Advanced Graph Theory
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Lecture – 01
Graph Theory: Introduction

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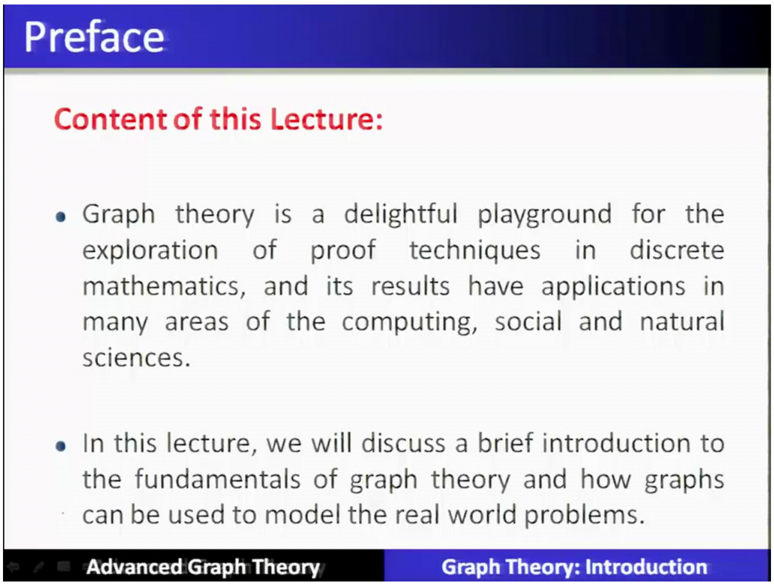

Graph theory introduction. So, content of this lecture. So, graph theory is the delightful playground for exploration of proof techniques in discrete mathematics and it is results have applications, in many areas of computer science social and natural sciences.

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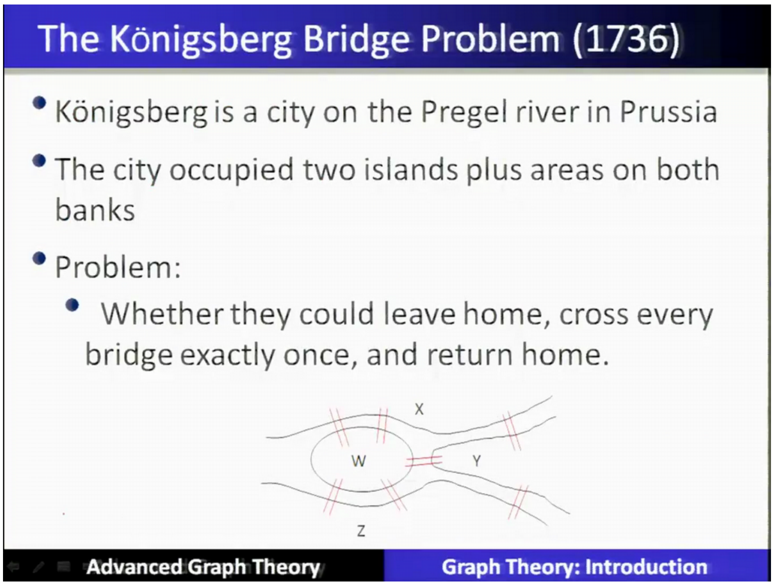

In this lecture we will discuss the introduction and fundamentals of graph theory and how the graphs can be used to model the real word problems? So, the graph theory is evolved out of a practical problem of a Konigsberg bridge problem. In 1736 Konigsberg is a city on Pregel river in Prussia, the city occupied 2 islands plus areas on both banks.

So, the problem was stated as under, whether the residents of that particular city they could leave home cross every bridge exactly once and return home back. So, in this particular figure can see that X, Y, Z, W they are basically the islands and these islands are connected by the bridges, which are marked by the double lines which are shown in red color.

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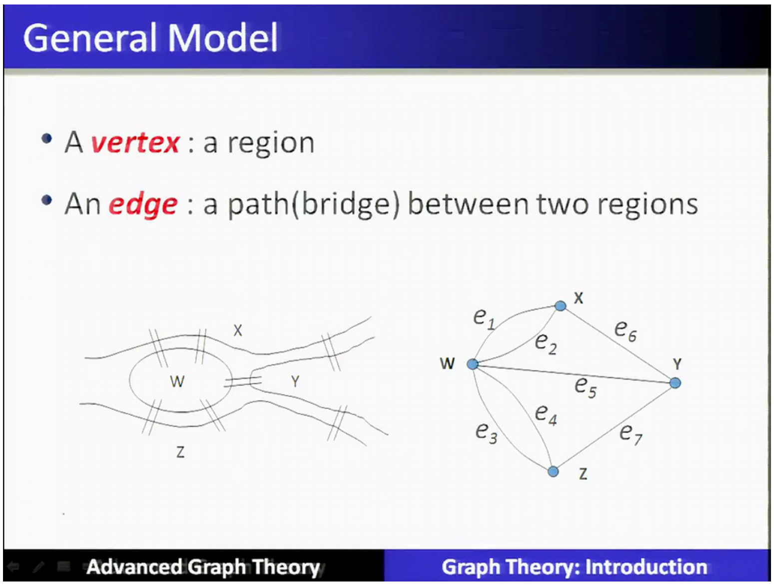

Now, this particular problem can easily be analyzed, if they can be modeled as a form of a graph so; that means, every region or a island, if we can represent as a vertex and then if there is a bridge between the 2 regions connection or connectivity, then basically we can place an arc.

So, this particular map is basically converted in the form of a graph. So, here you can see in a particular graph, now once a problem is converted in a graph form then this particular problem can be analyzed, using the model in a in a graph theory. So, this is the way the most of the practical problems or the applications, if they converted into a form of a graph then basically the problem can be analyzed and solved.

In this particular lecture, we are going to see how different applications can be modeled in this form of a graph and that is called a graph model. Once a problem is modeled in the form of a graph, then basically we are going to study in this part of the course. The theory that is a graph theory and which is based on the proofs and various techniques and the algorithms this tone these particular methods.

So, as far as the algorithms are concerned, which are basically covered in the in this graph theory, is going to exploit the structural property of a graphs and how they are going to be represented in a computer? That we are going to see, how the graph can be represented in a computer form in the memory? So that the algorithm can exploit; this particular model that is called a graph model.

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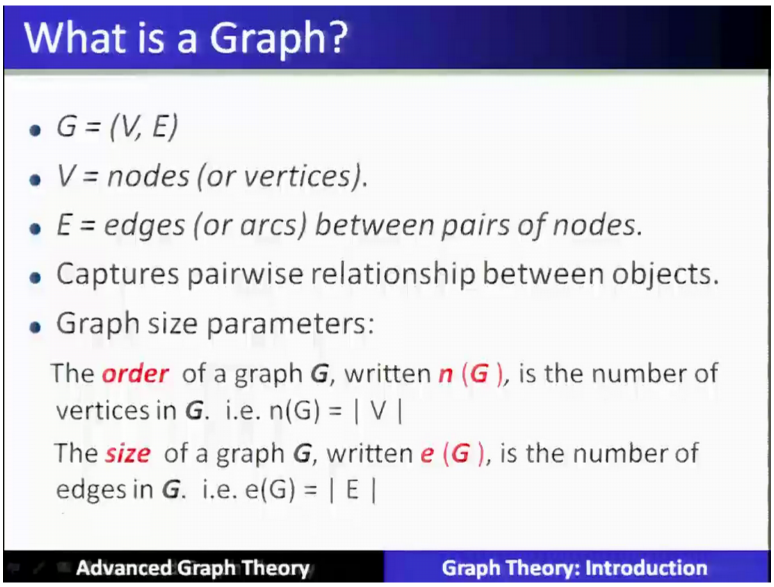

So, let us see in more detail about the graph. So, what is the graph. So, graph is a pair of set of vertices and a set of edges. So, graph G is represented as V comma E, where V is a set of nodes and E is a set of edges. So, set of edges is nothing but the pair of nodes. So, this particular graph will capture the relationship between the vertices, also called as a objects and these arcs represents the relations between the object. So, different applications will pose these relationships, and we are going to get a graph as a model of different applications.

So, this particular graph once it is defined as V comma E and represented as big G then, this particular graph has different parameters and P basically called a graph with those parameters like the order of a graph G, we can represent as the total number of vertices present in the graph.

So, when a similarly the size of a graph which is represented as the number of edges in a graph, is basically the cardinality of the edges and order of a graph is the cardinality of the vertices. So, they are basically the graph parameters.

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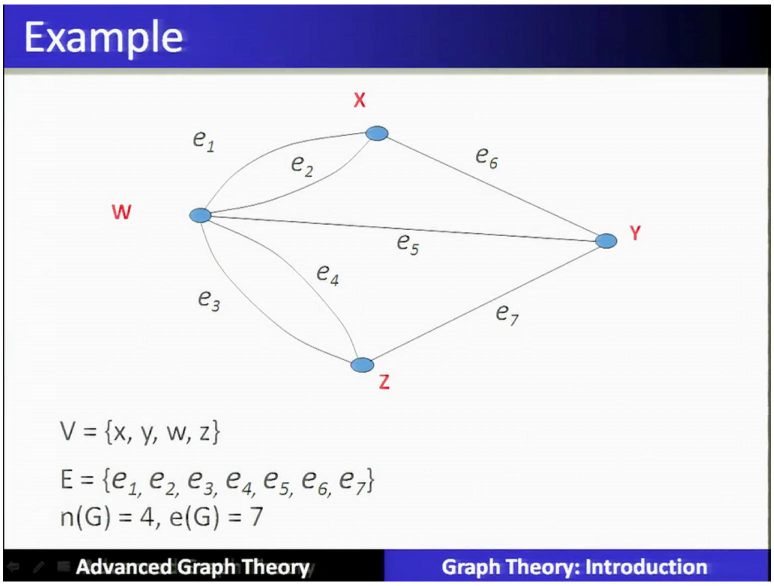

So, in this particular example of a Konigsberg bridge problem, which is converted in the form of a graph. We can see the set of vertices, they are number 4 different vertices the order of the graph is 4, similarly the size of a graph is 7 because there are 7 different connections or 7 different bridges, which are connecting those vertices.

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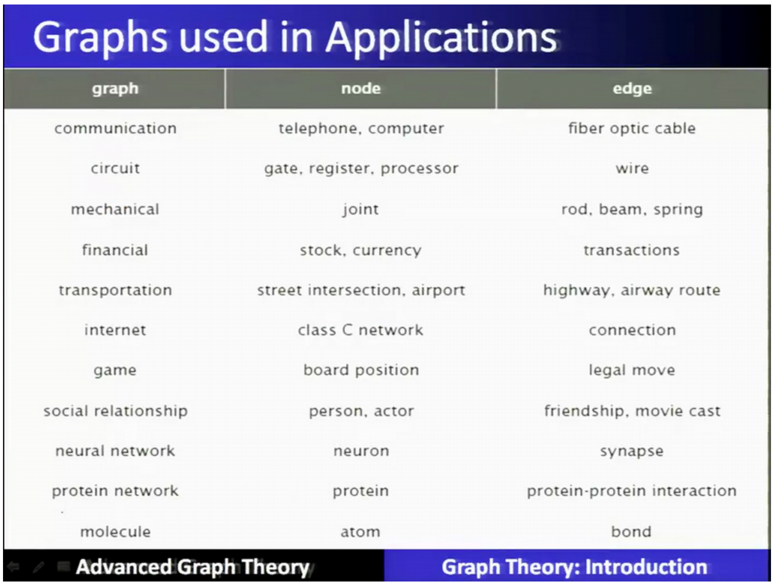

graph	node	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

Like we have seen a Konigsberg bridge problem, that particular problem was modeled as a graph. So, graphs we can now see it is being modeled for different other applications. So, the applications and how the graphs are being formed, we are going to now discuss briefly all of this.

So, as far as if it is a communication network, then the graph which we form out of a communication network, where the nodes represents basically the telephones or basically the other instruments, equipment's and edges are nothing but the cables which connects these particular end equipment's.

So, that is called basically the communication graph, similarly let us say the transportation graph comprises of the nodes, nodes are nothing but the intersection points of a road or basically the airports. And the edges are nothing but the arcs which connects these intersections or basically different airports connected by the flight. So, this is called basically the transportation network, whether it is a road transportation network you can represent as a graph, similarly the air line or air connectivity of a particular nation you can represent also in a form of a graph.

Internet also you can represent in a form a graph, where the node represents a classy network and the edges becomes the arcs, or the connections between these classy networks. And this all will you all into a internet and that can be analyzed.

Similarly the social network where the nodes represents the actors, or a persons and the arcs represents the friendships or the relationships or acquaintances. So, such networks are basically which are used in a social network, that is called or social graph. Similarly, neural networks also can be basically modeled in the form of a graph, where the nodes are nothing but the neurons and the arcs or snaps.

Protein networks, also similarly we can be defined. So, you see that there are applications almost most of the applications, where the problem you can model in the form of a graph and do the analysis and solve the problem. So, once a problem is modeled in the form of a graph, then you can visualize the problem and applying the theory of a graph theory which we are going to cover at this part of the course can be used to analyze the problem, and solve problem more accurately why because? Lot of theorems and the proofs are available. So, those can be used to solve the problem correctly.

Not only that, we can also develop the algorithms once the graph be represented in a in a computer. How it is represented in a computer? That we are going to see. Once the graph is represented in the form of a computer and the theory using that that particular theory we can exploit the structural properties of a graph, and we can come out with an algorithm. So, in this part of the course the algorithms which we are going to discuss, will exploit the graph structural properties to solve the problems, which are which are there in the applications.

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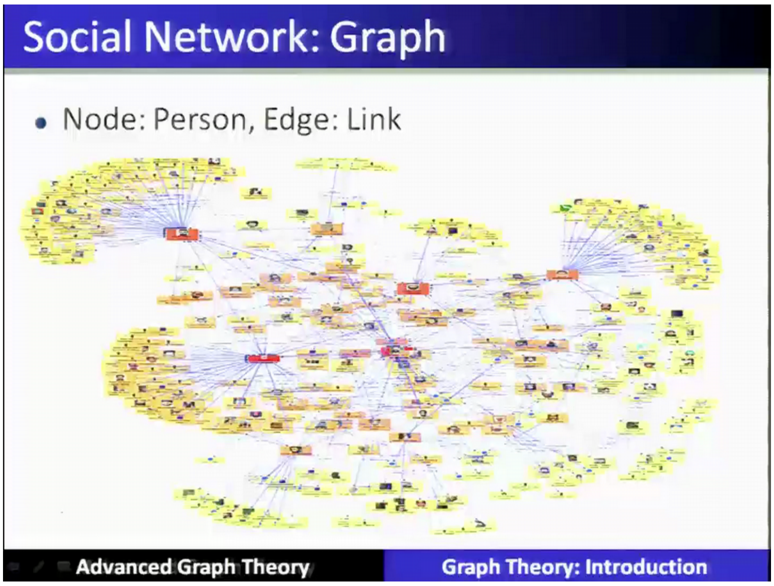

Let us see the example of a social network. So, what is the graph here in the social network? So, the nodes represents the persons or the people and the edges will represent the link between the people, or relations between the people. So, the social network graph will look like as per shown in the figure, it is a big graph which will form and which will keep on evolving, where the nodes will come and new nodes are joining the network. So, network social network keeps on growing in this particular manner.

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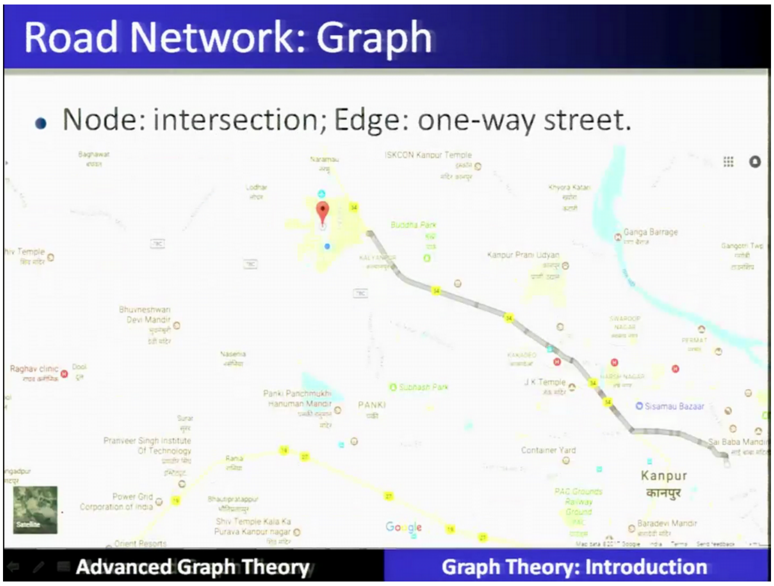

Road network, how it can be modeled as a graph? We can see over here. Where the nodes are nothing but the intersections, and the arcs or the edges are nothing but the road segment which connects these intersections. So, we can form or we can evolve a road network and this can be modeled as a form of a graph, ecological food web also you can represent in the form of a graph as a model.

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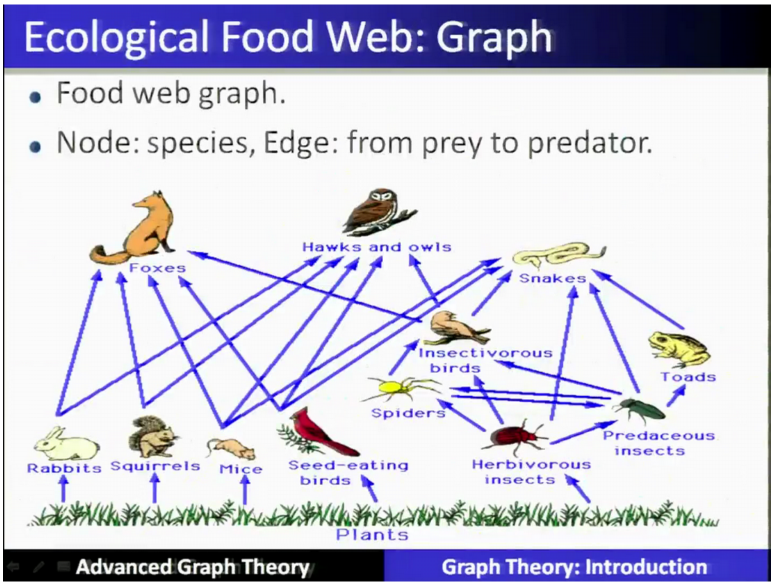

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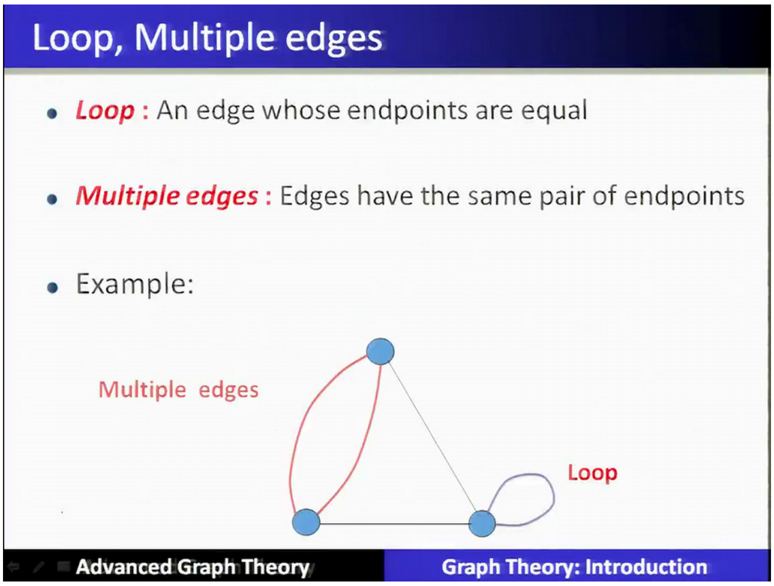

So, wherein the node will represent different species and the arcs and the edges are nothing but the prey to the predator. Now, coming back to our graph. So, we can see that if the edges or the arcs which will basically, have the same end points or the end points are equal in the arc or the edges then it is called a loop. And if let us say the edges or the arcs have the same end points, more than one edges having same end points called multiple edges.

So, it is shown in this particular figure you can see here, that this particular arc is basically the multiple edges and if let us say the edge, is meeting at the same vertex then it is called a loop.

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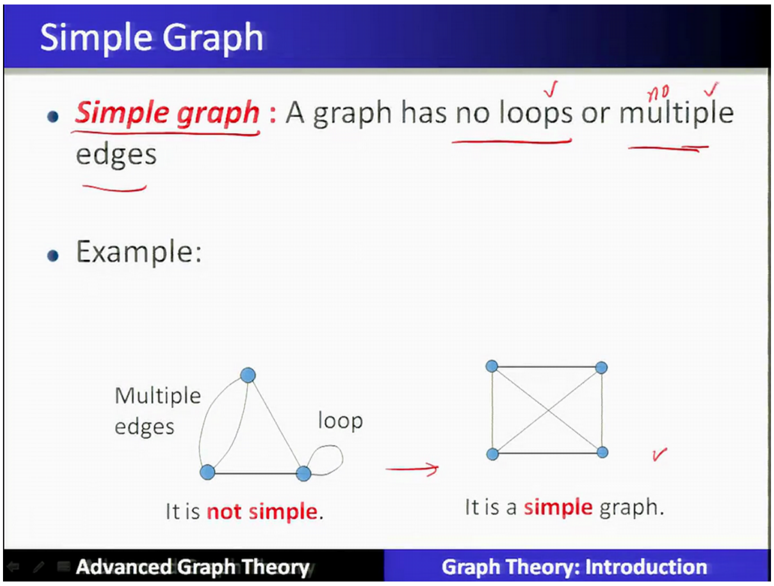

Now, a graph which has no loops and also which has no multiple edges, such graphs are called as simple graphs, some people call them as a regular graphs also. So, in this example on the right side this is the simple graph, why because? It does not have any loop nor it has any multiple edges, hence it is a simple graph example which is shown over here.

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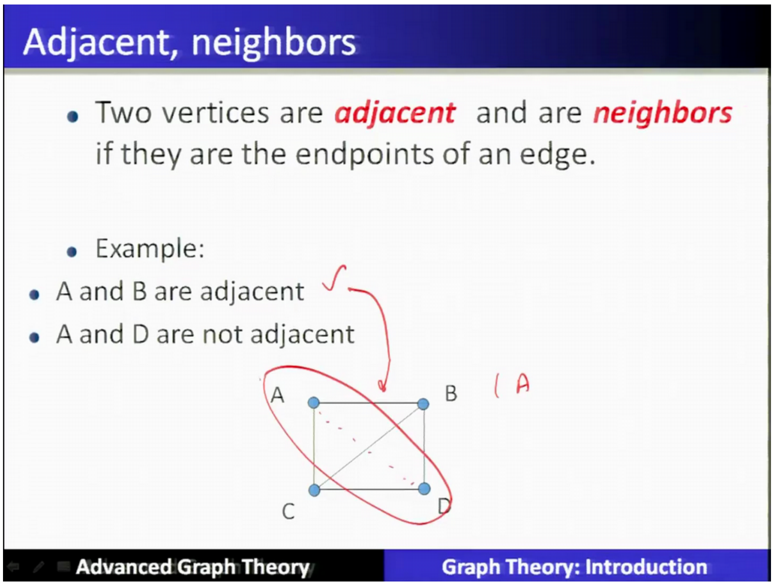

Now, furthermore we are going to discuss, if the edges are present in the graph what does it means. So, 2 vertices if they are having an edges, when they are adjacent and also, they are called as a neighbors, if they have the end points connected by an edge.

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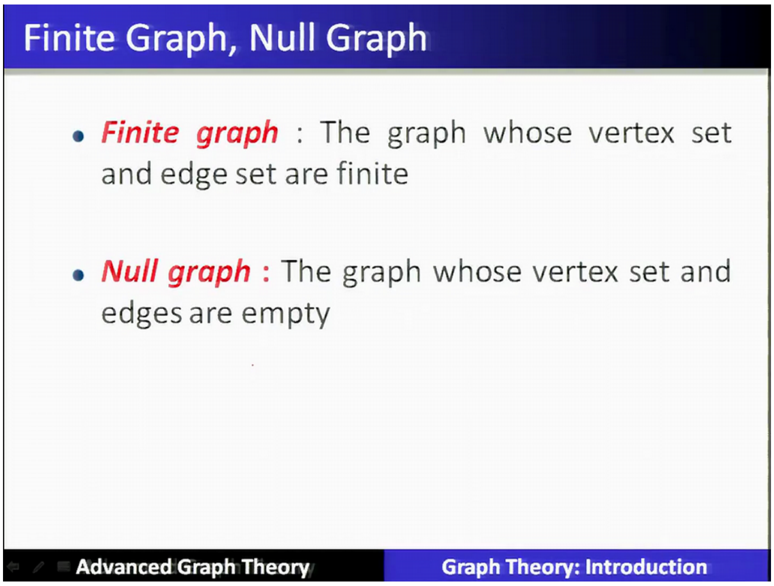

So, here this example a/ and b/ they are called adjacent. So, a/ and b/ are adjacent why because there the edge present over here. Now, a/ and d/ there is no edge, hence they are not adjacent. So, finite graph a graph whose vertex x and edge sets are finite, that is what we are going to basically assumed in this particular discussions.

So, null graph is the graphs whose vertex sets and the edge sets are empty. So, null graphs are a trivial graphs, we are not going to cover up those particular null graph as the model in this theory of a graph theory. Furthermore, the set of pair wise adjacent vertices, is called a click these are some of the terminologies, which are going to be widely referred throughout this particular lecture.

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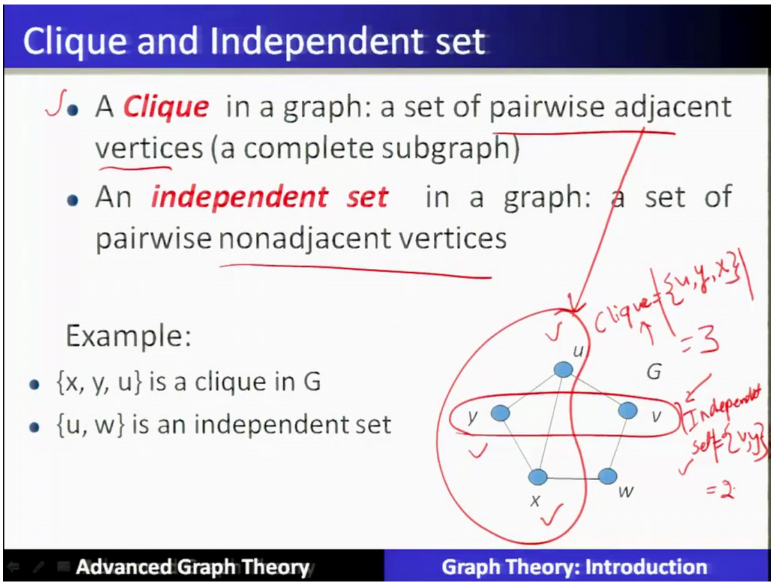

Now, a set of pair wise non-adjacent vertices is called independent set. So, in this example we can see that u x and y they are set of pair wise adjacent vertices, hence this is a click. Let us say that these 2 vertices V and y they do not have any edge, hence this will form a set of pair wise non-adjacent vertices and it is called independent set. Now, you can see in this particular graph you cannot find a click, which is bigger than this particular size that is more than the size 3.

Similarly, you cannot find an independent set, in this particular graph example which is having the size more than the size that is more than 2, hence they are basically the biggest or largest clicks, similarly the independent at it say largest side independent set. Now, a graph where in the vertices are partition into 2 disjoint independent sets. Independent sets I have already given you the example, we do not have a mutual edge or they are pair wise non-adjacent vertices.

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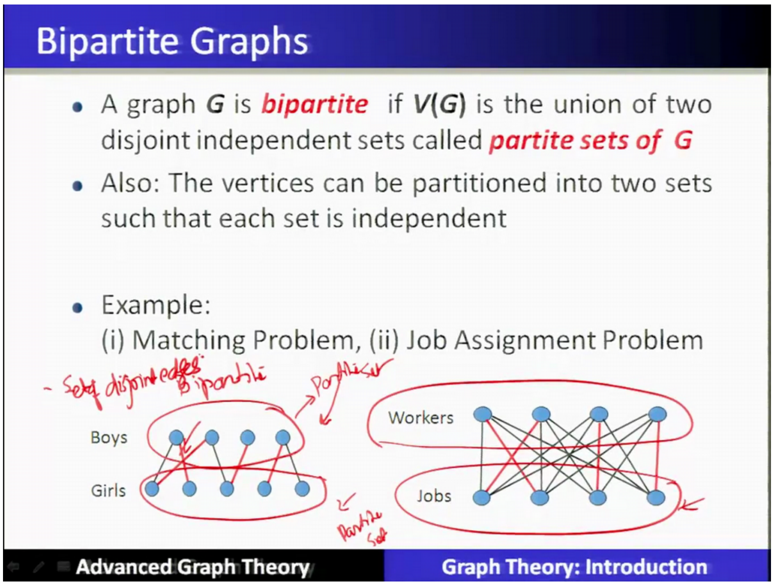

So; that means, if the vertex V G is partitioned as 2 disjoint independent sets, in a particular graph then that graph is called a bipartite graph. So, partite sets of a graph means, the independent sets of a graph. So, if the graph has 2 partite sets then the graph is called bipartite graph. Example we can see here, all the boys are represented in the form of a set, and all the girls are represented in another partite sets and the graph which we form is called a bipartite graph.

So, this is partite set 1 this is another partite set. Similarly, if you are solving the job assignment problem, that is the placement problem. So, here in this particular applications, we have the set of jobs which can be represented as 1 partite set and the set of workers which can do a particular job, will form another partite sets. The graph which will form out of their relationships is called basically, the bipartite graph. So, bipartite graph is going to be very useful wherever this kind of applications, they are 2 different partite sets are there and we require a graph to be analyzed, with such a relations that is called a bipartite graphs.

So, these particular example, you can see that it is written as a matching problem; that means, boys and girls you want to find out a match, match means the set of edges which will basically this edge will include one boy and a particular girl. So, set of such disjoint edges, is called basically the matching. So, matching problem is the set of disjoint edges. So, you have to find out the set of disjoint edges.

So, that it will solve some problem in an efficient manner in optimal manner. Similarly, job assignment problem also is to find out those set of match or edges, which will basically assign the workers to the job. So, that it will give the maximum profit to the company. So, that is the job assignment problem, similarly the matching problem is to match the boys and girls to for a particular ah purpose or a parti particular applications.

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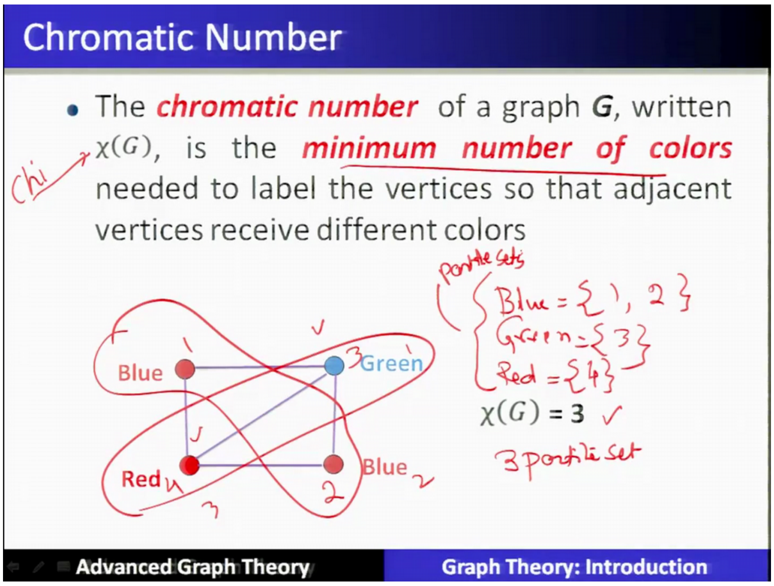

Now, with this partite set we have seen the 2 partite sets is called a bipartite graph. Similarly, if there are more than 2 partite sets then we have to define a parameter or that is called the chromatic number. So, chromatic number of a graph G is written by kai of g. So, this is a Latin symbol. So, it is a kai of G is nothing but a chromatic number of a graph, is the minimum number of colors needed to label the vertices. So, that the adjacent vertices should receive different colors.

Take this example over here, in this particular graph. These 2 edges which are these 2 vertices which are basically adjacent. So, they are receiving the different colors, these set of vertices they do not share an edge. So, they are non-adjacent. So, they receive the same color. So, how much minimum number of colors required to color the graph. So, that no to adjacent vertices receive the same color is basically, you can count 1 2 and 3; 3 different colors are required. So, the chromatic number of this particular graph is 3. So, you can partition this graph into 3 partite sets.

What are these partite sets? So, all the blue colored will form one partite sets, 1 and 2 let us say, then there will be a green is another partite set with only 1 vertex let us call it as 3 and then, there will be another set partite set let us call it as a 4. So, it will have the 3 partite sets. So, this particular chromatic number, is nothing but it will partition the set of vertices of a graph into 3 partite sets. So, this particular graph you can see, is a generalization of a bipartite graph which has only 2 partite sets, here you have obtained more than 2 partite that is a 3 partite sets. And that particular value is called the chromatic number of graph that is represented by kai of G.

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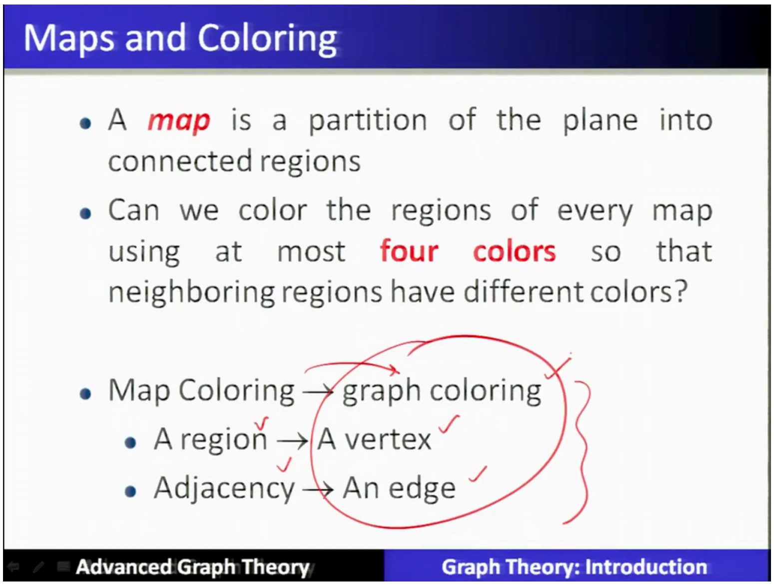

Now, the maps and coloring of a map is not a trivial problem, and it is a well-studied problem in a in a graph theory. So, map is a partition of a plane into the connected region, you might have seen the atlas as an example. So, how are you going to color this particular map, using at most 4 color is a well-studied problem. So, the coloring of a map is nothing but coloring these regions of in this particular map, using at most 4 colors. So, that no neighboring regions receive the same colors.

So, any 2-neighbor neighboring region they are going to receive a different colors. So, how using 4 colors, you can solve this particular map coloring problem. This particular map coloring problem can be represented as a graph coloring problem, in the following manner.

The regions in a map, you can be representing as the vertices of a graph and the adjacency relation between the regions, if you can represent as a edge of a graph, then this map coloring problem can be represented in the form of a graph, and map coloring you can represent in the form of a graph coloring; that means, you can find out the chromatic number of a graph and you can check whether it is at most 4 or not. So, that particular problem you can solve using, map coloring problem can be solved using a graph coloring problem.

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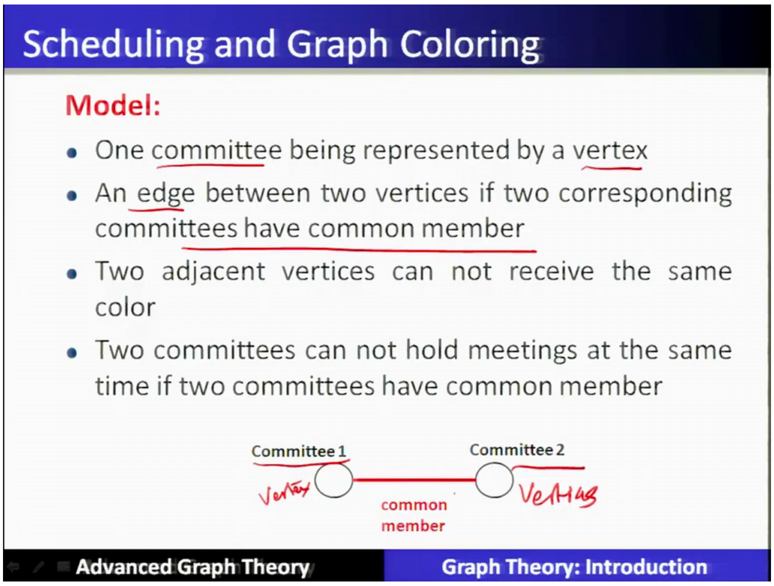

Now, we are going to see more applications here, and how the graph theory as a model of a graph? Can solve this particular problem efficiently, or can analyze the problem to the satisfaction of the solution.

For example, let us take the scheduling problem. So, we want to find out how to schedule the meetings of different comities, with a minimum number of time slots and this is called as scheduling problem. So, this particular problem you can model in a form of the graph wherein the committees are represented in a form of a vertex, and if there is a member present between these 2 committees or a particular member, which is common into these 2 different committees then you place an edge.

So, edge in this particular graph will represent, the committees having a common member. So now, here we can see that this particular committee is represented in the form of a vertex. So, different committees are represented in the form of a vertices of a graph, and between committee one and commit to do there is a member present, which is common. So, basically, they are represented in a form of an edge.

So, this having formed a graph out of the entire problem setting, we can find out the chromatic number of that particular graph, having known the chromatic number of a graph we can find out the how many minimum number of time slots required? To schedule all the meetings. And hence you can solve the problem in this particular way.

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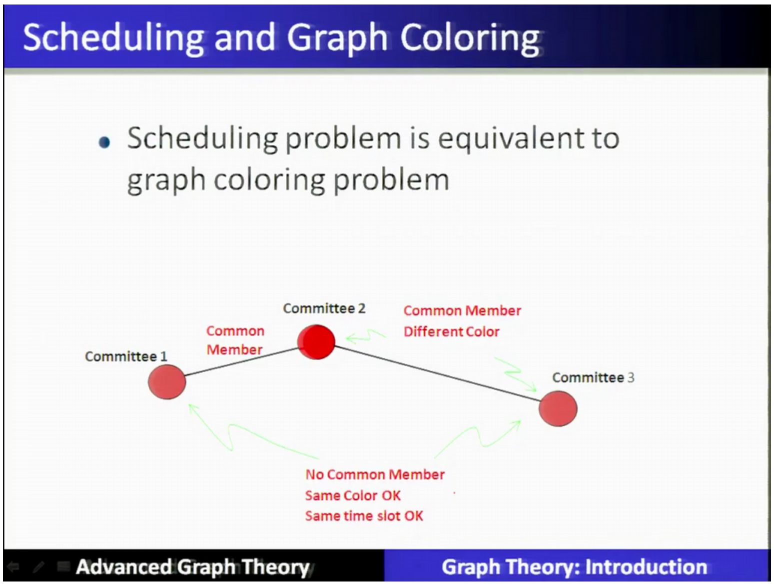

So, therefore, the scheduling problem basically, making it as an equivalent to a graph coloring problem. So, if you know how to solve the graph coloring problem we can solve this scheduling problem very well.

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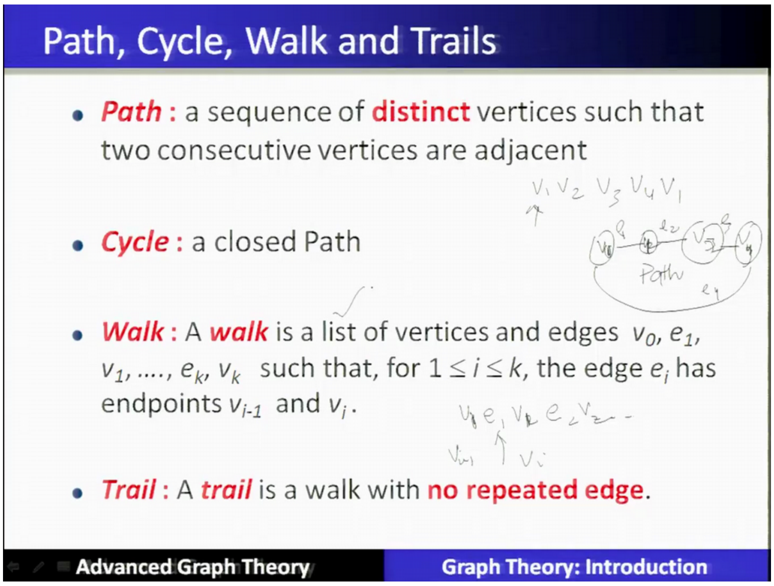

Now, let us quickly take up some of the definitions, those definitions we are going to quickly see their path, cycle, walk, trends path. A sequence of distinct vertices, shows that 2 consecutive vertices are adjacent for example, v 1 v 2 v 3 and v 4 there are 4 sequence of vertices. All these vertices are distinct, in the sense they are not repeated again and any of these 2 vertices are adjacent; that means, v 1 and v 2 they are having an edge, similarly v 2 and v 3 is having an edge v 3 and v 4 is having an edge. So, this is called a path.

Cycle is a closed path; that means, after v 4, if there is an edge to v 1 then it is called a cycle. Walk, a walk is a list of vertices and edges such that for a particular edge e I, it has the end points in v I minus 1 and v I. So, for example, if this edge is e I e 1 e 2 e 3 and e 4. So, you can write down the vertex v 1 e 1 v 2 e 2 and so on. So, for a particular edge e I it has basically. So, here we have to write down v 0 v 1 v 2 v 3 to have this particular notations v 0 v 1 v 2.

So, a particular edge e I has the end points as v I minus 1 and v I. So, having this kind of list of vertices and edges, is called a walk. A trail is a walk with no repeated edges. So, if the edges are distinct in a walk then it is called a trail. So, in this example a d then c then b then e this is a path sub graphs.

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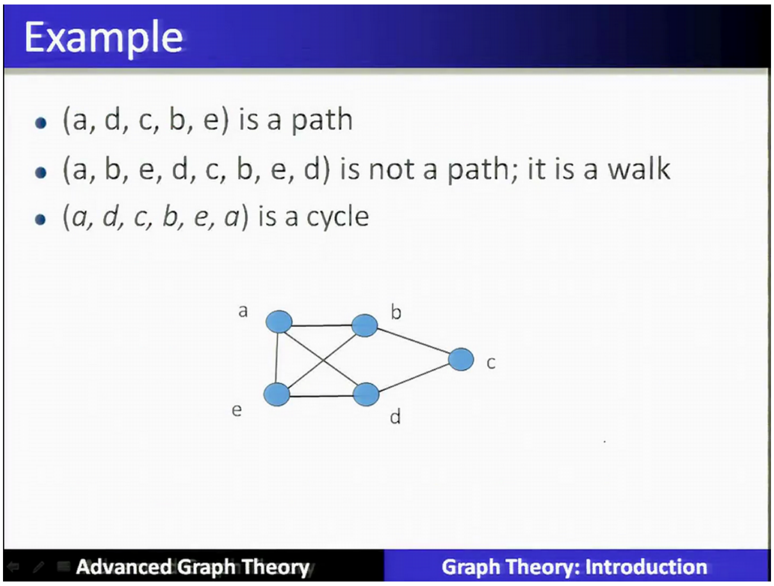

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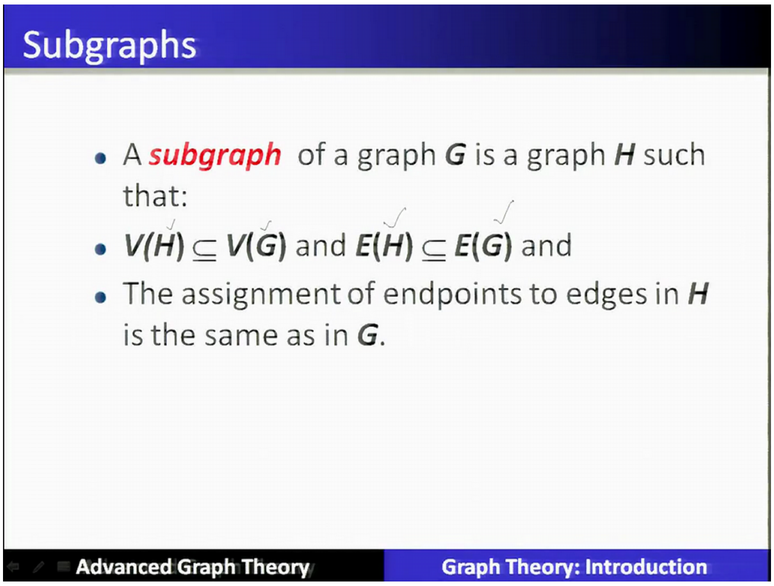

Sub graph of a graph G is a graph H such that the set of vertices of that sub graph H is a subset of vertices, of that particular graph G. And the set of edges of the subset of a sub graph H is a sub set of edges of a particular graph G.

And the assignments of the end points to the edges in H and is the same as in G, then it is called a sub graph of a, then it is called a sub graph of a graph G and that is represented as an H.

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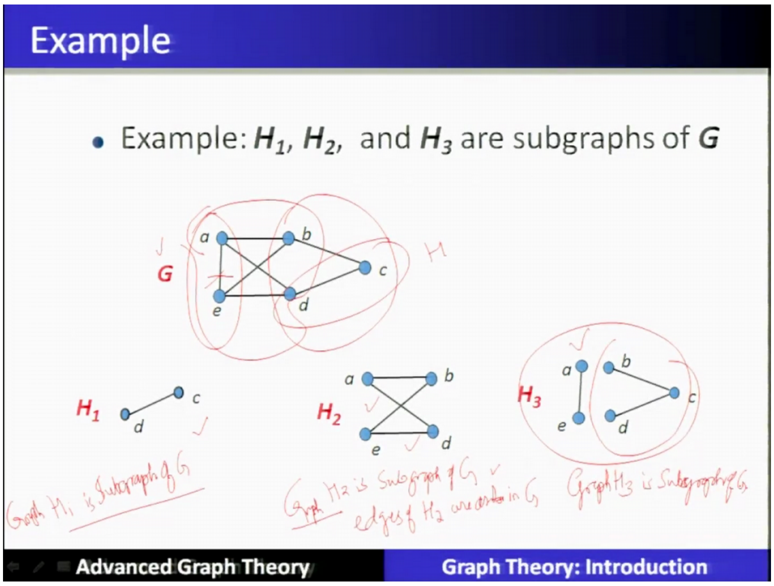

So, this particular example we can see here, that if this is the graph which is given then H 1 is a sub graph of G. So,. So, H 1 is H 1 is the sub graph of G. So, H 1 you can see c and d. So, this becomes a sub graph of G. So, all the incident edges to these vertices, is basically of a of a sub graph H 1 is also present in the main graph hence, it satisfies the definition of a sub graph H 2 is also is a sub graph of G.

So, so H 2 is a graph and it is also a graph. So, H 2 is a sub graph of g now we can see a b and e d. So, a b and e d. So, the edges which are present this is not present. So, all the edges which are present in H 2 are also present in G, hence this particular end also the set of vertices which are present in H 2 are also basically, thus the subset of the vertices of g hence H 2 is a hence the graph H 2 is a sub graph of G.

We can see that a e this edge, is also present vertices are also present in G, similarly b c d this particular component is also present. Hence the entire graph, that is the graph H 3 having 2 components is a sub graph of G.

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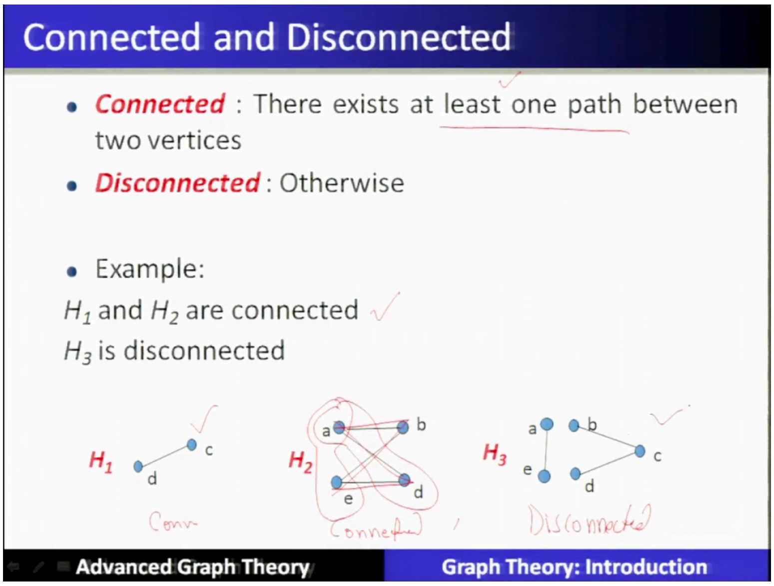

So, we have seen the the definition of a sub graph, and some examples for it. Now, we are going to see the definitions some more terms and definitions, here we have seen that the graph H 3 has 2 components is not a connected graph. So, therefore, we have to see what do you mean by a connected graph. So, a graph is connected if for every pair of vertices, there is a path then that graph is called a connected graph, hey otherwise it is disconnected. So, the example of H 3 is a disconnected graph and this is a connected graph and this is a connected graph.

So, if it is connected graph between any pair of vertices, there is a path similarly between a and d there is also a path for example, a is connected to e via d or e is connected to e via b. So, there is a path between a to e. So, there are more than one path, but says that at least one path exist between any pair of vertices. So, that exist hence H 2 is basically a connected graph. So, H 1 and H 2 they are connected graph whereas, H 3 is basically a disconnected this, is very important definition to be noticed.

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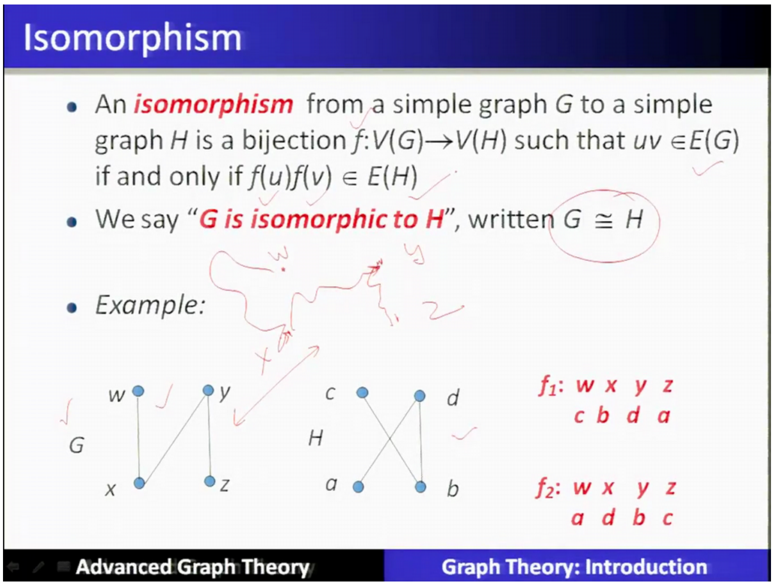

Now, another important concept here is called isomorphism; that means, the graphs you can form the way you like to form. Take this example the graph G, you can represent this particular graph g in another way so; that means, w x y and z you can represent using an arc like this, and this arc will also go like this and this is an arc. So, both these graphs are isomorphic in the sense that, this particular connection between w and x, either you can represent using a line or you can represent as an arc the way you like to have, all these connections are preserved in the other graph.

So, all the relations and all the vertices are preserved then it is called isomorphic. So, this is called as G is isomorphic to another graph, is represented by this particular notations. So, whenever they there is isomorphism between 2 graphs, that is there exist a bijection function f, which will basically do a mapping from the set of vertices of G 2 a set of vertices of the another isomorphic graph that is H.

Such that, any edge which is present in G will implies or by implies that, but edge in that corresponding mapping to the to another graph H should also be present as an H so; that means, all the adjacency relations are preserved by that bijection function. So, if you can come out with a bijection function, between 2 such graphs then those graphs are isomorphic and they are written in this particular manner.

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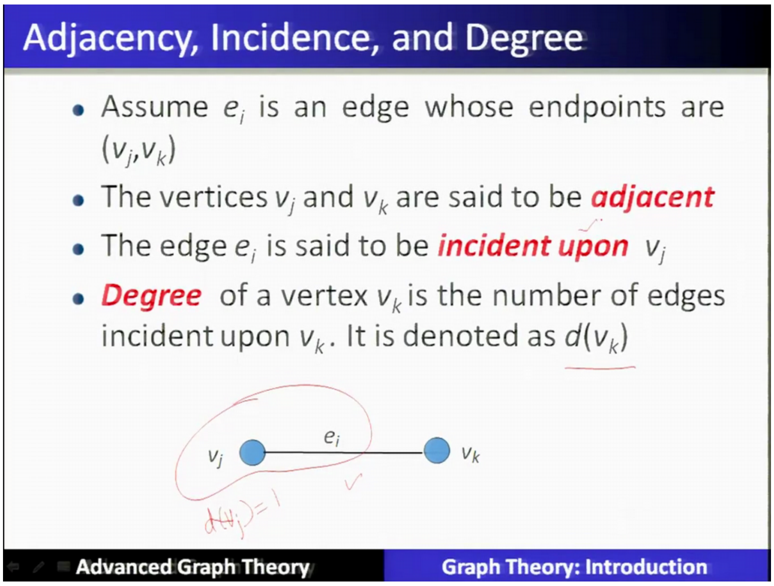

Few more definitions, as I told you that the adjacency relations are preserved by a function in isomorphic, isomorphic graphs or using isomorphism. So, we are going to see adjacency incident and incidence and degree, these 3 definitions now the vertices j and k they are called adjacent. If they have an edge which connects both these particular vertices, that edge is said to be a incident upon.

So, this particular vertex and this edge you can say that this edge is incident on this particular vertex. So, degree means degree of a vertex v k is the number of edges, which are incident upon v k and is represented by the degree of v k. So, let us take the degree of v j, is nothing but now many edges which are incident, here we can see only one edge e I is incident. So, the degree of v j becomes 1.

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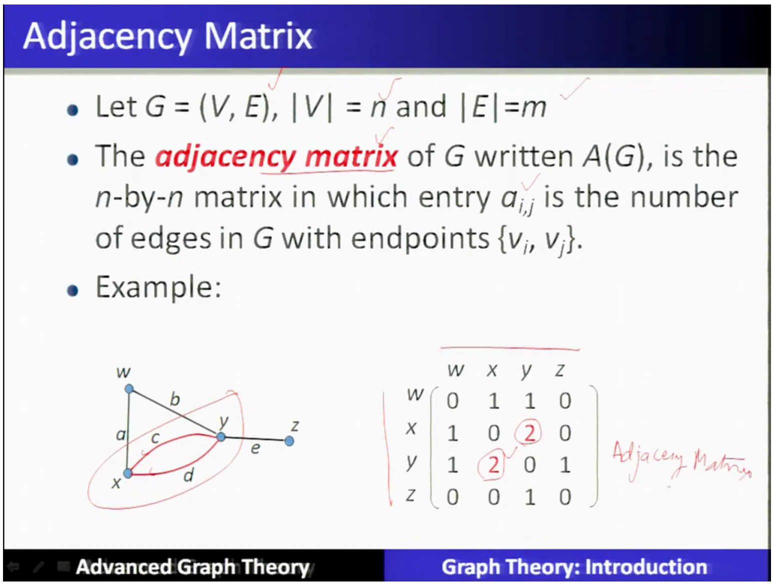

So, whenever we are counting the degree we are basically finding out the incident, how many edges are incident? On that particular vertex and that becomes a degree of a vertex. I told you that the graph, can be represented in a suitable form. So, that it can be stored in a computer memory. So, we are going to see what are those suitable forms? One way you can represent a graph that is called adjacency matrix.

So, given a graph G V E having an n vertices and m edges, we can form an n by n by m matrix and that matrix is called adjacency matrix. Wherein each element a I is the number of edges in G with the end points between v I and b j.

For example, y and x y and x. So, y and x if you take this particular example, x and y. So, between y to x y to x there are 2 different edges which are basically having the connectivity. So, a I j entry will be 2 similarly, x to y x to y is also to y because, from x you are having 2 different edges between these end points v I and v j. So, in on the row the set of vertices, similarly on the row also set of vertices these vertices how many edges? Basically, are connecting them how these edges are? How these vertices are? Connected and this particular connectivity of this representation of a graph is called adjacency matrix. So, this is the adjacency matrix representation of this particular graph G.

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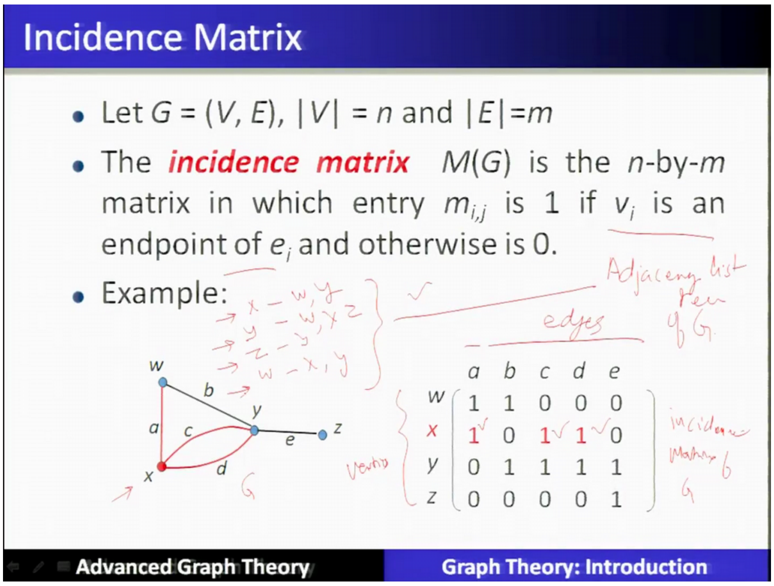

Now, another way we can represent and store the graph in a computer form or in a memory computer is called incidence matrix. Incidence matrix incidence I told you that a particular vertex v I, how many edges are incident? That relation is being modeled. So, on one side it will be a set of vertices, on the other side it will be a set of edges.

So, for example, x is a vertex. So, you can see that a incident on H or a incident on x. So, it is 1 then c is incident on x. So, it is also 1 and d is also incident on x. So, it is one. So, such representation is called the incidents matrix of this particular graph G. There is 1 more way to present the graph that is called the adjacency list, adjacency list representation of a graph for example, this is the graph. So, for every vertex x y z w you can find out the list of nodes or vertices which are adjacent.

For example, x will have it is adjacency list as w then y then only w and y. It is basically adjacent 2 similarly y is concerned, it is adjacent to w then x and then z z is adjacent to y and w is adjacent to x and y. So, such representation is called adjacency list representation of a particular graph.

Now, you can all 3 representations, you can easily express as a data structure in any of the programming language, matrix you can also express and adjacency list you can express in the form of a link list, representation and basically the graph is represented in this way and thereafter you can apply the graph theoretic algorithms, which we are going to discuss in this part of the course.

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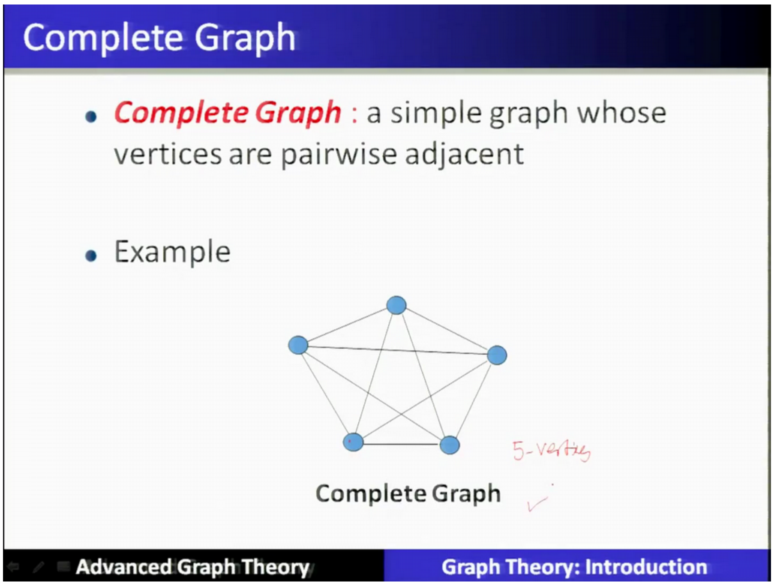

We have seen one graph, that is called a bipartite graph we are going to see another graph which is called a complete graph. So, complete graph is a simple graph whose vertices are pair wise adjacent. So, here in this particular graph having 5 nodes, all these vertices are basically pair wise adjacent. So, pair wise adjacent means this vertex is adjacent to any of these pairs you see that there is a connection. So, also all the vertices hence it is called a complete graph.

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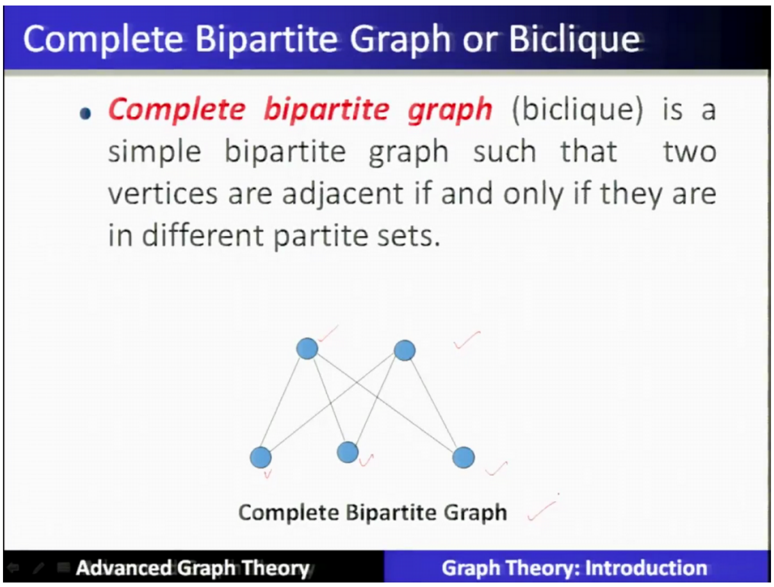

Now, complete bipartite graph it is also called a biclique, if it is a bipartite graph and having the adjacency relation to all other vertices. So, for example, this vertex is adjacent to all other vertices on the other partite sets, similarly this vertex is having the adjacent to the other partite set, similarly this one similarly these vertices also hence it is called the complete bipartite set.

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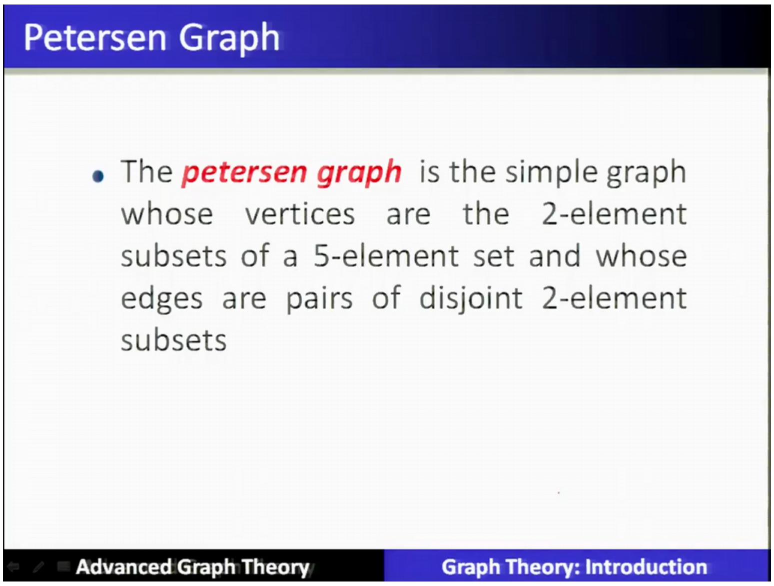

So, complete bipartite set is a simple bipartite set simple bipartite graph such that 2 vertices are adjacent, if only if they are in different partite sets. Now, another special graph which we are going to just mention is called a Petersen graph. So, Petersen graph is a simple graph whose vertices are 2 elements of sets of 5 element sets and whose edges are, the pair of disjoint 2 element subsets.

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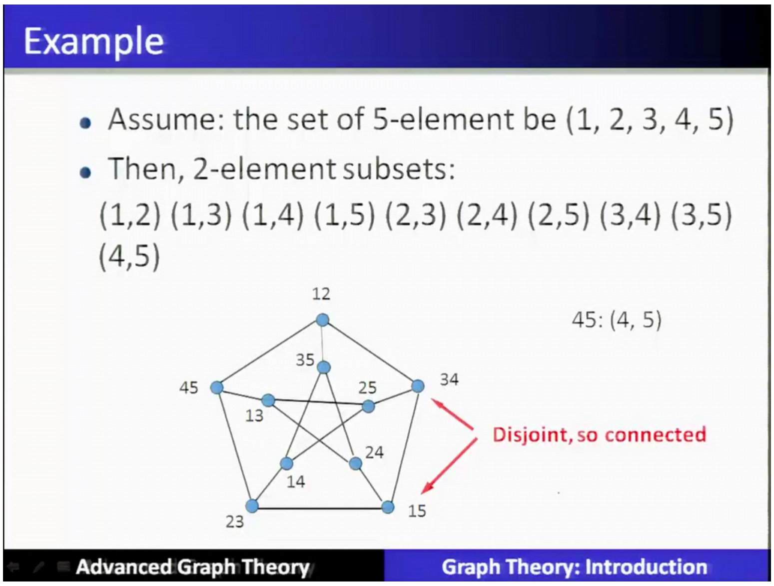

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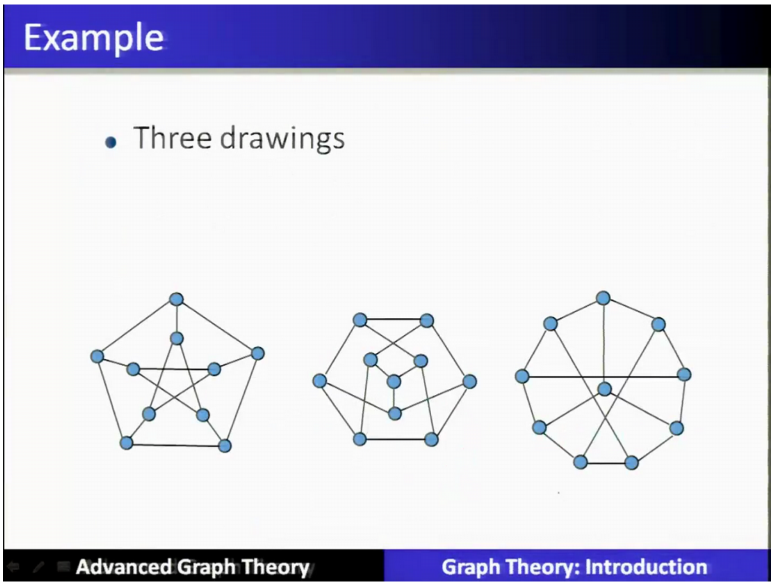

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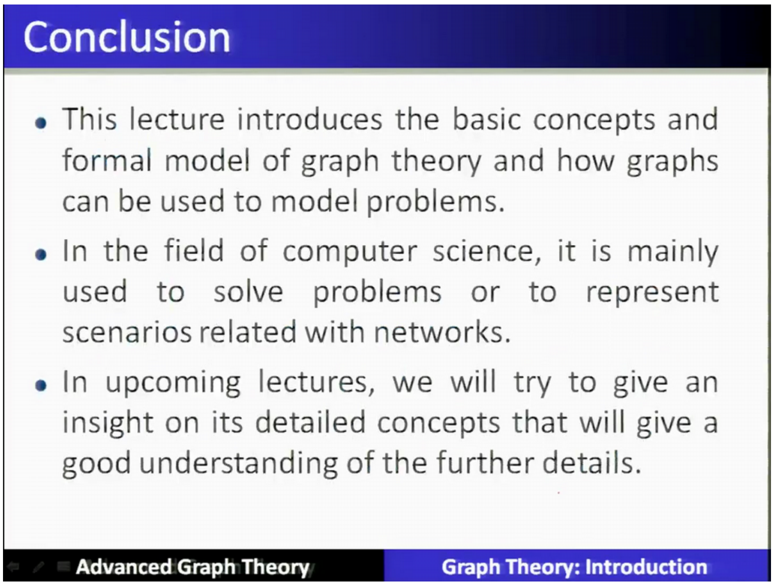

So, this is called a Petersen graph this is the example of a Petersen graph. So, here this is the diagram of a Petersen graph and different drawings of a Petersen graph are also given. So, let us conclude this particular discussion. So, in this lecture we have given you a very, very important notion, how of problem different applications you can model into a form of a graph, that is called a graph model of a application and you can analyze thereafter.

So, here we have introduced the basic concepts in the formal model, of that particular graph theory and how the graphs can be used to model various problems. In the field of computer science, it is the graph theory is mainly used to solve problem or to represent the complex scenarios and most of these scenarios will evolve, towards a network. And hence, represented in a simple form that is a graph you can do the illustration you can view that and you can basically do the analysis, and solve the problems underlying that applications.

So, in the upcoming lectures we will try to give an insight out on it is detailed concepts, that will give a good understanding of for the details. In the sense having expressed the problem in the form of a graph model and having represented the graph in the form of either the incidence matrix, adjacency matrix or adjacency list that is the computer representation. So now, we can exploit the structural properties of graph to solve various problems, using the algorithms and using the theory that we are going to cover up in the further classes.

Thank you.