Randomized Algorithms
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Lecture – 13
Introduction to Probabilistic Method

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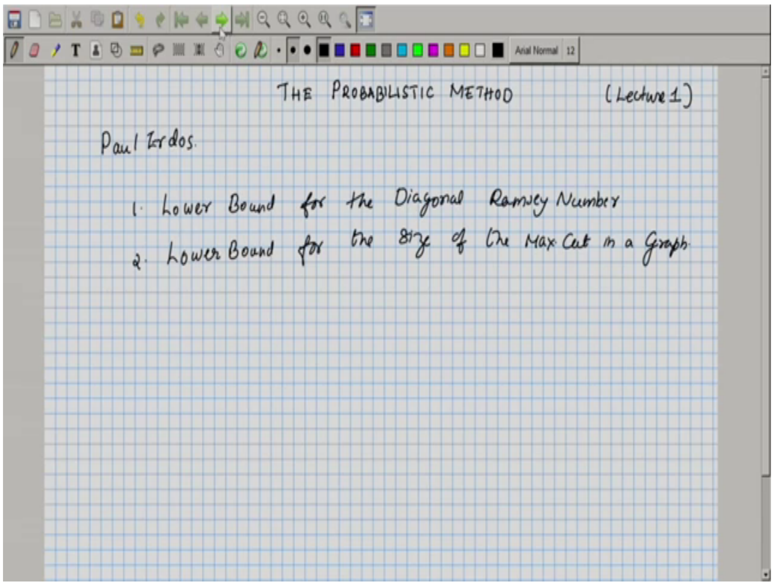

Welcome to the first lecture on Probabilistic Method. This is a method pioneered by Paul Erdos. And it is found a lot of applications in combinatorics and computer science, theoretical computer science. We shall begin by discussing two problems. One is the lower bound for the diagonal Ramsey number, and the second problem that we will address today is the lower bound for the size of the maximum cut in a graph ok. So, we shall learn probabilistic method by means of many, many examples. This is a two simple problems for which probabilistic method gives reasonably interesting bounds ok.

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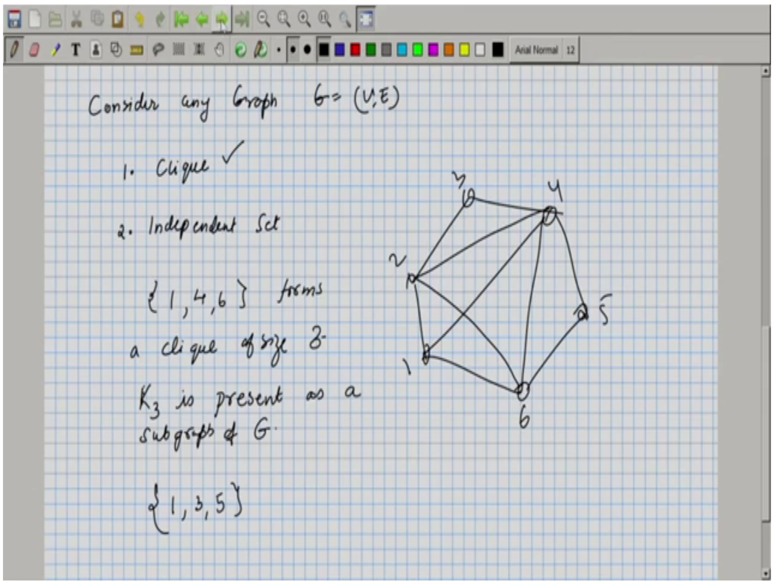

So, let us understand what is this diagonal Ramsey number. So, consider any graph G ok. So, this graph can, so we are looking for two substructures in this graph, one being a clique and the second being an independent set. So, clique is a set of vertices such that so if G equals V, E we take a subset of vertices such that all of them are connected to each other that is called as a clique. For example, if we look at this particular graph with 6 vertices, and if I look at vertices 1, 4 and 6, there are all connected to each other that is going to be a 3 clique, so that is a clique of size 1, 4, 6 forms a clique of size 3 ok.

Now, you can simply say that K 3 is present. K 3 is a complete graph on three vertices as a. So, we can say K 3 is present as a sub graph of G. If we had let us say 2 connected to 4 and 2 connected to 6, then 1, 2, 4, 6 forms a clique of size 4, so that is the definition of what is a clique. Now, if you look at three vertices 1, 3 and 5, you can see that it is in some sense the compliment of the clique, none of those edges are pressed. 1, 3 is not an edge 3, 5 is not an edge and 5, 1 is not an edge. So, those kind of collection of vertices are called an independent set. So, independent set is a collection some subset of vertices of the graph such that for any two vertices in that set, they are not connected by an edge, so that is an independent set.

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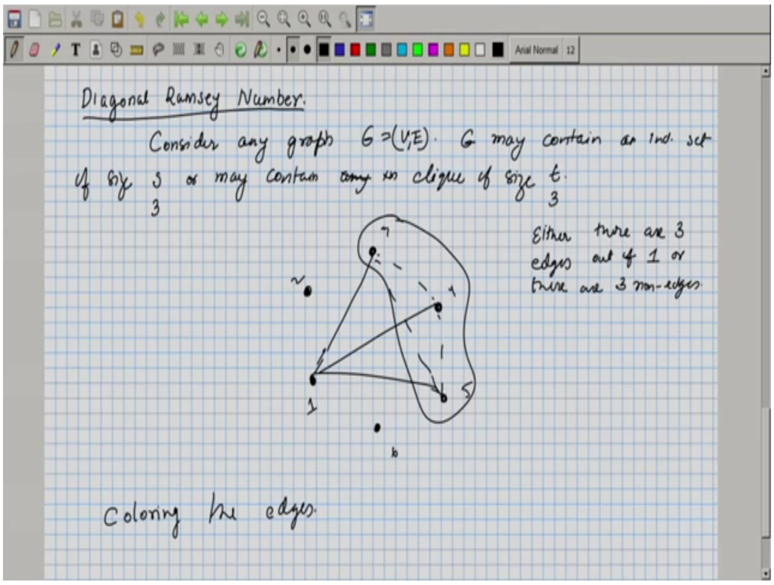

So, let us now defined what is called as the diagonal Ramsey number ok. So, consider any graph G ok. This graph may contain an independent set, G may contain an independent set of size let us say S or may contain a clique of size t ok. So, we are interested in finding, so this is not I mean if you take an arbitrary graph it may contain neither, but we are looking at the smallest number n such that if you take any graph with those many vertices, it will either contain an independent set of size S or a clique of size t ok. So, will there exists such a number, if there is one such number, and let us say if you take let us fix S to be let us say 3 and t to be 3, what sized graph should we take, so that it will invariably contain either an independent set of size 3 or a clique of size 3 ok.

One can show that if you take a graph on 6 vertices, this property will be true. So, let us just quickly see that. So, let us take any graph on 6 vertices and let this be one of the vertices of that graph ok. Now, there are six, five other vertices namely 1, 2, 3, 4, 5 and 6. Now, out of these 5 neighbours or 5 vertices in the graph, one may be connected to three of them or may not be. Let us I mean, so we can write the following, either there are 3 edges out of 1, or there are 3 non-edges. This is because the total number of vertices are 5 ok. So, number of edges plus number of non-edges is going to be 5. So, one of those numbers should be at least 3.

So, let us see without loss of generality assume that this is there are 3 edges, the non edges case is exactly similar, it is a symmetric argument ok. So, let us say these three edges were present. Now, if either 3 and 4 are connected, if I mean let us look at these dotted edges they are not edges. Let us consider those pairs of vertices, if even one of them is present, you automatically have a clique of size 3. So, in order to avoid a clique of size 3, all those three have to be not present in which case they form an independent set. And this tells us that if you take any graph with 6 vertices or more, it will invariably contain a clique of size 3 or an independent set of size 3.

We will generalize this ok. And instead of talking about independent sets and cliques, we will talk about colouring the edges. You may think of it as the edges that are present there have been coloured is in one colour, and the edges that are wrapped the pair of vertices which are not connected, they will be coloured use in a different colour. So, we will state the definition of Ramsey number in terms of colouring.

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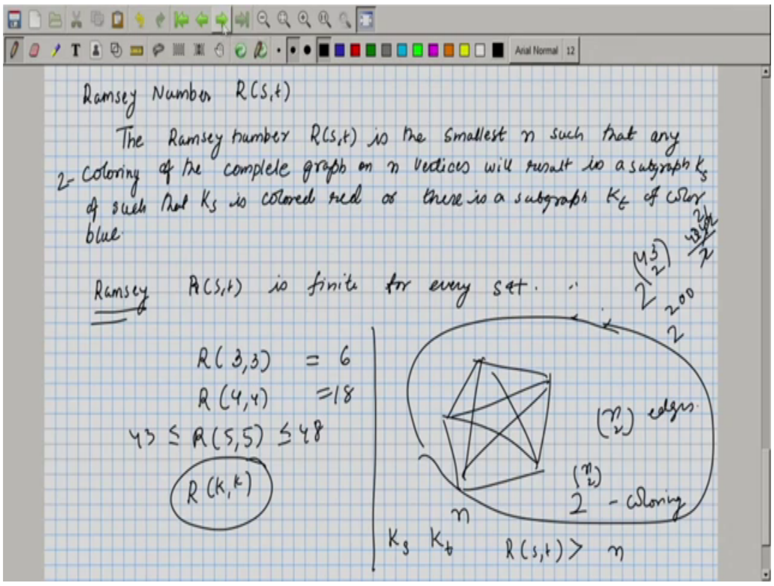

So, Ramsey number R s, t ok. So, this is defined as the Ramsey number R s, t is the smallest n such that any colouring of the complete graph on n vertices will result in a subgraph of say subgraph K s, so that is the complete graph on s vertices such that K s is coloured. See we are colouring. So, this any colouring here, we are two colouring. Any two colouring it is coloured and we will assume without loss of generality that the colours are red and blue, this coloured. So, there is a subgraph such that that is coloured red or there is a subgraph K t of colour blue ok. So, the smallest number n such that any two colouring of the complete graph on n vertices will result in either a subgraph of size s, which is coloured red or a subgraph of size t which is coloured blue ok.

So, clearly if there I mean the set of numbers which satisfy this property, they will be when if n is present, n plus 1, n plus 2 all those things will be present. The smallest such as what we are interested in ok. So, it was shown by Ramsey that, so Ramsey's result is that R s, t is finite for every s and t ok. So, now, what we are interested in is how large will R s, t be what is known us. We have right now seen that that R 3, 3 is equal to 6. R 4, 4 has been shown to be equal to 18. R 5, 5 it is not yet the exact value of R 5, 5 is not known. It is known that it is between say 43 and 48.

Now, how do we determine R 5, 5 or for that matter any R k, k. So, when s and t are equal that is what is our diagonal Ramsey numbers. So, how do we determine R k, k ok, exact values is very difficult even for small values of k. So, can we bound it ok. If we were thinking about computing R k, k, what should we do how expensive is it to compute it using computers.

So, let us just look at the following. Let us take a graph on n vertices ok, complete graph on n vertices. So, there will be n choose to adjust, and we will try all possible colourings, so that is going to be 2 raised to n choose to colourings 2 raised to n choose to colourings. We will try all possible colouring. If we can find one colouring such that, it avoids the read clique of sizes and the blue clique of size t, then we know that R s, t is going to be greater than that particular n ok. Clearly, we cannot carry out this computation because even if let us say n is say as small as 5, this gives us 2 raised to 5 choose to so 10 sorry not, so, this n is the I mean. So, here the R 5, 5 we know it lies between 43 and 48.

So, if we where to try to check if R 5, 5 is greater than 43, we could take the graph and 43 vertices and colour it using all possible colours that will give us 2 raised to 42, 43 choose to colourings ok, that is about you can say at least it is 2 to the power 200 different colourings that is prohibitively expensive. So, 43 into 42 by 2 that is 21, so 43 into 21, these many colourings are there 86. So, we cannot really run a bruit force algorithm to check whether R s, t is greater than a particular n ok. So, we will see how probabilistic method can be used to show that R s, t is at least greater than a certain number ok. We will use it to obtain lower bounds. So, let us see how this lower bounds is obtained.

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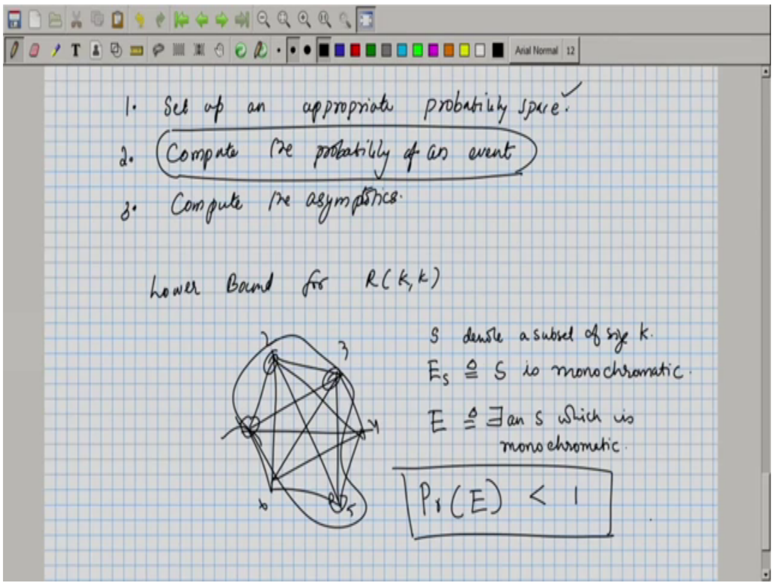

So, this is a kind of problem that we are interested in. So, we have a combinatorial property which is captured in terms of this let us say R s, t ok. So, is there a graph on n vertices, so if we take the complete graph on n vertices, can you obtain a colouring which avoids say cliques of size s which are monochromatic or cliques of size t whichever of the other colour. So, this is the kind of property that we are interested in. And we will use when we use probabilistic method typically the proof proceeds in two steps for three steps first.

We will have to setup an appropriate probability space, and then we will sample from this probability space, and compute the probability of an event ok. And in third step, we will compute the asymptotics ok. Initially, we will go through all these steps one by one. Later on, we will not explicitly do that ok. It will be I mean, but essentially that will be the steps that will be that we will be employing ok. So, in case of when we want to obtain lower bounds for R k, k diagonal Ramsey number ok. The probability space is the following ok.

So, let us look at the graph on n vertices ok, and we will choose a colouring of these things ok. So, the sample space consists of all the possible colourings that we can obtain for this ok. So, what are the probability of each colouring that we will assume that each edge? We are colouring the edges here. Each edge is being coloured either red or blue and that with equal probability, so that gives us a description of the probability space ok.

Now, in other words, we can think of this as if you are given a graph on n, so we want to analyse whether n for some value, whether it can be an upper can be a lower bound for R k, k. For this we will colour the edges of k n by tossing a coin. If the toss results in a head and that edge is coloured in red colour; if it results in a tail that edge is coloured in a blue colour coloured using blue colour.

Now, we can compute the probability of an appropriate event. We have coloured this thing thin k n randomly, and now we look at the following event there. So, so, let me describe a certain event. So, 1, 2, 3, 4, 5, 6 these are the vertices let me look at some k of these vertices ok. Let s denote a subset of size k ok. Let E s denote the event that s is mono chromatic ok. So, this will have a certain probability and let E denote the event that there exist an s which is mono chromatic ok.

So, this is the event that I am interested in and I will compute probability of E, if the probability of E is less than 1 then that means, when you randomly colour E there is a colouring which results in their being no monochromatic cliques ok. So, if we can compute this probability or we can show that probability of E is less than 1 then that would mean that n is a lower bound for R k, k. In other words the graph a graph on n vertices can be coloured avoiding monochromatic k cliques ok.

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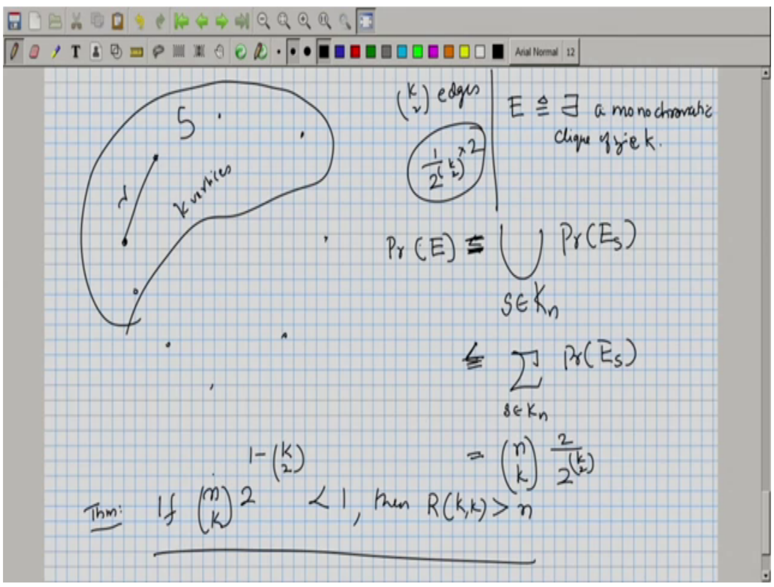

So, let us first compute that probability. So this is our graph on n vertices let us look at this particular set s ok. It has k vertices in it that means, the edges there are k choose to edges. All of them could be blue or all of them could be red let us compute the probability of that, well each edge is being coloured independently of the other edges.

So, if one of them is let us say, this is a red edge all the others have to be coloured in the same colour if this has to be monochromatic clique of size k and a monochromatic clique of colour red ok, so that happens with probability one by, so, there are two possible colours that we would have and each colour I mean each edge is being. So, probability that the edges are all coloured with the same colour is going to be 1 by let us say all of them being blue happens with probability 1 by 2 raise to k choose 2, and the colour could be either red or blue. So, this is the reason into two. So, probability of the event E ok, E was the following event there exist a mono chromatic clique of size k ok.

This probability is going to be less than union over all s belonging to K n ok. We will look at all possible subsets. Probability of E s and that is equal to, so, this is equal to and that is less than by union bound sum over s belonging to K n probability of E s ok. So, this is equal to there are n choose let us say so this is a subset of size k. So, there are n choose k possible subsets that we could choose and each of them for each of them then this is a probability that it is monochromatic. So, 2 by 2 raise to k choose 2 ok. So, this is the probability that this pair of vertices is monochromatic that there exist a set of monochromatic edges.

Now, if this quantity is less than 1, if probability of E is 1, so we can write this as a theorem. If probability of if n choose k times 2 raised to 1 minus k choose 2 is less than 1, then R k, k is going to be greater than n ok. So, that is our theorem. Now, we need to derive asymptotics that is for a given value of k, what is the best possible n that we can obtain.

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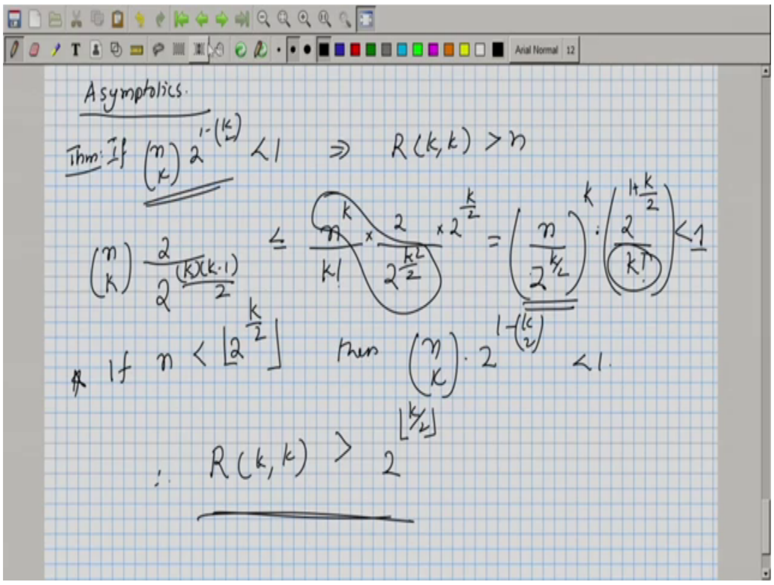

So, we will work out the asymptotics ok. So, our theorem was the following n choose K times 2 raised to 1 minus K choose 2 less than 1 implies R k, k greater than n ok. So, let us look at this quantity that is going to be n choose K and 2 raised to 1 divided by 2 raised to k choose 2 is k into k minus 1 by 2 is less than this quantity is less than. So, this is the form when n choose k you can approx you can say that it is all it is always less than n raised to k. There are k terms in the I mean when you expand it there are k terms in the numerator divided by k, factorial into 2, divided by say 2 raised to k, square by 2 and there is a minus k by 2, which will take up 2 raised to k by 2 ok.

I am rearrange this and write this as n by 2 raised to k by 2. So, this we will combine that is what I am writing n by 2 raised to k by 2 the whole raised to k times, 2 raised to k by 2 1 plus k by 2 divided by k factorial. Now, if you look at this quantity, if n is let us say less than 2 raised to k by 2 or let us say or 2 raised to k by 2, 2 raised to k by 2 depending on k being even or odd may not be an integer. So, if n is less than 2 raised to k by 2, and this quantity it is going to be less than 1 and 2 raised to 1 plus k by 2 has k by 2 terms, whereas this has k terms most of them like accept the term 1 and 2 everything else is greater than 2. So, you can say that this quantity for reasonable values of k, this is going to be a much smaller than 1. So, this is the overall quantity we can say that it is less than 1 if this condition is true.

So, if so, we can write it as if n less than 2 raised to k by 2, then n choose k times 2 raised to 1 minus k choose to is less than 1. Therefore, we can say from the theorem R k, k it is going to be greater than 2 raised to k by 2 ok, so that is the first lower bound that we have obtained. We can use more complicated constructions and more complicated sample spaces to provide an improved slightly more improved bound than this.

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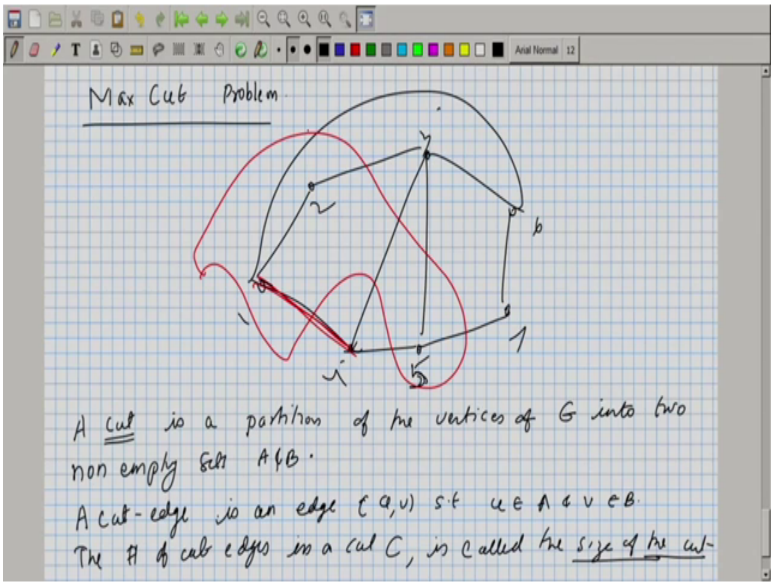

So, now we will look at yet another problem. The next problem that we will look to illustrate the power of probabilistic method is something called as Max Cut problem ok. So, let us understand what is a cut. Let us take a graph ok. So, this is a graph with 1, 2, 3, 4, 5, 6, 7 vertices. A cut is a partition of the vertices of G. So, if your graph was G, a cut is a partition of the vertices of G into two non empty sets A and B. Let us call them as A and B. There is also a different notion of an s t cut in which case where as s and t are vertices. In that case we will ensure that s belongs to one partition and t belongs to the other partition but so this is what a cut is. You could choose any collection of vertices in that full form cut.

For, for this graph let us say we will look at vertices 1, 2 and 5. So, if you look at this vertices 1, 2 and 5 in one set that forms a I mean 1, 2, 5 in one being a and 4, 3, 6, 7 being the other set that is our cut. Now, if you look at the edges of the graph, there are some edges which lie completely inside one portion. They are some which lie in the completely in the other portions, and then there are edges which go from one side to the other. For example, if you look at the edge 1, 4 ok. So, 1, 4 is an edge which crosses the cut that means one of the edges or one of the vertices of the edges in A and the other is in B. These are what is called as a cut edge.

So, a cut edge is an edge u, v such that u belongs to A and v belongs to B. The number of cut edges in a cut C is call; so if you name the cut by C, the number of edges in the cut is called the size of the cut ok. So, here in this cut the number of cut edges is going to be so, 1, 4 is a cut edge. 2, 3 is a cut edge. 3, 4 is not a cut edge, because, these are both in the same component ok. So, here 1, 2, 3, 4 and 5, 6, so there are 6 cut edges, you can verify whether it is 6 or something else. So, we are interested in finding given an arbitrary graph, we are interest in finding the cut of the largest size ok. This is a computationally difficult problem it is known to be n p hard, but can be find good approximation algorithms to compute the cut edges and the maximum cut. What bounds can we give to the size of the maximum cut ok.

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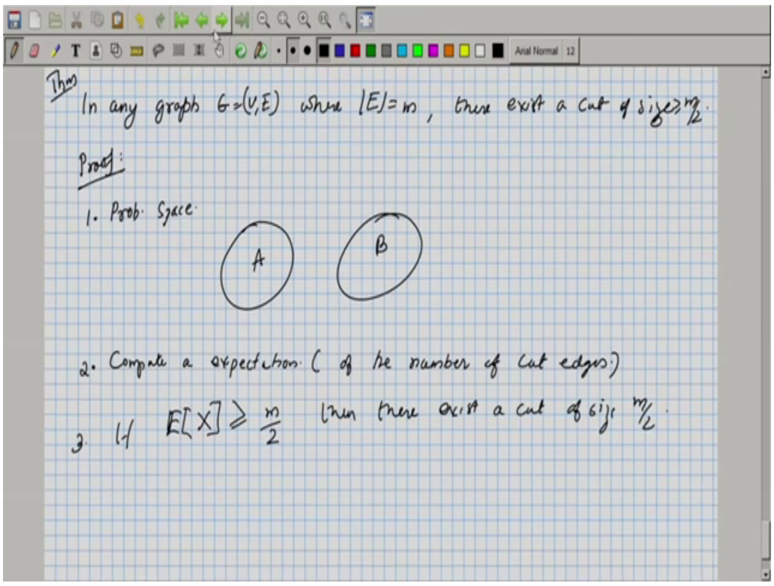

So, we will prove the following theorem in any graph G equals V comma E where size of where let us say there are m edges, there exist a cut of size m by 2, that means, you can always find a cut in any graph such that half of the edges cross the cut ok. Proof is very simple ok. The we will just trace the steps of probabilistic method one more time, because we have to design an appropriate probability space ok.

The probability space for us here it is going to be we will choose the cut randomly ok. So, how do we choose the cut randomly, each vertex goes into so there were two parts A and B, and we have to obtain a cut A and B. These A and B has be chosen appropriately. So, we will randomly put a vertex into A or B. We will toss a coin, if it is heads, the vertex will go into A; if it is tails, it will go into B ok. So, this will clearly obtain a cut in the graph ok.

Now, we are interested in the number of, so after this we have to compute a certain probability. So, what probability are we interested in. So, here we are interested in computing the probability, we will compute an expectation ok, when we take this random cut, how many edges are there in the cut, what is the expected. So, we will compute the following expectation we will compute the expectation of the number of cut edges ok.

And then we can resend that if the number of if the expectation if expectation of let us say if X is a random variable which denotes the number of cut edges if expectation of X is greater than or equal to m by 2 ok, so of size greater than ok, if expectation of X is greater than or equal to m by 2, then there exists a cut of size m by 2 ok. So, it is existence is proved that means when if all the cuts, where of size less than m by 2, then invariably the expectation if you randomly choose a cut the number of, if the expectation is less than m by 2, if you had randomly chosen a cut that will have edges strictly less than m by 2. And therefore, the expectation cannot be greater than m by 2.

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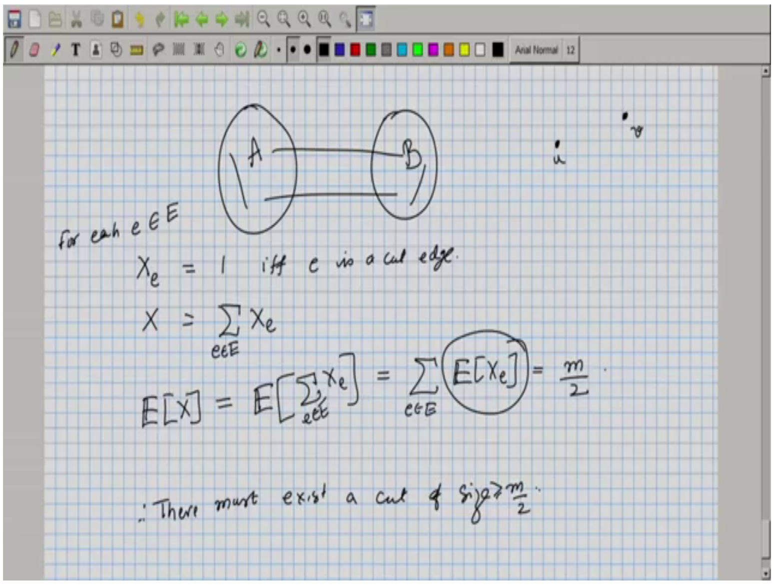

How do we calculate the expectation of the number of edges in the cut, so, let us do the following. So, this is our randomly chosen cut and this is let us say the edges in the graph. We will have a random variable X e for each e belonging to the edge set, we will have a random variable X c. And this X e is one if and only if e is a cut edge. So, let us look at the random variable X. X is just the sum of all edges belonging to sum over all edges belong into e, X e ok. This will give us this sum overall edges X e will give us a number of edges are there in the cut.

We are interested in computing the expectation of X. So, this is equal to by linearity of expectation. By linearity of expectation, you can take the expectation inside. So, this is just sum over e belonging to E expectation of X e. And expectation of X e, so what is the probability that one edge is a cut edge and let us to vertices u and v. Let us say u was in one side it was chosen to be in one side only if v is in the other side, we will have this as a cut edge and that happens with probability half, because in order to choose the side for vertex v we would have tossed a coin and the probability of that returning the suitable side is just half.

So, this expectation is going to be half, this is going to be half. So, this is going to be sum over all edges half, so that is going to be equal to m by 2. So, we know that the expectation of X is m by 2, therefore there must exist a cut of size greater than or equal to m by 2, OS that brings us to the end of the first lecture. We will see more advanced examples in the coming lectures.

Thank you.