Advanced Graph Theory

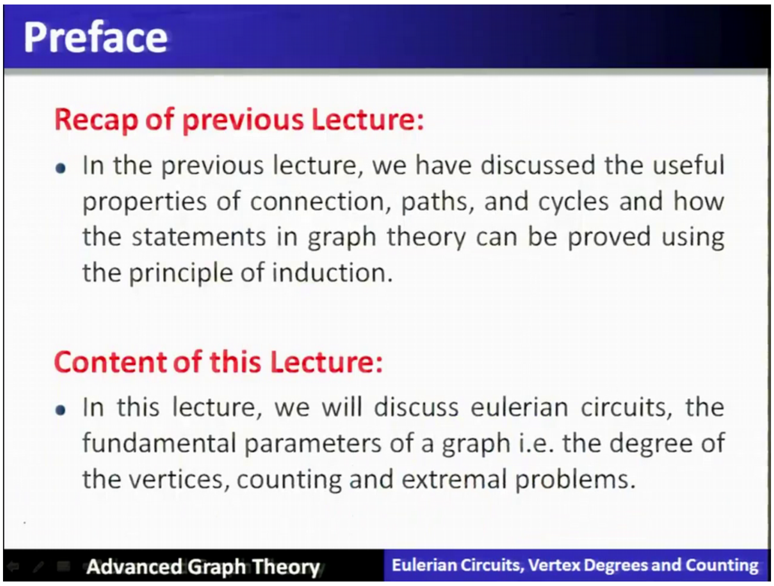
Dr. Rajiv Misra
Department of Computer Science and Engineering
Indian Institute of Technology, Patna

Lecture – 03

Eulerian Circuits, Vertex Degrees and Counting

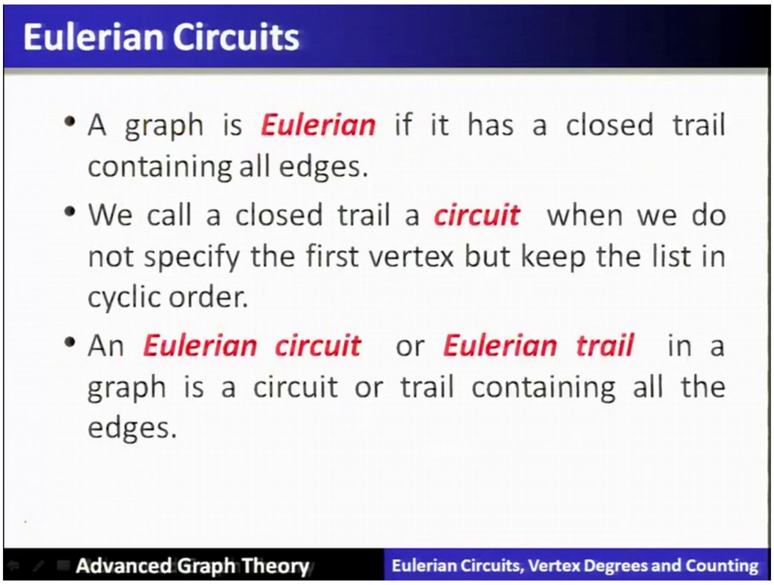
Lecture 3, Eulerian circuits, vertex degrees and counting.

(Refer Slide Time: 00:19)



Recap of previous lecture. In previous lecture we have discussed the useful properties of connection, paths, cycles, walk and how the statements in a graph theory can be proved using the principle of induction, and the principles of counter positive and so on. Now content of this lecture; in this lecture we will discuss Eulerian circuits, the fundamental parameters of a graph that is the degree of the vertices, counting an extremal problems.

(Refer Slide Time: 00:51)



So, Eulerian circuits background if you recall it is about the Konigsberg bridge problem.

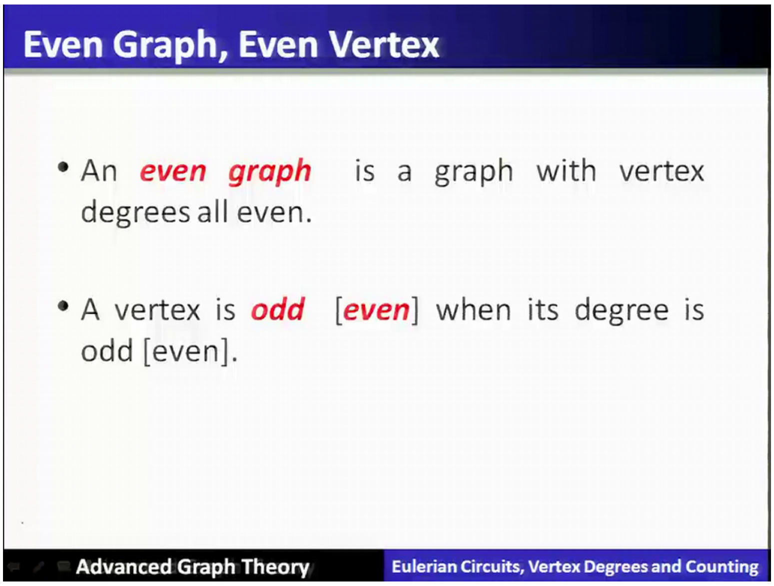
So, the famous mathematician Euler himself has given these particular proofs and ideas. So, we will see here in this part of the discussion. So, how that particular Konigsberg bridge problem which was posed and how this particular problem was solved by systematically going through all the background that is the definitions and then finally, in the hand of this particular lecture we will be stating up the theorem, which himself Euler has given and, but the proof Euler could not give; so, another mathematician Howard sir has given the proof. So, we will be looking up that proof in the end of this particular Euler's theorem.

So, let us begin with the Eulerian circuits. So, a graph is Eulerian, if it has a close trial containing all the edges. Trail we have already given the definition; that trail is a walk in which all the in which the edges should not repeat. If the edges are not repeating in the walk that does not mean that the vertices are also not repeating vertices are also not repeating vertices can repeat. And that particular walk in which only the edges should not repeat that is called a trail. A close trail means from the point where it is starting, if it is finishing that trail it is called a closed trail.

So, a graph is Eulerian if it has a closed trail containing all the edges. So, we call a close trail of circuit when we do not specify the first vertex, but keep the list in a cyclic order.

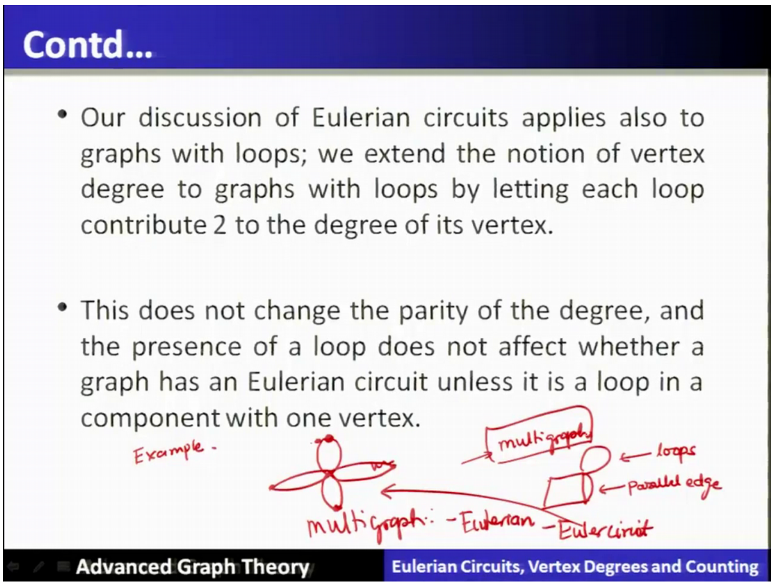
So, an Eulerian circuit or a euilerian trail in a graph is a circuit or a trail containing all the edges.

(Refer Slide Time: 02:55)



Let us take some more definitions. So, an even graph is a graph with the vertex degrees all even. So, if all the vertices having an even degrees, then that graph is called a even graph.

(Refer Slide Time: 03:12)



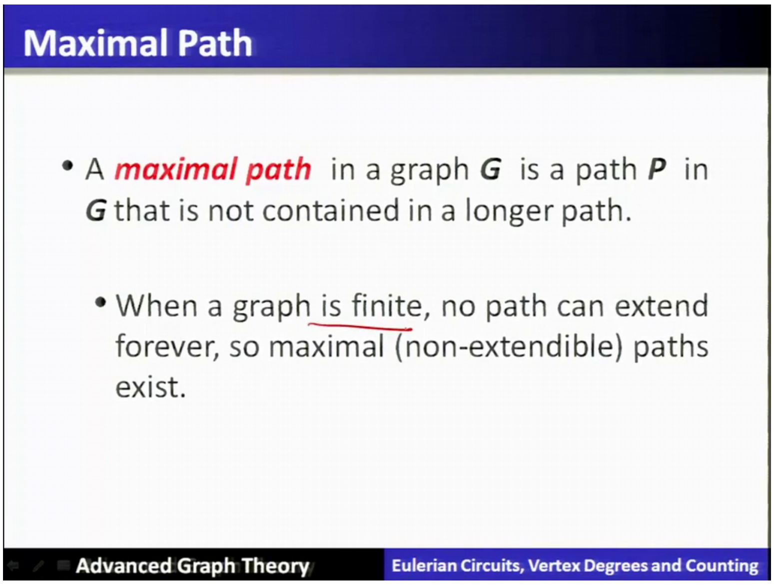
A vertex is odd when the degree is odd or even. So, our discussion of Eulerian circuit applied to the graphs with the loops, we extend the motion of the vertex degree to the graphs with the loop by letting each loop contribute 2 to the degree of it is vertices. So, this discussion as far as the Eulerian circuit is concerned, Eulerian circuits are basically applicable on a multigraph, as well. So, what is the multigraph? So, as you see that the graph which has basically the parallel edges; like, this, this is called a parallel edge or the graph which it contains the loops. This particular graph is called a mutligraph.

Whereas in the regular or a simple graph these parallel edges are not there and loops are not there. In this Eulerian circuits we will assume that the graph is given in the multigraph, why? Because the problems which you have seen in the Konigsberg bridge problem, similarly, the ah the mail transfer problem in which a postman delivers the mail on both the sides of a street will form basically the multigraph. And this particular problem is basically having practical relevance to establish whether the close trail containing con comprising all the vertices that is called Eulerian circuits are present in the graph or not.

Take the example; that if let us say there is a city , and the mail courier basically start from this point visits these set of houses, then go and visit these set of houses, and then visit these set of houses these set of houses, and return back to the same point this; particular kind of map of a particular locality can be represented in the form of a graph. So, where the vertices are present at all the intersections like this, and for each side of a street we can put an edge and when we complete this it will form a particular graph. So, this graph is basically a multigraph.

Now, the question would be whether this multigraph is an Eulerian graph or not. So, if it is a Eulerian graph then Euler circuit must be present in this particular kind of graph. So, that is why we say that this discussion; that is, the Eulerian circuits when we discuss it will include the multigraphs because multigraphs are present in most of the scenarios which we are going to use it for our different applications.

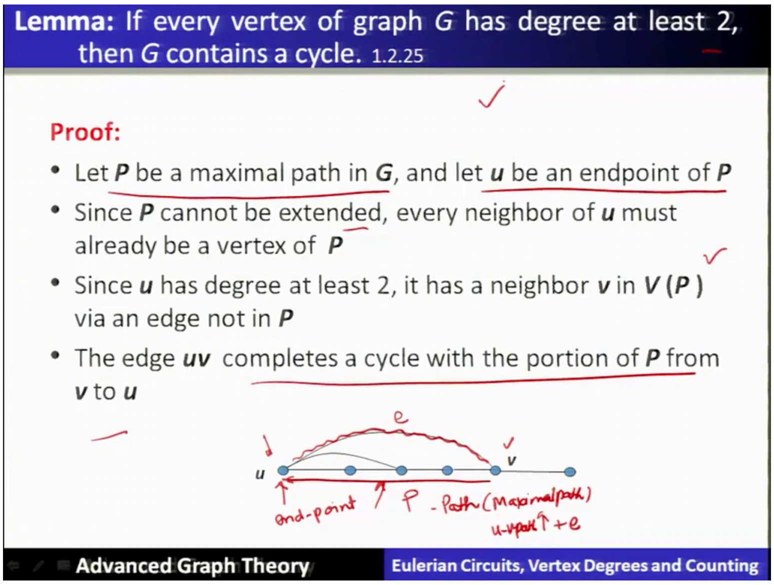
(Refer Slide Time: 06:50)



So, let us see some more definitions and then you will go ahead. So, maximal path; a maximal path in a graph is a path P in that particular graph G that is not contained in a longer path called a maximal path.

So, when a graph is finite no path can extend forever. So, a maximal path exist in a graphs which are finite, and that we are going to consider in our case that in the graph which is finite there exist a maximal path and we will use it in the extremal conditions.

(Refer Slide Time: 07:25)

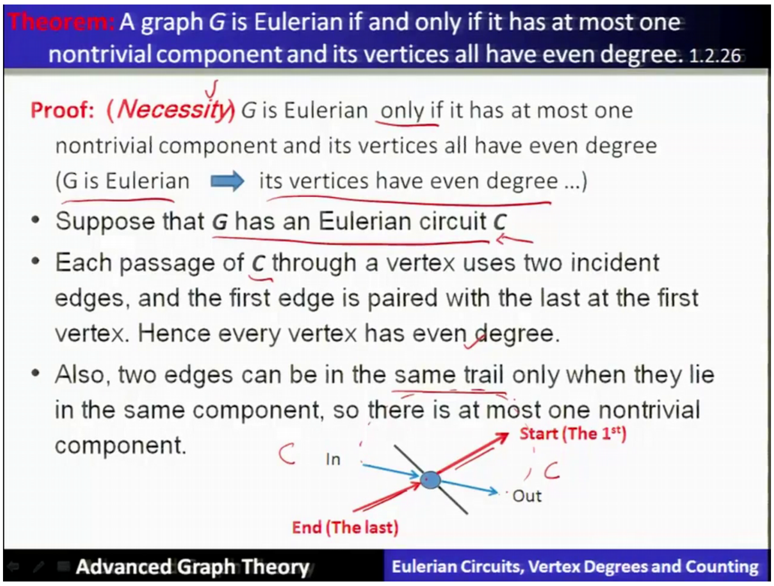


So, let us see the lemma, lemma is which states that if every vertex of a graph G has the degree at least 2 then that graph G contains a cycle. So, let P be a maximal path in a graph and let assume that u be the end point of that particular path. So, this is a maximal path. Maximal path is obtained by s v at the vertices it will keep on growing the path, and finally, a stage will come when you cannot extend the path further. So, that becomes a maximal path and let us assume that this P is a maximal path in a graph. Now since this particular maximal path cannot be extended. So, every neighbor or u. So, neighbors of u must already be the vertices of this particular path p.

Now, since u this particular node has the degree at least to from thy statement. So, it has the neighbor let us assume v in this particular vertices of a path, via an edge which is not in P this particular edge is not there in P; hence, this u v edge which basically will incident on you to increase the degree to 2, this edge u v and also the remaining portion of a path which is connecting u and v; that is, u v path portion of a path P, which is containing u v path plus e together will form a cycle.

So, hence this particular theorem will prove that if the graph G has the degree at least 2 then the G will contain the cycle.

(Refer Slide Time: 09:51)



Now, this is the important theorem which is given himself by the Euler, but the theorem Euler could not prove it. So, another math famous mathematician has proved this particular theorem after several years of basically existence of this particular theorem proof came quite late let us see this important theorem, which he states that a graph G is Eulerian if and only if has at most one non-trivial component, and it is vertices all have even degrees.

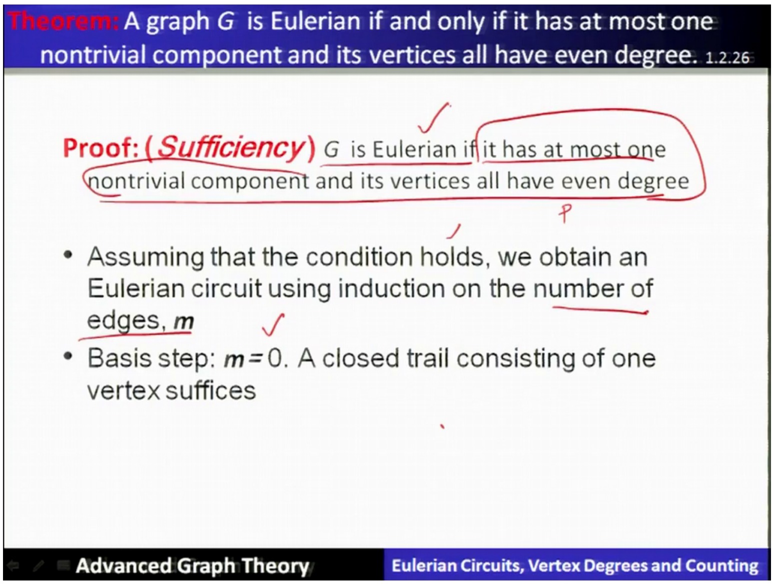
Now, this particular theorem will characterize the graphs, the class of graphs that is the Eulerian graph, with the properties which is stated on the other side, that it should have at most one nontrivial component, and it is vertices all have even degrees this particular property is very important, to characterize a particular graph as an Euler graph.

So now see first the necessity part of this particular condition one side, which it states that a graph G is Eulerian only if it has at most one nontrivial component, and it is vertices all have even degrees. So, that means, that if you are given a graph as Eulerian graph you have to prove all these properties. It will imply all these properties that it has one component, it has nontrivial component and all it is vertices have the even degree.

So, let us suppose that G is a given graph which contains an Eulerian circuit, and let us call that circuit as c. Now this particular circuit C when passes through the vertices it uses 2 incident edges. For example, of this particular vertex if that circuit passes C passes through it. So, it will incident through one edge and it will exit out from another edge of this particular cycle . Now and also the first edge and the last edge first edge is paired with the last edge at the first vertex. Hence, every vertex has the even degrees in this particular case of incidents, when the circuit passes through the vertices.

Also 2 edges can be in the trail only when they lie in the same component. So, there is at most one nontrivial component.

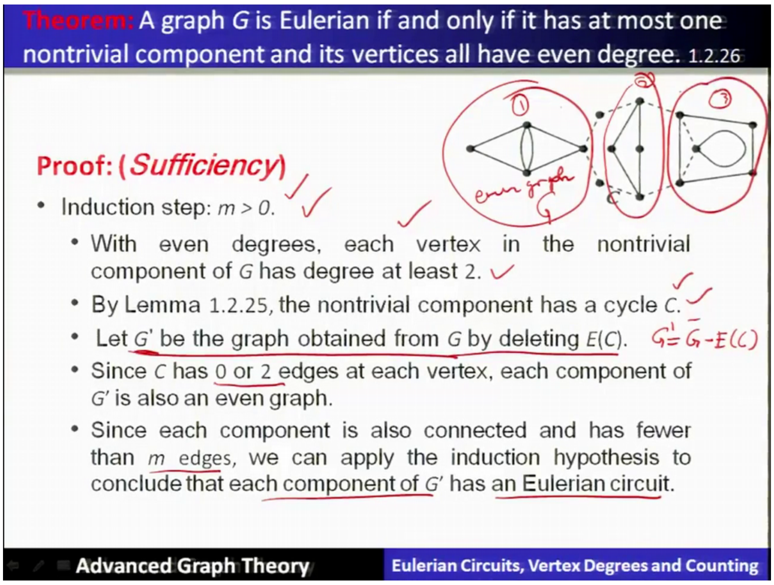
(Refer Slide Time: 12:50)



Now, we are going to see the sufficiency condition; states that, the other side of the proof that if these Euler if it has at most one nontrivial component, on all it is vertices have the even degrees. So, that means, now we will assume that this particular condition holds. Which condition? That, it has at most one nontrivial component and it is vertices all have the even degree.

So, let us assume that this particular condition holds. And we obtain an Eulerian now we have to obtain an Eulerian circuit using the induction on the number of edges m. So, when m is equal to 0 that is number of edges is 0; that is, called the base step of the induction. So, we will see that in a closed trial consisting of one vertex will suffice why because there is no edge; hence, this will be considered as an Eulerian circuit.

(Refer Slide Time: 13:46)



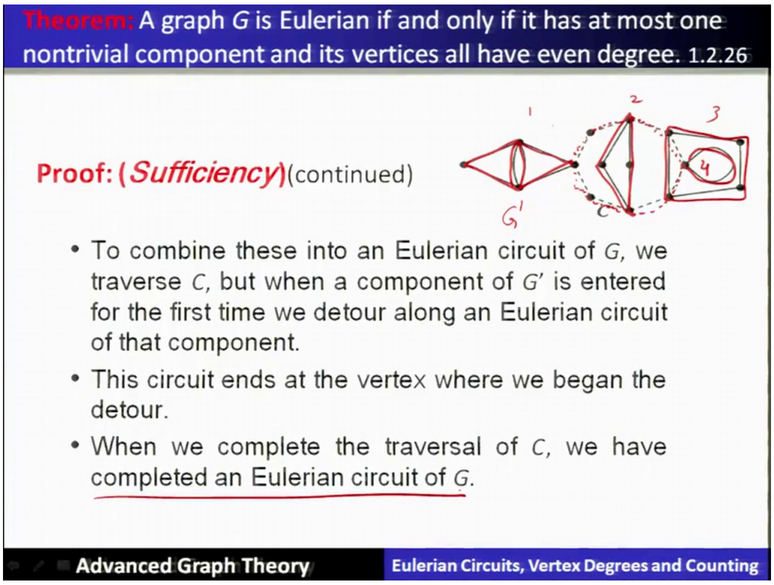
Now, this particular induction hypothesis, let us assume that it is true for n is greater than 0, with even degrees every vertex in a non-trivial component of G has degree at least 2. Now by lemma 1.2.25 the nontrivial component has the cycle C that we have already seen. In the previous lemma, that if the degrees are at least 2 then it must have a cycle in a nontrivial component. Let us assume that the cycle is C which exists, why because, we have seen that particular degree at least 2.

Now, let us remove the cycle from the graph G, and we will obtain a G prime via graph obtained from G by deleting that particular cycle. So, G prime is nothing but G minus e of C. So, here in this particular example if this is G and this C cycle which is shown as the dotted if you remove it, what will happen? It will be broken into component one component 2 and 3 components will basically be available, once it is deleted.

Now, since C that is the cycle has 0 or 2 edges at each vertex. So, each component of G prime is also an even graph, why? You can see that, this particular vertex which touches this particular cycle it has the degree 2, the other vertices which are not touching this particular cycle has degree 0 in this particular cycle. So, C has 0 or 2 edges at each vertex. So, each component of G prime is also a even graph.

Now, since each component is also connected, and has pure than images, we can apply the induction hypothesis to conclude that each component of G prime has an Eulerian circuit .

(Refer Slide Time: 16:17)

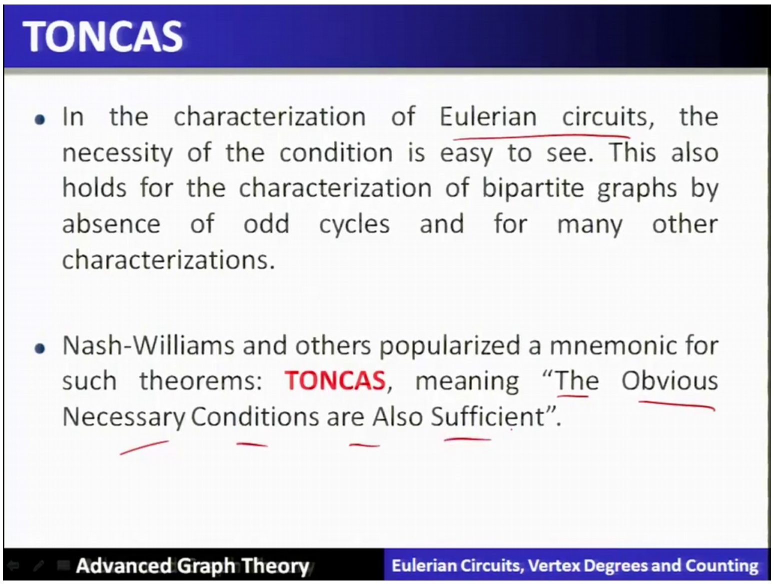


So, this particular components have an Eulerian circuit. Now further on to combine these into the Eulerian circuit of G v traverse C v traverse c, but when a component of G prime. So, this is one component, this is 1 component, 2 component, 3 component. But when a component is entered for the first time we detour the an Eulerian circuit of that component.

Let us take the example. We are basically traversing through the through the circuit of G, and once we enter for the first time in this component we detour, this particular component, and come back to the same point, and then go ahead when we enter for the first time to the second component we detour. And come back to the same point, and when we enter for the first time third component we detour, and so on. When we enter here the 4th component this is the 4th component. We come back to the same point detour this component is already visited and we come back again to the same point from where we have started. This completes an Eulerian circuit of G, the way we have detoured it.

So, hence what we have proved here is that we started by assuming that this particular condition holds and now we have obtained a Eulerian circuit hence this particular graph is Eulerian that we have proved. So, both sides we have completed the proof.

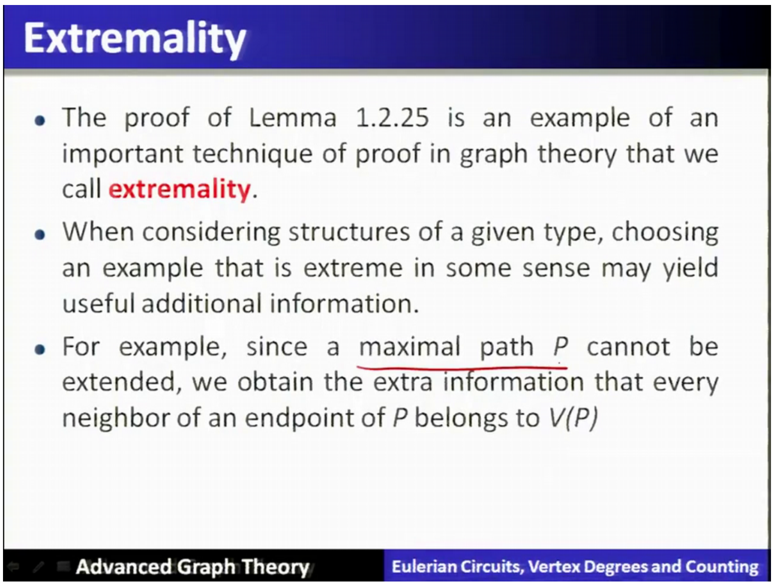
(Refer Slide Time: 18:08)



Now this particular theorem will characterize the graphs as Eulerian graphs. So, in the characterization of Eulerian circuit, the necessity of the condition is easy to see it is also holds for the characterization of bipartite graph by the absence of odd cycles and for many other characterizations. So, all these characterizations will basically will make the equivalent statement. And we are using them in our theory the graph theory.

Nash Williams and others popularized a mnemonic for such theorem for toncas; meaning that the obvious necessary conditions are also sufficient extremality.

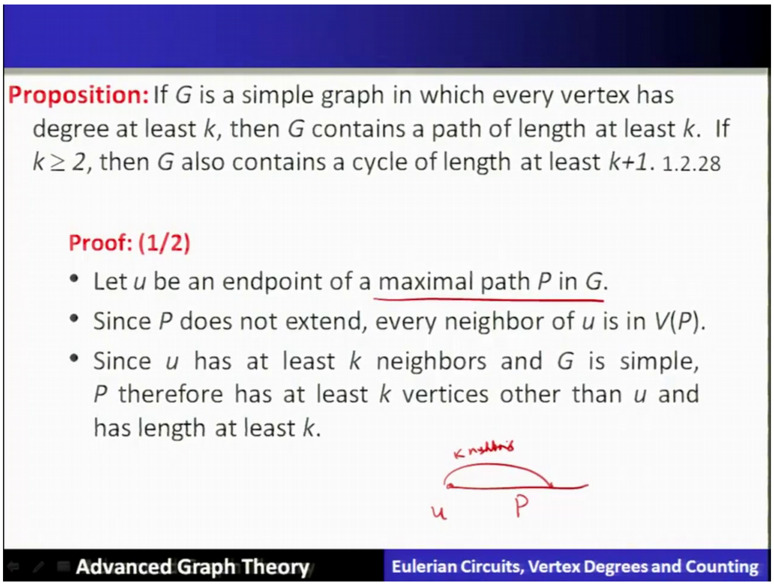
(Refer Slide Time: 18:51)



So, the proof of lemma 1.2.25 in which we have use the maximal connected subgraph or the maximal path that is an extremal condition is an example of an important technique of a proof in graph theory, and we call it extremality, when considering the structures of a given type choosing an example, that is extreme in some sense may yield useful additional information. For example, since a maximal path P cannot be extended and this property we have used in the previous theorem proof.

So, this particular extra information that every neighbor of an end point of a P belong to the path why because it cannot be extended further. So, this extra information is utilized in the proofs of the theorem, and this is called extremality property and that we have used it.

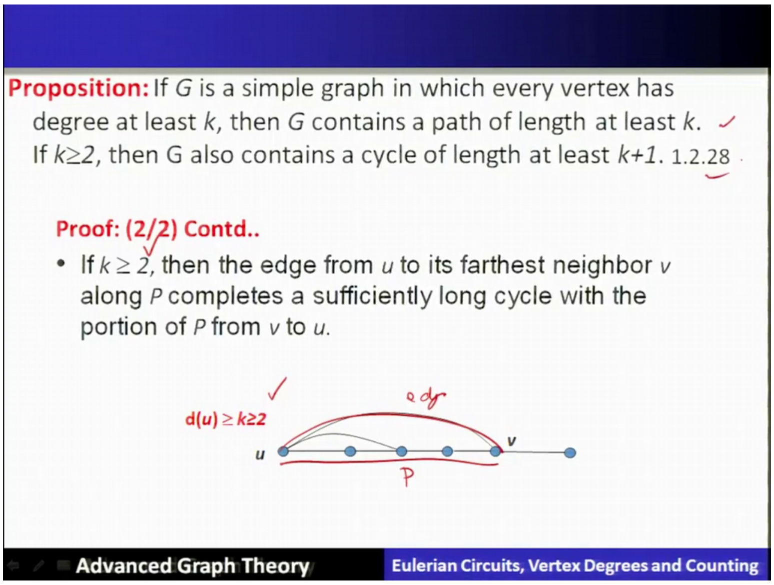
(Refer Slide Time: 19:52)



Now, let us see the preposition. If G is a simple graph in which every vertex has the degree at least k then G contains a path of length. At least k now if this particular k value is greater than or equal to 2, then G also contains a cycle of length at least k plus 1. Let us see the proof quickly.

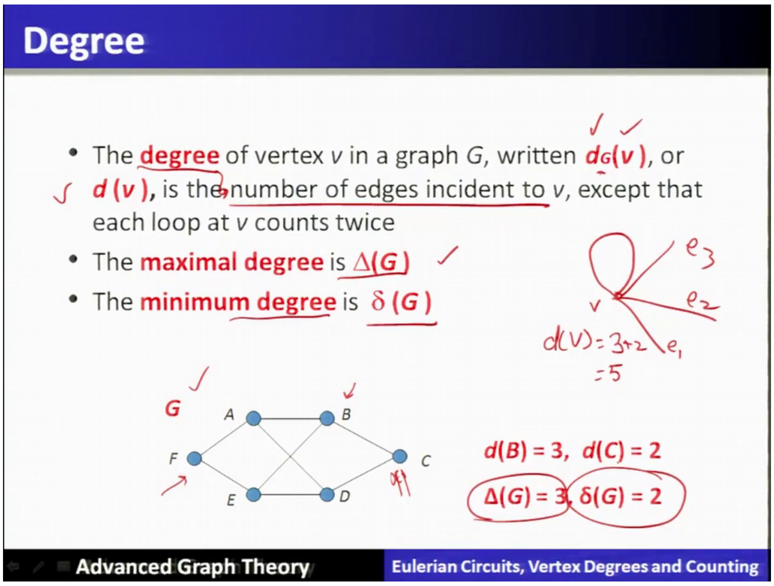
Let u be end point of a maximal path. So, we will assume a maximal path P, and u be the end point. Now since P cannot extend. So, every neighbor of u will be in v P. Now since u has at least k neighbors and G is a simple graph. So, so P therefore, has at least k vertices other than u, and has length at least k.

(Refer Slide Time: 20:54)



Now, if the k is greater than or equal to 2, then the edge from u to a farthest vertex to a farthest neighbor, let us say v along P this particular edge along P completes a sufficiently long cycle with a portion of P from v to u. And hence basically both the conditions we have proved this particular preposition.

(Refer Slide Time: 21:26)



Now, let us see the important notion of the degrees and their properties and some important lemma that is called a hand shaking lemma based on this particular property of a graph.

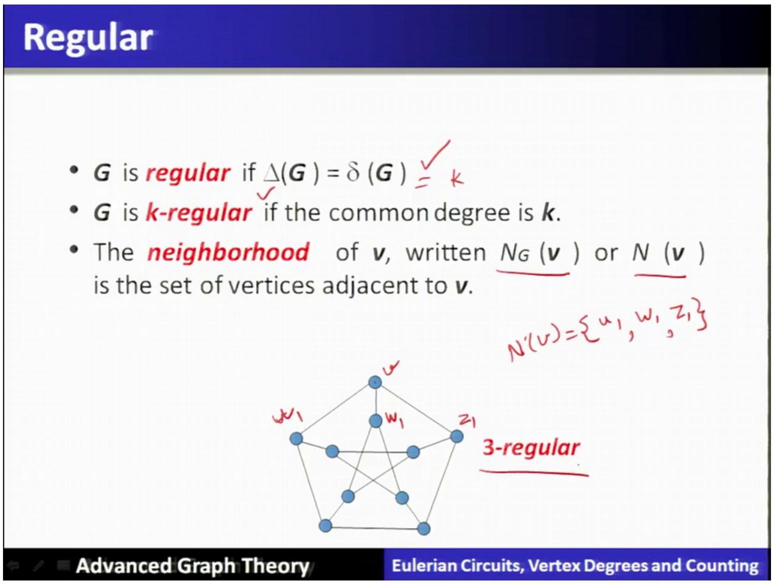
So, the degree of a vertex v is written as d of v since for the graph G we are talking about. So, we have to write on G. For it only one graph is basically considered in the problem, then we straight away directly write the v means the number of is the degree of a vertex.

Degree of a vertex means, the number of edges which are incident to that particular vertex. So, degree is nothing but the number of degree is the number of edges which are incident to that particular vertex. So, if this is the vertex how many edges which are incident. This is edge 1, this is edge 2, edge 3. So, the degree of this particular vertex is equal to 3 in this case why because 3 edges are incident , to this particular v; except that, each loop at v counts twice for example, if loop is also there in a multigraph, then it will count 2 to it. So, the degree then becomes 5 here in this particular case.

So, having defined the degree, then now we have to see another property of a graph. This is called a maximal degree. So, a particular vertex having the maximum degree will become a property of the entire graph called a called a maximum degree of a graph and is represented by the symbol big delta of G. Similarly, a particular vertex which is having the minimum degree in the graph will basically also induce a property into a graph that is called little delta of G is called a minimum degree of a graph, these are the properties of the entire graph.

So, here in this particular example. We can see that, the degree of B is 3, degree of C is 2. And there is no other vertex which is having degree more than 3. So, hence the maximum degree on this graph is 3, and C has degree 2. So, no other vertices is having degree less than 2. So, obviously, the minimum degree of a graph will become 2 here in this particular case.

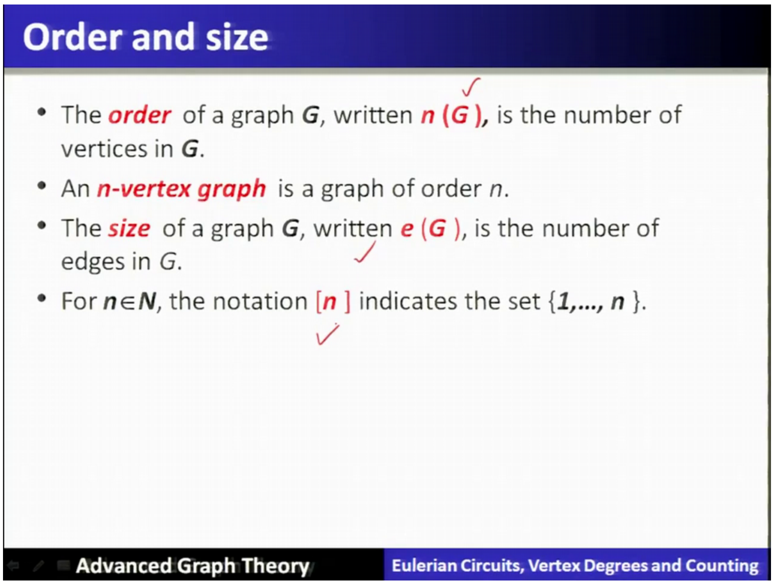
(Refer Slide Time: 24:01)



Now for a regular graph. Now if G is a regular graph, then this property holds that the maximum degree is equal to the minimum degree of a graph. Now if G is a k regular graph, then basically this particular common degree is equal to k.

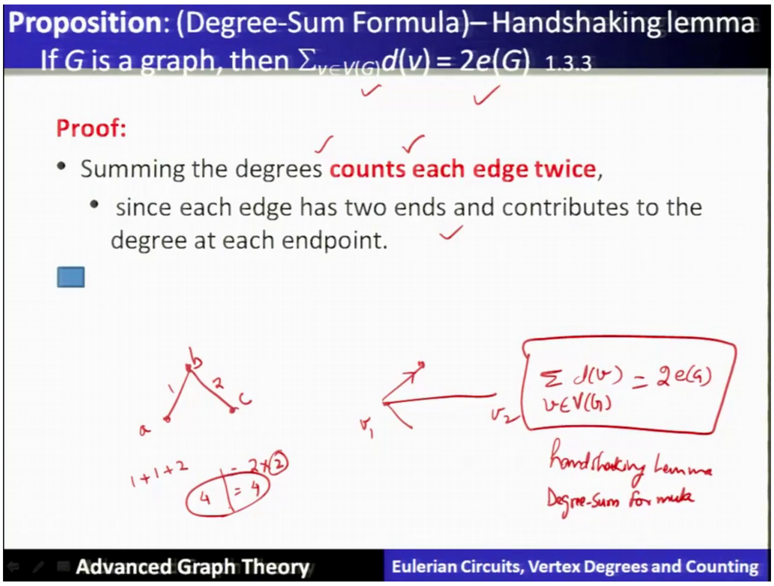
Now, the neighborhood of a graph is represented by N G of v or n of v is nothing but the set of vertices which are adjacent to v. For example, if this is vertex v. So, let us say this is u 1, this is w one z 1. So, the neighbors of v will be the these set of nodes. Now you see that the degrees of all the nodes is 3. Here in this case that is why this graph is a 3-regular graph.

(Refer Slide Time: 25:01)



Order and size of a graph. So, order of a graph is written in this particular form N G; that is nothing but the total number of vertices, which are present in the graph is called order of a graph. Similarly, the size of a graph is nothing but the total number of edges which are present in the graph called size of a graph. And the notion the order of a graph we can represent through the indices one to n.

(Refer Slide Time: 25:30)



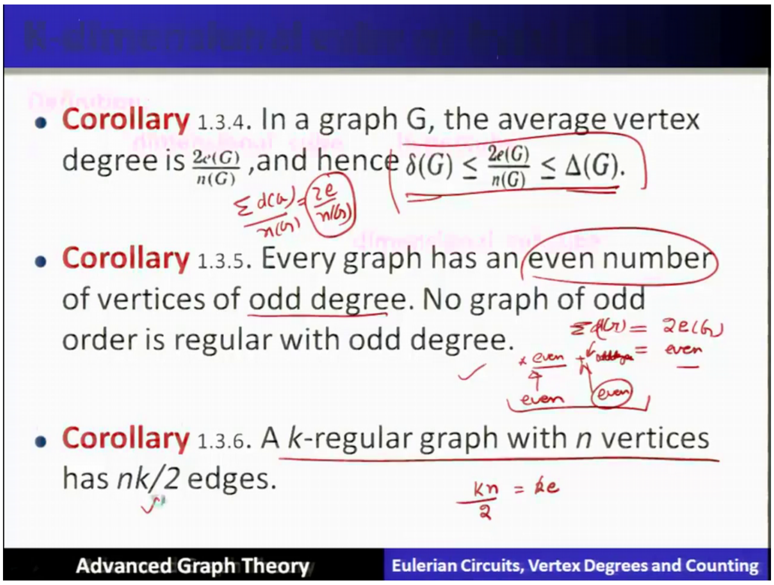
Now, we have to using the degrees of a graph, we will see a important preposition or important lemma that is popularly known as hand shaking lemma. So, this proposition is called degrees sum formula; which is stated as so, if G is a graph then summation of the degrees of all the vertices in G is nothing but twice the number of edges in the graph.

So, let us see the proof now if we sum the degrees then. So, this is the particular vertex v one and this is particular vertex v 2. So, whenever we are summing up the degrees of a particular vertex is nothing but the edges which are incident on that particular vertex. Similarly, this edge is incident on the other side of the vertex also , other side of the edge that is also another vertex. So, every edge is incident on 2 vertices. So, hence the degrees when you are sum is being counts each edge in a 2 times. Hence, each edge has 2 ends and contribute to the degree at both the ends.

Hence when we come up with the degree sum; that means, we have ended up counting the edges 2 times. Hence this becomes a hand shaking lemma. It is also called degree sum formula. If you want to see the example, let us take this particular graph which is having 3 verticesm a b and c, and there are 2 edges 2 edges means total number of edges are 2 2 into 2, that is 4. And if you count the number of degrees. So, this particular the degree of vertex a is 1 degree of vertex C is also 1 degree of vertex b is 2. So, that becomes degree sum will become 4. To sum of the degrees of all the vertices is nothing but twice the number of edges how many edges are there one and 2 there are 2 edges.

So, hence this particular theorem is always proved. So, if you count the total number of degrees if you count the degree sum of a particular graph, you will end up with the twice the number of edges which are present in the graph.

(Refer Slide Time: 28:28)



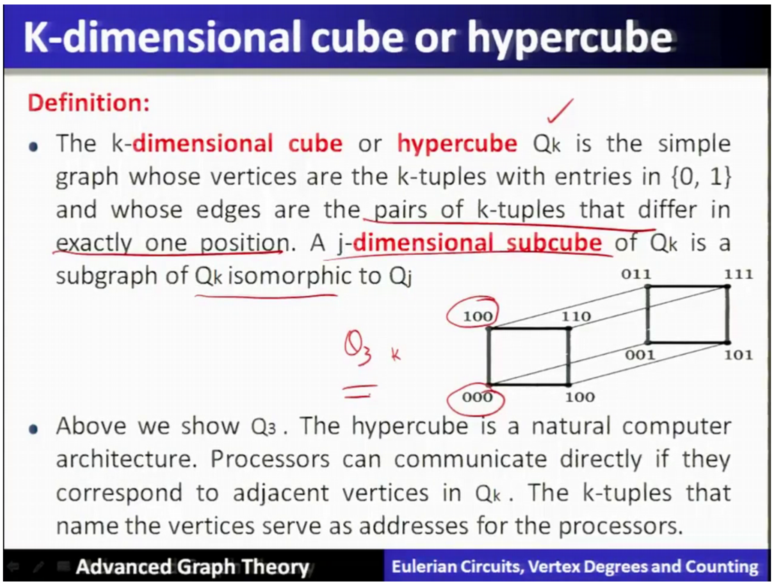
So, this hand shaking lemma has given some important corollaries let us see them. So, in a graph G the average vertex degree is 2 e by n g. So, 2 e means degree sum; is nothing but 2 e. And divided by total number of vertices that becomes the average vertex degree of a graph. Now, this degree of a graph is at least the minimum degree of a graph and basically at most the maximum degree of a graph; hence this corollary is proved.

The second corollary states that every graph has an even number of vertices of or degree. So, every graph has an even number of vertices of odd degree. Now you know that sum of degrees of vertices in a graph is, twice the number of edges of a graph, and this is an even number.

So, degree sum is an even number. So, there are 2 types of or there are 2 parities of the degrees in the graph one is the even degrees other is the odd degrees. So, the even degrees whether you multiply with even number or a odd number this particular factor will become an even number. In the odd degree, it has to be a even number to be multiplied. So, that the entire component will become even which is same as the other side. So, how many? So, that is some even number is to be multiplied.

So, even number of that is how it says that. So, every graph has an even number of odd degrees. And hence the corollary proves, and no graph of odd order is regular with the odd degree this is straight forward we can obtain another corollary which says that a k regular graph with n vertices. So, k regular graph with n vertices what is the degree sum k n, and that is that will count twice the number of edges. So, if you want to find out how many edges are there then you divide by 2. So, this will be total number of edges. So, if a k regular graph with n vertices are there, then it has n k by 2 different number of edges.

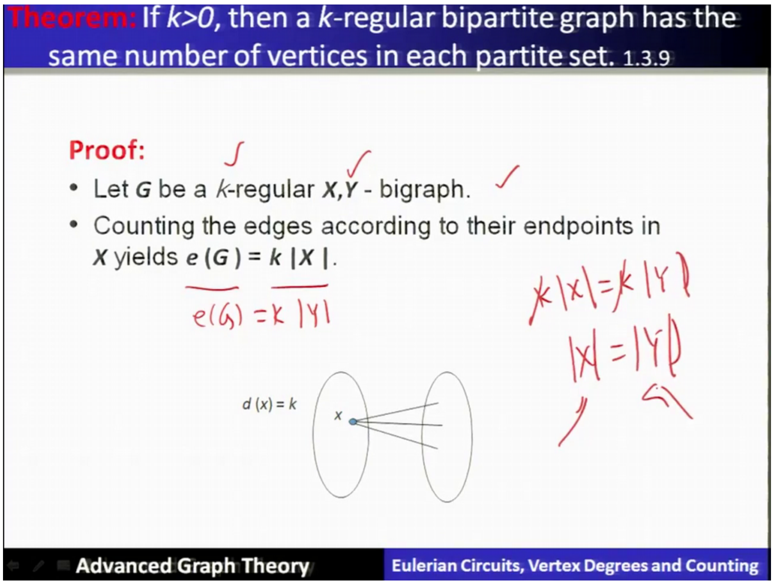
(Refer Slide Time: 31:06)



Now, we are going to see a k dimensional cube or a hyper cubic structure. So, a k dimensional cube or a hypercube Q k is a simple graph, whose vertices are k tuples with the entries in 0 and 1, and whose edges are the pairs of k tuples that differ in exactly one position. For example, so, this is the Q 3; Q 3 means here the vertices are represent by these 3 tuples 1 2 3. So, there is an edge, between these 2 vertices and they differ only in one bit or 1 place of this k values.

Now, j dimensional sub cube of a Q k is a sub graph of Q k isomorphic to Q j. Now above we show Q 3. So, that I told you Q 3 it is shown over here. So, the hypercube is a natural computer architecture. Processors can communicate directly if they correspond to the adjacent vertices in Q k. So, k to pull that name the vertices survive the addresses for the processors.

(Refer Slide Time: 32:27)

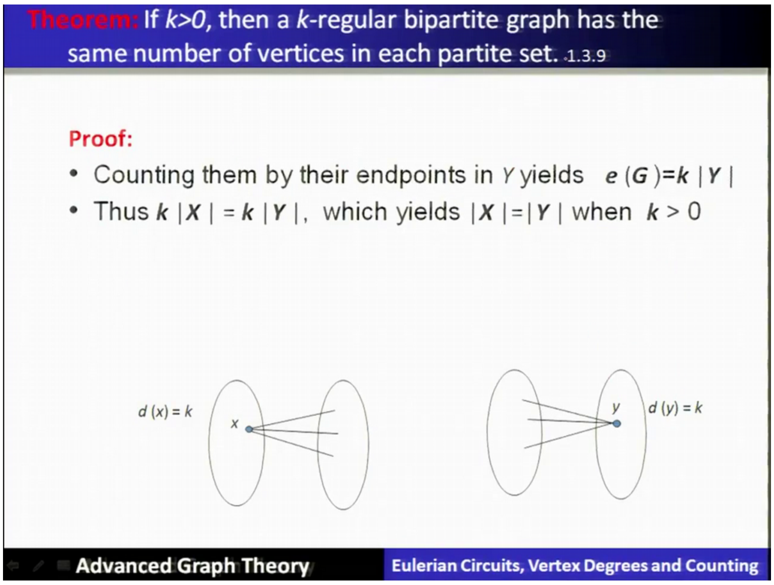


Now, let us see the theorem, which is quite simple theorem and which basically can be proved also in a very simple manner quickly we will see that. Now if k is greater than 0 or k is non-0 , then our k regular bipartite graph has same number of vertices in each partite set. Let us prove that. Now if G be a k regular X y by graph, then we can count how many edges according to their end points in x. So, if you if you see how many edges are there. So, so how many edges will be incident k number of edges are there because it is a k regular graph. So, it will be k times X, X means all the nodes which are present in this partite set X that is e G.

Similarly, if we count how many edges are incident on Y. So, that also e G will be k times the number of k times all the vertices.

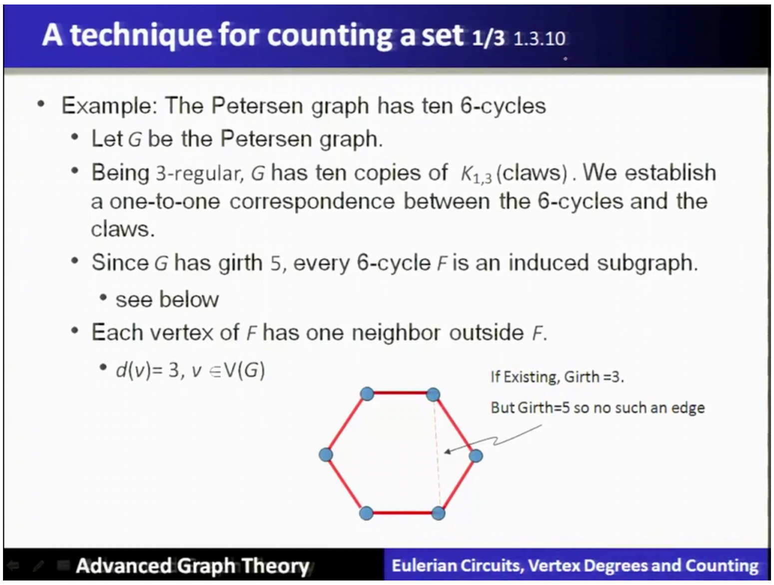
Now, edges have same. So, k times X is equal to k times y, and k and k will go off. So, X is equal to the y; that means, the partite set X is same as partite set Y. So, as the same number of vertices in each partite set hence the theorem is proved.

(Refer Slide Time: 33:51)



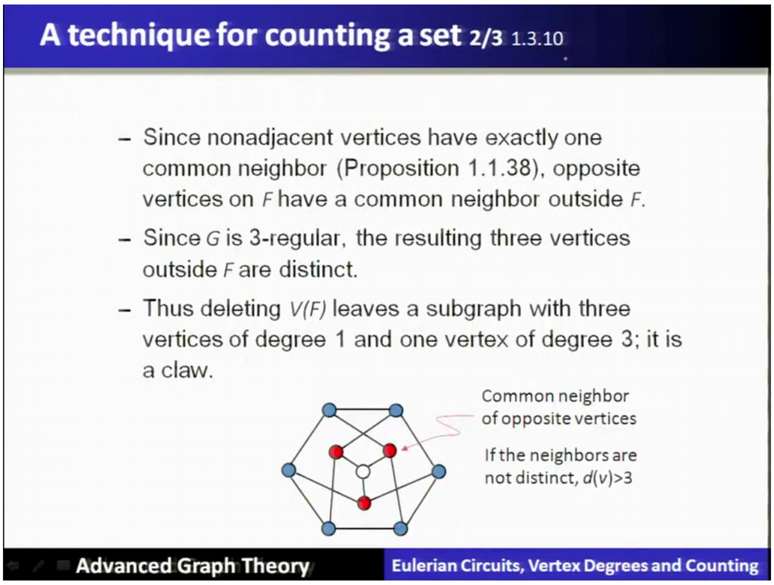
So, this is the theorem which we have already proved.

(Refer Slide Time: 33:53)



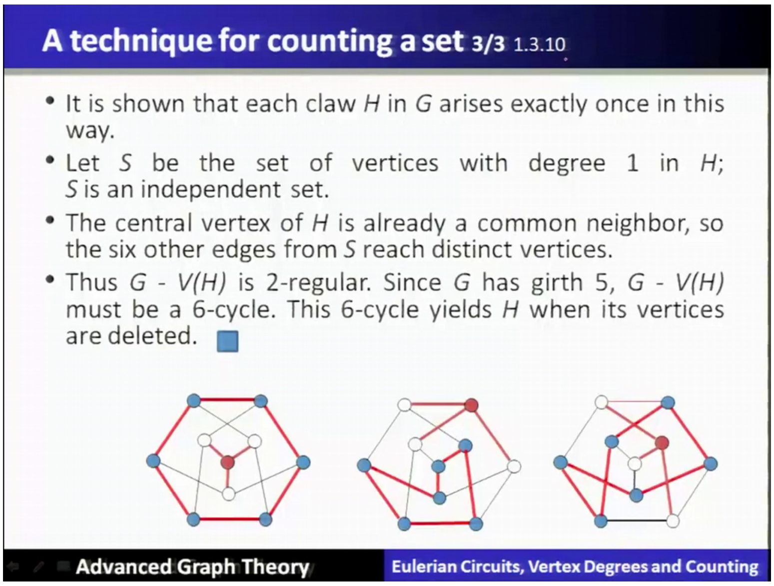
That now there are the techniques which will be applied to for a counting. So, this is the example in which the Peterson graph we can say that it has 10 6 cycles.

(Refer Slide Time: 34:09)



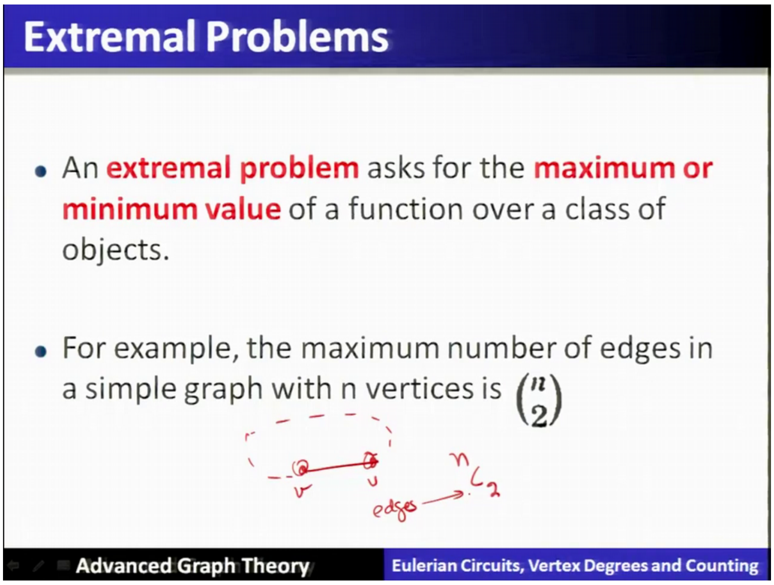
And we can definitely can see this there are 2 different ways. We can draw the Peterson graph 3 ways.

(Refer Slide Time: 34:11)



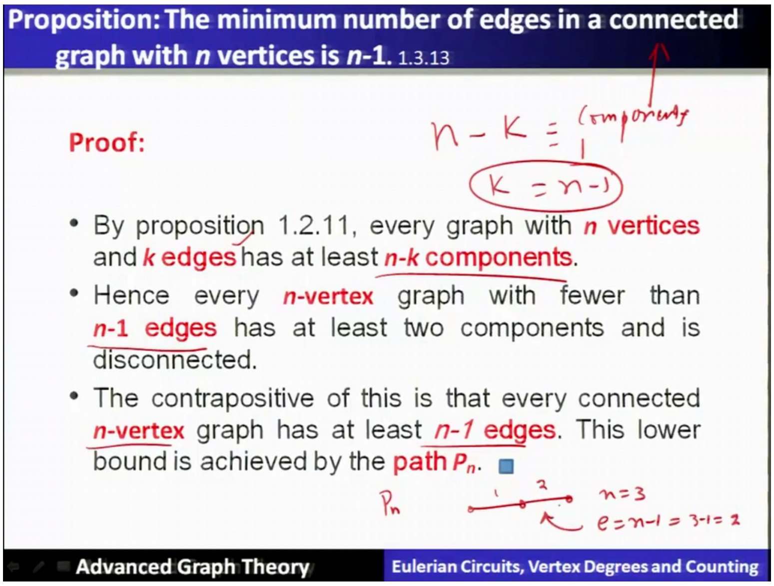
And we can find out using clock that, it has these many properties.

(Refer Slide Time: 34:22)



Now, let us see some more extremal problems. So, extremal problems ask the maximum or minimum values of a function over a class of objects. So, for example, the maximum number of edges in a simple graph with n vertices is n choose 2. So, that means, if you see the edge in a particular pair of vertices u v. So, out of or n vertices, every time you choose 2 vertices and place an edge; hence, n choose 2 times basically you will be or number of ways you will be adding the edges. So, the maximum number of edges which can be placed in a graph of n vertices and choose 2.

(Refer Slide Time: 35:15)

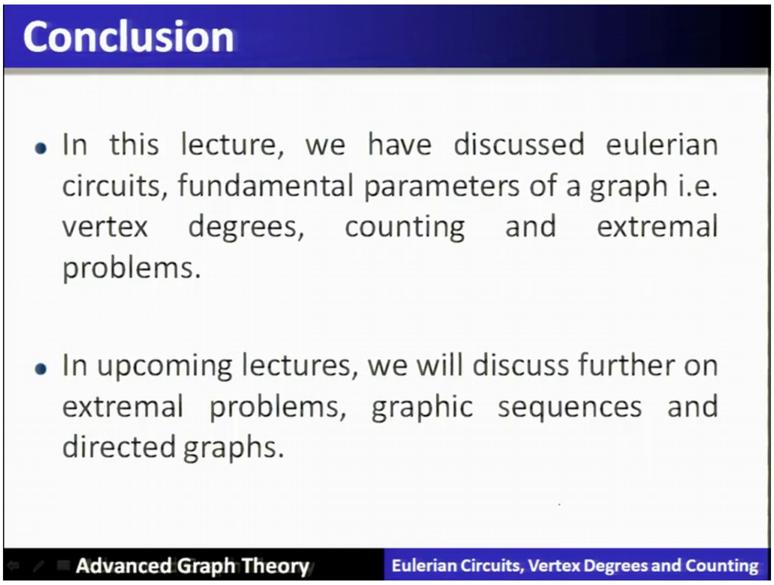


Similarly, let us see the how many minimum number of edges in a connected graph with n vertices can be there. So, with every graph with n vertices and k edges their we have proved earlier a particular lemma, that since that it has at least n minus k components. So, that means, with n vertices and k edges will have that many number of components; when you say a connected graph; that means, the number of component you want to make it as 1.

So, what is the value of k over here k is equal to n minus 1. So, that means, there are n minus 1 edges; minimum number of n minus 1 edges are present where the graph having n vertices as connected graph. And if the number of vertices goes less than n minus 1 then it will be disconnected.

The example of a graph having n minus 1 edges is called a path P n. So, we can see this is P 3. So, how many edges are present? 1 and 2, there are 2 edges are present. And it has 3 vertices n is equal to 3. So, number of vertices will be n minus 1; that is, P minus 1 that is 2. So, 2 verti 2 edges are present here in this particular graph.

(Refer Slide Time: 36:49)



So, conclusion in this lecture we have discussed Eulerian circuits, fundamental properties of a graph, vertex, degrees, counting and extremal problems. In the upcoming lectures we will discuss further extremal problems, graph, graphic sequence (Refer Time: 37:03) and directed graphs.

Thank you.