Introduction to Computer Graphics
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Lecture - 10
Curves

So today we are going to have a new topic. So far we have been basically talking about what is involved in the rendering pipeline. We have not really discussed choose in terms of what we need from modeling where we have considered very simple primitives like lines, circles when it came to drawing those primitives. Now we will actually start talking about how we can model various geometrical objects.

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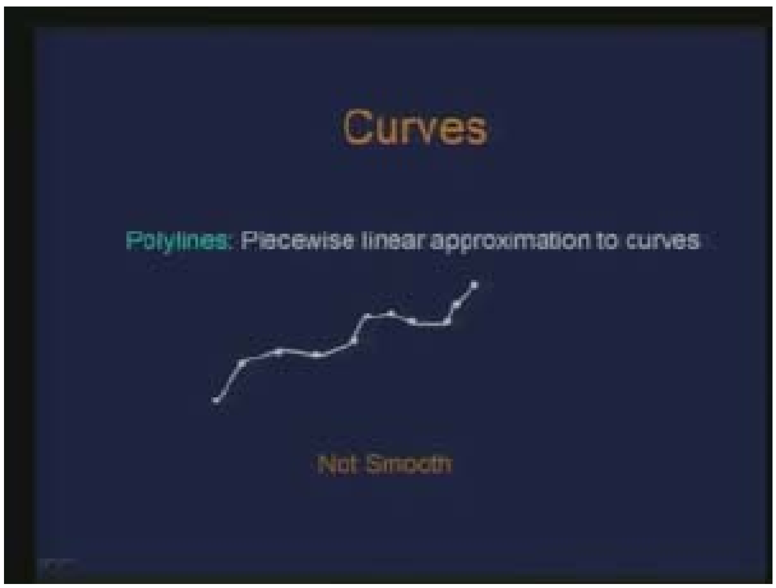

To start with we will look at the design of curves and one of the motivation here is that we would like to have the smooth curves passing through some data points or giving us a shape which we would like to use it for different models. So, if you look at as a first degree of approximation to be able to design a curve what you may do is just use collection of lines or polylines. These are nothing but collections of lines which you see here and the resulting shape is some sort of a capture of a curve linear shape. What we absorb immediately is that the shape is not smooth it is a collection of points joint by these lines and the resulting shape what we get is not smooth. So the motivation here is that we would like to design curves which would also have some smoothness. So smoothness now we are considering in a sort of a spoken language. Generally we refer to smoothness when we feel something really smooth. But later on we will look at the mathematical definition. So, if we look at some of the higher degree of approximation to be able to design curves we can look at various representations of curves.

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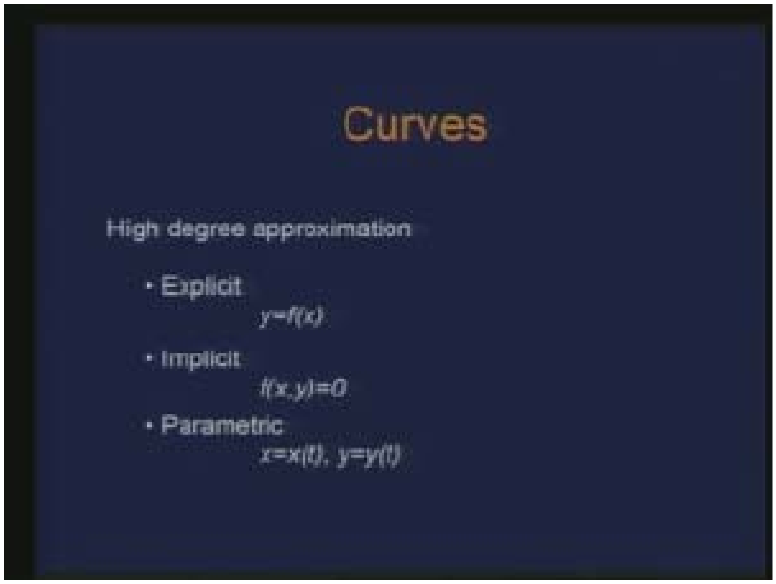

For instance there is this explicit form or explicit representation of curves where you consider y is equal to f(x). So for each point value x you have a value y. The other representation is an implicit representation where you actually get these points on the curve by solving an equation or set of equations. The parametric curves are curves where we have the values for x and y or the point in a Cartesian space given through a parametric x is equal to x(t) and y is equal to y(t) where t is a parameter.

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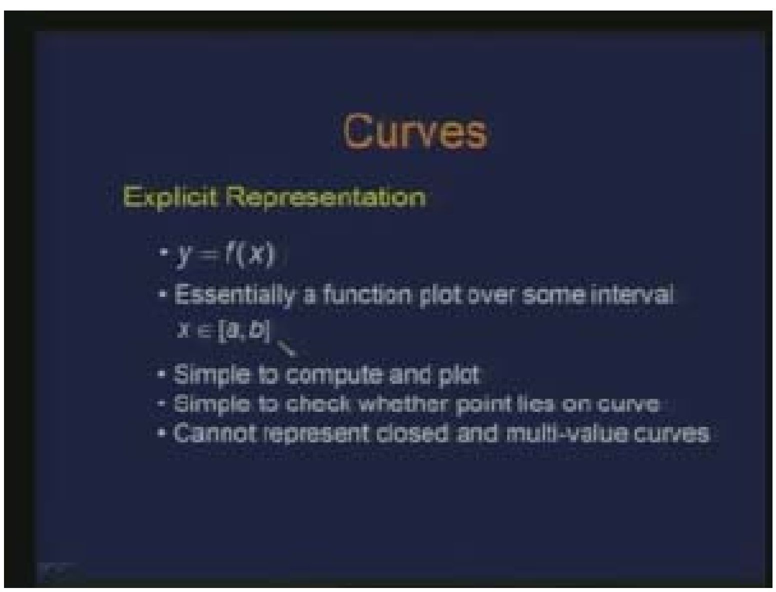

Explicit representation as we observe is actually the functional form where Y is equal to f(x) which is obtained by a plot of the function. For given x we plot y, the x is defined in an interval a and b. therefore when it comes to doing a computation for drawing or plotting it is fairly a simple thing to do, just run various values of x and obtain the values of y and plot them. Sometimes we would like to know whether a particular point lies on a curve or not. So in this representation it is fairly simple because it is going to satisfy this. But the problem is that you cannot have closed or multi value curves because it is a uni-valued function where for one value of x you have one value of y so we cannot have a multi value kind of a function or a closed curve even. Implicit representation:

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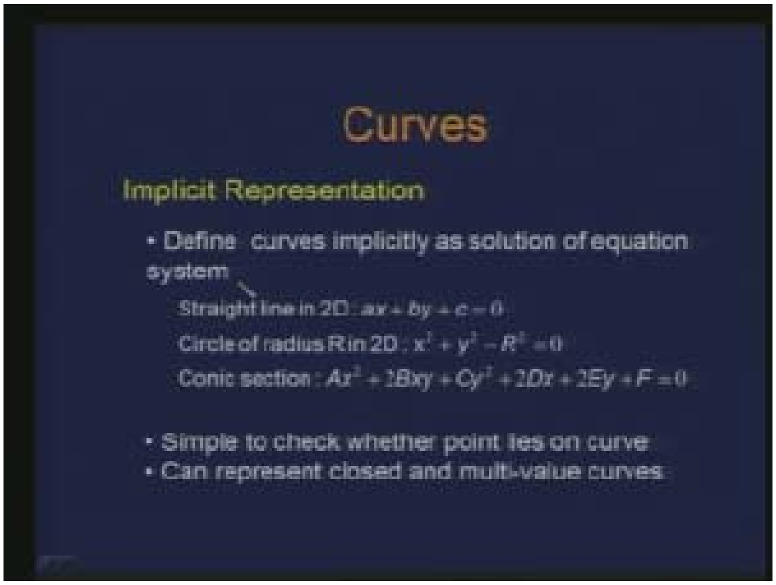

In the case of implicit representation you have the definition of curves given implicitly as solution of equations or solution of system of equations. For instance a line could be defined as ax plus by plus c is equal to 0, a circle can be defined as x power 2 plus y power 2 minus R power 2 is equal to 0 and a general conic section or a quadratic curve can be defined as Ax power 2 plus 2Bxy plus Cy power 2 plus 2Dx plus 2Ey plus F is equal to 0. Again if you want to check whether a particular point lies on curve is straight forward and here we have the advantage that this representation enables us to have closed and multi value curves.

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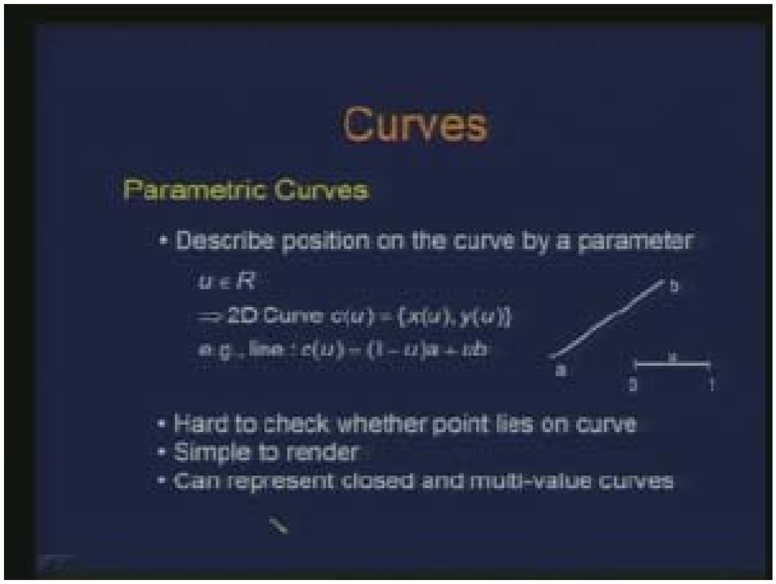

The next representation is the parametric curves. Here what we have is position on the curve is basically defined through a parameter. So a parameter u defined in R gives me the values of x and y for the various values of parameter u. So this is the curve in 2D. A simplest example we absorb is of line. So I define the parameter u between 0 and 1 which gives me a parametric definition of line between the end points a and b. So this is the parametric curve cu for the line between a and b.

Now if you look at from the point of view of determining whether a point lies on a curve or not it is not straight forward because you have to solve for the parameter u or whatever parameter you are using to be able to know whether the corresponding point x(u) y(u) is what you are looking for. It is not a direct analytical substitution. It is simple to render because all you have to do is change the parameter for the definition of the curve and then obtain the corresponding x and y values. This is referring to the planner curve but you can also extend this to the space curve. All you need is the third dimension Z defined through the parameter u. So the space curve is just a direct extension. And this also gives you the advantage of to be able to define the multivalue curve because you are basically traversing along the curve through the parameter. Here we shall see the definition of the parametric curves.

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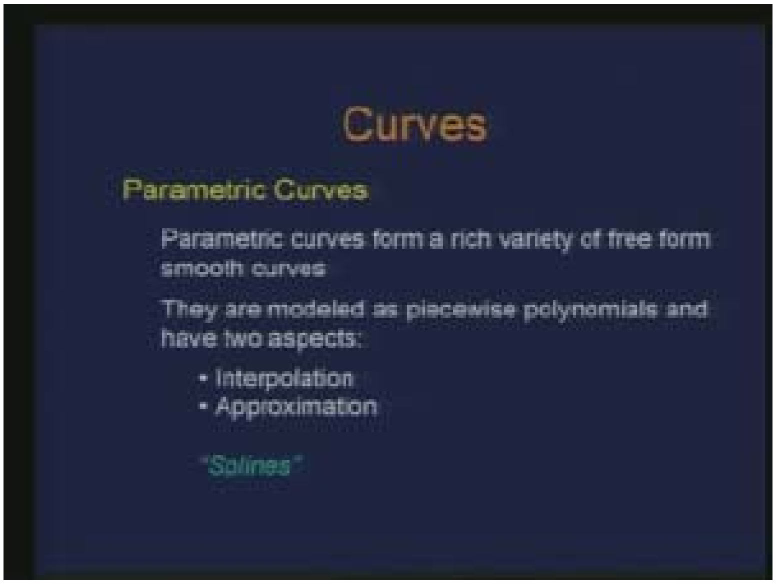

What it renders or gives us is the ability to have what we call as free form curves. Free form means there is no particular form but by the way we just draw it so it is called a free form. That free form nature of curves can be obtained through parametric curves. Generally they are modeled as piecewise polynomials and there could be two aspects looked at from the point of view of the design. So you may want to have curves which are interpolating a set of data. So you are given a set of data points and what you want is a smooth curve passing through those data points. The other way to look at is some sort of an approximation of a shape which is in your mind. So what you have as a designer is the shape of a curve or a surface and you want to design that shape using the control handles are the manipulators in the parametric curves. So you want to approximate a shape which is in mind for you to design. Often we hear the terms splines, splines are basically parametric curves.

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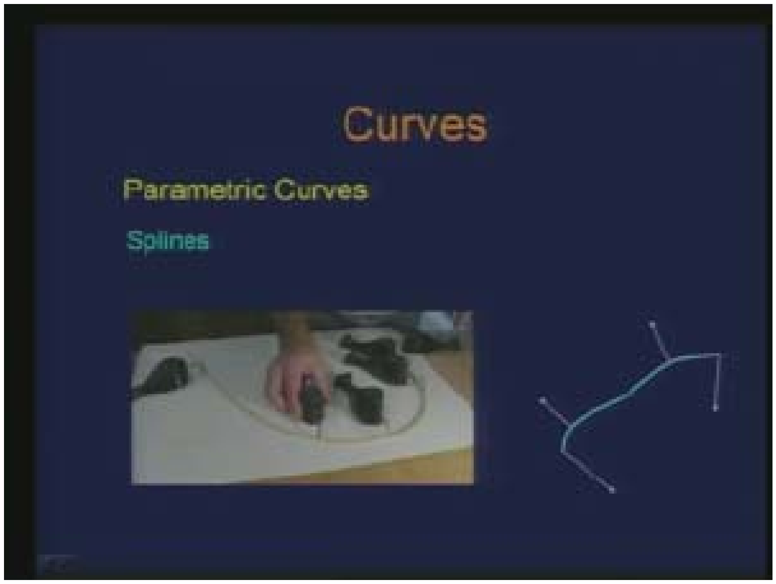

The word splines originated from, earlier this was required for design of the surfaces used in shift design where you had a smooth surface being designed for shapes. What they basically did in order to have that particular shape is they took a sheet and they put some weights at various points on that sheet and by changing the weights they obtained some shape which they wanted to have. It is a flexible sheet which they could manipulate the shape of which by attaching weights to this at various points. For instance here a similar example is being illustrated.

You have these weights which are in the shape of ducts attached to this wooden or a plastic flexible sheet at various points. Hence by manipulating the location and the amount of these weights you would see a different shape of the sheet. Just because these weights are going to change there locations and the amounts we will have the particular shape of that sheet referred to as splines.

Therefore you have this flexible kind of a sheet and these are some sort of weights attached which in turn governing the shape of this curve. This was done just by the mental mapping of the shape one wanted to have and adjust these weights. Hence this was the similar idea we wanted to use in reference to design of parametric curves.

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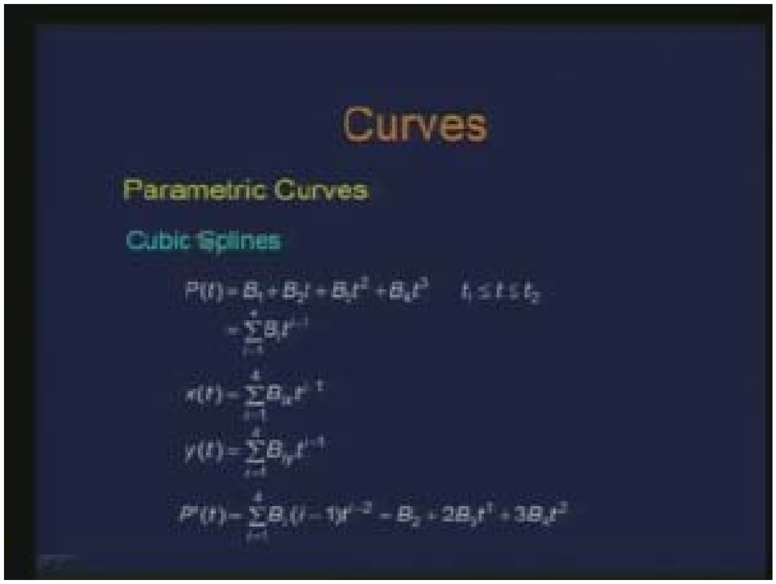

Now let us look at cubic splines which are defined as a cubic polynomial of the parameter. This is the cubic polynomial of the parameter t defined in an interval t_1 and t_2. I can write this as a summation of some coefficients B_{is} and the polynomial t. And this P_t which I obtain as the parametric curve has the components x_t or y_t and z_t. These are nothing but the same polynomial using in x and in y. You can consider these B_s to be some sort of a vector having x and y and z_s. And if I take the derivative of this polynomial P_t I get this which is given here as this. So these derivatives are in some sense capturing the tangent vector at some location. These derivatives are nothing but some sort of tangent vectors at some points of the curve. So with this familiarity of polynomials defining a parametric curve let us try to have a particular kind of splines which we can obtain.

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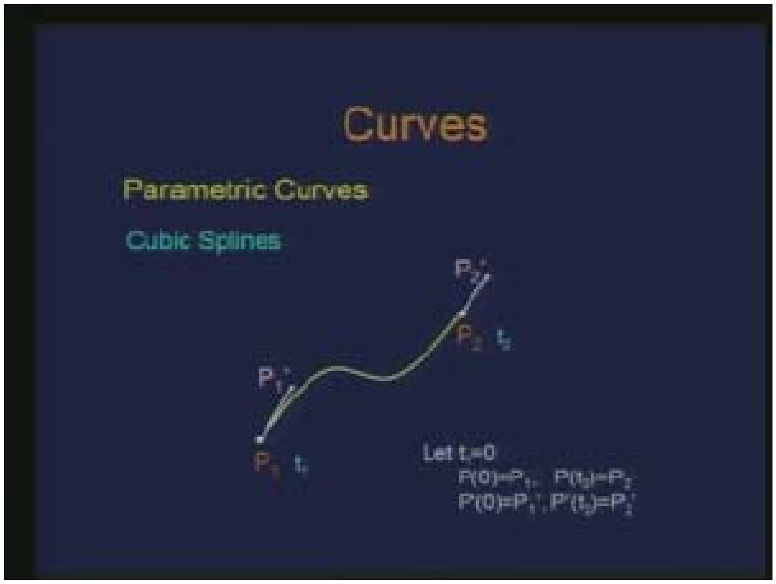

Given the two end points P_1 and P_2 the position vectors P_1 and P_2 the tangent vectors P_1 prime and P_2 prime at the two end points defined for the parameter value t_1 and t_2 how I obtain these cubic splines. From the user primes perspective what you have been given is the end points and the two end tangent vectors. So now without sort of loss of generality I can consider this to be 0 and some value there which could refer to t_2. So if I say t_1 is equal to 0 then the curve here the parametric definition of the curve at t1 is equal to 0 is nothing but P_1, the value of the curve at t_2 is nothing but P_2 these are the end conditions I have, the derivative at t is equal to 0 is nothing but the tangent vector at the end point P_1 and the derivative at t_2 is nothing but the tangent vector at the other end. So this is what is basically given to me as the end conditions.

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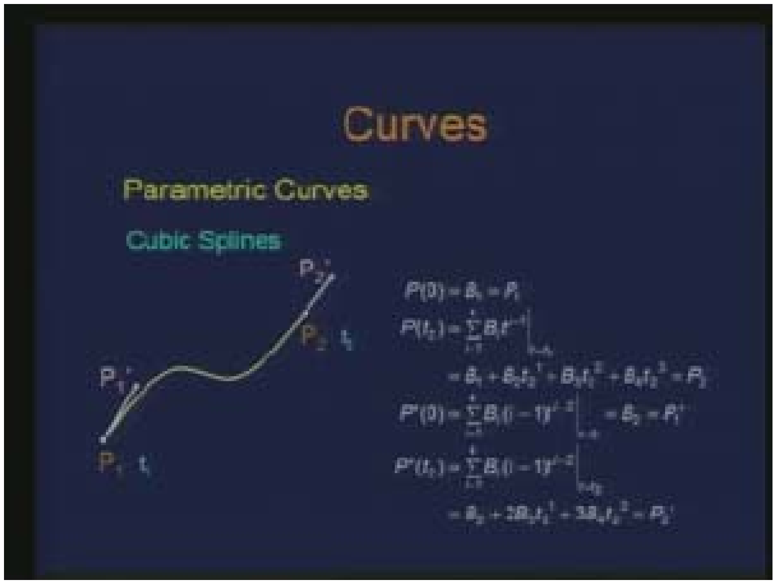

Now I have seen the definition of the parametric curve in terms of the coefficients B_{is} which is nothing but this. So, if I am trying to evaluate the curve at t is equal to t_2 all it is saying that I value at this t is equal to t_2 which is this. All I have to done is substituted t is equal to t_2 in that polynomial definition which I considered which is nothing but P_2. So here I am trying to find out the B_{is} given the end conditions to me because that is what the user has given to me in order that I design or obtain the parametric curve I should find out the values of B_{is}. Hence, for that I am setting up the system equations which I would solve. So, similarly I take the derivative at t is equal to 0 which is given as this which by substituting I obtain B_2 is equal to P_1 prime. Similarly, I take the derivative at t is equal to t_2 that is what I get and I substitute that as P_2 prime. So I know B_1 is equal to P_1 here, I know B_2 is equal to P_1 prime I just need to solve B_3 and B_4 from these two.

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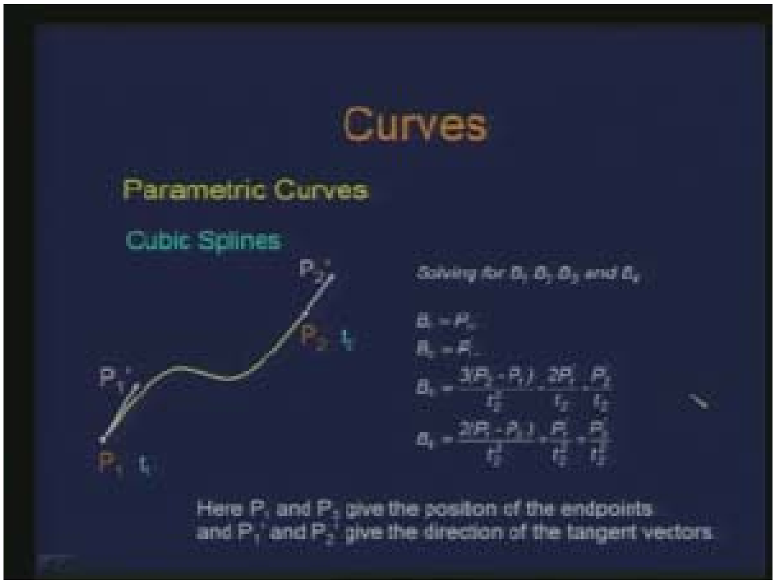

Therefore by solving B_1 B_2 B_3 and B_4 I get these. Therefore now I have basically the curve given P_1 P_2 P_1 prime and P_2 prime I have this spline curve. So having done to this extent how you would now extend the idea of having more than two points. So what I am trying to say is that we had the conditions at the two end points given the position and the end tangent vectors, now the general problem would be that I have a set of points and I would like to have a spline going through those points which I want to achieve. So the problem is that can I extend this idea which I did just between two points. The answer should be yes.

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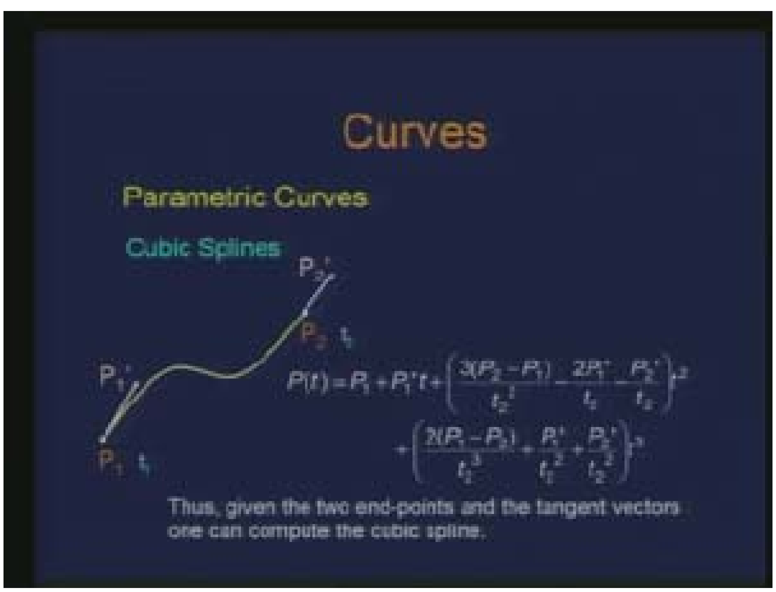

The entire curve is written in terms as P(t) in terms of the given values Ps and P primes position and the tangent vectors so the whole curve is known now.

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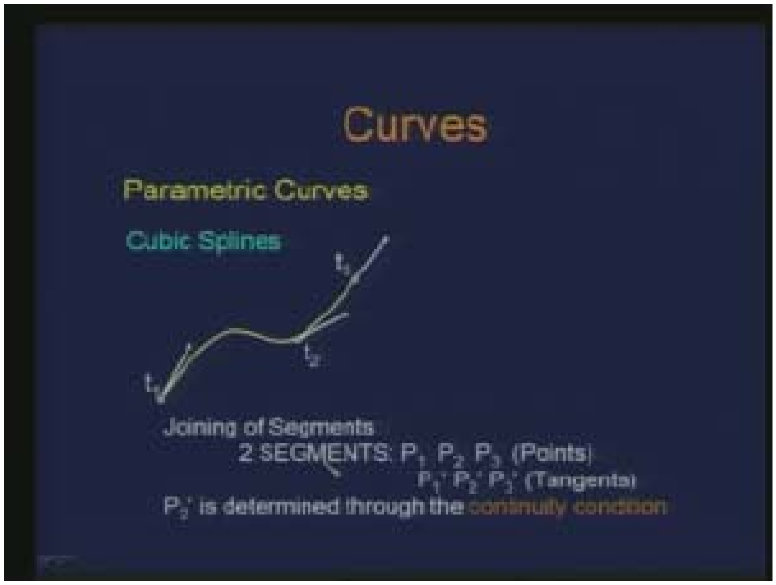

Now what we are saying is that first extend the idea from one segment to two segments and then we can generalize it. So I have one segment here and another segment here. The position vectors are P_1 P_2 P_3 this is the point information I have and the tangent vectors are P_1 prime P_2 prime and P_3 prime. Now for practical purposes what may happen is that for the two end positions it will be all right if you supply the end tangent vectors. It may be difficult to have the values of the tangent vectors at these points. So may be I should be able to find out these tangent vectors given certain constraint for the type of curve I want to obtain. So what we basically do is we find out the intermediate tangent vector which is P_2 prime add this point through some constraint of continuity about the curve. So we impose some constraint on the curve and find out this value.

Now if I look at a piecewise spline that which several segments of spline of degree k then it has continuity of order K minus 1 at the internal joints. So a simplest example is if I consider piecewise polyline at the junction of the two lines, it is a spline of a degree one first degree spline so if I look at the continuity at the junction point I have the continuity of 0 order which basically says that only the position match, the end point of the first segment is the same as the first point of the subsequent segment so the position at that points are the same so I call that as continuity of order 0.

So, for a general spline of degree k I can have the continuity of order k minus 1. So this is what I impose on the curve to find out the intermediate tangent vectors. So, for finding out the P_2 prime I say that P_2 double prime which is the second order derivative is continuous over the joints. basically I do the evaluation of P double prime t which is given as this and then I do the evaluation at t is equal to t_2 for the first segments so again I consider this is segment which could be defined within the parameter 0 and t_2, this is the segment which could be defined between 0 and t_3 so the first segment which I consider I evaluate P double prime for this point which is this and then for the second segment I do the evaluation of again P double prime at t is equal to 0 so I get these two equations and I want this to be the same and that is the constraint I am imposing. So, for the segment first I have this and this is the value for the second segment I just equate them. This gives me the extra equations. Now I substitute the expression for B_4 and B_3 to solve the values for P_2 double prime. Remember these are actually expressible in terms of the derivatives when we looked at the individual segments.

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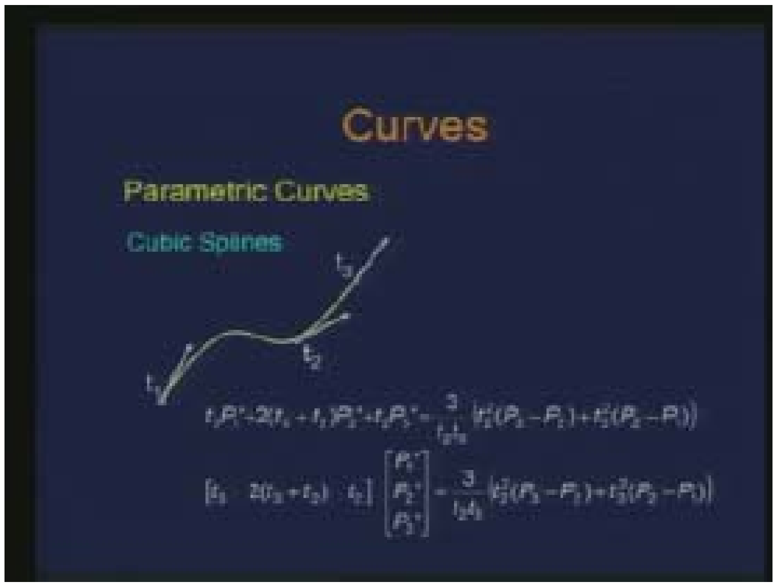

Therefore what it leads to is, if you do a rearrangement of the terms what you may get is something like this. So, there are terms containing the derivatives P_1 prime P_2 prime P_3 prime and this is what you have on the right side which I can in some sort of a matrix form, this is the multiplication to P_1 prime P_2 prime and P_3 prime and this is what I have at the right hand side. Now this is between two segments. If I was to solve only for the two segments given P_1 prime and P_3 prime I could get the value of P_2 prime as I require. Now in fact I can extend this to a general setting where I can consider any number of segments.

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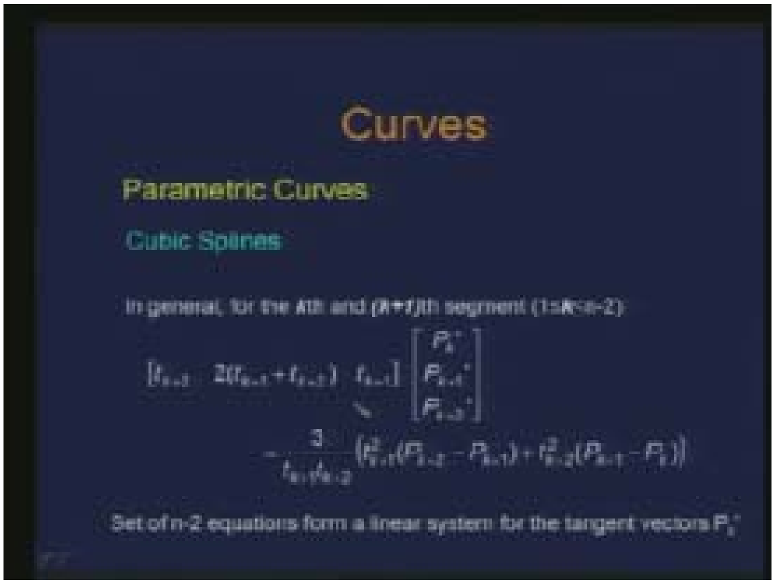

Now extending this for the Kth and Kth plus 1th segment where I have K going from 1 to n minus 2 this would be the equation. All I am doing is basically I was solving for P_2 prime but now I will solve it for pk plus 1 prime. Now this would basically enable be to set this n minus 2 equations for the various segments and then I can solve these pk prime. So basically this gives me this kind of a matrix where I have this n minus 2 entries and I have these tangent vectors and these are the corresponding right hand sides.

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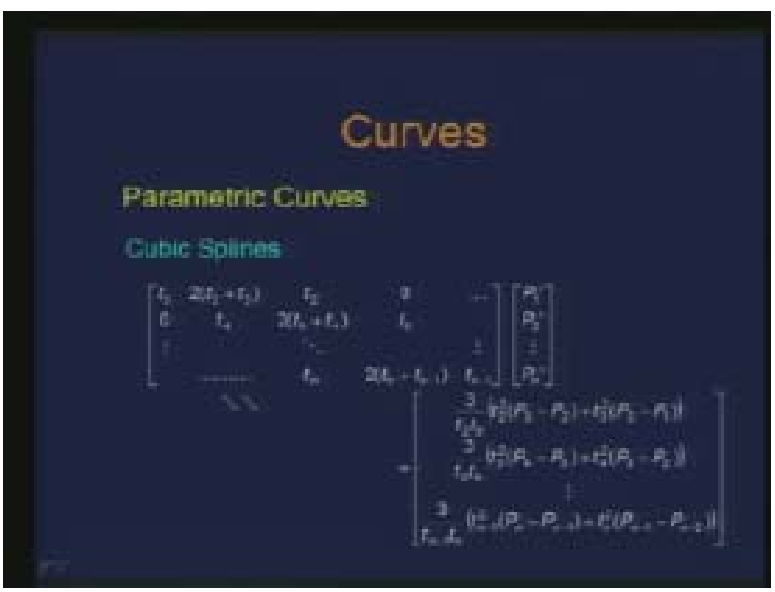

Therefore now what I am trying to solve is I am trying to solve these. So this is actually a matrix of n minus 2 rows. I cannot invert it as this is not a square matrix. But remember we are given these values the two end tangent vectors so I can actually have additional rows into the matrix without changing anything.

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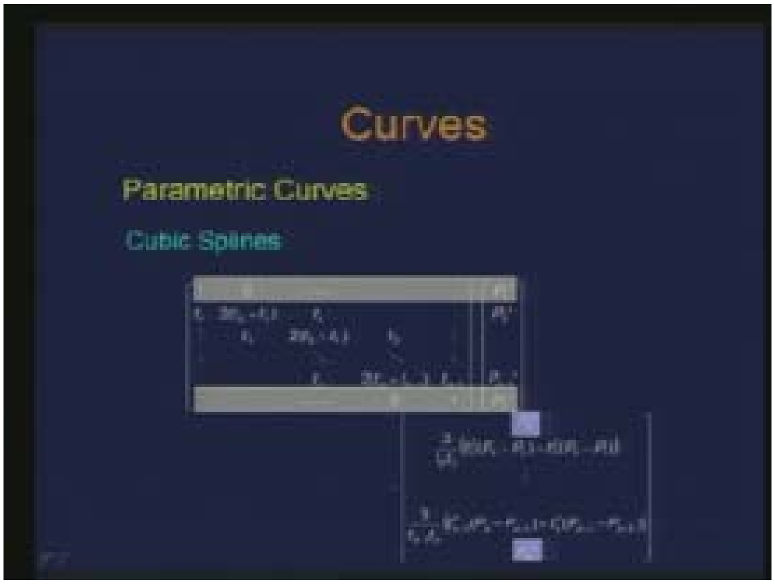

I can add this row and this row and correspondingly add these values from the right hand side. Now this sets up the square matrix here which I can easily invert. i may not choose to do the matrix in version as used in general case because this is the tri-diagonal system diagonally dominant matrix so there could be better ways to solve it. So we basically have set the system of equations for solving these derivatives or the tangent vectors and that is what we require.

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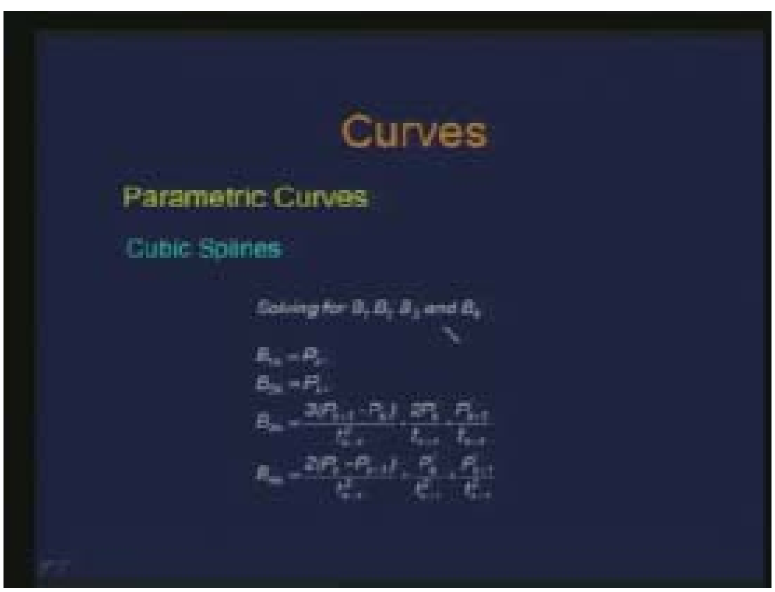

So now if I solve B_1 B_2 B_3 and B_4 for the Kth segment this is what I would get because I am interested in finding out these B_i case so that now I can do segment by segment plot of the curve. I can basically obtain the curve for each segment.

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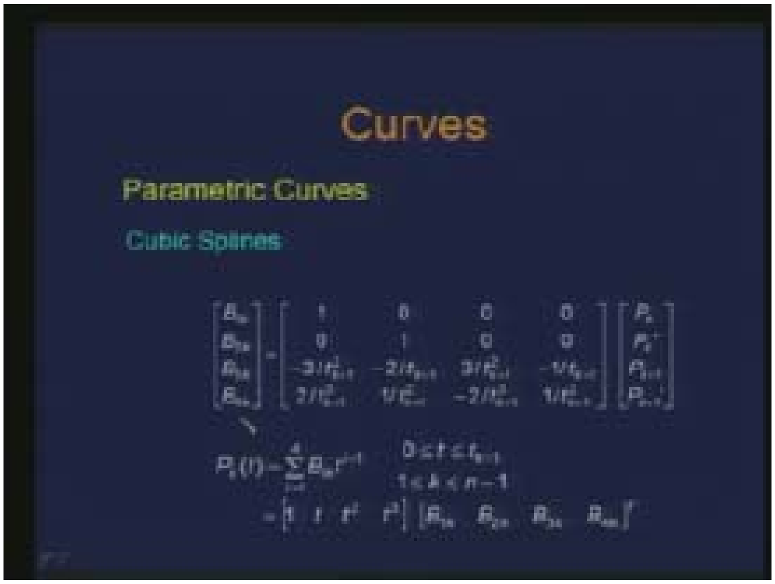

So just to have a different rearrangement of that what I have done is I have written these B_{ik} values again in some sort of a matrix which is basically the multiplication coefficients of these values. Therefore here again we see something which is pertaining to the position of the point and the tangent vectors. Now, again going back to the individual segment of the cubic spline which is defined as this I can rewrite this in this form which is the monomial form where I use the coefficients for 1 t t power 2 and t power 3 and these are the coefficients and again substitute for B_i case and that is what I get.

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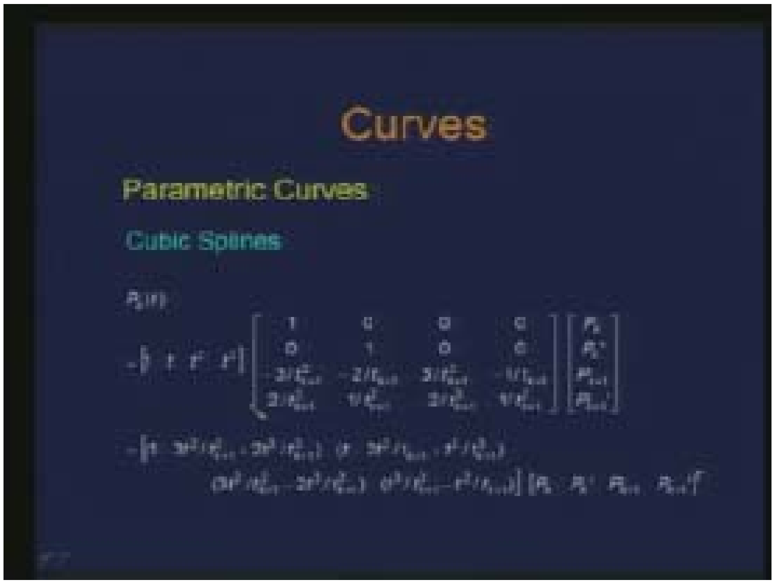

So this is the substitution for Bi case. So what I obtain here is some multiplication which is the multiplication of this and this with the information given as the position at the tangent vectors.

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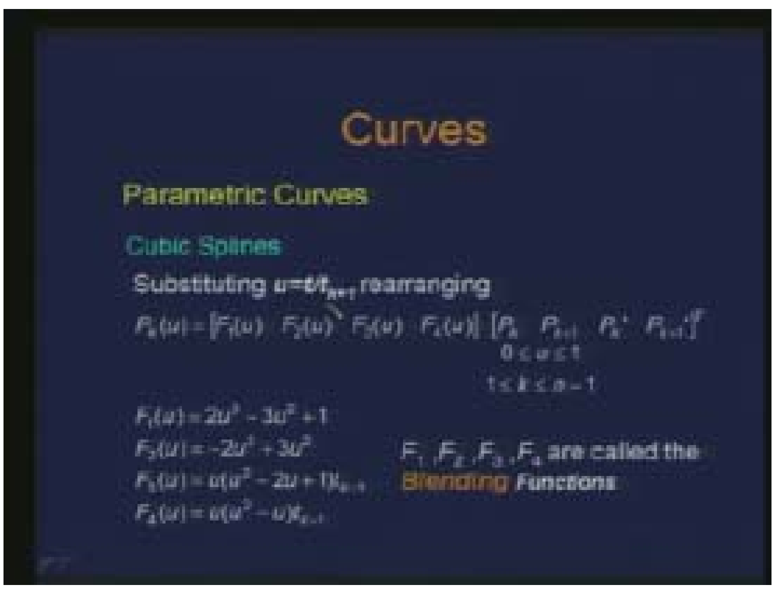

Therefore now if I substitute for some t by tk plus 1 as u which is some other variable then I can rewrite this as Pku as something like this where each of these polynomials F_1 F_2 F_3 and F_4 and we written in the form. Now I call that as some sort of a blending function. So what are they doing is they are trying to blend the information given to us in the form of position and tangent vectors. These are basically some blending functions or basis.

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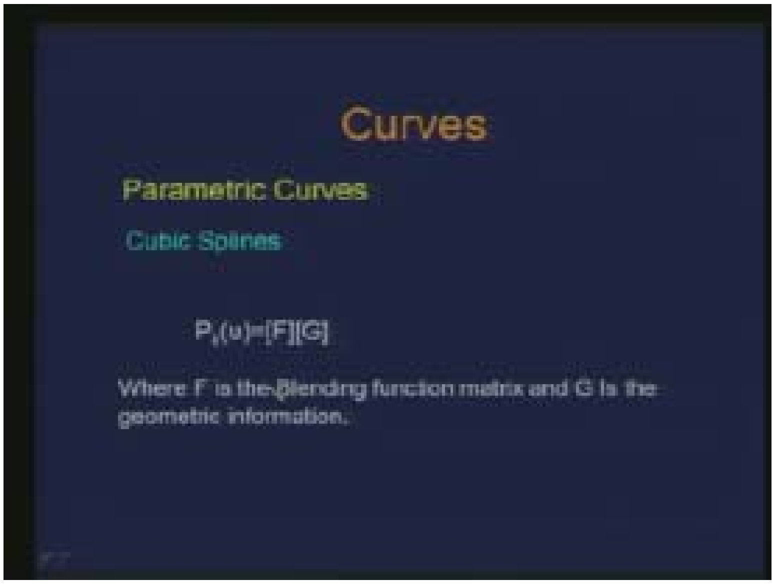

Now if you want to see the equation for a cubic spline what you absorb is that there is some blending function here giving through F and there is some G which is basically the geometrical information which was given to you. So the geometrical information which was given to us was in the form of position in tangent vectors. So all we are trying to say is that there is a set of these blending functions and by using these blending functions for the information which was given we can generate the cubic splines.

We have to look at the definition of the problem. The first which we considered was I was given two points and the corresponding tangent vectors at those two points and I could derive a spline satisfying these. So it is a curve satisfying these two end conditions, the position and the two end tangent vectors. Now generally what you have is you have a set of n points and you want to design a curve passing through these points with certain properties. And when I say certain properties they are reflecting towards the smoothness of the curve and the shape of the curve.

Now what we are trying to do here is we want to extend the idea which we considered for two points to this set of points. That is why I said it is a piecewise spline with the constraint of continuity which is captured through second derivative to be the same end internal points. And you also look at from the pragmatic point of view.

If I have n set of points I may be able to either deduce or supply the end tangent vectors at the two end points. But this is over killing if I require end tangent vectors for each segment but then how do I match them? If some segment is giving me individually the end tangent vectors and they are not the same then there is no continuity. So from the point of design this is a desirable thing to have. So what we have basically established is, there is some sort of the blending of information given as some geometrical information to us to be able to get the d-splines which we have basically established.

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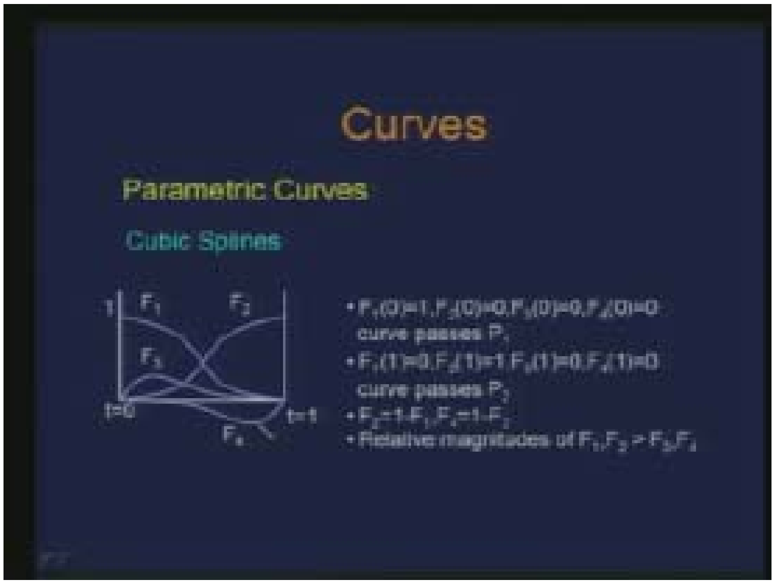

So if I make a plot of these F, F_1 looks like this, F_2 looks like this and this is F_3 and this is F_4 and these are just the plots for u ranging between 0 and 1 and here I have called this as t. Now if you look at the values of this function F_1 at 0 is equal to 1, F_2 at 0 is equal to 0, F_3 at 0 is equal to 0, F_4 at 0 is equal to 0 which means that the curve passes through P_1. Some sort of a cardinality you have if you look here you have the information of the position and then the tangent vectors. So when you use at t is equal to 0 only one of them is 1.

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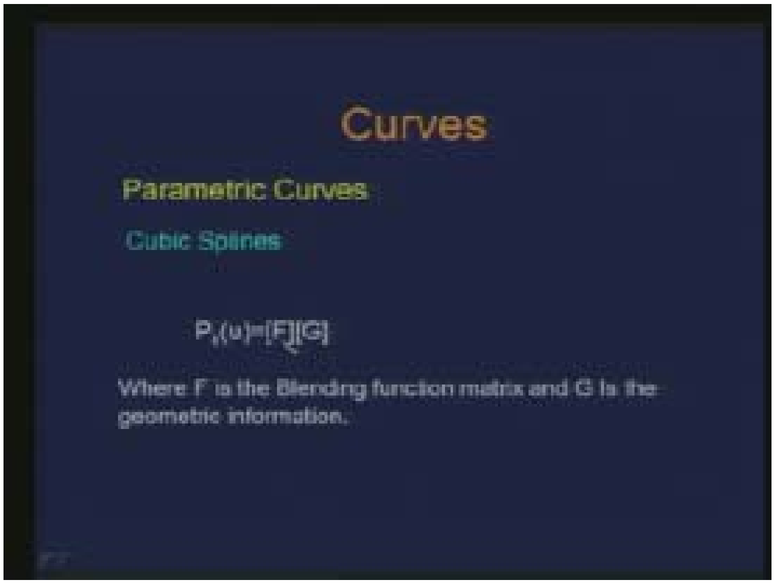

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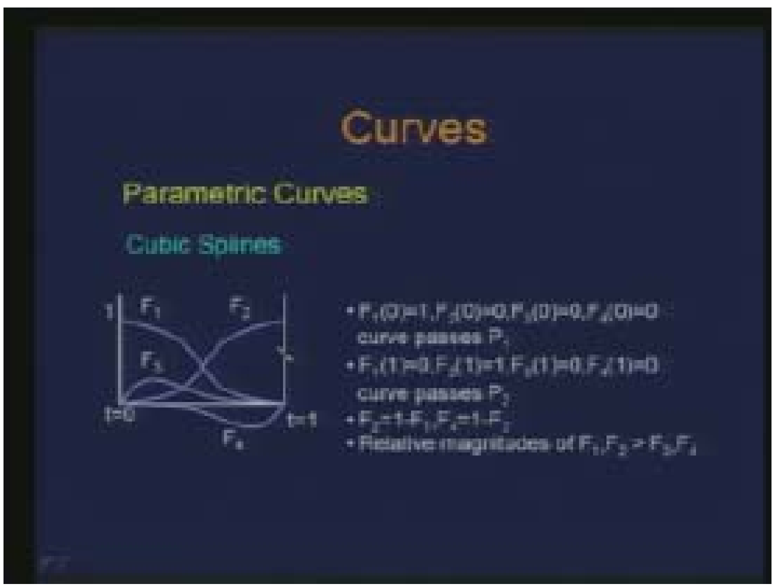

Similarly, if you evaluate at t is equal to 1 what you absorb is that F_2 is equal to 1 and the rest of them are 0 which says that you have the curve interpolating the point P_2. And you also absorb that F_2 and F_4. For instance, they are symmetric. So F_2 is equal to 1 minus F_1 and F_4 is equal to 1 minus F_3 they are all symmetric functions. And the other thing what you absorb is the relative magnitudes of these. So you absorb that F_1 and F_2 are fairly larger than F_3 and F_4 what it does mean? If you look at from the influence point of view the F_1 and F_2 are trying to do blending of information coming as position vectors whereas F_3 and F_4 are the functions related to the end tangent vectors. So the sensitivity of the curve with respect to the position vector and the end tangent vectors are basically demonstrated by this relative magnitude. Therefore the change of position is sort of more influential than the end tangent vectors towards the shape of the curve.

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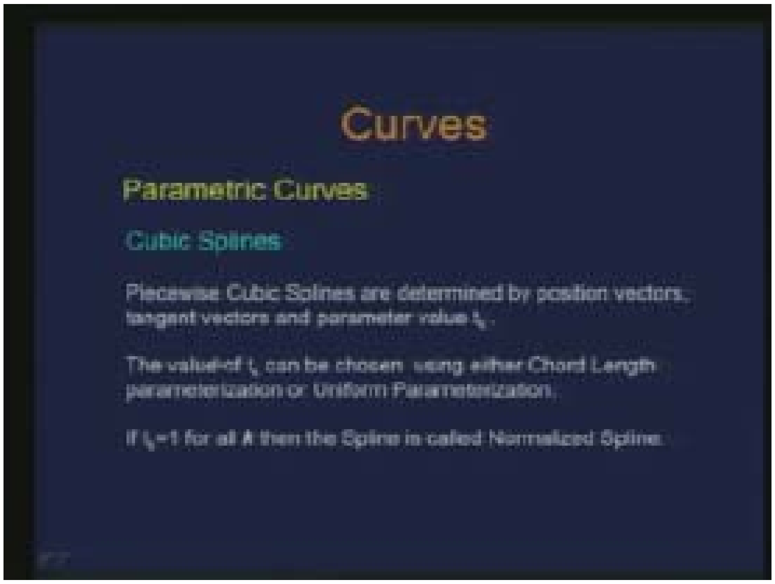

Now what we also see is that we can obtain the whole piecewise cubic spline by the position vector, the tangent vectors and the parameter values because this is also required you need these parameters the tk values. So the value of tk can actually be chosen either as a chord length parameterization or also as uniform parameterization that means just take equal between 0 and 1 and 0 and 1. If I consider tk is equal to 1 for all K then it has a uniform parameterization and the kind of spline we obtain is called normalized spline. Therefore all tks are basically considered as 1.

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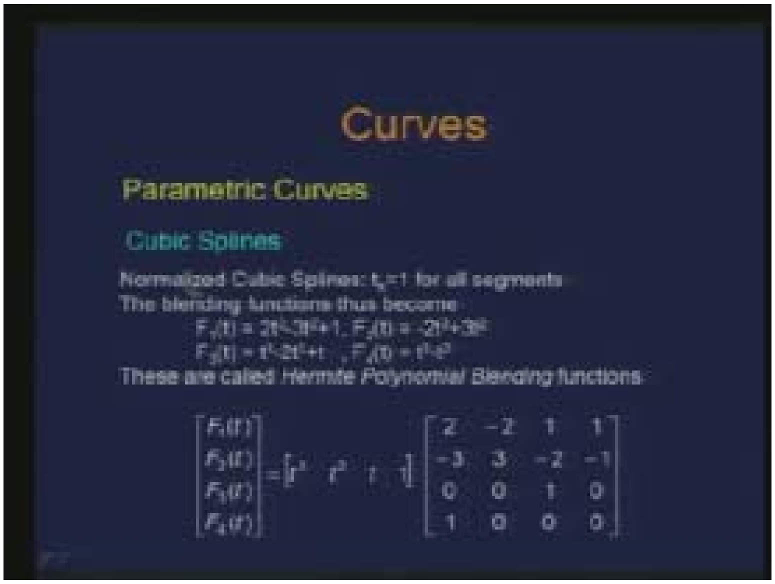

If you do that then the evaluation of these functions basically looks like this and if you write it in this matrix form then the blending functions what we obtain are called as hermite polynomial functions. These are just special cases for cubic splines.

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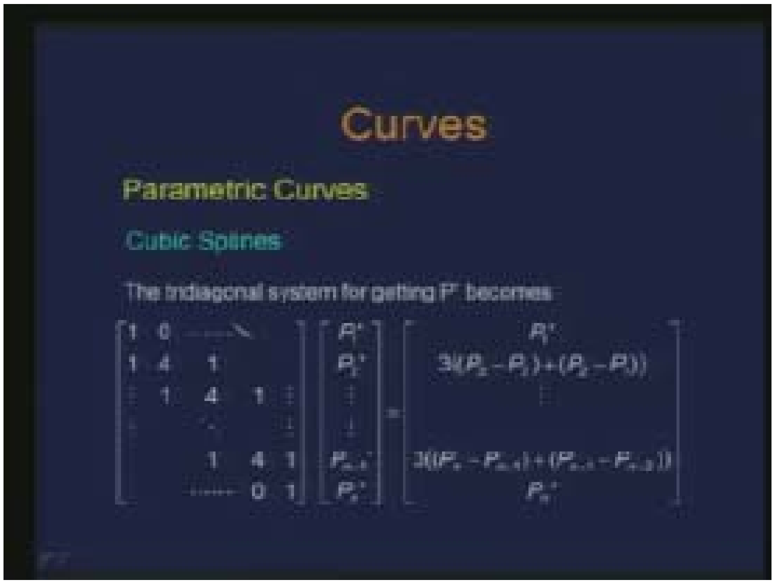

Now, if you look at the total set of equations we have this is how your tridiagonal matrix will look if you have normalized spline. In this case it turns out to be a constant matrix which could be beneficial for the computational point of view because if you have to invert this you just have to invert it ones. It is a constant measure. So these are basically normalized cubic splines. Now there is one thing which we also have looked at is the end conditions which are given to you.

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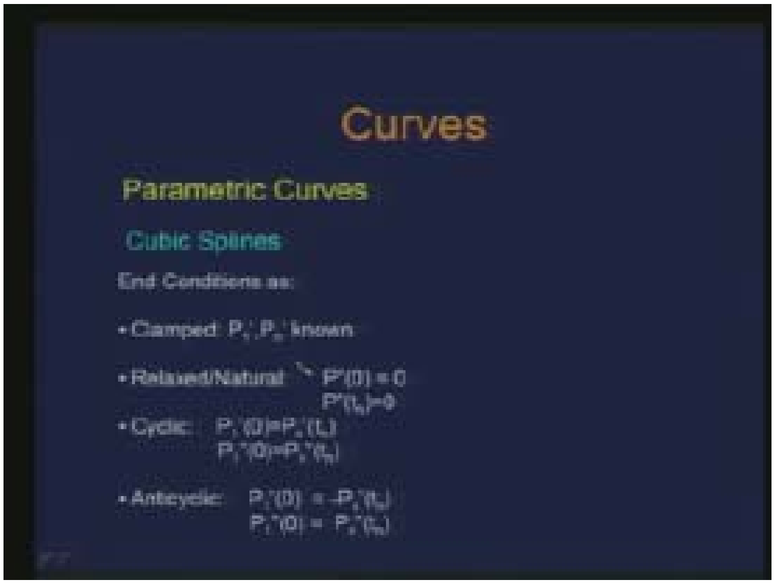

Therefore, we consider that the first and the last tangent vectors were given and we obtained the intermediate tangent vectors. So, if we have that as an end condition we call that as clamped end condition. So you are basically clamping the two ends. You may have end conditions saying that the second derivative at the two end points equals to 0. We call it as relaxed or natural end condition. Now with this clearly there will be a change in the first row of the matrix we have seen. This is going to give you the additional equations which you need to incorporate in the set of equations.

Similarly, we have the cyclic end conditions where we say that the two ends match in terms of the first derivative and in terms of the second derivative. So this is relevant in closed curve situation. If you have a closed curve wherever you are closing the curve these two end conditions match. And we also have the anticyclic conditions where we would like to have the first derivative and second derivative as the complement of these, this is a minus here, where would you have that as a desirable situation?

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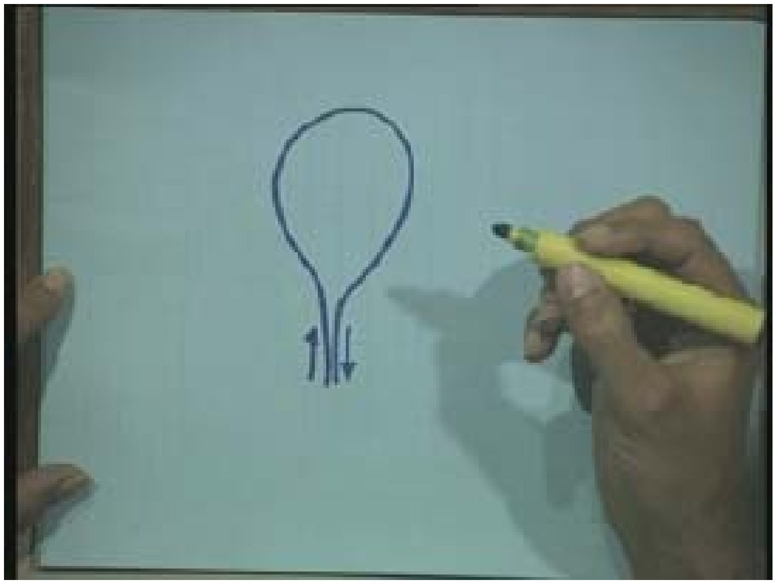

You play any game with a racquet, tennis or square or badminton you notice the shape and you probably see there is the head part of the racquet, you go in one direction and come back in the opposite direction, that is where you need to have the anticyclic end conditions, the signs are just the reverse. So that basically gives us another end conditions which we may use it for the design of curves.