Logic for CS
Prof. Dr. S. Arun Kumar
Department of Computer Science
Indian Institute of Technology, Delhi

Lecture - 23
Predicate Logic: Proof Theory (Contd.)

So, you are doing predicate logic proof theory there is some small changes from last times lecture. So, I think you should just quickly go through the changes because, of certain proofs I had to make some changes.

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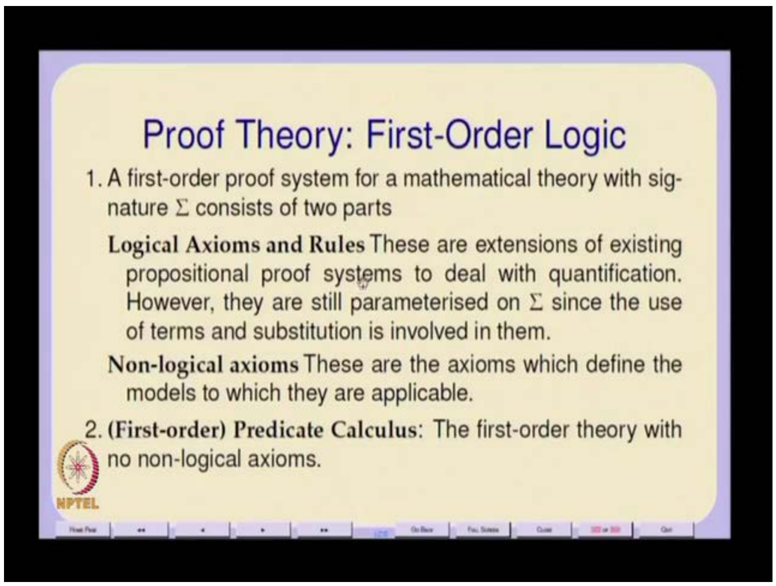

So, let us go to let us quickly go through last times lecture and then we will come to this. So, you are so as a matter of terminology so you the Proof Theory for any sigma First Order Logic essentially consists of some logical axioms and rules and, some non-logical axioms. So, of course it is possible to separate them out. So, we the first order theory with no non-logical axioms is called a predicate calculus first order predicate calculus. And, it is a good idea to look you are at properties of sort of predicate calculus before we go on to the first order any other first order theory. Because, all are the first order theories are essentially depend on the signature and the axioms through the non-logical axioms. So, for us this one just look you areas at logical validity itself for all possible kinds of different signatures. And, so we will actually look at first order predicate calculus before we go on to any first order theory.

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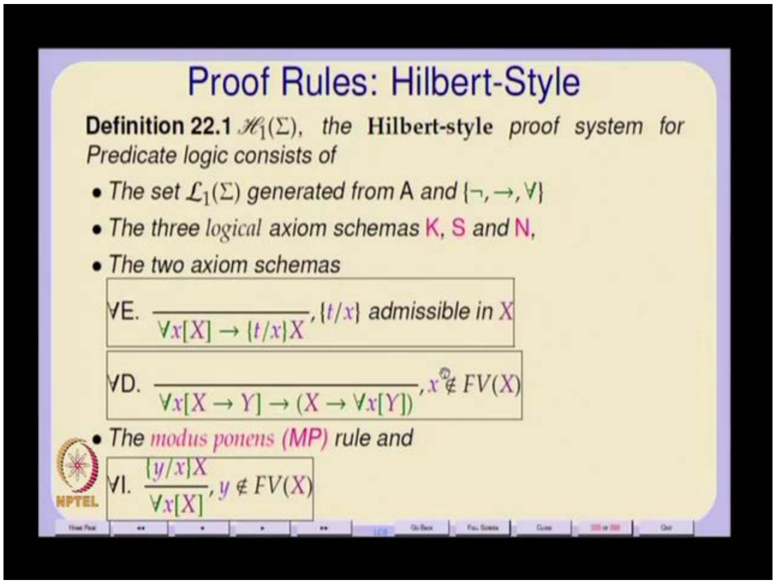

And, then you are so there is a small change here this for all D there used to be quantifier for all x. So, I actually said that for all x I can distribute so at this arrow but now I am actually putting some restrictions this is. Because, of some proofs there is some problem with some proof so it had to be in this form. So, if you if X does not have small x occurring as a free variable then what it means is that rather than distribute this quantifier for all x you can push it in. So that so then this capital X becomes a quantifier free and. So, you are so it is add a certain an extra power the purpose of certain proof rules. And, it is validity is not in question because if x is anyway not a free variable of x then even if you quantify it over x it does not matter but, this form has some advantages.

The other thing is this I have also made this I think I do not know I think it was not there in this form when I spoke last time. But, this universal is introduction universal quantifier introduction or what is known as universal generalization. So, I am writing it in this form essentially what I am saying is if you ever I think I had it in this form last time also so there Is no problem. So, essentially this means that if you can this Y is of completely arbitrary variable then you can quantify for x. Then, notion of arbitrary needs to be captured somehow symbolically and that is that can be a problem but, you are we went through all this is all fine you are.

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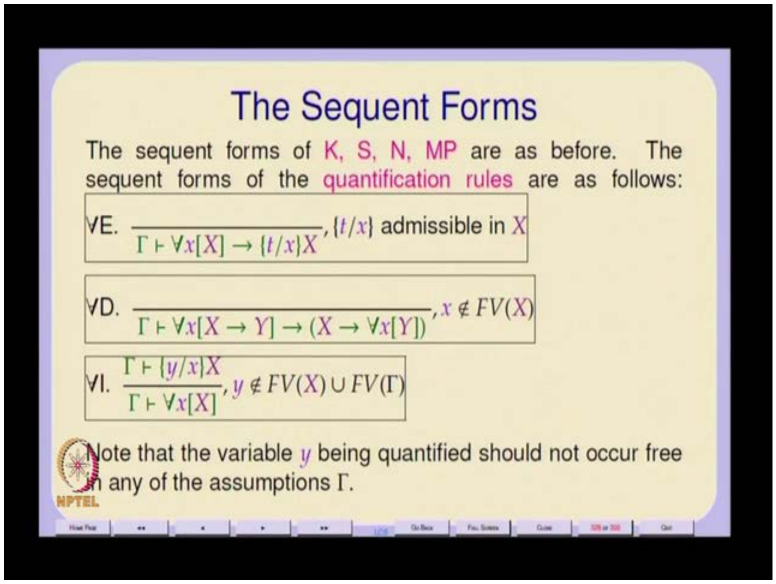

Of, course we have to look you at the sequent forms of this, rules because and this, the Sequent Forms are essentially of this kind. Now, that in the sequent form what we are saying is we assume there is some set of assumptions gamma for any first order theory it will be the axioms non-logical axioms might form gamma. So, we have to look you are at so this gamma is therefore important while we are dealing with predicate calculus it does not whether we look you are at use as the sequent form or the non sequent form. But, the moment we talk about the first order theory we are talking about a set of non logical axioms which will be there in gamma. Here, again there is a small change here in the universal generalization.

So, now if you have a collection of formulae in gamma and if y occurs so y obviously might occur free in this y for x of x. So, however if y occurs free in some of the formula in gamma then there is no guarantee that y is an arbitrary y meaning of the word arbitrary has to be captured somehow. If, that y may not be arbitrary then you cannot necessarily generalized on it if however y occurs free only in this expression capital X with y by substituting x. And, this proof goes through then it does not depend on any particular value of y. And, therefore then it can be generalized you are so, I mean that is the intuitive reason in a formalization one has to worry about what are the free occurring names. So, if y is a name occurring in more than one of these assumptions I mean one or more of these assumptions and it also occurs free in this hypothesis. Then, it means that y is some particular kind of y I mean like Socrates for example S.

Then, it is not you cannot directly generalize then because it might because under certain interpretations. So, this may not for example then if you generalize then you may you may not preserve validity. Because, that if that y stands for only certain particular values for which this predicate is true that it does not generalize to all values. And, that is the intuitive reason why we require that this y should not be a, or should not occur free anywhere in gamma here. Any, way if y occurs bound anywhere in gamma it does not matter because that those bound ys are different from any free y. So, the notion of y being completely arbitrarily is only guaranteed if it occurs free in this formula. And, if occurs free in other formula also in gamma that means your it is like this you take the identity element for example. So, predicates dealing with the identity element cannot be universally generalized. Because, usually in any algebraic structure there will be a single identity element all other elements are not identity.

The identity might have a variable name or it might be even if it is not a constant this, the set of predicates assumptions. That you might with term free variable y might imply that y should stand only for the entity element in which case you cannot generalize arbitrary even that is the intuitive reason. Why you require that this y should not be a free variable anywhere in either in x or in any of the assumptions in gamma? If it does occur then you are not necessarily preserving validity if you generalize on y. Whereas, it does not occur free then this y replacing small x and capital X and occurring free is essentially an arbitrary y it is not a particular y. And, hence can be generalized that is the intuitive idea why you need to be careful about the occurrences of free variables. So, in this sequent form we will actually use universal generalization in this way the.

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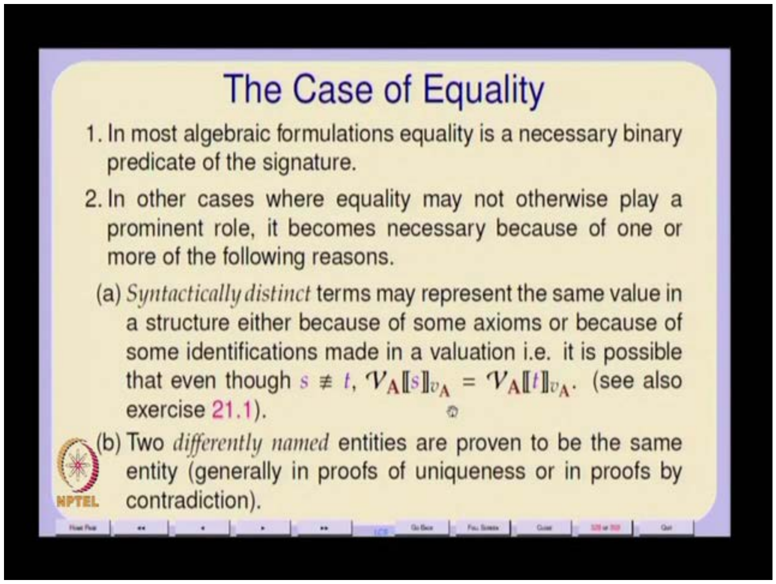

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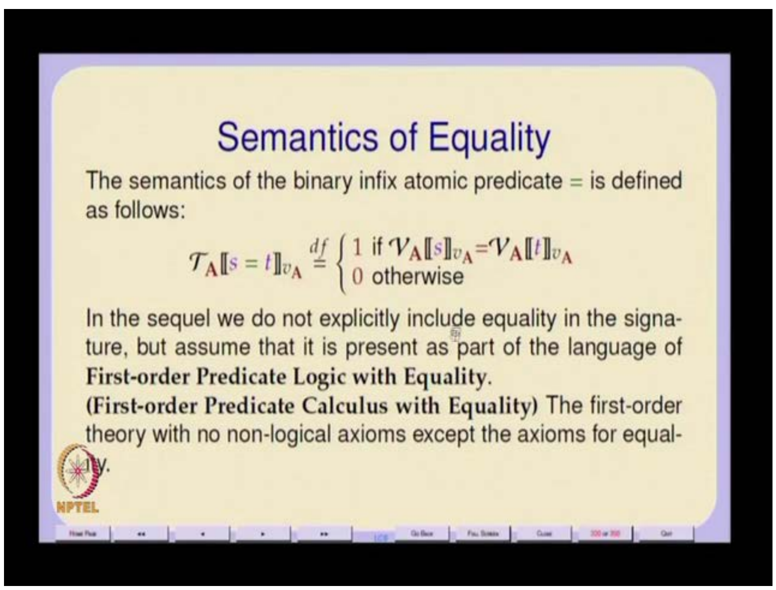

So, then there was this Case of Equality so the case of equality of course. So the Semantics Of equality is something that we will fix because it is something that is going to be there in all algebraic structures. But, however keeping in mind the fact that, syntactically distinct terms under certain interpretations might lead to the same value in the structure. We, have to define the semantics of equality essentially as this.

So, S and T might be two distinct terms and under any interpretation A with the valuation VA this, equality is true if the value of S and the value of T are the same in the structure A and, otherwise it is false. Since, equality is sort of special we can also talk about first order predicate calculus with equality. So, which means that we have to look you are at the so I mean it is a debatable question whether the equality axioms should be considered non-logical or logical with equality lies somewhere interface between the logical and the non-logical. So, but I mean so because of this we might actually so this you have actually sort of predicate calculus with equality. Where equality is regarded as one of the logical and special binary relations binary predicates that should be available let us say.

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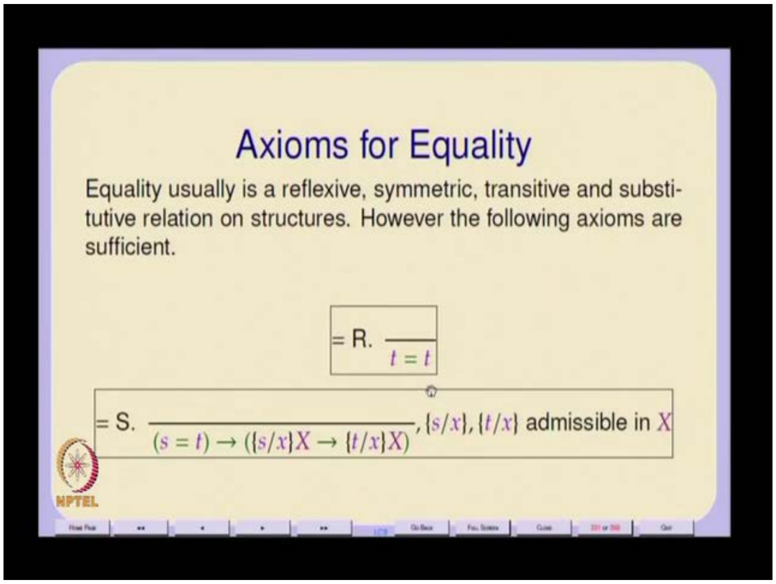

So, may not be true always so we let us so that is called first order predicate calculus with equality. So, let us look you are at the Axioms of Equality and I think that is so firstly of course I made a correction instead of x equals x I decided it to make it for all the terms t so, there is it reflectivity property. The second thing is that I had this notion of replacing 0 or more occurrences. Now, I have made that more uniform through this substitutivity axiom. So if S for x and t for x are both admissible in X. Then, s equals t arrow s for x of X arrow t for x of X. So, now its uniform substitution and all occurrences all free occurrences of x will be substituted this has some advantages. So, substitutivity is a, this axiom and its actually very powerful axiom.

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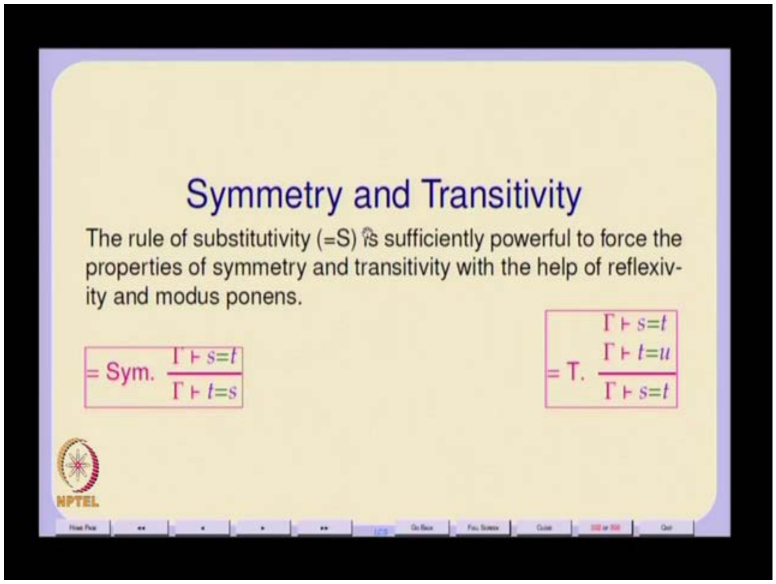

So, one thing is that normally any equality relation is also an equivalence relation. So, what you expect is that, what you expect is that the properties of symmetry and transitivity should also holds. But, this substitutivity axiom in this form is so powerful that we can actually use symmetry and transitivity as derived rules basically.

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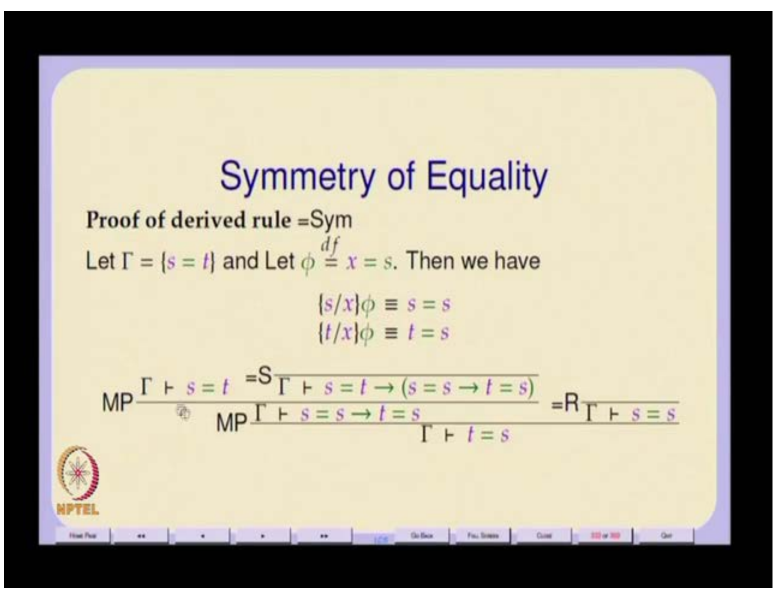

And, So here is a Symmetry rule and so essentially if I take s is equal to t as an assumption. And I take this formula x equals s one thing of course is that x might occur free in s for all we know is it can x occur free in s x could occur free in s. I mean there is this question of whether x is requirestly defined in which case whether you are finding in a fixed point or so on forth. But, clearly syntactically speaking there is no reason why x should not occur free in s.

Which case it can also be substituted so which means that you might actually substitute a free a term containing a free occurrence of x for occurrences of x in this predicate. And, that is not un-meaningful I mean not meaningless I mean in the sense it is still bears a syntactic meaning which is acceptable. What, I mean by that is that this kind of substitution does not get you into.So, this proof essentially shows that if I take this formula x equals s then by the symmetry rule and so on I get s equals s implies t equals s equal to s is of course is a reflexive rule. And, so and you have got an assumption s equals t so you can use modus ponens twice to get t equals s. So, can actually you can prove this symmetry so which means you can take symmetry to be derived rule rather than rather than fundamental rather than a basic rule.

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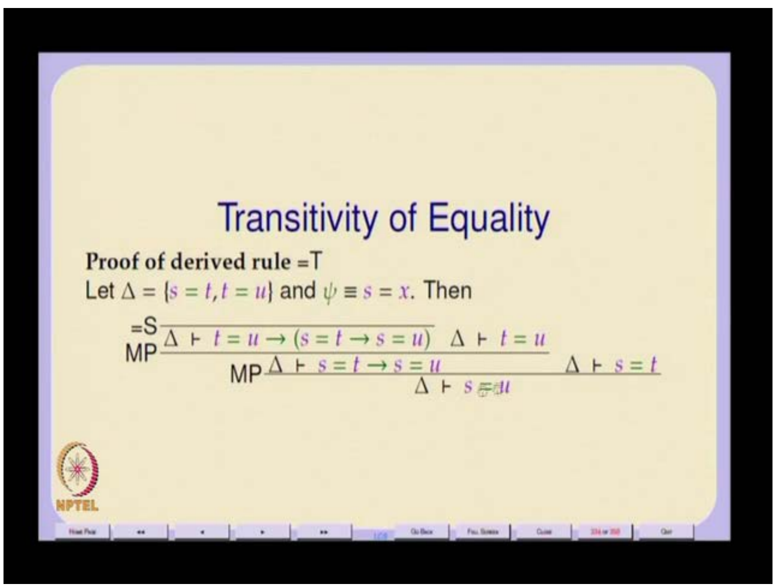

Transitivity is also provable actually as so here again we take here we take so there we to you are a formula x equals s in the case of symmetry here we take x equals s. So, you have given assumptions s equals t and t equals u. And, you have to prove s equals u so I am going to take x equals s as a formula. And, then it is very clear that I have t equals u in arrow s equals t arrow u basically I take this formula psi and replace t for x and u for x on this two sides. And, I have got the assumption t equals u so which means I can use modus ponens twice and finally i will get s equals u.

So, it is enough to have just these two it is enough to just have these two axioms for equality and the other two axioms can be. So, now today let us continue with predicate logic so we are going to obsess with free variables and bound variables for sometime because that is actually crucial for various things. So exactly, How can you capture the notions of arbitrariness? And, we will obsess with it also for existential quantifier later. Because, there is a question of determining whether a certain name is a constant or a variable Which should be treated like a constant and so on so forth. Whereas, lots of linguistic certainties which have to be taken care of and so we will do this.

The other thing this is that, after conversion need not be considered basic I mean some one thing you usually in all programming languages and the lambda calculus for conversion is considered an absolutely basic thing.

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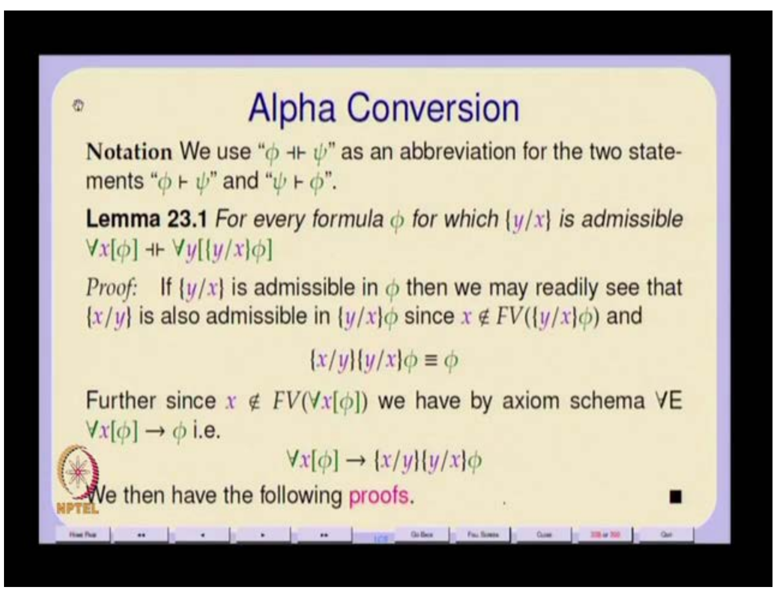

But, in a in a minimal powerful proof system Alpha Conversion can actually be proven and that is so that is the that is the first thing we will do now. However, in order to prove alpha conversion we will actually take two formulae phi and psi. And, this is this is fairly common notation which I am introducing now for the first time. Which this essentially says that psi can be proven from phi and phi can be proven from psi. So, basically it stands for these two statements so which means that you require two proofs. So, I mean this is like this is the proof theoretic analog of if and only if you like. So, for every formula phi for which y for x is admissible for all x phi proves and can prove can be proven from for all y, y for x of phi. So, the first thing to realize is that if y for x is admissible and phi then it, means that firstly x is not a free variable of y for x of y of phi. Therefore, x for y is also admissible for y for x of y of phi. And, actually if you take x for y in y for x for phi then you get back phi it is actually a syntactic identity.

So, these this left hand side and this right hand side after substitution is a meta syntactic operation it is not a syntactic operation so meta syntactic operation expressed in this fashion. So, these two are actually are syntactically identical formulae further the x is clearly not a free variable of for all x phi. So, which means that I can apply the universal instantiation rule for all E and for all elimination to get for all x phi arrow phi where of course I am going to take this phi as basically obtained from phi by two substitutions first y for x and then x for y. So, which means that I essentially get this as a, particular case of the universal instantiation formula.

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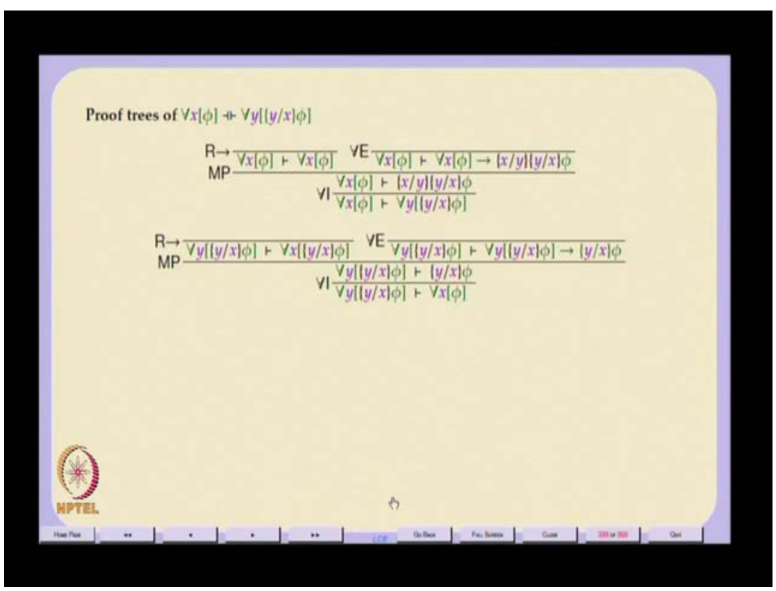

So, if I get this as a particular case of the universal instantiation formula then I have some trivial proofs easy proofs and have to make them small because otherwise it is because it is going beyond the screen. So, you can see that first of all for all x phi I can infer for all x phi and by quantifier elimination I actually get for all x phi arrow x for phi y for x phi. And, I can use modus ponens and get x for y and y for x phi. And, of course x for y is admissible in this entire thing and x and y are both are arbitrary essentially which means I can quantify over this. So, I can get this for all y y for x of phi.

Similarly I can start with for all x y for x of phi I get this I also get for all x y for x of phi arrow y for x of phi. And, then by modus ponens I get y of x of phi and since y is arbitrary and is not does not occur free anywhere in the assumptions or here, or anywhere in the assumptions. Therefore I, can generalize on that and I get for all x phi. You, can some of these proofs are settle especially when you are looking at minimality you have to look at these proofs in some certain fashion.

Student: what is it for all y?

You are you are this is that no this is that the propositional logical reflexivity this is derived by applying the deduction theorem this rule r arrow. So, if you want I could have called it R or the other no better I think to regard this as part of that a sweet where we said that any assumption is also provable from the assumptions. So, that is you can you can think of it that way if you like so maybe it is not you are so maybe I should not have called this r arrow I should have called it I mean whatever the monotonicity any assumption is also provable from this from the same assumption. It actually with that with that if you look at so I think that is the best of given that any formula phi belongs to the assumptions gamma. Gamma proves phi we had that in propositional logic and we had that actually stated in a very general fashion for all theories. So, there is absolutely no reason why we cannot apply that here. So, maybe I should not so this justification is probably not R arrow that should have been the justification from formal theories what is, that.

We, have those things like monotonocity and so on right was it here, I thought I had it somewhere you are I have forgotten at the moment where it is and if it is not there it should be included actually but you are so this all I am saying is just writing the assumption out here. And, that is true for here also I am writing the assumption out here. Any, proof any formal proof we had this notion of formal proof where each step of the proof is either an assumption or an axiom or obtained from some previous steps by the application of the rule of inference. So, this is the case of the step being an assumption that is all that is to do. So, this is quite justified maybe this R arrow should not have been there you are then, these two proofs go through quite easily you are. So, actually alpha conversion is something that can be proven by this.

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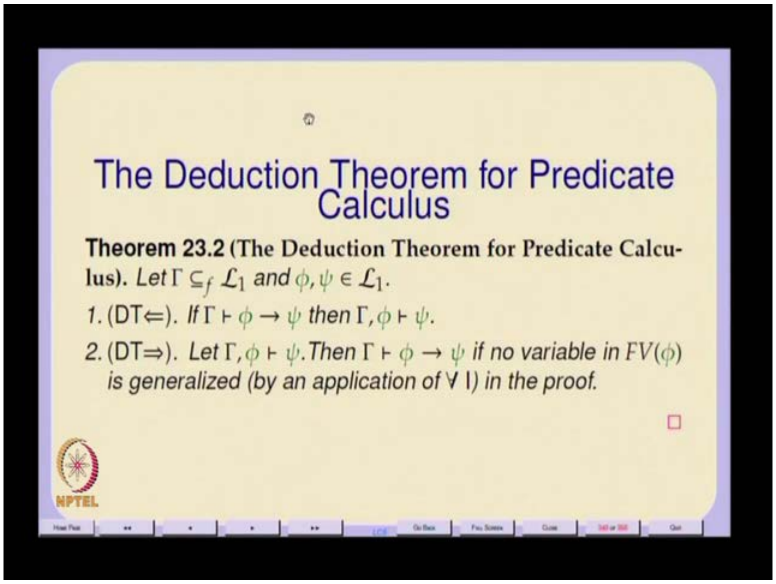

So which brings us to the Deduction Theorem so now the deduction theorem for Predicate Calculus is a little complicated by few variables. And, precisely the notion of arbitrariness actually when is a certain name arbitrary that is the question that basically we have because if you are ever going to generalize. Then, you have to be sure that you do not generalize on some constant you would not replace the constant by a variable and put a universal generalization. You, have to generalize on some variable symbol that is somehow guaranteed to be completely arbitrary. So, we take this in analogy with what we did in preposition analogy. So, essentially I am stating this deduction theorem that was in the case of preposition logic if and only if thing with two proofs that, so I am stating this separately. So, one part is that if you do have from a set of assumptions gamma if you do have phi arrow if you can prove phi arrow psi. Then, it is perfectly safe to pull phi to the left of the turn style. So, and the proof of this is exactly as in the case of the proposition of propositional logic. Because, of after all the proof involved just using the k axiom I think you are the using the k axiom.

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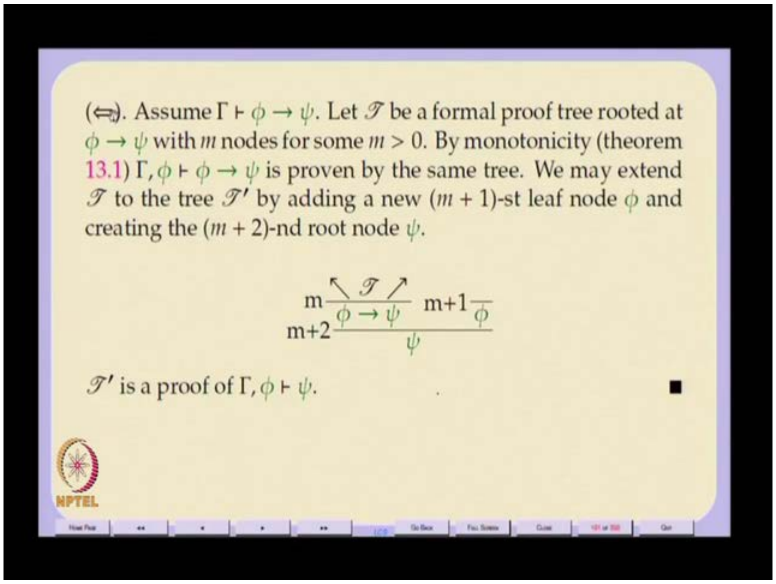

So, this is the left arrow so what we are saying is you can actually not the k axiom with you are by adding we use the monotononicity by adding an extra assumption like phi to the set of assumptions gamma you are not losing anything in the proof. So, you add that extra assumption phi and then you have phi arrow psi. So, if you have proof of phi arrow psi from gamma then you have a proof of phi arrow psi from gamma comma phi. And, of course in you also have phi and therefore you can use modus ponens and get psi. So, this is left arrow part is actually quite trivial and exactly as in the case of propositional logic it is the right arrow part. Which is slightly difficult you are the arrow part essentially says it. Supposing, I have an assumption phi that assumption phi might have some free variables no I do not know this status of those free variables they could be arbitrary some of them could actually represent some constants.

And, therefore for only certain particular valuations phi might be true. But, still if I am assuming phi and if I can prove psi then I can push phi onto the right hand side provided no free variable or phi is generalized anywhere in the proof and that is. So, if so essentially if no free variable of phi has been generalized anywhere in the proof and psi also has that free variable as if that variable free. Then, there is a good chance that free variable is not an arbitrary variable might be a particular might be something that is true only for certain values. So, which means that it is safe to move phi to the right hand side provided again actually there is more to it no free variable or phi should also occur free in gamma. I mean any free variable that is generalized should not occur free in gamma otherwise, you cannot apply this gamma rule. And, that actually will distinguish between what is arbitrary and what is not arbitrary.

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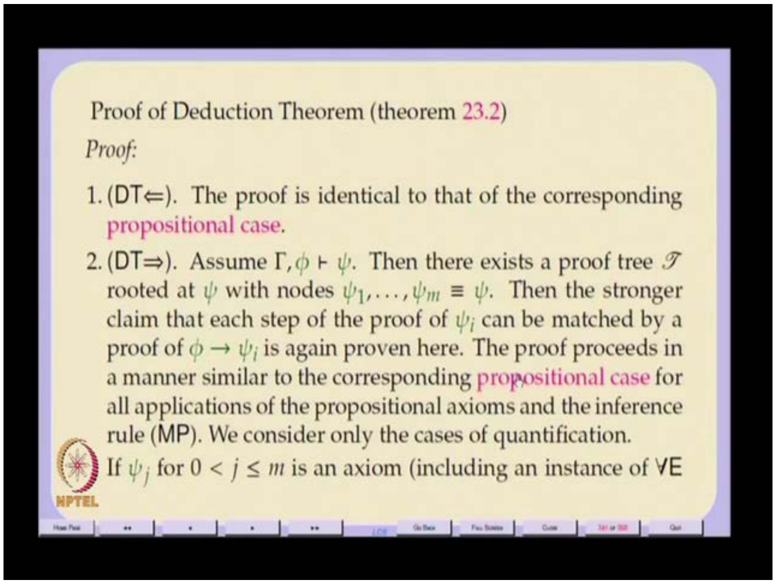

So, let us look you are at the proof of this so we are not going to go through the entire proof because of for most of the cases like the axioms. For example if, you the proof is exactly as of the propositional case so in we used for example you take so let us just worry about the quantifiers and so on. So, you take some supposing your proof actually has some m formulae m nodes it is a proof tree with m nodes the last one being the formula psi which is to be proven from the assumptions gamma and phi. Then, essentially what we did in the propositional case was that we proved that every step corresponding to the corresponding to each gamma phi proves psi i.

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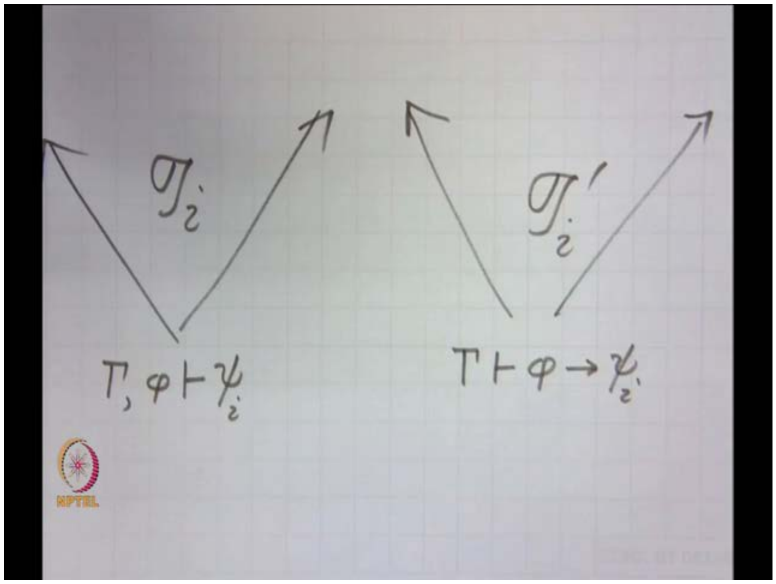

There, was a proof tree so essentially what we proved in the propositional case was that if I had some tree Ti in which i prove psi i. Then, I can construct a proof tree Ti prime rooted at this is what we did in the proposition proof.

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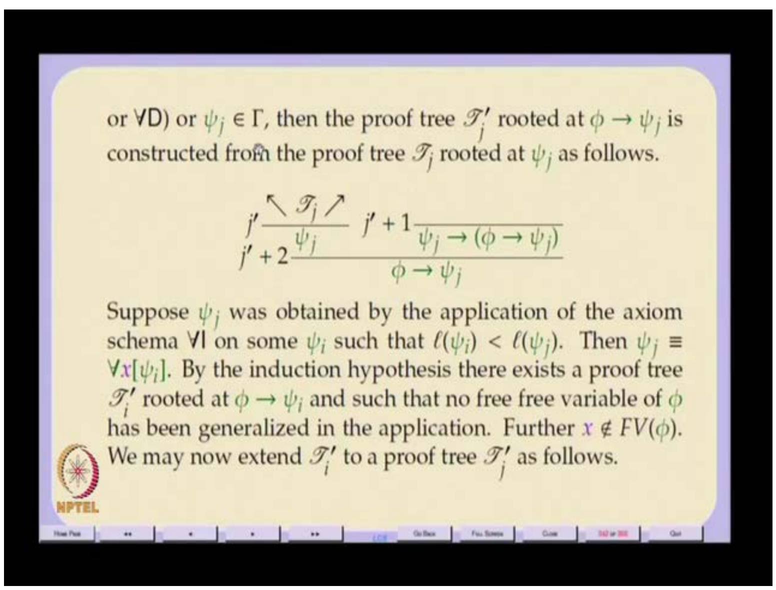

We showed that every step of this proof for every proof tree of this kind there is a proof tree of this kind. So, I will go back to the propositional proof and if you see that one of the things that we did in the proposition case was that for so take something like any axiom for any axiom actually and in this particular case it does not matter. Because, we had those three axioms k s and n and now we have two more axioms for all E and for all D these two axioms. But, regardless of what axiom you are using you take any axiom for any step for any tree Tj rooted at sign j this you can use the k axiom to get psi j arrow phi arrow of psi j. And, then you can use modus ponens to get phi arrow of psi j.

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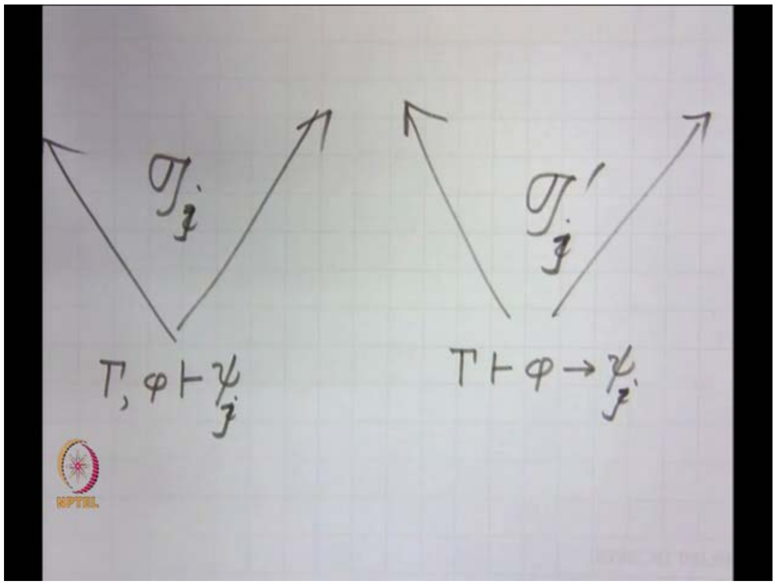

So, the fact that we added some two extra axioms does not change the proof in anyway because the proof that you can construct Tj prime given t j given t j is just is just an application of the k axiom and modus ponens we also had a proof from modus ponens. And, that proof is exactly as in the case of propositional logic and there is no change. The only thing therefore now to worry about is what happens for an application of the rule for all i so, if you have an application…

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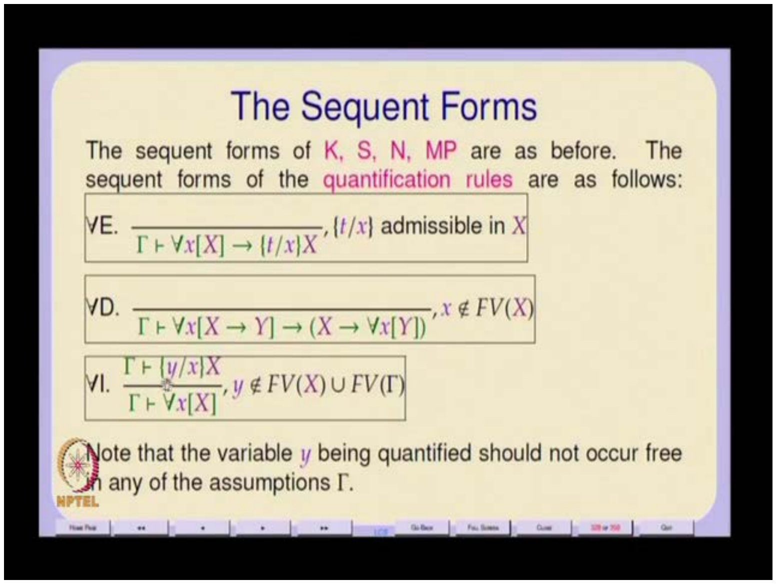

So, this is another rule so these two are axioms so it does not matter so, now we have to worry about this rule. So, if you do have this rule then we look you are at this way. So, if you have that means you up you inferred psi j by an application of for all i on some psi i. And, this proof is by induction on the depth of the proof tree essentially on the levels of the proof tree.

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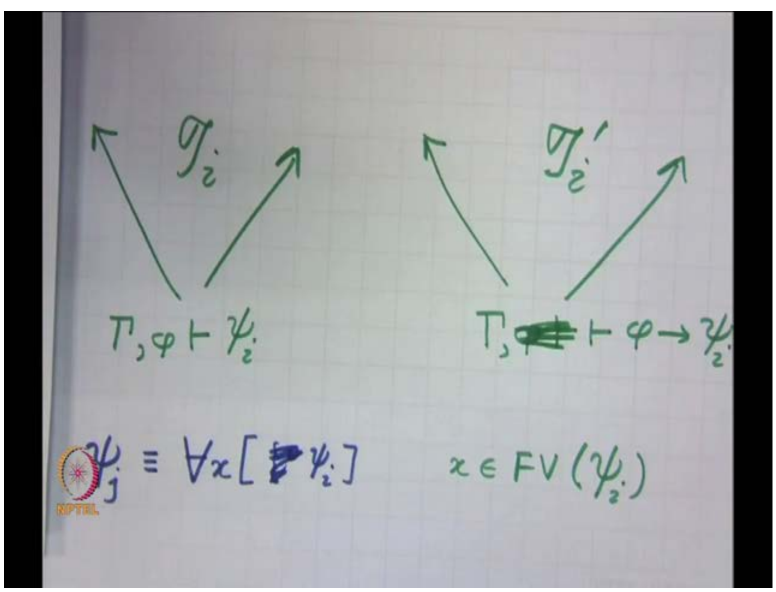

So, which means that by the induction hypothesis i can actually assume that there is a proof tree Ti rooted at gamma comma phi psi i. And, correspondingly by induction hypothesis i can assume that i have already constructed the proof tree Ti prime. And, which has phi arrow psi I given that psi j would therefore if psi j was obtained from psi i by an application for all i. Then, essentially what I have is there is some variable x. Such that well psi i is here, let us say is that fine and x may have occurred free in psi i. So, x maybe free in psi i you are and basically now what we have to show is that we can construct this proof tree Tj prime provided all these conditions are satisfied and what are these conditions.

This conditions are just this so one thing is if you prove this so one thing that your assumption tells you is that no variable in phi has been generalized. So, if you generalized on this x the x is not a free variable of phi first thing. And, further if you actually apply this generalization x is also not a free variable of gamma anywhere.

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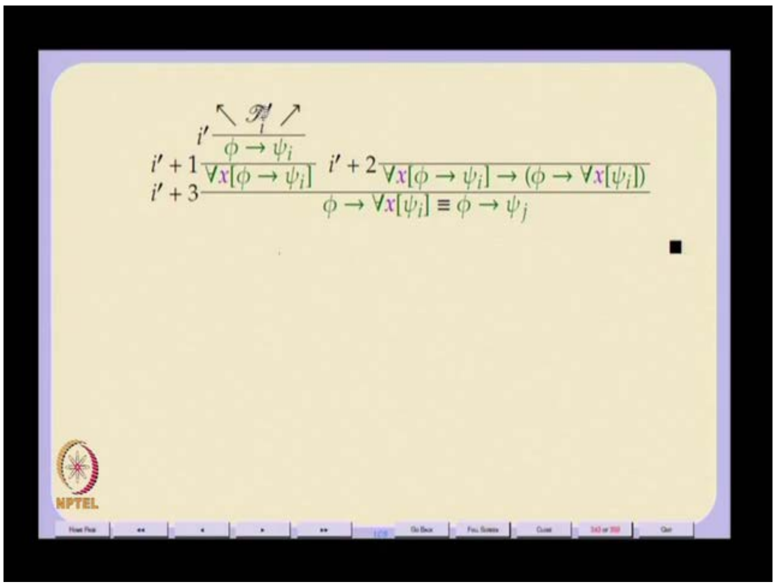

So, then what you have is I start with this Ti prime which proves phi arrow psi i. And, I generalize it to get for all x phi arrow of psi i. And, I can apply universal generalization here because x is not a free variable of phi and it is not of a free variable of gamma. And, so I can do this generalization I have from my new axiom I add a new axiom which says for all x phi arrow phi arrow all xi x is not a free variable of phi. So, this axiom so this is why i had to change that axiom for all D. So, this is an application for all D since x is not a free variable of phi is because x is not a free variable of phi that you could do this generalization. But, given that x is not a free variable of phi therefore you can push this for all x inside and you can get this.

And, when you get this therefore this then I can apply modus ponens. And, I get for all x psi i I get phi arrow for all x psi i and this is just phi arrow of psi j.So, if x had been a free variable of phi then all this things could not have been done so this deduction theorem actually is slightly settled. So, this deduction theorem in this fashion so actually be able to apply this deduction theorem most of time you would not be able to looking at it in all it is certainties. I mean there are there is one thing is clear the deduction theorem holds for propositions.

So, one thing is clear that if you are you take any general theory all you send all your formulae are likely to be closed formulae because, you are trying to prove general theorems. So, when you prove general theorems there are no free variables. So, the deduction theorem in its propositional formed becomes directly available.

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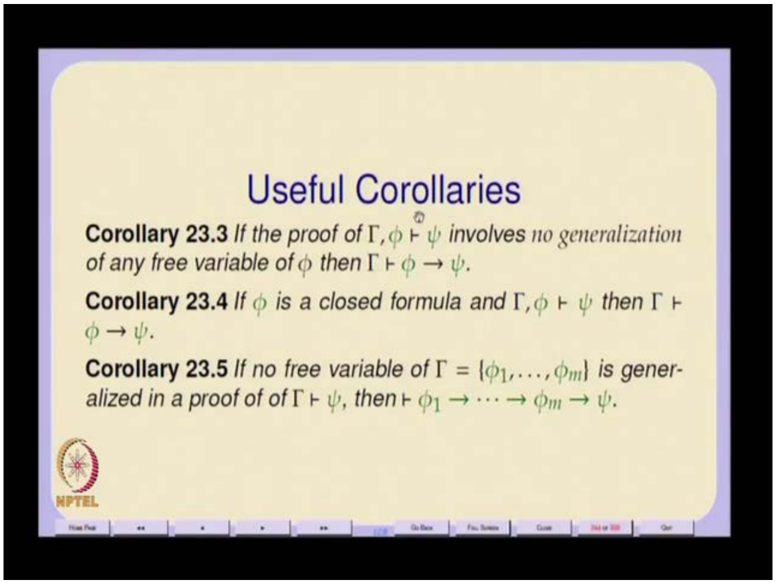

So, actually so Corollary of this is the following if phi is a closed formula and from gamma phi you can prove psi then, you can assume that you can prove from gamma phi arrow psi. So, even if psi is not a closed formula it does matter it does not matter. But, most importantly if you are proving very general theorems if all the formulae that you are taking as assumptions are closed formulae like the group theoretical axioms and trying to prove general properties of groups. If, all of them are closed formulae then your deduction theorem is anyway applicable without any problem. It, is only when your formulae actually use some free variables that you have to be careful about whether you are following in the conditions of the deduction theorem before you use it. So, essentially in order to prove if phi one to phi m are all closed formulae then I can either decide to prove this or I can decide to prove take all of them as assumptions. And, prove psi from these assumptions and both of them are equivalent.

And, so the other thing is supposing your proof actually is a proof of some very specific object which is being which is called x let us say. And, your proof is all about that then anyway you would not generalize on it. So, you take any proof which does not have a, involve any generalization there again the deduction theorem is applicable so that is the first corollary. So, if the proof of psi from gamma comma phi involves no generalization of any free variable then from gamma you can claim that phi arrow psi is true if phi is a closed formula. Then, you can claim that from gamma and phi if you proved psi and prove that gamma proves phi arrow psi. And, if all your there are no free variables anywhere then you can apply this deduction theorem. The only case where you have to worry about, whether the deduction theorem is at all applicable is a case when a generalization has taken place in the proof.

So, it requires going through the proof tree to find out whether there is any occurrence of universal generalization. And, if you are doing an occurrence of universal generalization you have to check basically whether the free variables in this assumption phi have been generalized or not. If, none of the free variables phi has been generalized then again you are safe you can just apply the deduction theorem. If, a free variable of phi has been generalized sort of or you are using some if a free variable of phi has been generalized in you cannot push it. But, remember that normally what you want to do is you want to factor everything which appears on the left side of arrows as an assumption.

So, which means you are looking at all free variables of phi 1 to phi n in order to have these two equivalents it requires that, any proof which involves proving phi from gamma does not generalize any of the free variables in phi 1 to phi n if it does not in. But, if it generalize something else may be generalizes a free variable in psi which is not present in phi 1 to phi n then you are still safe. So, there are various conditions it becomes very settle at this point you know. So, under what conditions you can actually check whether the deduction theorem is applicable. Now, actually it is now the question is whether I should go into soundness or postpone that. So, there is of course a question of existential quantification also so which but that may be let us do this soundness. So, you are first proposition you are I have not actually I have not spoken about, existential quantifier. But, the point is that you are existential quantifier is going to be defined by the Demorgan rule.

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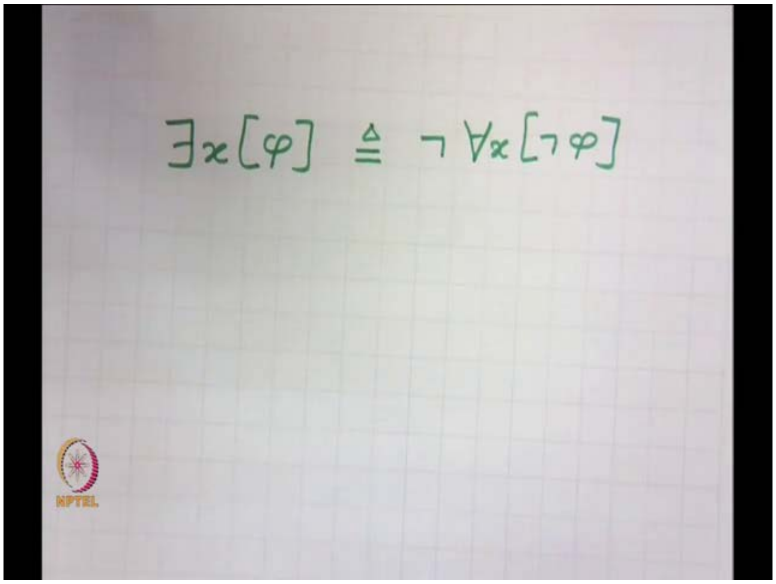

You are essentially called in a Hilbert style proof system your essential stating that this formula is being is going to be defined as naught of for all x naught phi. So, you are going to use this definition. So, the notion of arbitrary in particular then take a certain meaning here usually with the existential quantifiers. And, we have to be able to justify some of the proof methods that we use for existential quantifiers from the Hilbert style system itself. But that is a, little complicated so I will postpone that in the mean time just let us look at this it is soundness. Firstly of course that you take anywhere well formed formula which is an instance of a propositional tautology no valuation can change it no model can change it is always going to be true. So, tautologess of this is preserved by the by that propositional tautologess of this form. So, you can essentially take so this proposition I am not even going to prove.

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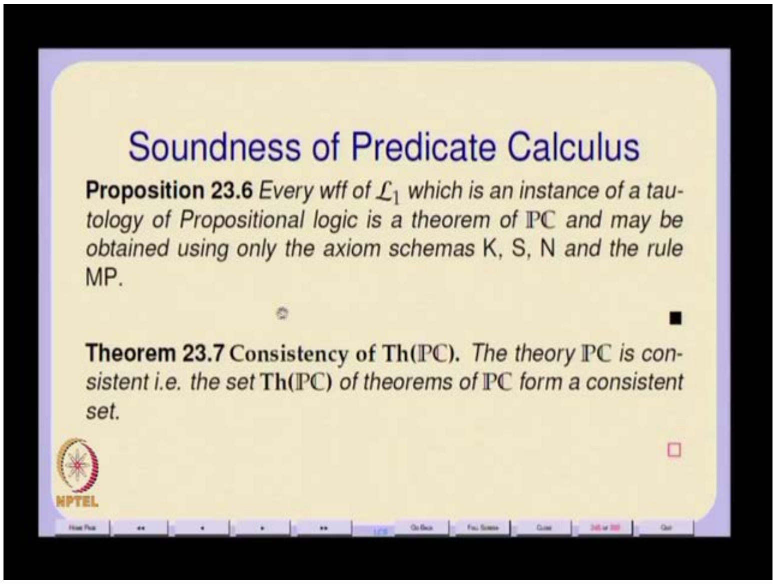

Essentially what, we are saying is so that is equivalent to saying that you take any predicate formula which can be derived by just using K, S, N and MP it is always going to be valid. So, all propositional of this forms with corresponding propositions replace the predicates do not change the truth value ever and their validity can be proved. So, in that sense K, S, N and MP are sound rules for predicate logic. So, the other interesting thing is that you take so, I am going to use Th of so I am looking at predicate calculus itself as a theory. You, know with theses axioms K, S, N and MP for all i for all e for all D. And, the PC stands for all the theorems of this theory.

So a formal theory take a set of all theorem of this theory so whatever is provable by the Hilbert in predicate calculus by the Hilbert style proof system. So, this theory is first of all is consistent that means all the theorems that you prove have truth values. They have models and valuations have interpretations in which they can be made true it is important. Because, the consistency of this set of formulae is required to show that your proof system is itself is sound.

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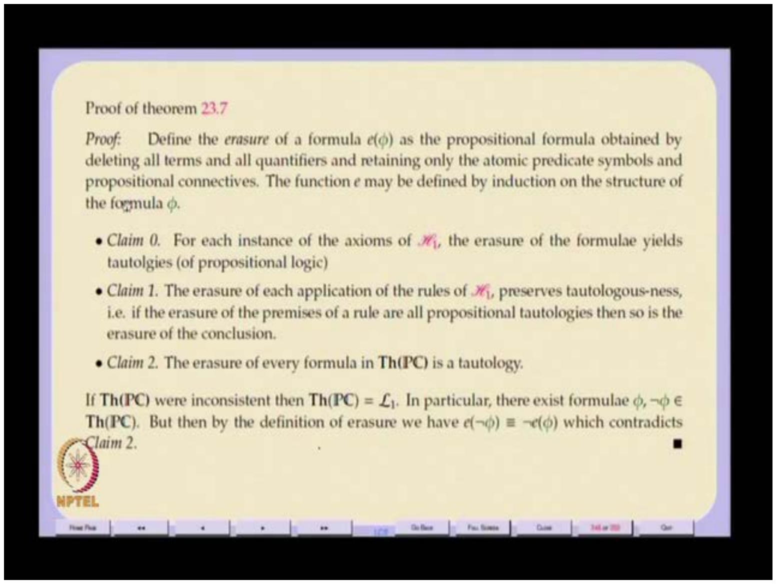

So this, the consistency of this is just depends upon these claims so I take any of the any of the theorems any of the formulae in the theory of predicate calculus some formula phi. And, I define what is known as its erasure you are the erasure of the formula is just that I remove everything that is violet in color from it I remove all the terms I retain only the atomic predicate symbols. And, I remove all the quantifiers if I remove all the quantifiers I also removed all the variables if I removed all the terms i have removed all the variables.

So, the quantifiers are anyway of no use I removed all the quantifiers. When, I remove all the terms and all the quantifiers whatever, I left with I am left with something that looks purely propositional. It just looks like a, proposition with propositional atoms and propositional connectors is like naught and arrow and so and so that is all so it looks like a purely propositional formula. So, the erasure of a formula e phi is just a purely propositional form with propositional connectors no quantifiers no terms nothing. Now, you take any of the theorems of first order logic so you take anything that is provable in the Hilbert style system H1.

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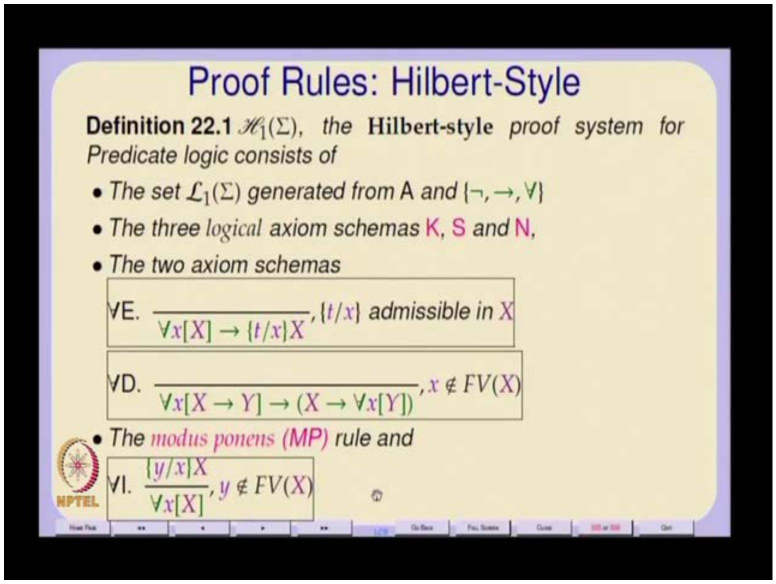

The erasure and take the erasure of any of the formulae that you get as theorems the erasure is always a tautological form for all the axioms. So, for K, S and N it is easy to show that they are just tautological forms the only thing therefore is to show for these two. But, for these two you can see I mean if I erase if x is a purely propositional atom. And, I am going to erase this for all x and I want to erase all this p for x because I am erasing all the terms there all the violet terms. So, then what am I left with I am just left with x arrow x which is total of this form similarly in the case of all D when I do the erasure I am just left with x arrow y arrow x arrow y. Which, is also a to toutologus form.

So, the erasure ensures that you will have only totals of this forms and in fact. So, the modus ponens rule of course preserves total of this form that is not the problem the problem is with universal generalization. Here, again when I do the erasure, What do I get from x? I get x which preserves tau tautologies if x is a tau tautology then x would also be a tautology. So, all so the erasure is such that all the axioms and the inference rules of H1 of this system H1 the axioms only create total of this forms the rule of inference preserve total of this forms. So, if they had a hypothesis which a toutologus then the rules would ensure that the conclusions are also tautologus the erasure is of the conclusion is also toutologus.

So, claim 0 essentially says that the erasure of all the axioms gives you tautologus forms claim 1 says that the erasure in every rule preserves total of this form. So, if the hypothesis was a tautologus form then the conclusion would be also the tautologous form. And, then claim 2 says that the erasure or every formula and PC is the tautology because every proof just preserves total of this forms. So, now you take the erasure of the theorems of PC you will get only tautologies. So, all the sentences in theory of PC you apply erasure on all of them you will get only tautologies. Now, we know that that has to be consistent. But, now but that does not mean that theory of PC itself will be consistent it is only the erasure.

Since, it produces only tautologous forms it will be consistent. So, now we argue why contradiction suppose theory of PC contains supposing it were inconsistent then firstly the theory of PC would be the whole language L1. And, in particular there will exists two formulas phi and naught phi belonging to the theory of PC. But, if phi and naught phi both long to the theory of PC then there erasure should both be tautologies. But, naught does not get erased. So, you will get the erasure of naught phi is just naught of the erasure of phi you are so you will get a contradiction in propositional logic. Which, is not possible and therefore we know that propositional logic is sound and creates only tautologies. So, therefore the theories of PC is consistent.