Advanced Graph Theory
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Lecture – 19
Planar Graphs

Planar graphs.

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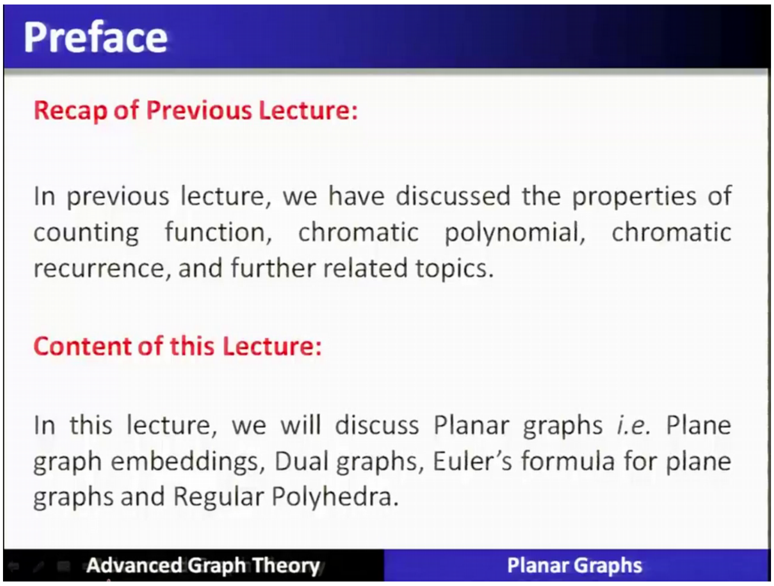

Recap of previous lecture: we have discussed the properties of counting function, chromatic polynomial, chromatic recurrence and the further related topics. Content of this lecture: in this lecture, we will discuss planar graphs, that is, the plane graph embeddings. We will also cover dual graphs, Euler's formula for plane graphs and regular polyhedra.

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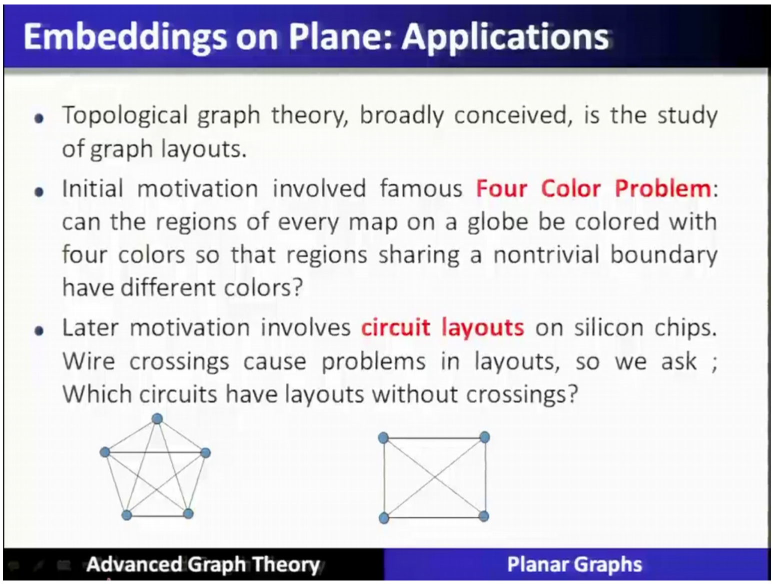

Embedding on a plane, let us look into the applications of this concept. Topological graph theory, broadly conceived, is the study of different graphs layouts. Hence embedding on a plane is widely used in a different field that is called topological graph theory. Now, initial motivation of this particular putting graph on a plane involved in a famous four color problem which basically was a question that can the regions of every map globe is colored with four colors. So, that the regions shearing non trivial boundary have different colors. Later motivation of graph embedding on a plane involves circuit layouts on a silicon chip. Wire crossing cause problems in a layouts. So, the question is asked which circuits have a layouts without crossings.

The two example graphs are shown over here are you can see that there are the embeddings of a graph which has the crossings which are shown by the red dots can be redraw this particular graph so that these edges should not cross other then at the end vertices. Similarly, the second graph also has these particular crossings. So, edge crossings are not required in the circuits. So, hope we can plot the graph without crossings and that is called embedding on a plane.

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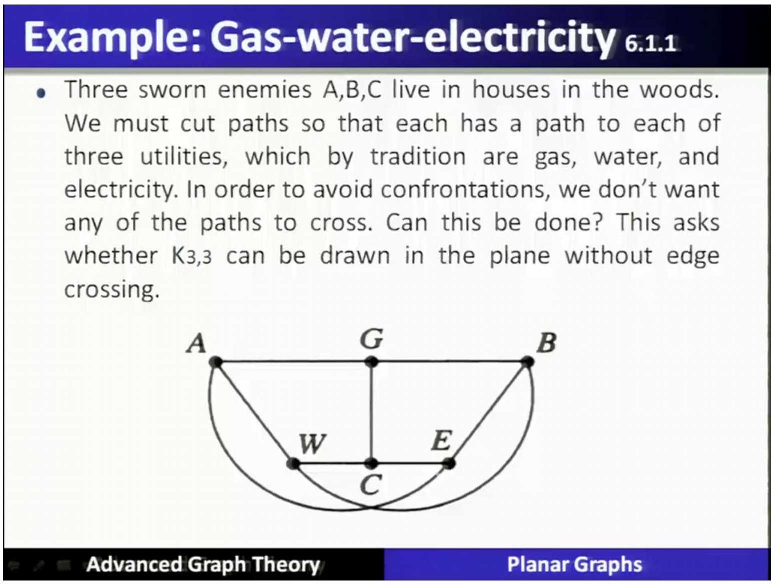

Another example is about laying down of gas-water-electricity. In the three different houses let us say ABC, so, we have to cut through the path and we have to do this. So, that these paths do not cross each other. Now, the question is can we do this? So, that is this particular question can be answered if we can draw K 3, 3 graph without edge crossings. We will explore in this particular lecture, what are the conditions and what are the different graphs which we can draw on a plane without edge crossings, hence they are called the planar graphs.

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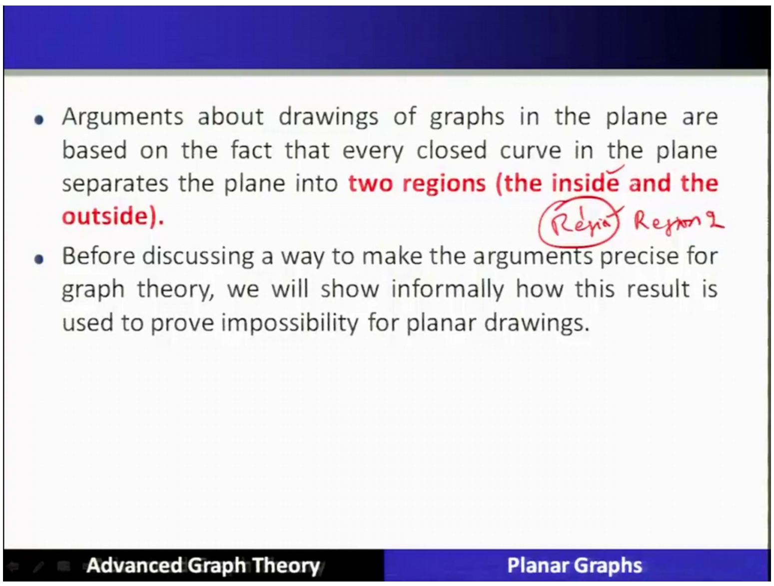

So, the arguments about drawing graph in the plane are based on the fact that every closed curve in the plane separates the plane into two regions. For example, this is the closed curve it separates into two regions, this is region 1 and this is the region 2 one is inside and the other is outside. So, before discussing a way to make the arguments precise for the graph theory we will show informally how this result is used to prove impossibility of for the planar drawings.

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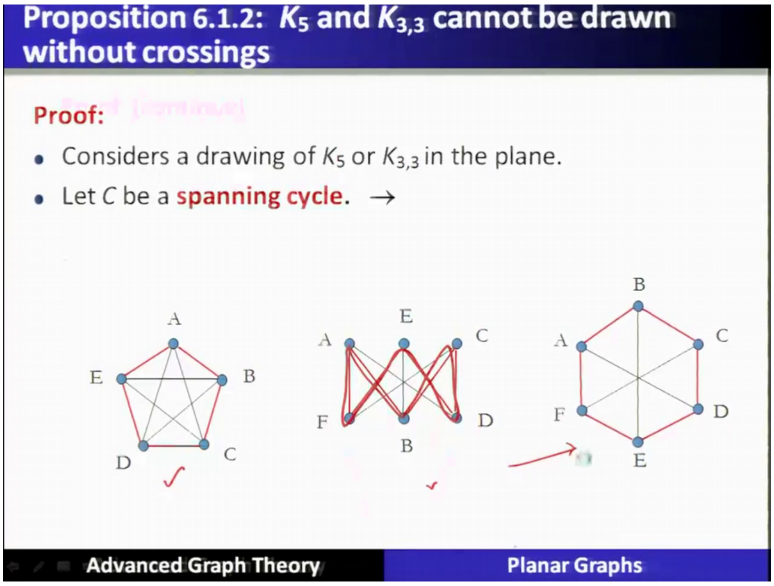

So, proposition K 5 and K 3, 3 cannot be drawn without crossings. So, K 3, 3 graph we have seen earlier in the pipe line problem let us see whether here it is the preposition that is cannot be drawn without the crossings. So, let us see these particular two graphs K 5 and K 3, 3 and we consider the drawings. So, in these particular drawings of let us say K 5 and K 3, 3 there exist a spanning cycle which is shown by the red color. Here also there is a spanning cycle which starts from A and come backs to A again which can be shown over here in another embedding and which is shown as a spanning cycle.

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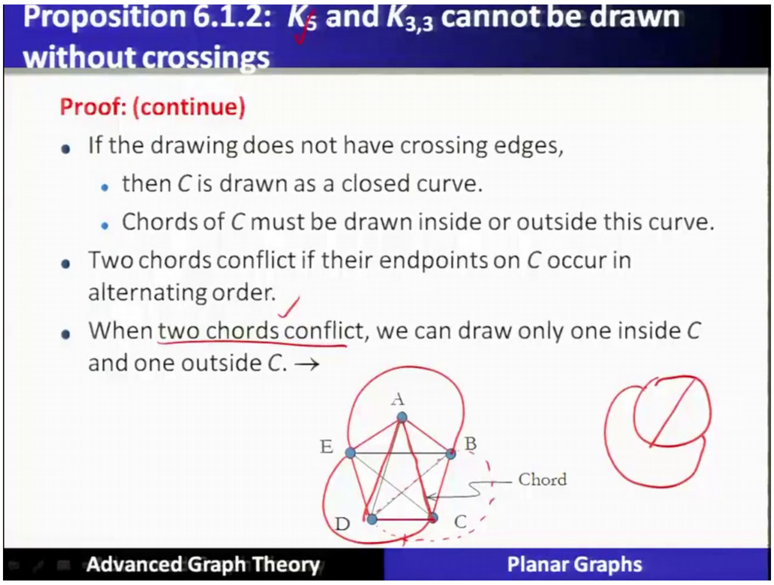

Now, if the drawing does not have the crossings then this cycle spanning cycle is drawn as a closed curve. The chords of C must be drawn inside or outside this particular curve to chords conflict if their endpoints on that spanning cycle occur in alternating order when two chords conflicts and we can draw one inside the other is outside. What it is saying is that if this is a cycle and if there are two edges which are crossings then what we can do is we can redraw the other edge, one is inside other is outside hence this crossings can be removed if there are two chords conflict we can obtain this particular drawing. Let us see with this concept can be draw this particular drawing that is K 5 using this particular concept.

Now, here we see that this is the spanning cycle, let us draw one chord let us re draw another chord. So far there is no crossing problem, now to take this particular chord we can draw from outside and if we consider this one we can also draw from outside, what about this particular line. So, if you take from outside, this will conflict. So, it says that only if the two chords we can resolve by such a drawing, but a three chords are conflicting we cannot resolve, they have to cross. So, hence K 5 cannot be drawn without edge crossings.

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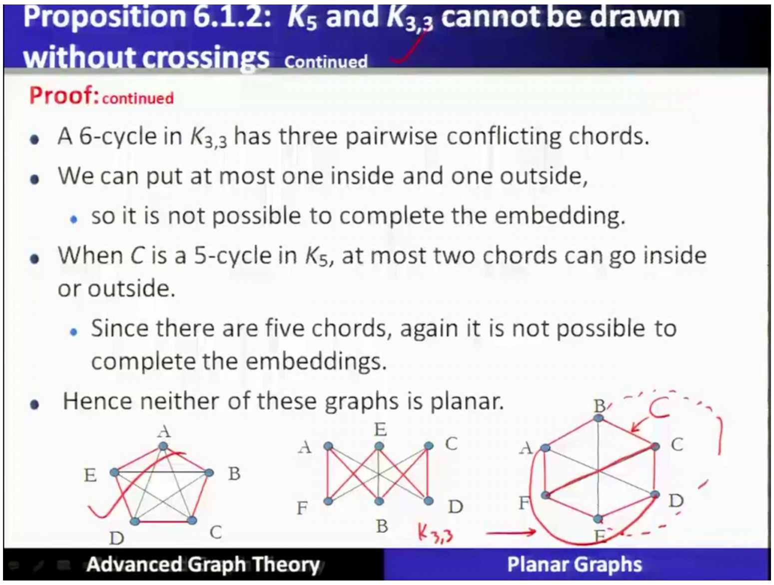

Let us see about K 3, 3 graph. Now, this is K 3, 3 graph we have already seen this is K 3, 3. It can be redrawn in this way also. Let us consider this particular drawing of K 3, 3 we have a spanning cycle C which is shown as the red color, now let us see the chord. First chord we draw inside the other chord we can draw from the outside hence the conflict we can remove, remaining the third chord if we draw from this side even then it will it will cross to the outside chord. If we draw inside then it will cross with the inside chord. Hence, we cannot draw K 3, 3 without the crossings. So, therefore, these graphs are not planar.

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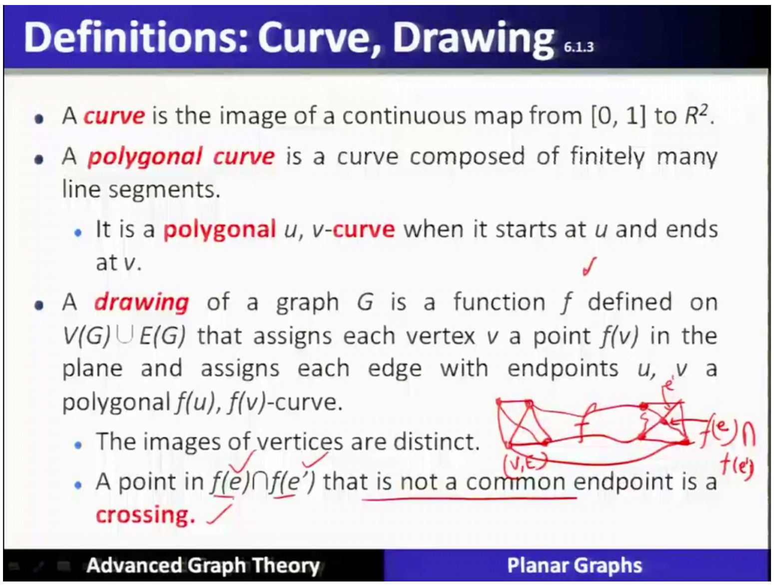

So, a curve in a drawing; so, a curve is a image of a continuous map from 0, 1 to R square of polygonal curve is a curve composed of finitely many line segments. It is a polygonal u, v curve when it starts at u and ends at v. A drawing of a graph G is a function f defined on the vertices and the edges union that assigns each vertex a point f v in the plane and assigns each edge the endpoints u, v with a polygonal f u curve.

The images of vertices are distinct the point in f e intersection f e prime that is not a common end point is a crossing. Meaning to say that if let us say this is the graph with a set of vertices and set of edges given then we require a function f which is nothing, but a drawing of this particular function f. So, f will draw will map every point and when there is an edge between these two points so, f will draw a curve polygonal curve.

Now, if there is a point when while drawing the edges e and e prime if there is intersection if they are basically in the sense there is a drawing of this is e and this is e prime. So, f of e and f of e prime intersection if it is other then the end points if there is a crossing which is which is not empty then this common end point is called a crossing in this particular drawing.

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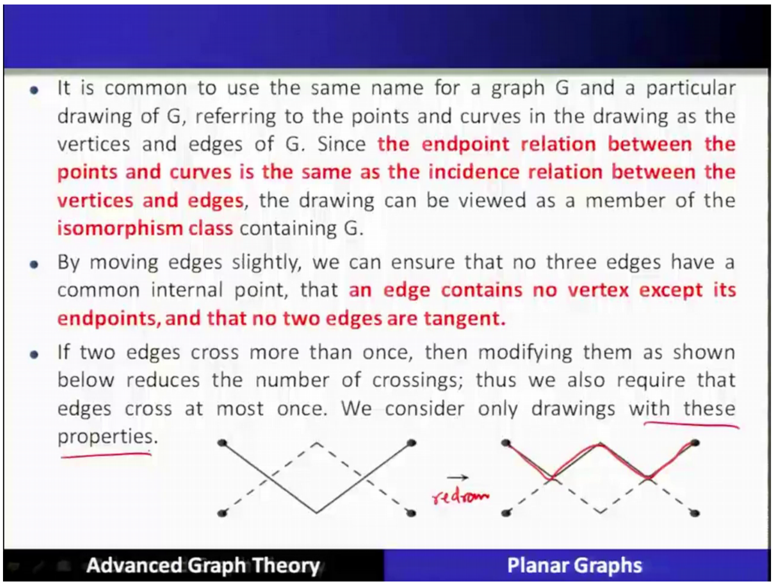

So, it is common to use the same name for a graph G and a particular drawing of G referring to the points end curves in the drawing as the vertices and edges of G. Since the end points relation between the points and the curves is the same as the incidence relation between the vertices and edges the drawing can be viewed as a member of isomorphism class containing G. By moving edges slightly we can ensure that no three edges have a common internal points that an edge contains no vertex accepts it is end points and no two edges of are tangent.

So, if two edges cross more than once, then modify them as shown below this can be re drawn so that they will not cross. Hence, we consider only the drawings having these particular properties that to our best we will consider the drawings without crossings.

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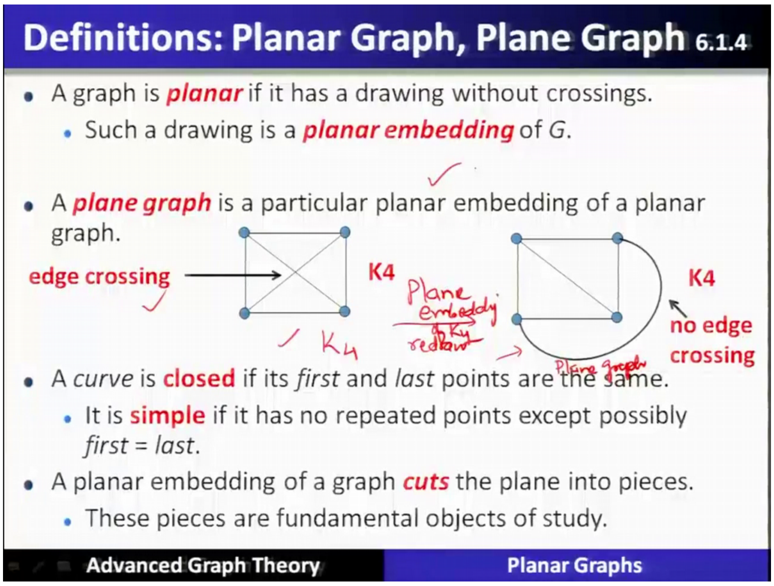

Definitions for a planar graphs and plane graphs: a graph is planar if it is has the drawings without crossings. Such a drawing is a planar embedding of a graph. So, a plane graph is a particular planar embedding of a planar graph. So, for example, if this is a graph which is called K 4, here there is a edge crossing shown. Now, we can redraw this particular graph using points and curves. So, all four points will have and this straight line are or a curve we can draw this particular edge we can draw from outside as we have seen in the previous slide. So, we can we have obtained a planar or a plane embedding of given graph K 4. So, this is called a plane graph, why because we have obtained an embedding without edge crossings.

So, a graph a plane graph is a particular planar embedding of a planar graph. So, a curve is closed if it is first and last point are the same. It is simple if it has no repeated points accept the possibly the first and last if planar embedding of a graph touch the plane into the pieces. These pieces are the fundamental objects of the study here in this topic.

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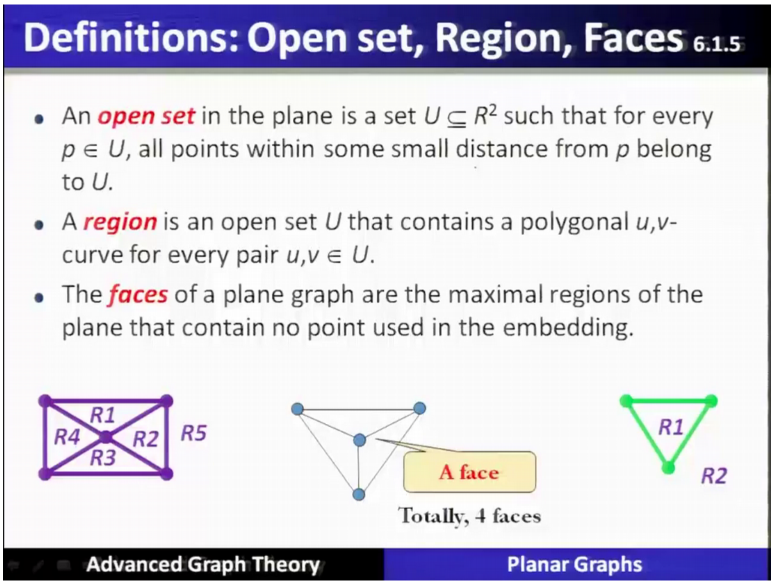

A definition, open set, region, faces; open set in the plane is a set in the 2 dimensionally face such that for every point all points within a small distance from p belongs to U. So, the region is an open set U that contains a polygonal u, v curve for every u, v which contains in U. The faces of a plane graph are maximal regions of a plane that contains no points using used in this embeddings.

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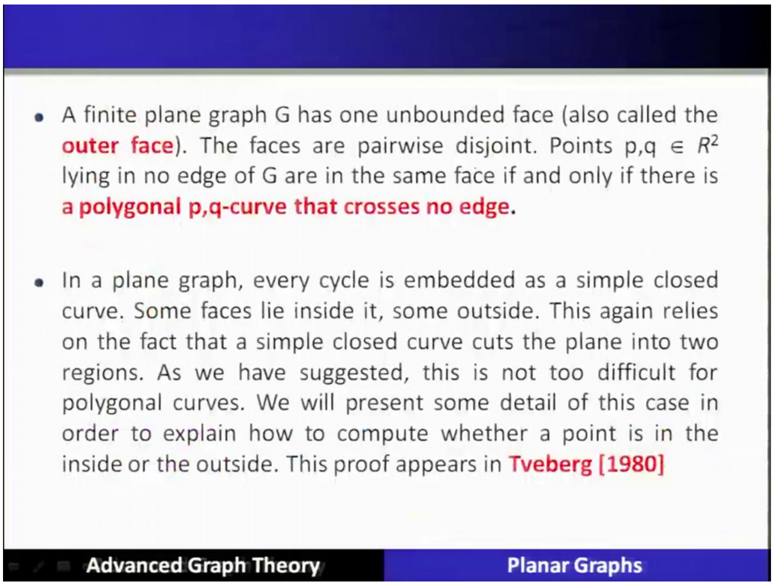

So, a finite plane graph G as one unbounded face also called as outer face. The faces are pair wise disjoint points p, q in 2 dimensional face lying in no edge of G are in the same face if and only if there is a polygonal p, q curve that crosses no edge. In the plane graph every cycle is embedded as the simple closed curve. Some faces lie inside it, some outside. This again relies on the fact that a simple closed curve cuts the plane into two regions. As we have suggested, this is not too difficult for polygonal curves. We will present some detail of this case in order to explain how to compute whether a points are inside or outside.

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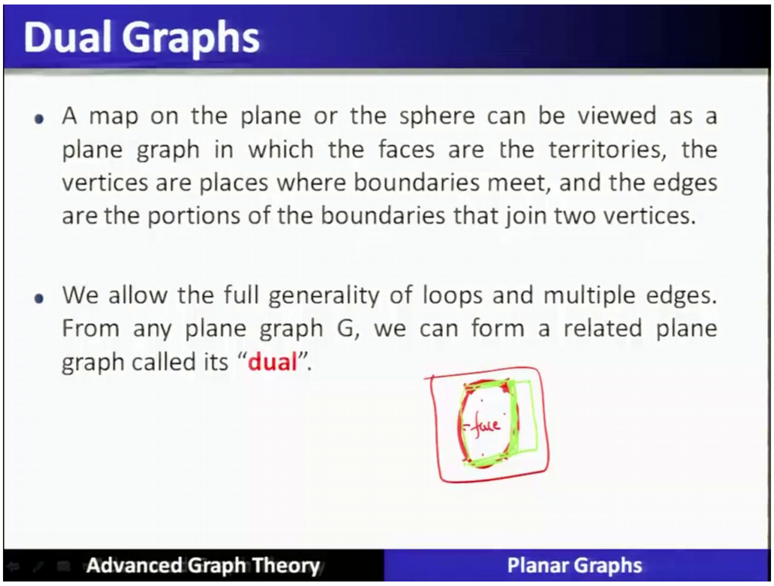

Dual graphs a map on a plane or a sphere can be viewed as a plane graph in which the faces are the territories, the vertices are the places and where the boundary meets, the edges are the portions of the boundary that join two vertices. So, again let us see that this is the map. So, we can draw a map on a plane or a sphere, this is called a face this is these are called faces this is called face it defines the territory of this particular region. The vertices are the places we can define the boundary let us say that comprises of. The vertices are the places where these boundaries meets and the edges are the portions of the boundary that joins the two vertices.

So, all these particular terms we will see now we allow the full generality of loops and multiple edges. So, from any plane graph G we can form a related plane graph which is called its dual.

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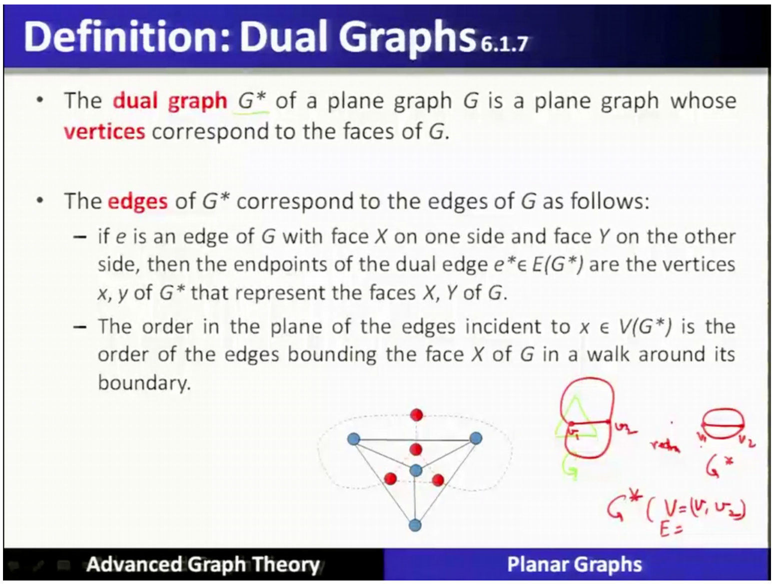

So, dual of a graph which is denoted as G star is a plane graph G is also a plane graph whose vertices corresponds to the faces of G. As per as the edges of this dual graph corresponds to the edges of G as follows; if e is an edge of G which face X on one side and which face Y on the other side then the end points of the dual edge are the vertices x, y of the dual graph that represent the faces X, Y of G. The order in the plane of a of the edge incident to x is the order of edges bounding the face X of G in a walk around its boundary.

Let us see through an example we can redraw another example then I will explain it. For example, this is the graph G what it says that if we draw a dual of this particular graph G prime. So, it as the faces this is face 1, this is face 2. So, the faces of G represent the faces of represent the vertices whose vertices corresponds to the faces of G. So, the faces of G become the vertices of G star, let us say v 1 and v 2 are the vertices of as per as the edges of dual graph is concerned we will draw the edges if E is an edge here this is an edge. If E is an Edge with face x on one side and face y on the other side then the end points of the dual graphs the dual edge are the vertices x and y of G.

So, you see that the v 1 is the vertex on the face on one face of G and v 2 is another vertex on another face of G and there is a edge. So, this edge we have to place between v 1 and v 2 which will cut through these particular lengths of this particular face one side of a face. Now, there is another side of a face and there is an edge. So, we have to draw another edge around that and the third edge. So, this particular red graph again I will redraw G star that is the dual nothing, but v 1, v 2 it has a multiple edges and loops also. So, this example shows it is a multiple edge, loops will come whenever there is a cut edge.

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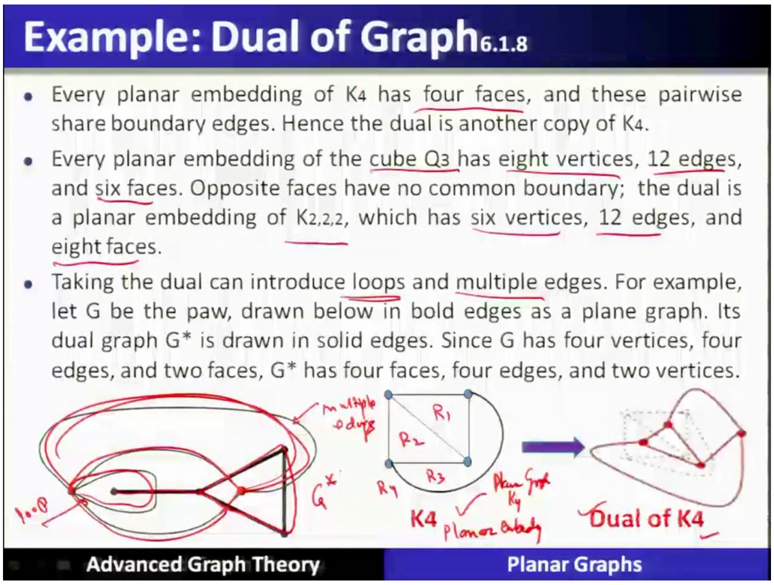

So, every planar embedding of K 4, so, this is the K 4 and we will see the plan we this is the planar embedding or this is the plane graph of K 4. Now, we can see that there are four faces this is region 1, region 2, region 3 and region 4 has four faces and these pair wise share the boundary edges. So, these boundary edges they share. Hence the dual is the another copy of if you draw the dual of this particular graph. So, for every face there will be a vertex. So, you see that there are four different vertices of a dual graph and this boundaries that means, which are bonded by an edge in G there will be also an edge which will connect those particular. So, that is shown over here a dual graph.

So, every planar embedding of a cube similarly has the cu cube has eight vertices, twelve edges and six faces. The opposite faces have no common boundary. So, the dual of this cube is a planar embedding of K 2 comma 2 comma 2 which is the six vertices which is having the six vertices because it was having six faces and twelve edges are common in both G and it is dual and eight faces why because it has eight vertices. Taking the dual can introduce the loops and multiple edges that we can we have seen. So, if you draw a paw this is the paw graph shown by the bold lines. So, here it has two faces, two regions, two vertices will be there in the dual graph and for every edge there will be a edge in the dual graph. So, you can see that this cut edge will form a loop.

Hence whatever we have discussed that loops will form and multiple edges are also formed. So, here the example shows the multiple edges this is the loop and this is the multiple edges in the dual of in the paw graph.

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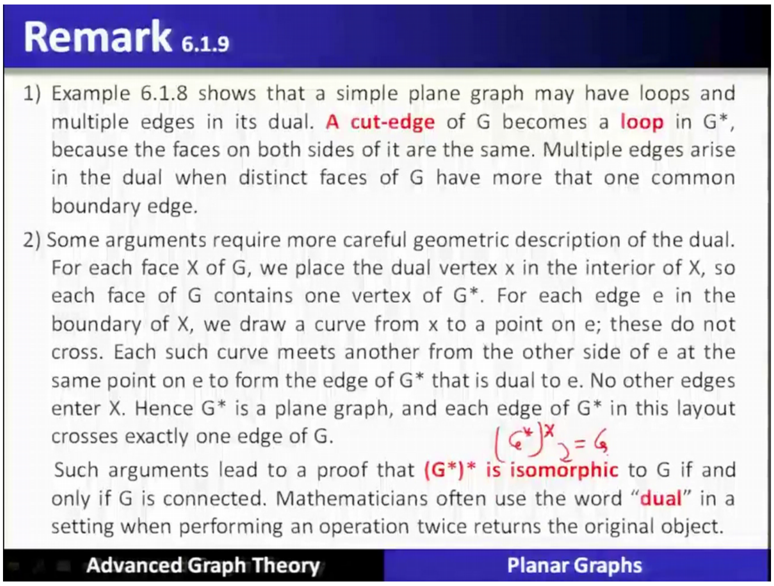

So, the example shows that the simple plane graph may have loops and multiple edges in it is dual. The cut edge of G becomes a loop in the its dual graph the because the faces on both the sides of it are the same. Multiple edges arise in the dual graphs distinct faces of G have more than more than one common boundary edge. Some arguments require more carefully geometric description of dual for each face X of G we place a dual of x in the interior. So, each face of G contains one vertex and so on.

So, with these arguments we can show that the dual of a graph if we take the dual of it that will be the original graph G. Hence dual of a dual of a particular graph is are isomorphic to a G graph.

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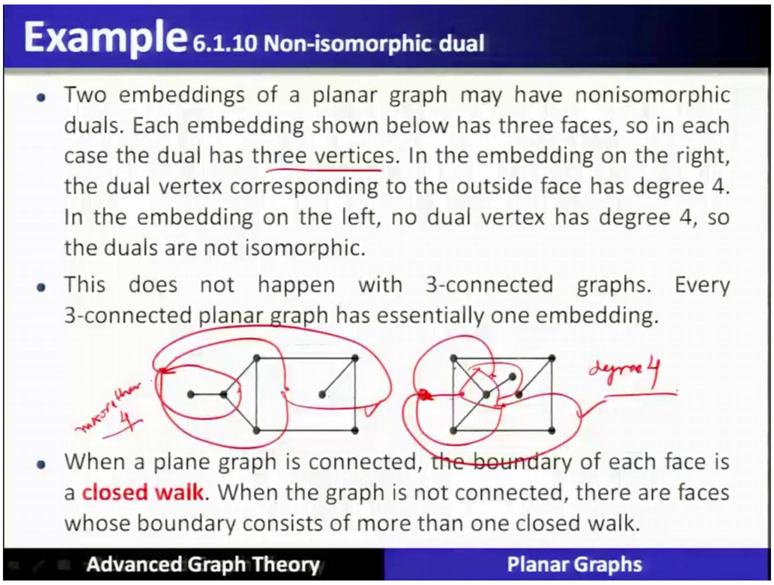

Now, we can see that for the given two isomorphic graph if it take their duals not necessary they will be also dual. So, this will be a example of a non isomorphic dual. So, here we have seen two different embeddings of a plane planar graphs, but if we take their dual they may be non isomorphic. So, here we can see that. So, this particular graph has three different faces; so in both the cases the dual as three vertices. So, vertices now, in the embedding on the right side the dual vertex corresponds to the outside face has the degree 4. So, this will have the degree 4 why because this is one region one, two.

So, we can see we will see that it has the degree four, but as for as this is concerned this will not have degree 4 here in this particular outside case one, then two, then this will form three, then this will form four and five. It is more than more than four and here the dual will be of degree 4. This is one here we will have two, then three and then four. So, hence both are non isomorphic duals why because here the degree is having 4 in this dual and this is degree of more than degree 4 this is degree 4. So, when a plane graph is connected the boundary of each face is closed walk when the graph is not connected there are the faces whose boundary consist of more than one closed walk how many edges are? There six edges are there.

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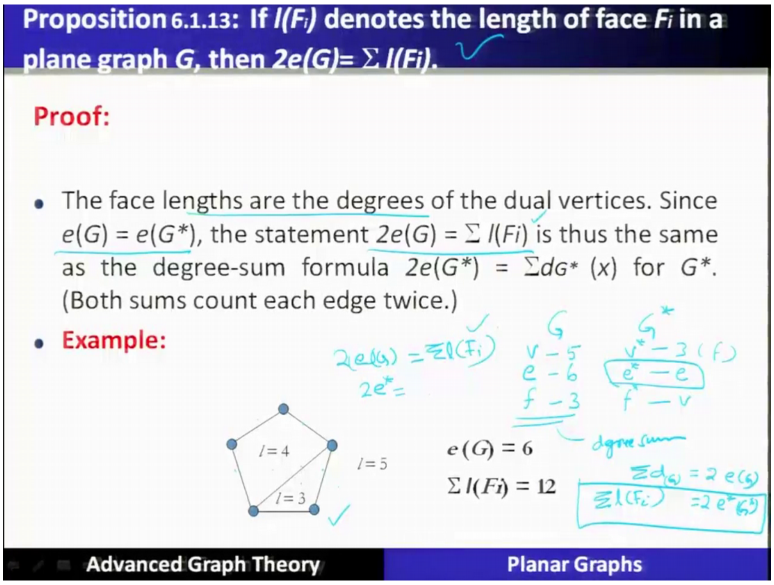

As far as it is dual is concerned. So, the number of vertices number of vertices in the dual will be equal to the number of faces. So, dual will have three faces and as far as so, this is nothing, but F of G; so the edges are the same as e; that means, 6 in this particular case and the faces of a dual is the same as the number of vertices. So, in the original graph you know that the degree sum formula is applicable. The degree sum formula says that the sum of the sum of degrees of that particular graph is nothing, but two times the edges of the graph.

Now, here we see that this particular edge e is same as e star. So, that becomes 2 e star in the dual graph G. As far as the degree is concerned here what we can do is that the degree is nothing, but the sum of the lengths of the faces. So, the sum of the lengths of the faces will become the degree. Hence this particular formula we have derived using degree sum formula. So, let us see the proof what it says the face lengths are the degree that we have already shown of the dual vertices. Since, e of G is same as e of G star. So, the statement two times number of edges that we have seen in the previous one that is nothing, but summation of the lengths of all the faces it choice the number of degrees that we have seen.

So, this will be going to be equal to the degree sum formula because e is same as the dual graph and the sum of the lengths of the faces is nothing, but the degrees of the dual graph.

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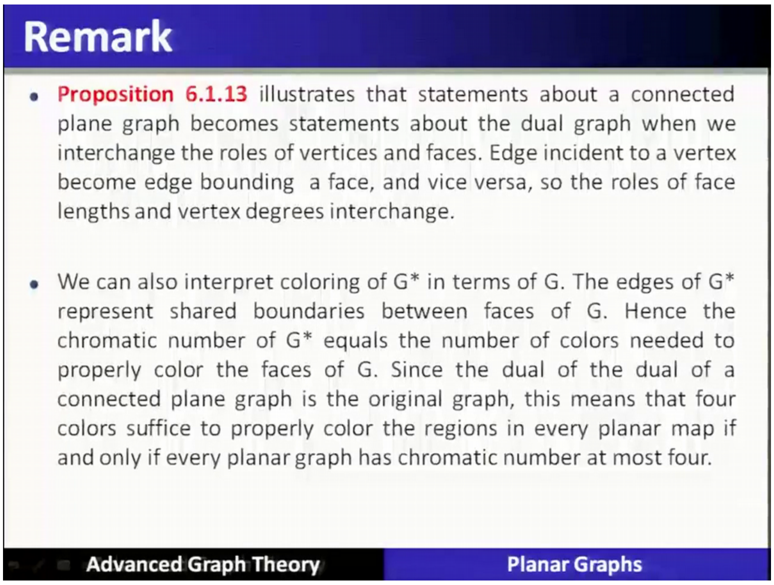

So, the proposition 6.1.13 illustrates that the statement about the connected graph becomes the statement about the dual graph when we interchange the roles of the vertices and faces the incident the edge incident to the vertex becomes the edge bounding of a face and vice versa. So, the role of face lengths and the vertex degrees into change.

We can also interpret the coloring of the dual graphs in terms of G. So, the edges of dual graph represents the shared boundaries between the faces of G. Hence the chromatic number of the dual graph equals the number of colors needed to properly color the faces of G. Since the dual of the dual of a connected graph is the original graph this means that four colors will suffice to properly color the regions in every planar map if and only if every planar graph as the chromatic number at most four.

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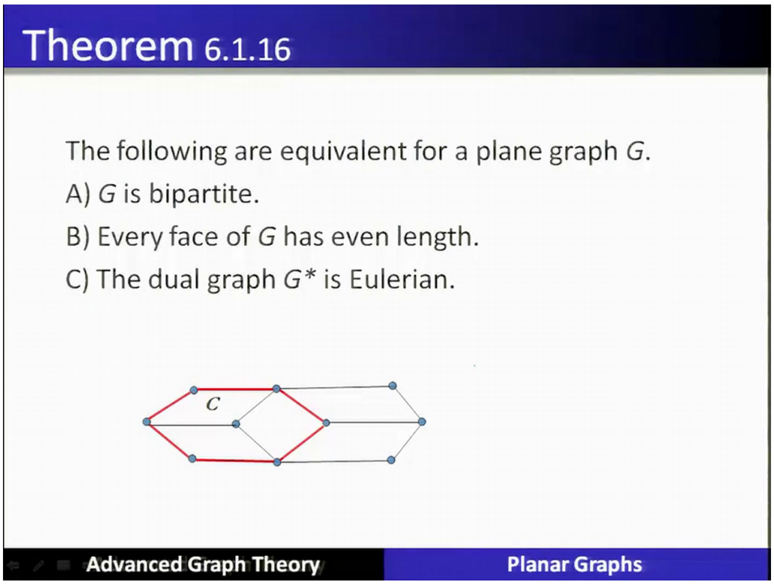

So, let us the characterization of a planar graphs. So, the following are equivalent or the plane graph that is the G is bipartite, second equivalent statement says that every face of G as even length, third statement says that the dual graph G star is Eulerian. So, a will derive A is equivalent to B.

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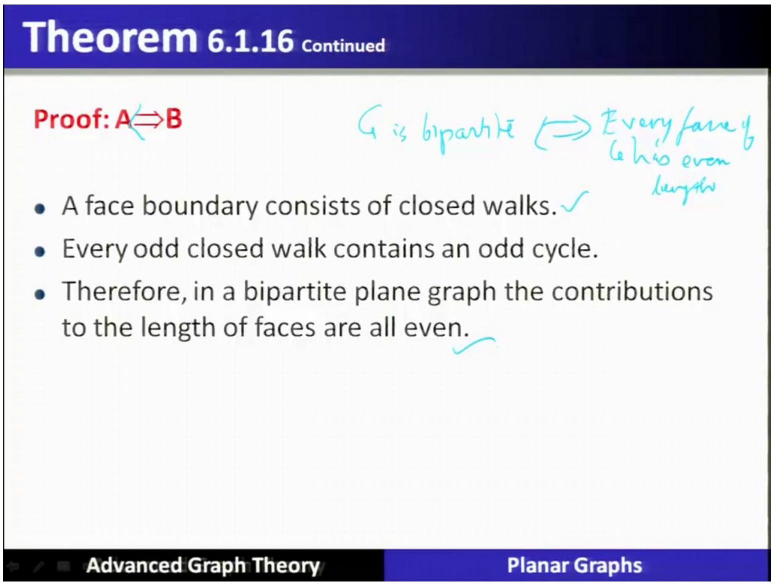

So, A means G is bipartite and this is equivalent to saying that every face of G has even length. Now, let us see that the face boundary consist of the closed walks. So, every closed walks contain a odd cycle, therefore the bipartite plane graph the contribution to the lengths of the faces of are all even. Hence A is equivalents to B.

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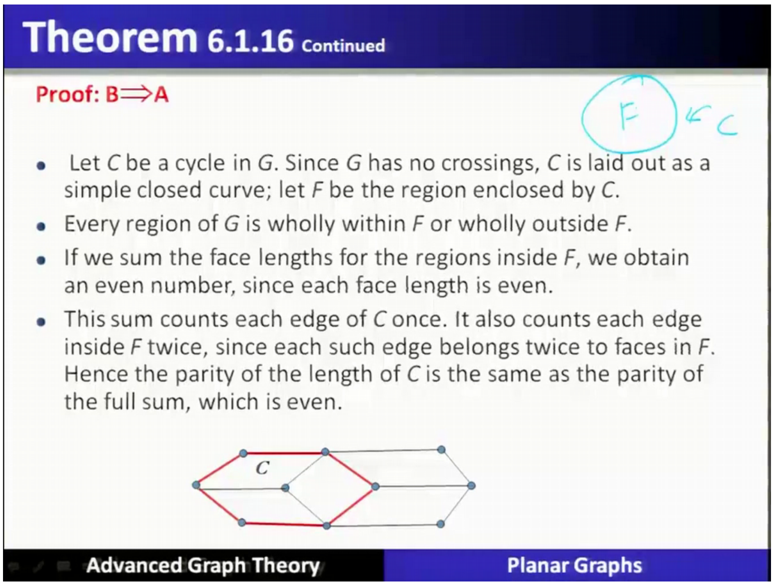

Now, B is equivalent to C let us see be a cycle in G since G has no crossings C is laid out as the simple closed curve. So, the F be the region bounded by the cycle C. Now, every region of G is only within F or outside F. If we sum the face lengths all the regions inside we have obtain the odd number and the even number since each face has the length even. This sum counts each edge of C ones, it also counts each edge inside F twice. Since each edge belongs twice to the faces enough hence the parity of the lengths of C is the same as the parity the full sum, which is the even.

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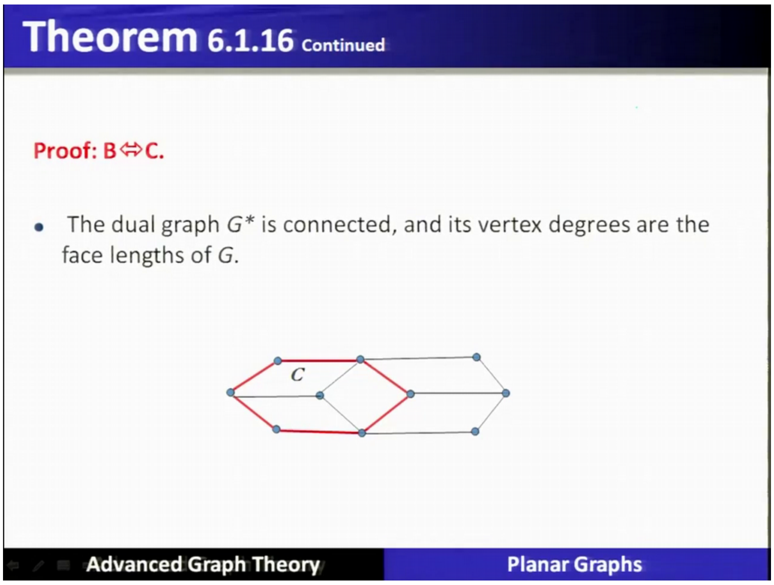

Now, B is equivalent to C, the dual graph is connected and it is vertex degrees are the face lengths, hence B is equal to B is equivalent to C.

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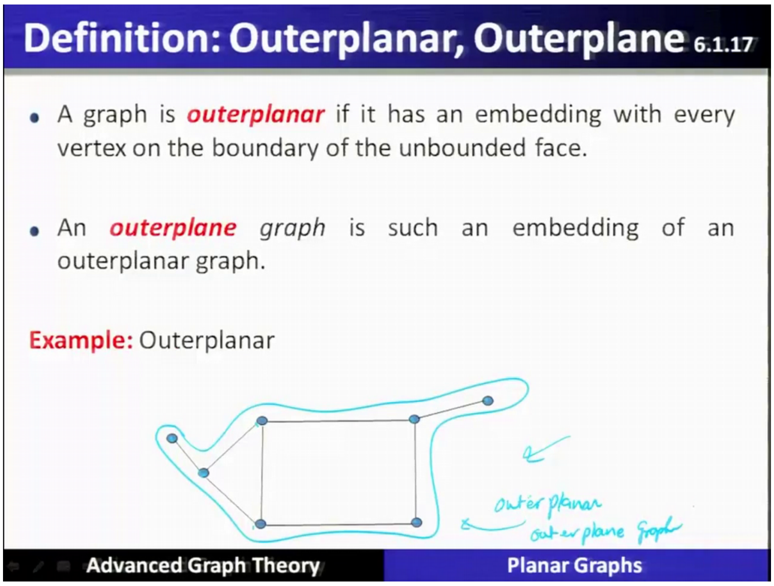

Now, we will see the definition of the outer planar a outer plane graph. A graph is outer planar if it as an embedding with every vertex on it is boundary of an unbounded face. An outer plane graph is such an embedding of an outer planar graph, let us see in this particular example. So, this is the planar embedding here we can see that all the vertices are lying on the outer boundary outer boundary means there is no vertices is in side this particular bounded region. Hence it is called a outer planar graph.

Outer planar graph and we have got this kind of embedding this is called a outer plane graph. A particular embedding if the graph as a outer plane a graph is outer planar if it as an embedding with every vertices on the boundary of an unbounded face or a graph is called outer planar of it as an outer plane graph as an embeddings.

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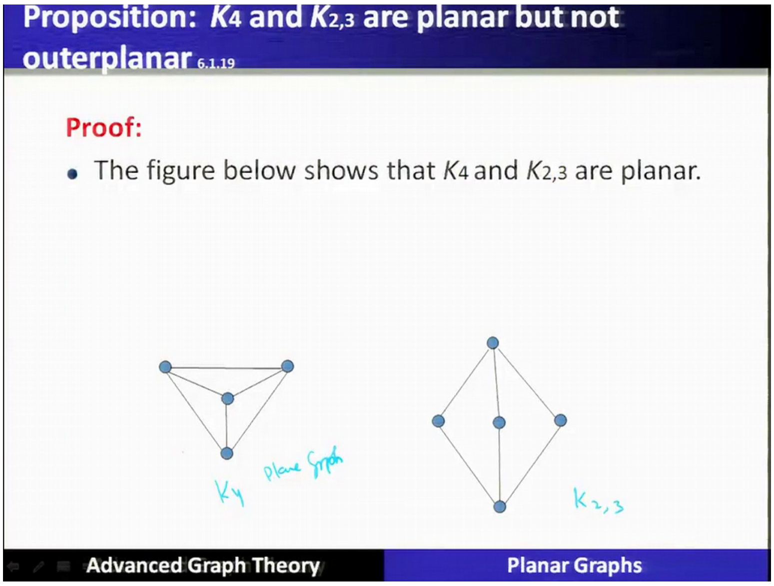

Proposition K 4 and K 2, 3 are planar, but not outer planar we can see here in the figure that this is K 4 and this is K 2, 3. So, they are plane graph why because it follows this particular it is a plane graph embeddings. So, it is a planar graphs both are planar. Let us see whether it is a outer planar or not we cannot obtain a outer plane embedding of this particular graphs. Why, because this particular vertex is inside it cannot cross outside, if it cross outside then this edges will cross it will not a planar. So, there is no outer plane embedding of these particular planar graph hence it is not a outer planar, similarly the other one.

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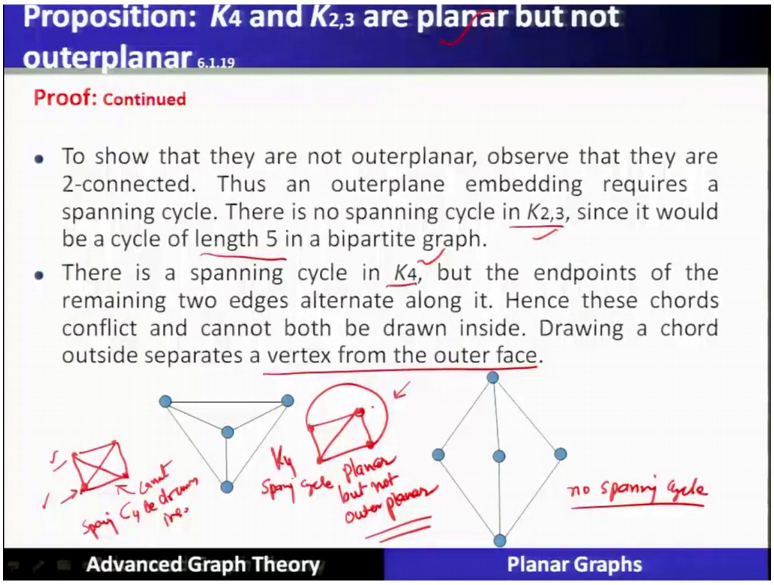

To show that they are not they are not outer planar observe that they are 2-connected. Thus an outer planar embedding requires a spanning cycle. There is no spanning cycle in K 2, 3 there is no spanning cycle in K 2, 3 since it would be a spanning cycle of length 4 5 in a bipartite graph that would be length 5 is an odd cycle, hence K 2, 3 does not have any spanning cycle. However, there is a spanning cycle in K 4 why because there are four vertices.

Now, if there is a if spanning cycle let us let us understand like this it is a having four cycle. Now, this is a spanning cycle C 4. Now, here in to complete this K 4 let us assume that one chord will be inside K 4 the end points of the remaining two edges, this particular edge this alternate along this particular cycle. So, these the remaining cycle so, the remaining, but the end points of the remaining two edges alternate along it. So, this is alter, they will alternate. Hence these chords will conflict cannot both be drawn inside.

So, drawing a chord outside separates a vertex from the outer face. So, if the if this particular chord is drawn then it becomes a planar, but separates a vertex from the outside face because this will be creating a inner face. Hence this is not a outer plane. To show that they are not outer planar observe that they are 2-connected thus a out an outer plane embedding requires a spanning cycle there is no spanning cycle in K 2, 3 hence it would be a cycle of length 5 in a bipartite graph.

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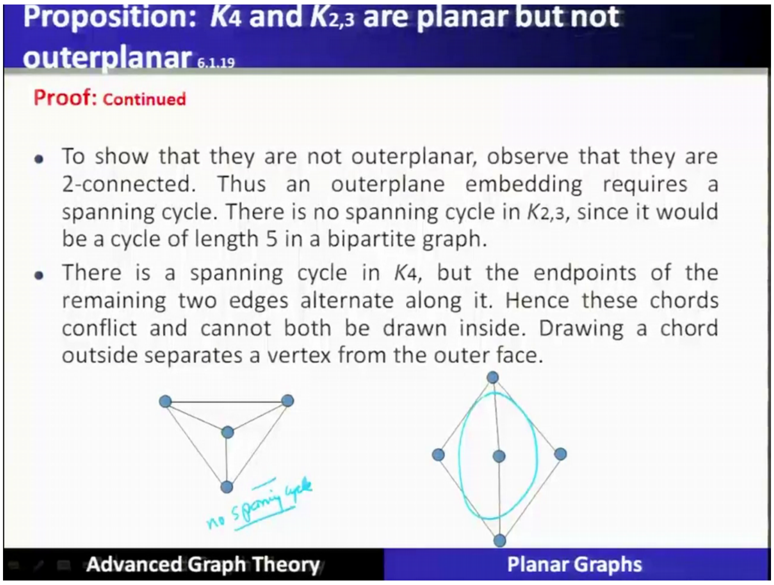

Now, there is a cycle In K four, this cycle is there, but the end points of the remaining two edges alternate along it. Hence the two chords conflict and cannot be drawn inside drawing of a chord outside separates the vertex from the outer face.

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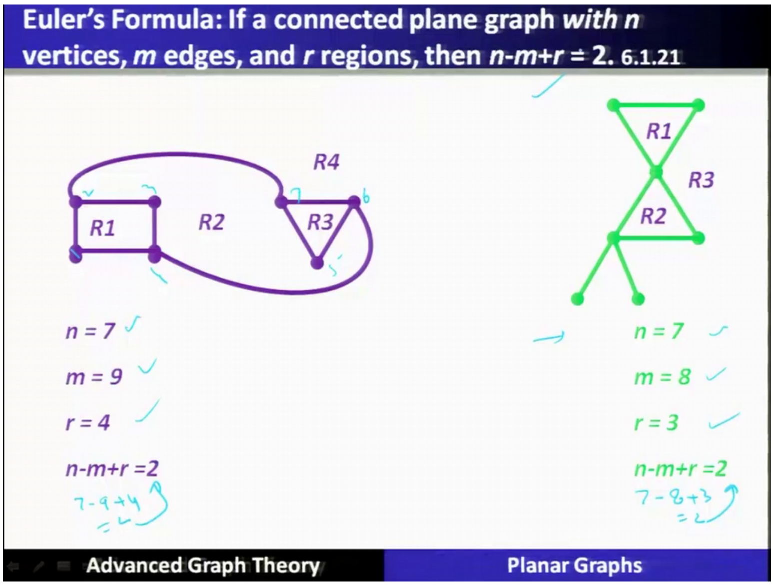

Euler's formula if the connected plane graph with n vertices, m edges and r regions then n minus m plus r is equal to 2, this particular formula is well known as Euler's formula for plane graph. So, it states for a connected plane graph having n vertices, m edges and r regions then this particular formula will be satisfied that n minus m plus r is equal to 2.

Let us take this particular example graph this example graph as 7 vertices one two three four five six seven. It has 9 different edges and 4 different regions, R 1, R 2 and so on up to 4. So, n minus m plus 4 becomes 2, hence this particular equation is satisfied. Let us take another example graph here we have 7 different vertices 8 different edges 3 different regions which are shown R 1, R 2 and R 3 if we plug in to this particular formula, 7 minus 8 plus 3 that becomes 2 and hence it is satisfied.

So, it is not a coincidence, but all such connected plane graph will satisfy this particular formula and this particular formula is called Euler's formula.

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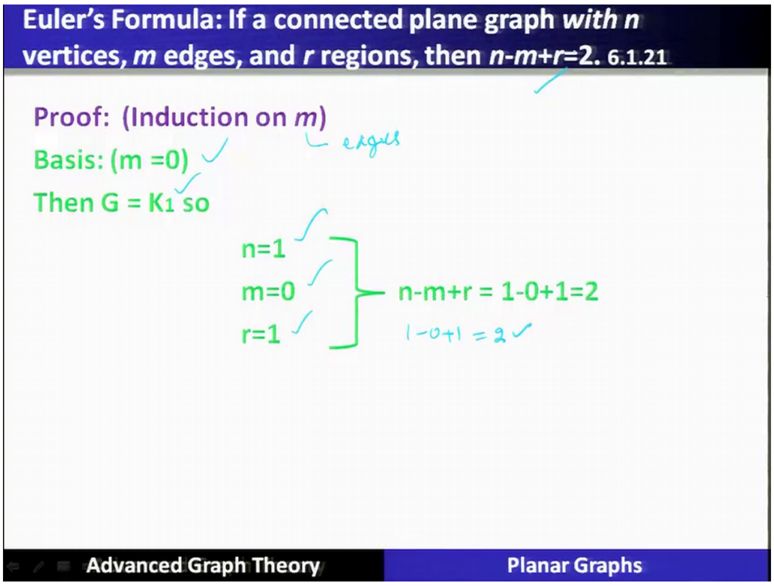

Let us see the proof by induction on the number of edges. Now, for the base case let us consider the number of edges as 0. So, if there are no edge present then it would be isolated vertex and that is nothing, but a K 1 graph. So, in K 1 graph the number of node is 1, the number of edges is 0 and the number of region is 1. So, let us include it in this particular formula. So, 1 minus 0 plus 1 that becomes 2. Hence, this particular formula is satisfied for the base case.

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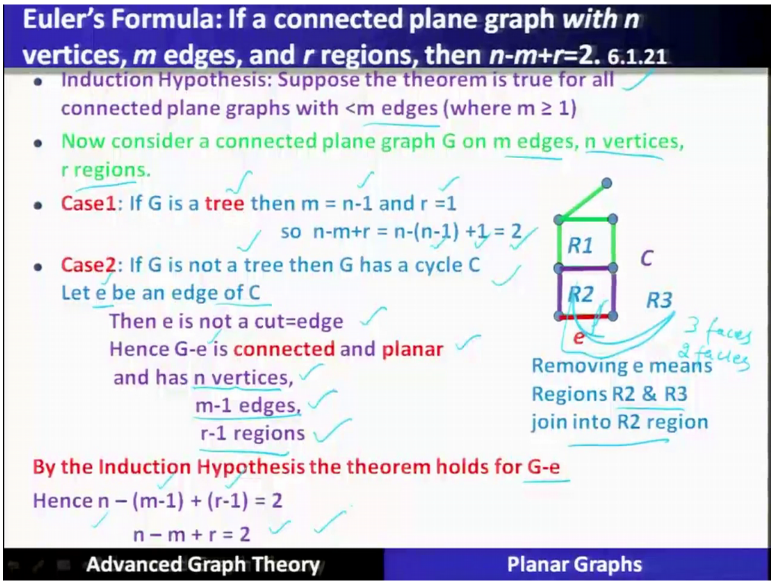

Let us see the induction hypothesis. Suppose, the theorem is true for all connected plane graphs with less than m edges there m is greater than or equal to 1. So, now, we have to prove for the general case. Now, consider a connected plane graph G on m edges and vertices and r regions, there are two cases. Case 1 says that if G is a tree. Now, you know a tree does not have any boundary region the number of region will becomes 1 for the tree and if the tree has n nodes it will be having n minus 1 different edges. So, if you plug in to n minus m plus r the value of m that is n minus 1 and also the number of region has 1. So, this comes out to be true n hence if the graph is a tree this particular formula is satisfied.

Now, let us say that if the graph is not a tree then it has a cycle. Let us consider the cycle C and on a particular cycle C there is an edge which is called e. Now, if this edge e is not a cut edge why because if a edge is on the cycle then in the previous videos we have seen that it is not an cut edge, hence e is not a cut edge. Hence if we remove e from a cycle then the graph will still be connected. Hence G minus e is still connected and a planar as well which as n vertices why because after an edge is removed number of vertices will be intact to values n and number of edges will be reduced by 1.

How about the faces of the regions? Now, we know that if we reduce an edge from a graph the number of region will also be reduced by 1. Take this particular example graph in this example graph if this edge is removed then R 2 and R 3 will be in the same region. Hence, the number of region will be reduced by 1. So, this graph will have before removing the edge it have 3 regions and after the removing of this edge the number of region is reduced by 1 that will have 2 regions, so, now number of regions will reduced by 1.

Now, by the induction hypothesis this particular formula or the theorem holds for a smaller graphs that is G without e. Let us see that if hence if we plug in these values over here n as n and m as m minus 1 and r as r minus 1. So, that becomes n minus m plus r is equal to 2, hence this particular theorem is proved.

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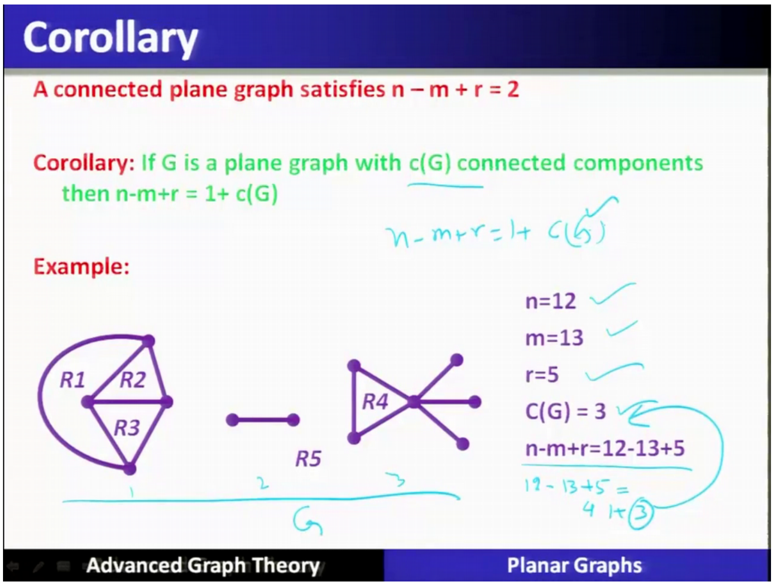

Now, there is a corollary which says that if G is a plane graph with c of G is number of connected components, then this formula can also be generalized for the disconnected graphs, that is, n minus m plus r is equal to 1 plus number of connected components in that particular graph G. Let us take this graph which is having three components one, two and three components. Now, this particular graph has one two three four five different regions it has 12 nodes and 13 edges. If we plug in to this particular formula so, 12 minus 13 plus 5 that becomes plus 5. So, that becomes 4. So, 4 means 1 plus 3, so, 3 is the number of connected components.

So, this particular formula is satisfied here in this particular example hence this particular corollary is being taken into an account with the Euler’s formula.

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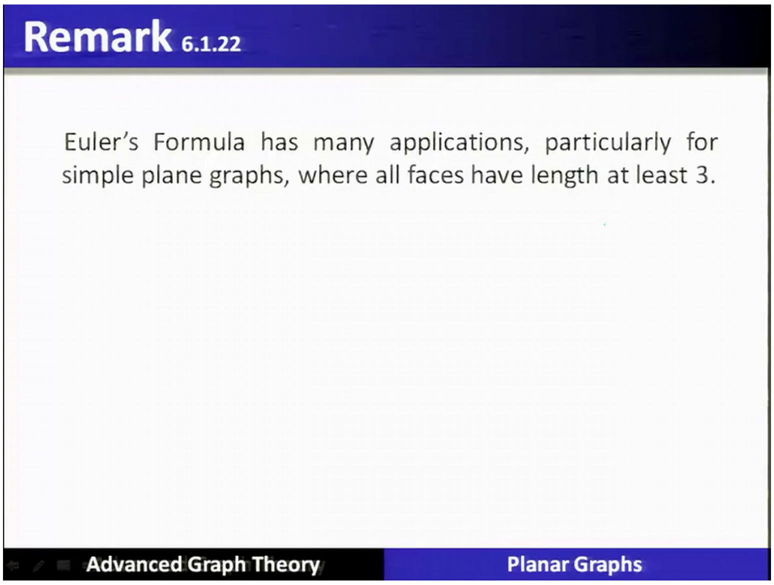

So, Euler's formula has many applications particularly for simple graphs where all faces have the length at least 3.

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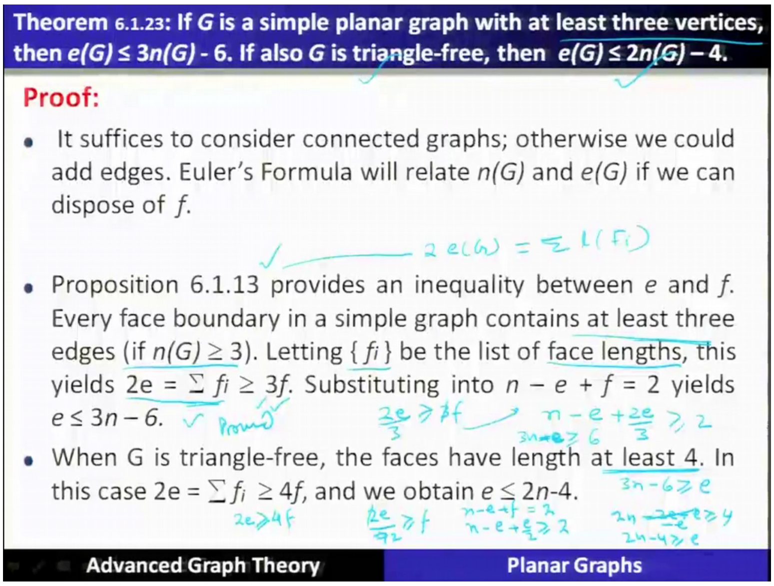

Let us use this particular remark as a theorem. So, if G is a simple planar a graph with at least three vertices then e is less than or equal to 3n minus 6. Let us prove this now it suffices to consider the connected graph otherwise we would add the edges Euler's formula will retreat and will relate n and e if we can dispose of f.

So, the proposition which we have seen previously provides an inequality between e and f. So, again of you recall twice the number of edges is equal to the summation of lengths of the faces that is given that we have already seen. Now, every face boundary in a simple graph contains at least three edges that we have already taken. So, therefore, the number of nodes should be at least three, so three or more. So, that is basically the requirement for this particular theorem.

So, let f i be the list of face lengths. So, this will yield if you take the sum of the faces the degree sum or the proposition will give 2e is equal to sum of all faces face lengths. So, f i's are all face lengths. Now, you know that every face has length 3 that we basically 3 edges in a simple graph. So, a every face contains the lengths 3 and f is the total number of faces. So, this particular formula if we substitute for f then it will 2e is greater than equal to 3f then f is bounded by 2e by 3 if you substitute here n minus e plus. So, this becomes 3m 3n minus e is greater than or equal to 6. So, that becomes 3 n minus 6 is greater than or equal to e, that we have proved.

Now, another part of this particular theorem says that if G is triangle-free, then the faces have the lengths at least 4. Hence 2e is greater than 4 times f, so, 2e by 4 that is f. So, if we plug in this value that is n minus e plus e by 2 is greater than or equal to 2. So, 2n minus 2e plus e is greater than or equal to 4 this becomes minus e. So, this becomes 2n minus 4. So, e is less than or equal to 2n minus 4 that we have proved in this particular theorem.

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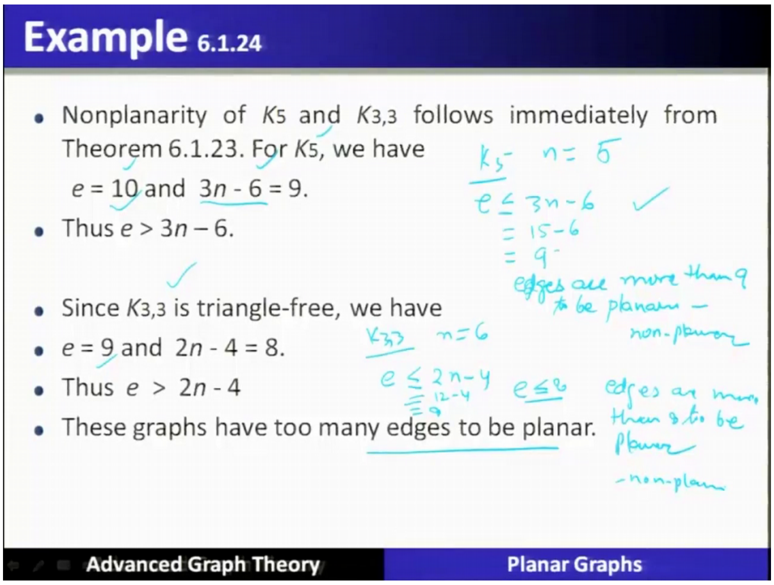

Now, we can prove the non planarity of K 5 and K 3, 3 using the this particular theorem. Now, for K 5 we can see that total number of edges is 10. So, if we plug in that particular 3n minus 6 is equal to that is. So, e should be less than or equal to 2n minus 6. So, here in K 5 n is equal to 5. So, that comes out to be 9, but how many edges are there here present the number of edges are more. So, it should be 9 or less than 9 since the number of edges are 10. So, 10 is not less then so, it is basically greater than 10. So, here the number of edges are more. So, edges are more than 9 required for a graph to be a planar one, hence it is non planar.

Similarly, e should be less than or equal to 2n minus 4 and here in K 3, 3, K 3, 3 we have n is equal to 6 so that comes out to be 8. So, e should be less than or equal to 8, but the total number of edges present is 9. So, here the also the edges are more than 8 to be to be planar, hence this is the non planar graph according to the Euler’s theorem.

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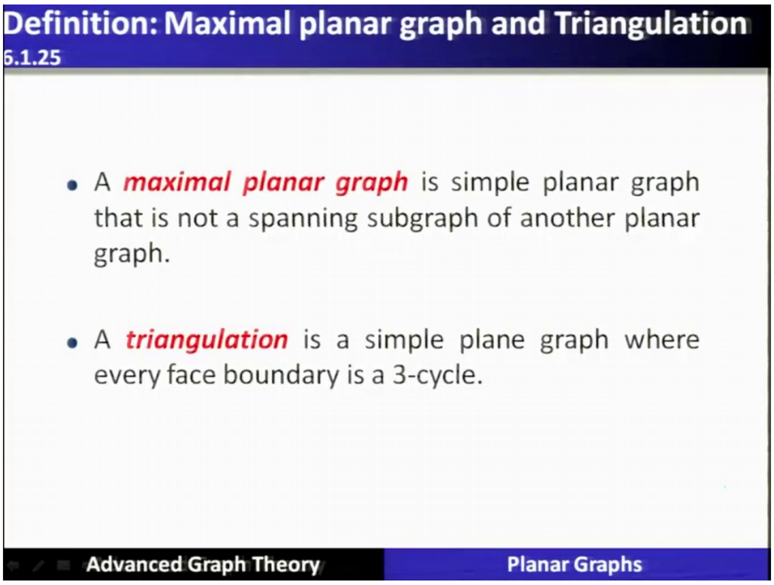

Now, the maximum planar graphs and a triangulation; a maximal planar graph is a simple planar graph that is not a spanning sub graph of another planar graph. A triangulation is a simple plane graph where every face boundary is a 3-cycle.

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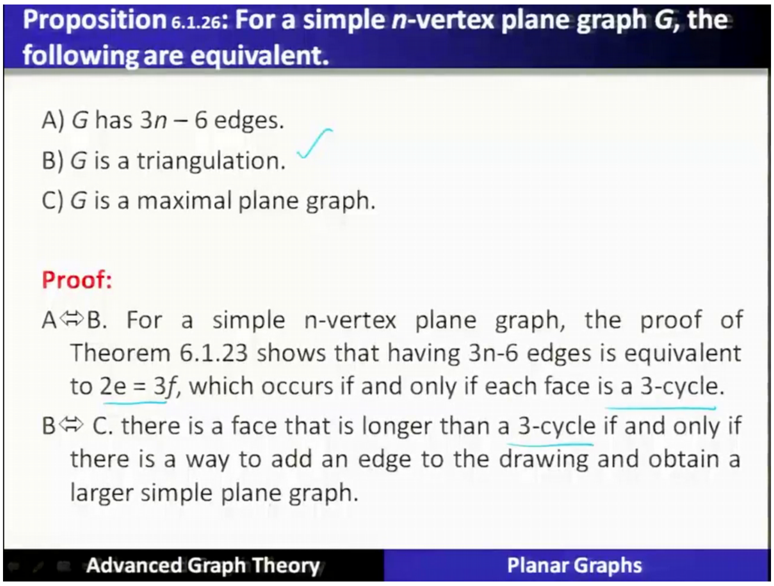

Now, there is a proposition which will make three different statements which are equivalent. So, for simple and vertex graph the following are equivalent G has 3n minus 6 edges, G is a triangulation and G is a maximal plane graph. A is equivalent to B, now for a simple and vertex plane graph the proof of the theorem shows that the graph is having 3n minus 6 edges is equivalent to saying that 2e is equal to 3f that we have seen in the previous proof of the theorem, so, which occurs of and only if every face of a 3-cycle and that is nothing, but a triangulation. Now, B is equivalent to C, now, there is a face that is the longer then 3-cycle if and only if there is the way to add an edge to the drawing and obtain a larger a simple planar graph hence this particular condition that is the maximal planar graph and the triangulation is equivalent.

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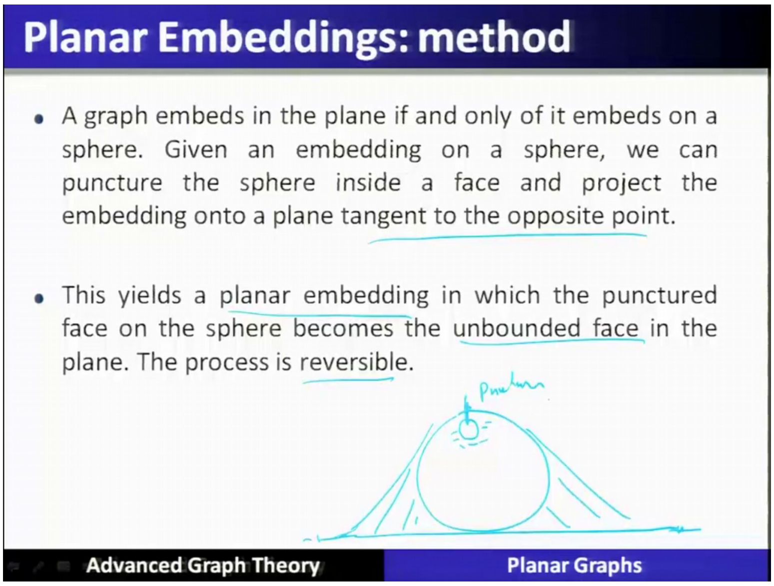

Now, planar embedding we have to see a method to do this. So, a graph embeds in a plane if and only if embeds on a sphere. So, given an embedding on a sphere we can puncture the sphere we can puncture the sphere inside a face and project, place a bulb and take this particular embeddings on a plane, this is called a stereographic projection. And, project embeddings on to the plane tangent to the opposite points. This yields a planar embedding in which the punctured face this face will be unbounded face in the plane and this process is reversible; that means, if we can have this kind of the plane graph we can wrap around we can do we can obtain a embedding on a this sphere, this is called planar embeddings.

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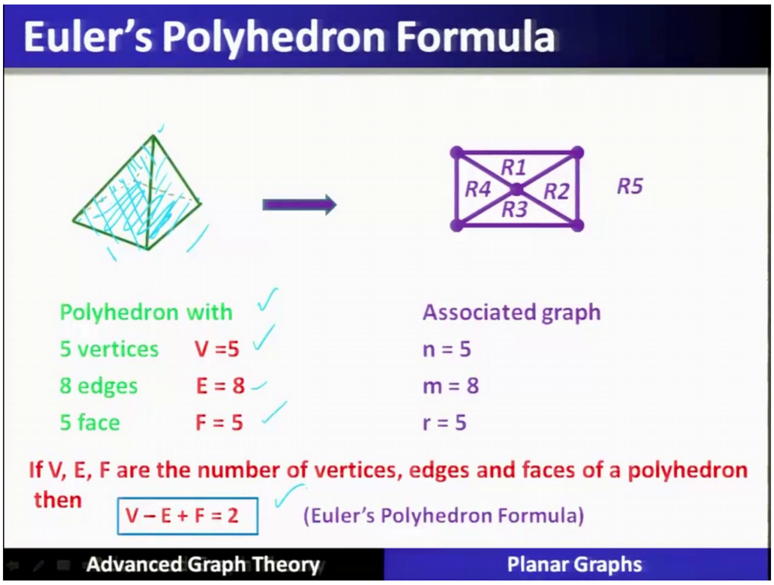

Now, with this let us see an Euler's polyhedron formula. So, polyhedron are basically here in this particular example polyhedron which we defined has 5 faces or have 5 vertices 1 2 3 4 and 5 vertices are there it has 8 edges 4 are in the base 4 are on the upper and it has 5 faces. So, this is face number 1, this is face 2 and that will be on other side face 3, then the bottom will be face 4 and outside will be face 5.

Now, if VEF are the number of vertices, edges and faces of a polyhedron then again same Euler's formula also applicable is called Euler's polyhedron formula.

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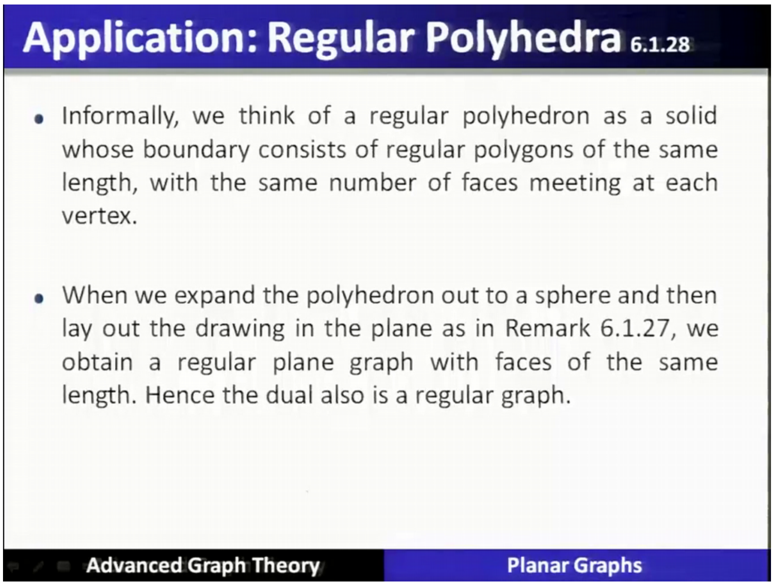

Now, the application for a regular polyhedron let us see; informally, we think of the regular polyhedron has a solid whose boundary consist of the regular polygons of the same length, with the same number of faces is meeting at each vertex. When we expend the polyhedron out to a sphere and then layout the drawing on a plane using the previous remark, we obtain a regular plane graph with the faces of the same length. Hence, the dual also is a regular graph.

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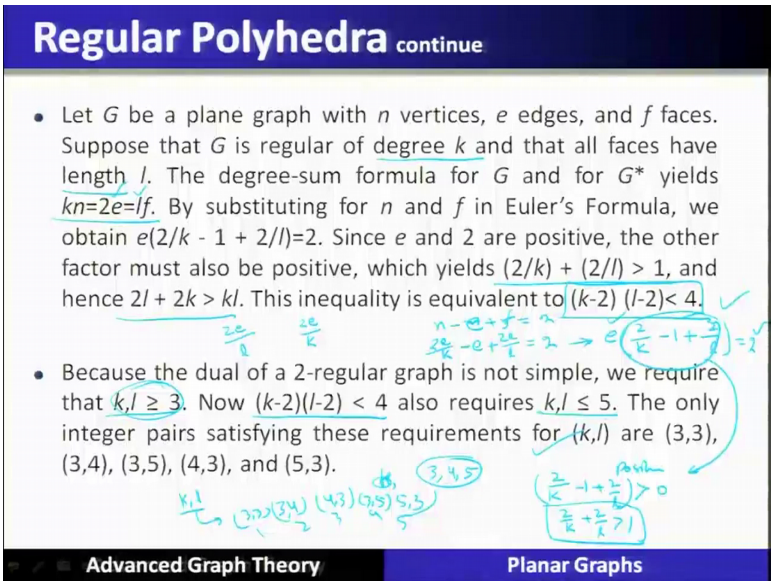

Now, let G be a plane graph with n vertices, e edges and f faces. Suppose, G is a regular graph of degree K and that all the faces have length l. If we do the degree-sum formula for G and for the dual which for it will yield to kn, k times n will be the degree-sum that is nothing, but twice the number of edges and that is nothing, but the since every face have all the faces has length l and there are f number of faces, so lf. Now, by substituting n and f in the Euler's formula n minus m plus r is equal to 2. Here we call it as e and instead of it is f. So, let us see if we substitute for n and f for Euler's formula. So, n if we substitute 2e by k so, instead of f we see 2e upon l is equal to 2.

Now, if you take if you take e outside this will become 2 upon k minus 1 plus 2 upon l is equal to 2. Now, this is positive and this is positive e and 2 are positive. Hence this entire thing becomes positive. Therefore, 2 upon k minus 1 plus 2 upon l is greater than 0. So, 2 upon k plus 2 upon l is greater than 1. So, this particular formula we have obtained and hence n this particular term is nothing, but 2l plus 2k is greater than kl and further we can obtain k minus 2 times l minus 2 is less the 4.

Again, the dual of 2-regular graph is not simple, we require then k and l to be greater than or equal to 3. Now, suppose k minus 2 and times l minus 2 is less than 4, this also requires that k and l should be less than 5. Therefore, satisfying both k and l should be greater than 3 and k and l less than or equal to 5. So, 3, 4 and 5 there are three different possible values for k and l are, so, k and l will have the possible values are 3, 3 then 3, 4 then 4 4, 3 then 3, 5 then 5 and 3.

One two three four five there are only 5 possible values of k and l is possible. Now, k is that that particular degree and l is the length of these particular faces.

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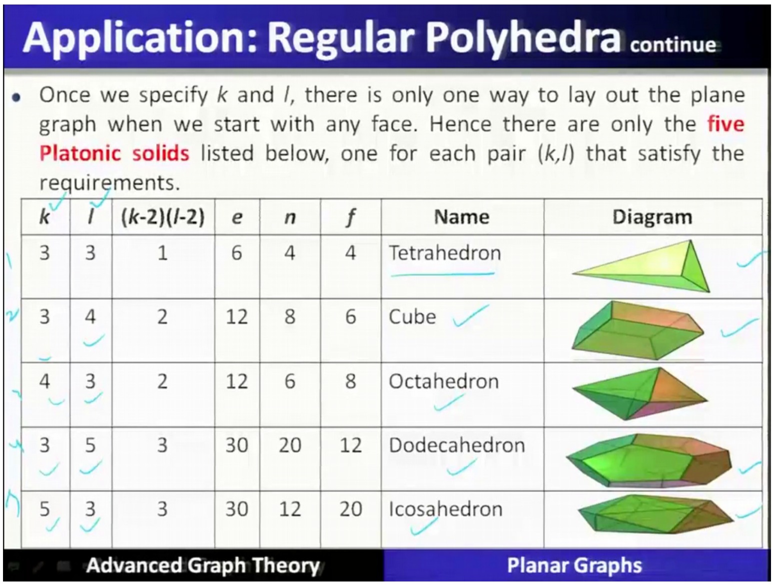

So, once we specify k and l there are only one way to layout the plane graph when we start with any face. Hence there are only five platonic solids listed below, one for each pair k and l that satisfy the particular requirement and they are called regular polyhedras. So, re regular polyhedra they are only five different platonic solids are possible that are shown over here.

So, we have considered all k and l different values which I have told you. There are five different pair scandal and that comes out to the number of different edges nodes and faces and that is given names to every such k and l pair that is the first one when k and l is 3 that is called tetrahedron and this is the diagram for it. When it is k and l value is 3 and 4 that becomes a cube. So, the cube is shown over here. Similarly, when k and l is called to 4 and 3 then it becomes octahedron and there all regular polyhedral, when k and l is 3 and 5 n the it is dodecahedron that is shown over here and when 5 and 3 these values are when icosahedrons it is figure is shown over here.

So, there are only five platonic solids and that we have derived through and they are regular polyhedras.

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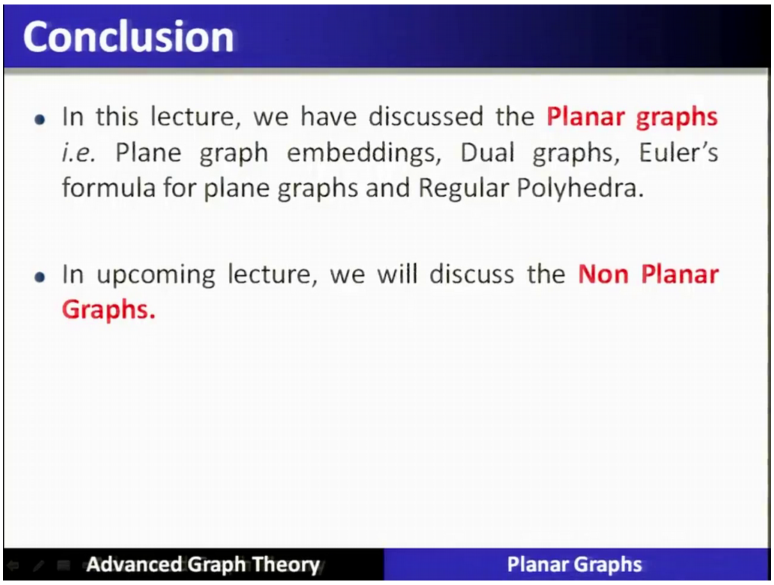

Conclusion: in this lecture, we have discussed the plane graphs that are the planar embedding of a planar graph. We have also discussed the planar graphs, dual graphs, Euler's formula for plane graphs and regular polyhedra and in upcoming lectures. We will discuss non planar graphs.

Thank you.