are 2 trees 1 is shown by the red the other tree is shown by the green color let us assume that this is T this is T prime. So, if we remove this particular edge, so this particular tree is the red color will be disconnected here, but if we add another edge e prime which is out of the another tree then the entire graph again it can be connected 1 these 2 components will be connected, let us see the proof by this particular corollary.

By corollary 2.1.5a, says that every edge of T is a cut edge that is quite obvious. So, when we remove an edge e then let us assume that there are 2 disconnected components u and u prime, when that particular tree will remove this particular edge now since T prime is, is another spanning tree. So, T prime is the connected, so T prime has an edge e prime with an n vertices in u and u prime so; that means, from P minus e if we add an edge e prime then this particular graph will be connected 1, and also it has n minus 1 edges why because 1 edge is route 1 edge is added. So, number of edges is n minus 1 still the graph is a spanning tree.

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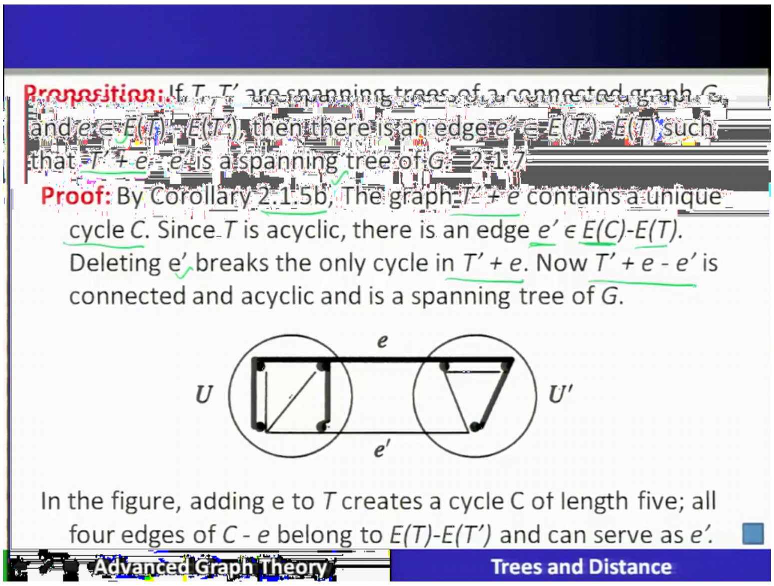

So, another preposition says that if T and T prime are the spanning trees of a connected graph and e is an edge which is there in T, but not there in T prime then there is an edge e prime which is there in T prime, but not there in T such that in T prime, if we, if we add an edge and then remove an edge from T prime then it will become a spanning tree..

So, T prime if you add an edge it will become a cycle that you know that we have seen in the previous corollary that in a tree if you add an edge it will result into an exactly 1 cycle. Let us say that this cycle is C, now since T is acyclic there is another edge e prime which is there in the cycle, but not there in that particular tree T hence if we remove that e prime will break only the cycle in T prime plus e. Hence, T prime plus e minus C will become connected and acyclic and is also a spanning tree of that particular graph G another preposition.

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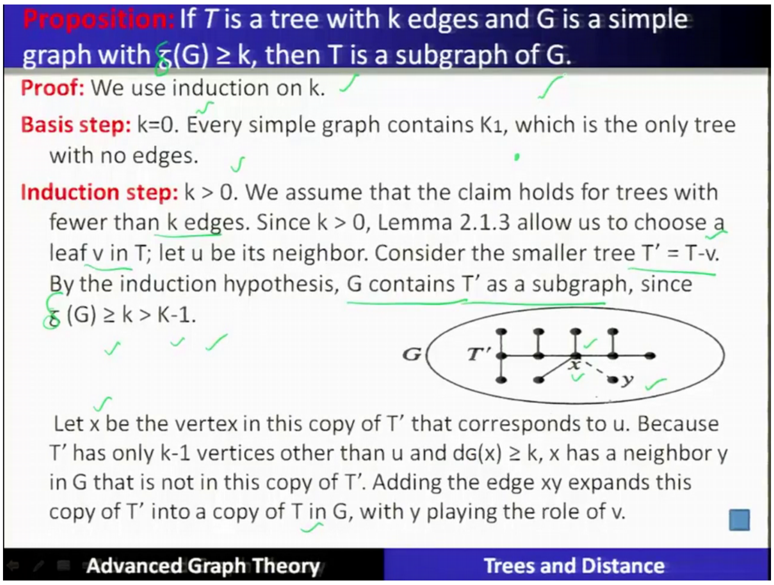

If T is a tree with the k edges and G is a simple graph with little delta that is the minimum degree is at least k then T is a sub graph of G. So, this particular preposition can be proved with the help of induction on k, when k is equal to 0 then every simple graph contains k 1 given means isolated vertex is a k 1 which is only a tree with no edges, so hence the base case is proved.

Now, induction means we assume that this particular induction hypothesis says that if for k greater than 0 let us say it holds. So, we assume that the claim holds for the trees with the fewer than k edges, since k is greater than 0 lemma 2.1.3 allows us to choose a leaf v in T let u be its neighbor consider the smaller tree T prime which is obtained by removing a particular vertex v and. So, by induction hypothesis thy smaller tree T prime, so G contains that a smaller T prime has a sub graph. So, this particular induction hypothesis since this particular little delta of G is greater than k and k is basically greater than k minus 1.

Now, let us go ahead to prove this particular induction step, since let x be the vertex in this copy of T prime here that corresponds to u because T prime has only k minus 1 vertices other than u, and the degree of x is greater than k that we have already seen. So, x has the neighbor y in G that is not in this copy of T prime adding an edge x y will expand this copy of T prime into a copy of T in G with y playing a role of v, hence this particular condition is condition in the sense this induction is proved for a bigger values.

Distance and increase and graphs, when using graph to model the communication network we want vertices to be close together to avoid the communication delay we measure the distance using the lengths of the path distance increase and graph.

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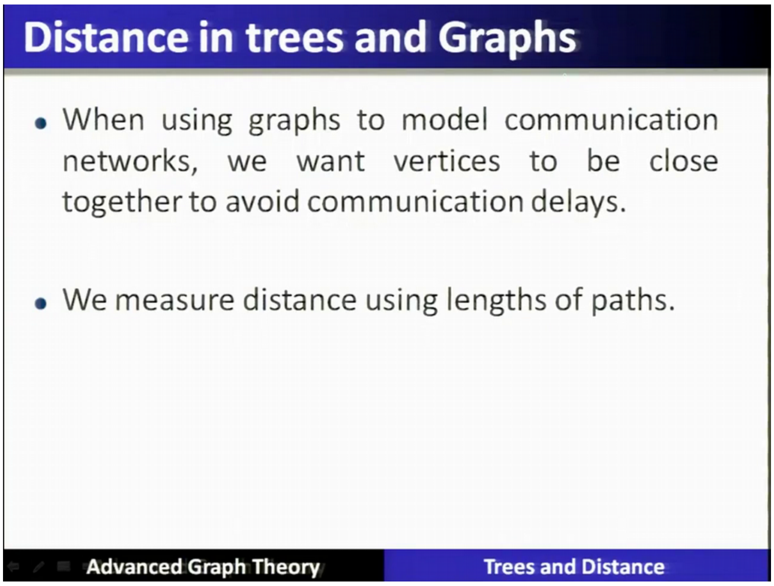

So, if G has u v path then distance from.

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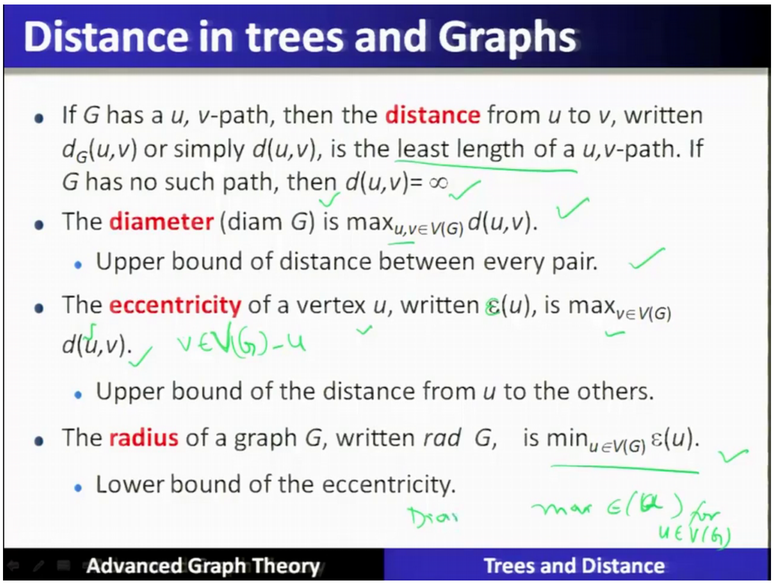

u to v is written as d of u v is the least length of u v path if there is no such path exist then this particular distance will be very high value that is infinite. The diameter of a graph G is nothing, but a maximum of the distances between any 2 pair of vertices and that is the maximum of T u v available. So, that becomes a parameter of the graph diameter of a graph..

So, upper bound of the distance between any pair of the vertices similarly eccentricity of the particular vertex is written by epsilon u is nothing, but the maximum distance from that particular vertex to any other vertex that becomes an eccentricity. So, the radius of a graph is the minimum eccentricity, and similarly the diameter of a graph also we can say that the maximum eccentricity which is there in the in the graph, so diameter equals the maximum of the vertex ec eccentricities.

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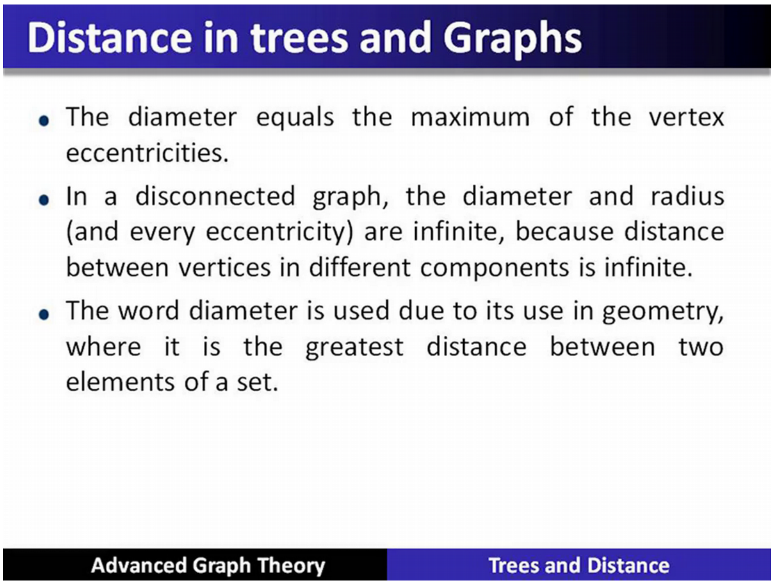

So, in a disconnected graph the diameter and the radius are infinite, because the distance between vertices in different component is infinite the word diameter is used due to its use in the geometry where it is the greatest distance between the 2 elements of a set..

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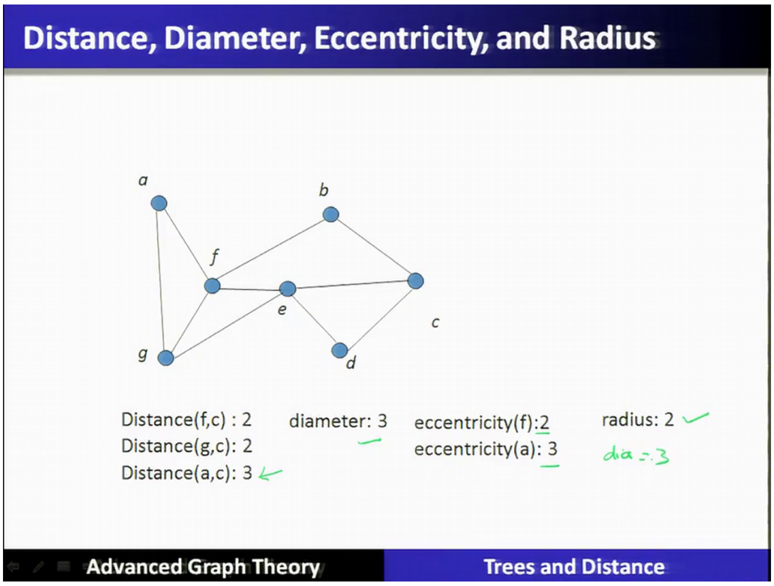

So, let us take this particular example, so this is the graph and if you calculate the distance between any 2 pair of vertices, what we find is that this particular distance is the maximum, so the distance. So, the diameter of this particular graph will become 3 eccentricities if we calculate the minimum eccentricity is 2, so that becomes the radius.

Similarly, the maximum eccentricity is 3, so diameter of a graph will become 3 the Petersen graph has the diameter 2.

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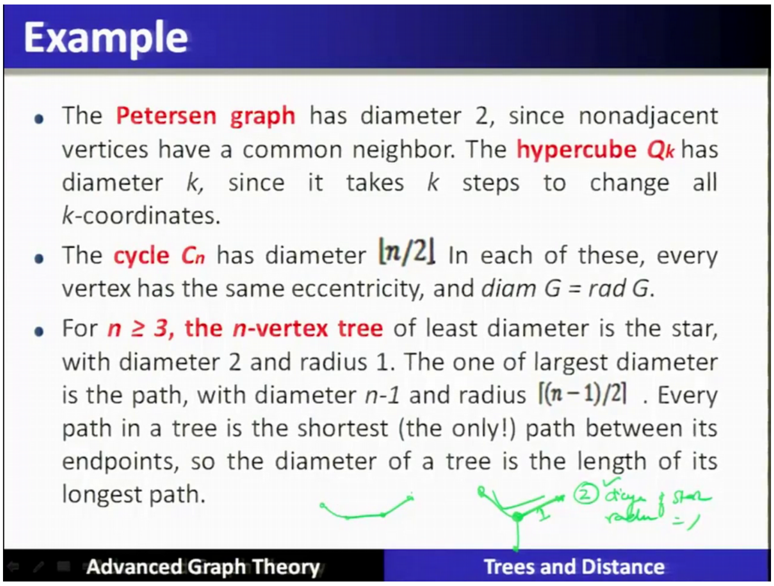

Since nonadjacent vertices have a common neighbor the hypercube Q k has the diameter k, since it takes k steps to change all the k coordinates. Cycle C n has the diameter n by 2 in each of these, every vertex has the same ec eccentricity, and the diameter of G is nothing, but the radius of for n is at least 3 the n vertex tree of the least diameter is thy star, with the diameter 2 and the radius 1. So, you can see this is the star between any 2 pair of vertices the maximum distance is 2.

So, the diameter of star is 2 now as far as the radius is concerned for this particular vertex the distance is 1. So, that minimum eccentricity is 1, so radius will become 1 here in this case now every path in a tree is the shortest the path between its end point. So, the diameter of a tree is the length of its longest path. So, path is a tree if this example we can see.

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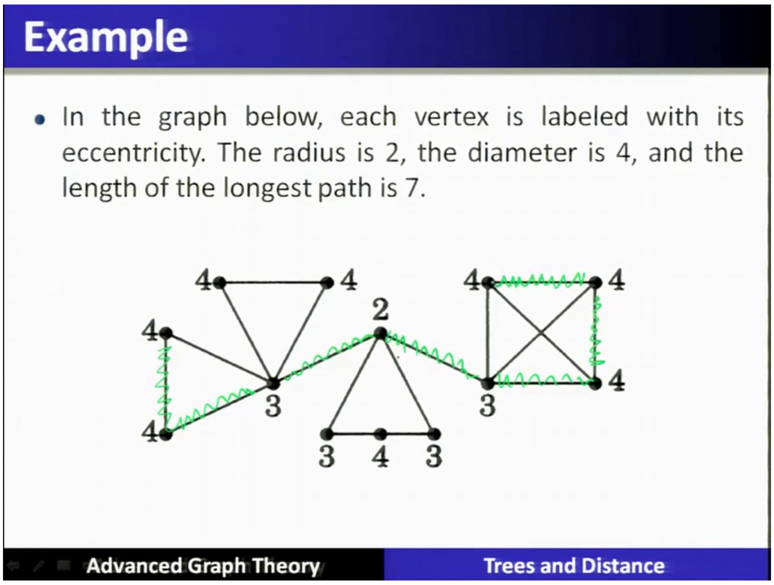

The diameter is 4 radius is 2 and the length of the longest path is 7. So, length of a longest path you can see 1, 2, 3, 4, 5, 6, 7, theorem if G is a simple graph..

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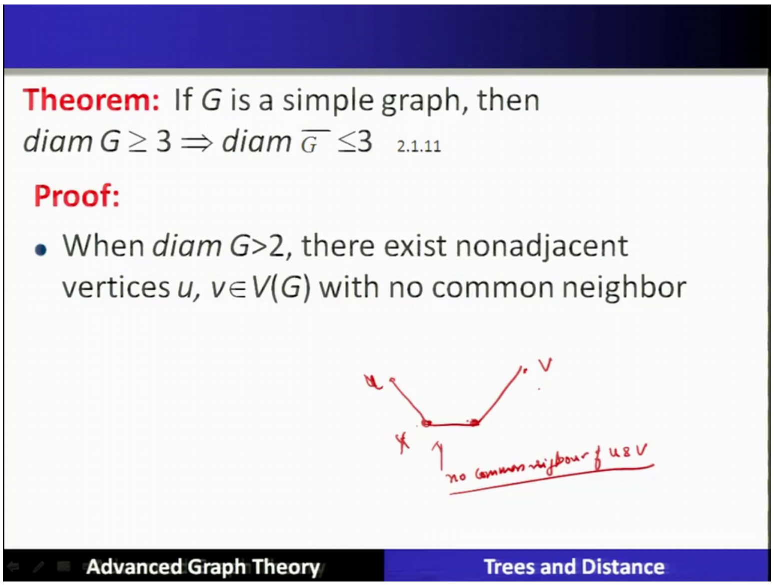

Then diameter of G is at least 3 this will imply that diameter of the compliment of G is at most 3 proved, when diameter of G is more than 2 there exist a non-adjacent, vertices u and v with no common neighbors take this particular example this is u this is v. So, they have this diameter is greater than 2. So, there will be there will not be a common neighbors.

So, there is no common neighbor of u and v here in this case why because it has 2 neighbors, now hence every x if we remove u v has at least 1 of these u and v as the non-neighbors. So, here this x will have v as a non-neighbor, so that is what is stated over here this makes x adjacent.

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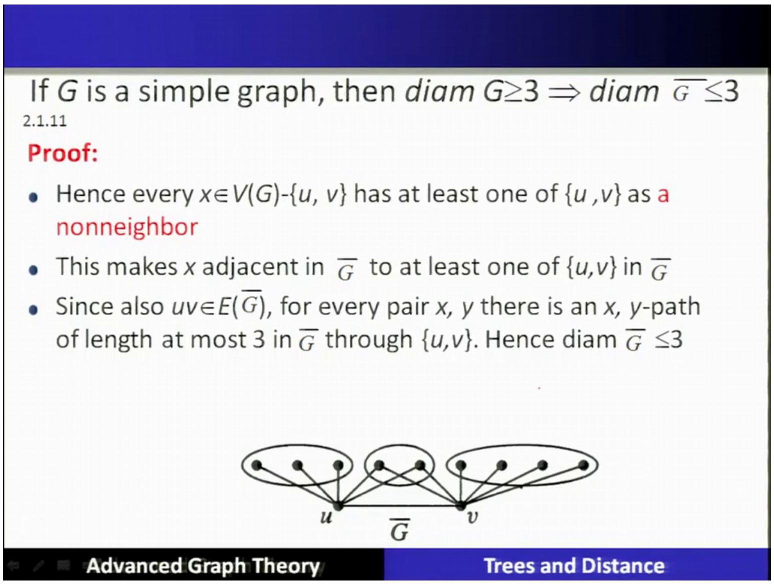

In G compliment to at least 1 of u and v in G compliment, since also u v has an edge in G compliment for every x y for every pair x y there is an x y path of length at most 3 in v compliment through u v hence the diameter of G prime is center.

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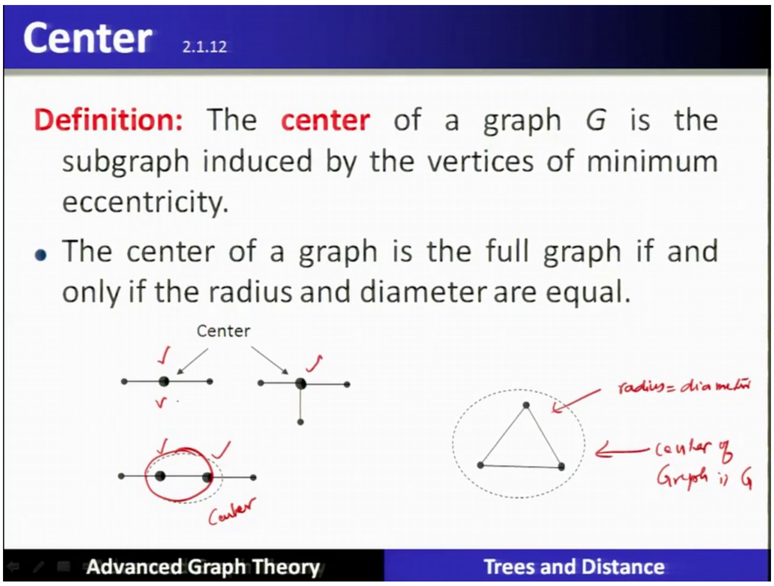

Definition the center of a graph G is the sub graph induced by the vertices of minimum eccentricity. So, the center of a graph is the full graph if and only if the radius and the diameter is equal. So, take this particular example here in this particular graph the radius is equal to the diameter, then the entire graph will be the center of a graph is G..

Now in these examples, the vertices with a minimum eccentricity is this vertex in this particular case here in this vertex and here there are 2 vertices with a minimum eccentricity hence, this is an edge which will be the center similarly here there will be a single vertex, now that there is a theorem.

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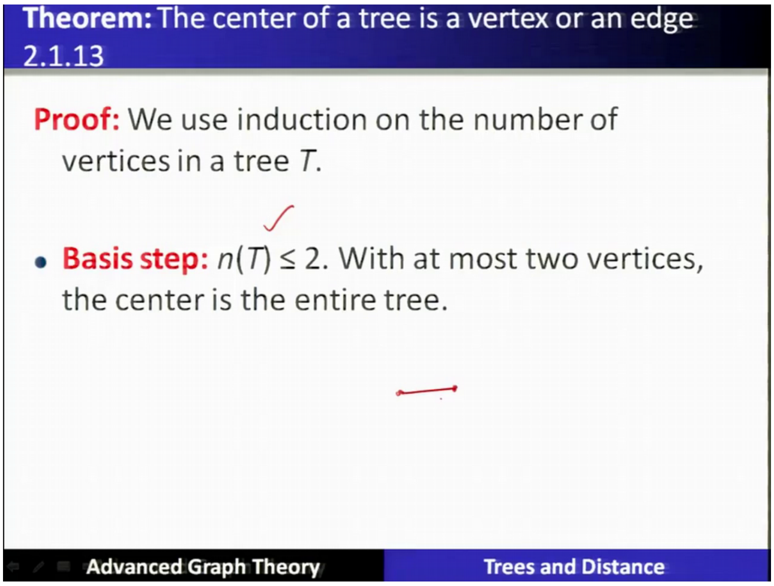

Center of or tree is a vertex or an edge, so the proof we use the induction on the number of vertices in a tree. So, basic step base case when the number of nodes in a tree is at least. So, at most 2; that means 2 so with at most 2 vertices the center will be the entire tree why because eccentricity of both the nodes is the same.

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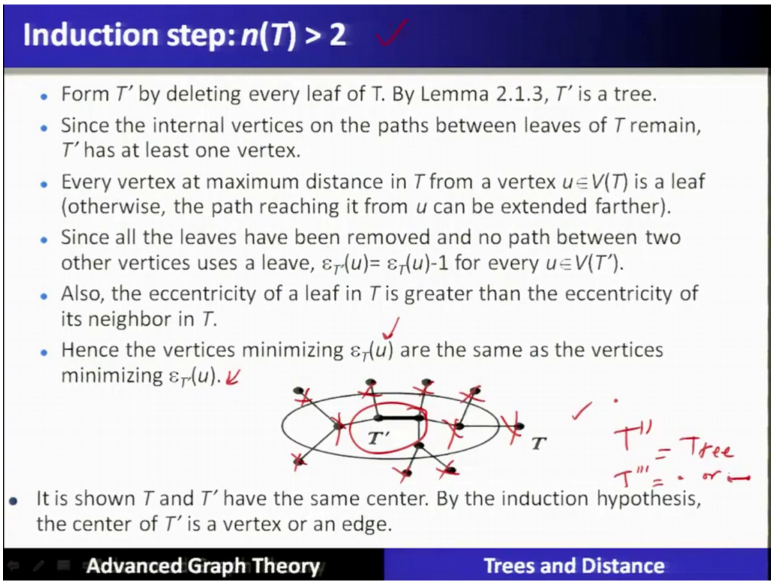

Now, let us assume that the induction holds through for the number of nodes which are more than 2, now form T prime by deleting every leaf of T by lemma 2.1.3, after le deleting the leaf the remaining after deleting a leaf from a tree the remaining tree that is T prime will also be a tree, since the interval vertices on the path between the leafs of tree remains why because only the pendent vertices will be removed. So, internal nodes comprising that particular path will remain, so T prime has at least 1 vertex.

Then every vertex at the maximum distance from maximum distance in T from the vertex u of that particular tree is a leaf. If not then it will complete a cycle that is; that means, the maximal path property since all the vertices since all the leafs have been removed and no path between 2 other vertices uses the leaf, hence the eccentricity of the remaining tree that is T prime will be reduced by 1, for every vertex which is present in T prime also the eccentricity of the leaf in T is greater than the eccentricity of its neighbor therefore, the vertices minimizing that eccentricity here for that particular vertex are the same as the vertices which will minimize the eccentricity, here in T prime for those set of vertices.

Thus we have shown that it is shown that T and T prime has the same center the by induction hypothesis. So, let us assume here that this is a tree if we remove this particular vertex they are called leaf vertices from T then it will become T prime in T prime also if we remove these leaf nodes then the remaining tree will be T double prime. So, d T double prime will also be a tree and so we keep on doing this iteratively till it we, we form a T prime. So, that either it will remain a vertex or an edge hence this particular theorem is been proved.

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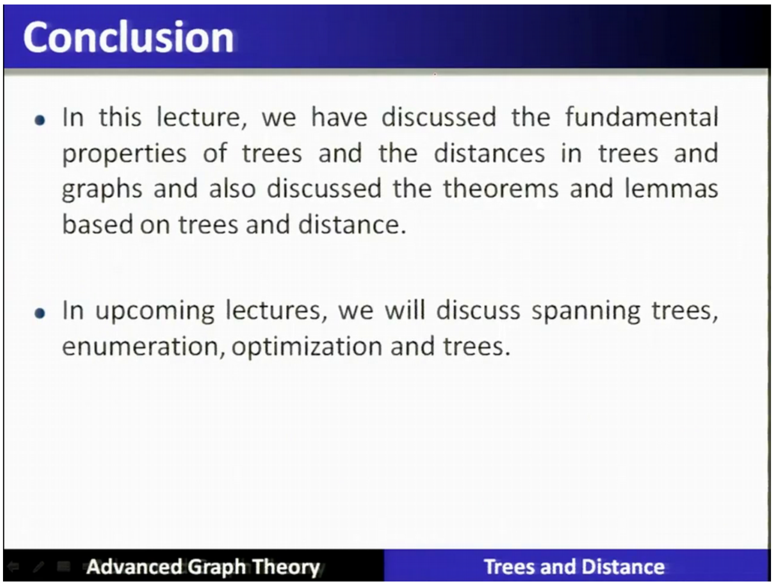

So, conclusion in this lecture we have discussed the fundamental properties of trees, and the distances in the trees and the graphs and also discussed the theorems and lemma based on trees and distance, the upcoming lectures we will discuss spanning trees, enumerations, optimizations of trees.

Thank you.