Time Series Modelling and Forecasting with Applications in R

Prof. Sudeep Bapat

Shailesh J. Mehta School of Management

Indian Institute of Technology Bombay

Week 07

Lecture 34: Further Extensions and Use Cases

Hello all, welcome to this course on time series modeling and forecasting using R. Now, again this week, we are discussing the idea of multivariate time series processes. Up to the last session, we discussed some examples of, let us say, multivariate time series processes. So, let us say VMA with a certain order, let us say VMA(1) or VAR with a certain order, right? And we also discussed the combination. So, let us say VARMA process of some orders (P, Q), etcetera, right? So, particularly, this session is not on the easier side, right? Because up to now, we have

spent a lot of time analyzing, modeling, and forecasting univariate time series structures. But the whole idea is, if you transition from a univariate situation to, let us say, even a bivariate situation or multivariate situation for that matter, then things become very difficult. So, let us say you have to bring in the ideas of, let us say, random vectors or matrix theory and so on and so forth. Because each component in the underlying multivariate structure would be either a random vector or a random matrix. Okay. All right.

So, again, just to give you a very quick review as to where we stand now. So, up to the last session or towards the end of the last session, we were talking about VARMA processes in general, which is nothing but a combination of an AR structure and an MA structure. And then we put forward a case as to when one can say that a VARMA process is stationary, right? And there, if you remember, we brought in the idea of the inverse of a matrix by writing down the inverse as 1 divided by the determinant into the adjoint and so on and so forth, right? So, the whole idea was by rewriting such a VARMA equation, right?

We can actually make sure about the stationarity of the underlying process in easier ways, right? So, here the first thing we will do in today's session is take up a concrete example from a bivariate situation. So, we will discuss a few things about this bivariate MA1 process. And then again, of course, this is nothing but a VMA—or rather, not V but VMA1 process. So, a vector moving average process of order 1.

But the only thing is that the vector only contains two elements. So, hence it is called a bivariate. And then again, why bivariate? Because here, if you see on the left-hand side, I have two random vectors. So, the first one is Y1 t and the second one is Y2 t.

So, I can give you a very quick practical example. So, let us say, think of a situation where you want to analyze two different time series processes. So, let us say, the stock price of Reliance versus the stock price of, let us say, Adani, right? And there you want to analyze how these two stocks co-behave or how they are interdependent, and so on and so forth, okay? And for doing all these things, we are proposing a model called a bivariate MA1 process, okay?

So, this is the numerical structure of that. So, the first thing we require is a random error vector. So, e1 t e2 t minus the coefficient matrix. So, 0.2 minus 0.4 minus 0.2 and then 0.6 multiplied by e1 t minus 1 and then e2 t minus 1, right? I mean, of course, this is structure of any VMA1 process.

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} - \begin{pmatrix} 0.2 & -0.4 \\ -0.2 & 0.6 \end{pmatrix} \begin{pmatrix} e_{1t-1} \\ e_{2t-1} \end{pmatrix}; \ \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

Further,

$$\Gamma(0) = \begin{pmatrix} 0.2 & -0.4 \\ -0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0.2 & -0.4 \\ -0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.64 & -0.92 \\ -0.92 & 1.28 \end{pmatrix}$$

$$\Gamma(1) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0.2 & -0.4 \\ -0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ -1.4 & 2.2 \end{pmatrix}$$

$$\Gamma(l) = 0 \ \forall \ l \geq 2$$

So, in general, VMA1 process is nothing but y t equals let us say e t plus the theta 1 coefficient, if you remember, applied on Et minus 1, isn't it? So, this is the VMA1 process, which we even discussed in the last session. And then we are simply writing down this structure in a numerical way. So, this is the actual numerical value of that. So, Yt equal to the E1t and E2t random vector minus the

Now, again this is nothing but my theta 1 matrix right. So, even as discussed in the prior session all these coefficients. So, theta 1 theta 2 or be it phi 1 phi 2 on the AR side. So, all these are matrices now ok. So, theta 1 coefficient or rather theta 1 matrix multiplied by the E t minus 1 vector ok.

And here, further, we are assuming that sigma is 4 1 1 4. Now, what exactly is sigma? So, sigma is nothing but the variance of my random error vector, ok. Because, again, remember, this E_t is what? So, E_t is not a univariate error, right.

So, E_t is a multivariate, or rather, a multivariate white noise process where the variance-covariance matrix is given by sigma, ok. So, this is that sigma. So, we are defining sigma to be 4, 1, 1, 4, alright so far. So, further from this expression, I can gain insights into some covariance structures and so on and so forth. So, again, if you remember towards the very end of the last session, we wrote down an expression for the autocovariance structure of a VMA(1) process, ok.

So, what would happen if you take L to be 0? So, we described some cases there, right. So, L to be 0 or L to be 1 or L is greater than or equal to 2, and so on and so forth, ok. So, firstly, what would happen if your L is 0, or rather, gamma of 0? So, gamma of 0 can be calculated easily by this expression.

So, whatever theta 1 you have here, right? So, this is nothing but theta 1 into sigma into theta 1 transpose. So, the expression is theta 1 into sigma into theta 1 transpose. And once you evaluate this, you will get 0.64, minus 0.92, minus 0.92, and then 1.28. So, this is nothing but gamma 0, and similarly, if you extend it to what is gamma 1.

So, gamma 1 is nothing but given by this expression, which is sigma into theta 1 prime. So, this is my sigma 4114, and this is my theta 1 prime. So, 0.2, minus 0.4, minus 0.2, and then 0.6. And then this is nothing but 0.4, minus 0.2, minus 1.4, and 2.2. And for all further orders, whenever L is bigger than or equal to 2, my gamma L happens to be simply 0.

So, this is just an empirical way of how you visualize a bivariate process and then sort of write down a few initial values of the covariances there. Now, similarly, further let my matrix V take this form. So, my V is 0.6400, and then 1.28, and then here, why do we require V? We require V such that we can actually find out autocorrelation matrices. So, my row 0 has this structure.

$$V = \begin{pmatrix} 0.64 & 0 \\ 0 & 1.28 \end{pmatrix}$$

Autocorrelation matrix is thus given by,

$$\rho(0) = V^{-1/2} \begin{pmatrix} 0.64 & -0.92 \\ -0.92 & 1.28 \end{pmatrix} V^{-1/2}$$

$$\rho(1) = V^{-1/2} \begin{pmatrix} 1.6 & 0.2 \\ 2.2 & 2.9 \end{pmatrix} V^{-1/2}$$

So, V minus half, and by the way, what is this matrix? This matrix is nothing but what we saw on the first slide, which is right here. So, this matrix is nothing but gamma 0, in other terms. So, essentially, my rho 0 is nothing but V to the power minus one half multiplied by gamma 0 multiplied by V to the power minus one half. So, essentially, my rho 0 takes that form, and then rho 1 takes this form, and so on and so forth.

So, hopefully, the idea from a moving average perspective should be clear now. So, we described starting with a VARMA process, then we took some special cases, discussed the vector moving average process of a general order Q, and then we discussed VMA1. And then, in the last slide or the initial few minutes of today's session, we took up a practical example—a numerical example—and explained VMA, or rather, a bivariate MA process with order 1. Now, of course, the other side of the coin is explaining vector autoregressive processes of a certain order P. And then, this model structure is given by VAR(P). So, a vector autoregressive process of order P, or in short VAR(P), is given by this structure.

$$Y_t = \delta + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \cdots + \Phi_p Y_{t-p} + e_t$$

So, y_t equals some delta plus phi_1 multiplied by y_t minus 1 plus phi_2 multiplied by y_t minus 2, and so on and so forth, up to phi_P multiplied by y_t minus P, plus the white noise error vector. And again, as before, all the phi_i coefficients, where i goes from 1 to P, are nothing but k-dimensional square matrices. So, all these coefficients are nothing but k-dimensional square matrices. My e_t is nothing but a k-dimensional vector of residuals or, in other words, a vector of purely random processes at time t, and delta is nothing but a vector of constants. So, here, just to remember one important thing: we are not anymore in a univariate kind of structure.

So, if you are transitioning to a multivariate structure, the constant term would be a vector, all the coefficients would be matrices, my error or ET would again be a vector, and so on and so forth. So, hence my phi i's are nothing but k-dimensional square matrices, my E t vector is a k-dimensional vector of errors or residuals, and my delta is nothing but a vector of constants. And then I can sort of write down the same VAR process of order P in a concise form. So, the concise form takes this expression. So, phi b applied on y t should be equal to delta plus e t. And what exactly is the phi b coefficient?

$$\Phi(B)Y_t = \delta + e_t$$

$$\Phi(B) = I_k - \Phi_1 B - \Phi_2 B^2 - \cdots - \Phi_p B^p$$

$$E(e_t) = 0,\ \ E(e_t e_t') = \Sigma,\ \ \ E(e_t e_{t+l}') = 0, \quad \forall\, t, l \neq 0$$

So, the phi b coefficient is nothing but i k. So, i k is what, by the way? So, i k is nothing but an identity matrix of order k. So, you have 1s on the diagonal and all the off-diagonal entries are 0s. So, my phi b coefficient becomes i k minus phi 1 b minus phi 2 b square minus phi 3 b cube and so on and so forth up to phi p into b to the power p. And all these are sort of corollaries. So, the expectation of E t is 0, the expectation of E t into E t transpose is sigma, and then the expectation of E t into E t plus L transpose is 0.

So, what do you mean by that? So, we are assuming that the means of my random error vector is 0, right? And the variance-covariance matrix of the errors is given by sigma. And any of the cross-covariances, so all the cross-covariances are 0 because they are assumed to be independent. So, something like E t into E t plus L transpose would be 0, and this should be true for all t and L which are not equal to 0, or rather for all t, but then all L excluding the value of 0.

Okay, so now we will again try to sort of rewrite the initial equation. So, now the very initial equation that we saw in the VAR structure could be slightly modified and then rewritten as probably something like this. So, on the LHS what we have is yt minus mu equals phi 1 yt minus 1 minus mu plus phi 2 yt minus 2 minus mu and then dot dot dot so on and so forth up to phi p yt minus p minus mu plus my error vector Ep, okay. But here, just a couple of things to note is that my delta. So, you do not see any delta here, right.

$$(Y_t - \mu) = \Phi_1(Y_{t-1} - \mu) + \Phi_2(Y_{t-2} - \mu) + \cdots + \Phi_p\big(Y_{t-p} - \mu\big) + e_t,$$

Where $\delta = \Phi(I)\mu$, or $\mu = \Phi^{-1}(I)\delta = \Psi(I)\delta$. Under this condition one can have the following MA representation,

$$Y_t = \Phi^{-1}(B)\delta + \Phi^{-1}(B)e_t$$

$$= \mu + e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \cdots = \mu + \Psi(B)e_t$$

So, you do not see a delta because I am expressing my delta to be nothing but phi i applied on mu or rather my mu becomes phi i inverse applied on delta, right, which is nothing but given by this expression. So, psi i applied on delta. Now, the whole idea of sort of rewriting the initial equation is that under this condition, one can actually have an MA representation. So, one can actually have an MA representation which is given by that form. So, yt becomes phi inverse b into delta plus phi inverse b into et.

And if you sort of re-express these, the inverses, they take this form. So, yt is nothing but mu plus et plus psi 1 et minus 1 plus psi 2 et minus 2 and so on and so forth up to infinite. So, this can be written down again in a concise form in this manner. So, mu plus psi b into et. Now again, how are all these things happening?

You have to sort of go back a slide or couple of slides and then borrow some equation from there and then see that how are we writing that equation in something like that form. Because again, if you go back a slide, this is the initial equation we have, right? So, y t equals delta plus all of that, right? Now, the first thing I am doing here is that subtracting mu throughout. So, what would happen if you subtract mu throughout?

If you subtract mu throughout, then on the LHS, I have y t minus mu equal to phi 1 yt minus mu minus mu plus phi 2 yt minus 2 minus mu, etc. Right? And here I can write down some concise form. So, my delta becomes phi i into mu or rather my mu becomes phi inverse into delta. And this phi inverse I am denoting as a psi which is psi i into delta, ok.

And under all these conditions, I can express my yt as a ma infinity structure, alright. And then similarly, I can put forward and then give some expressions for the autocovariances matrices and so on and so forth. But just one last point to mention, if you go back a slide, then so in other words, in the last session, we saw that I can actually express a VMA1 process as a VAR process of order infinite that I can actually express a VMA1 process as a VAR process of order infinite. So, this we saw in the last class. Similarly, I can actually express a VAR process of order P as a VMA process of order infinite.

So, you can actually, strangely, see that a particular vector moving average process of order 1 can be re-expressed as a vector autoregressive process of order infinity. And similarly, a vector autoregressive process of order p can be expressed as a vector moving average process of order infinity. So, you have these two sort of back-and-forth relationships between the AR part of the model and the MA part of the model. So, now once you sort of write down this expression, I can now talk about its autocovariance function and so on and so forth.

$$\Gamma_y(l) = E[(Y_t - \mu)(Y_{t+l} - \mu)']$$

$$\text{Let } \delta = 0 \Rightarrow \mu = 0. \text{ Thus, } \Gamma_y(l) = E[(Y_t)(Y_{t+l})']$$

$$= \Phi_1 E[Y_t Y'_{t+l-1}] + \Phi_2 E[Y_t Y'_{t+l-2}] + \cdots + \Phi_p E\left[Y_t Y'_{t+l-p}\right] + E[e_t Y'_{t+l}]$$

$$= \Phi_1 \Gamma_y(l-1) + \Phi_2 \Gamma_y(l-2) + \cdots + \Phi_p \Gamma_y(l-p) + \Sigma_e$$

$$\Gamma_y(0) = \Phi_1 \Gamma_y(-1) + \Phi_2 \Gamma_y(-2) + \cdots + \Phi_p \Gamma_y(-p) + \Sigma_e$$

$$= \Phi_1 \Gamma_y(1)' + \Phi_2 \Gamma_y(2)' + \cdots + \Phi_p \Gamma_y(p)' + \Sigma_e$$

Thus,

$$\Gamma_y(0) = \Phi_1 \Gamma_y(-1) + \Phi_2 \Gamma_y(-2) + \cdots + \Phi_p \Gamma_y(-p) + \Sigma_e$$

$$\Gamma_y(1) = \Phi_1 \Gamma_y(0) + \Phi_2 \Gamma_y(-1) + \cdots + \Phi_p \Gamma_y\big(-(p-1)\big)$$

$$\cdots$$

$$\Gamma_y(p) = \Phi_1 \Gamma_y(p-1) + \Phi_2 \Gamma_y(p-2) + \cdots + \Phi_p \Gamma_y(0) + \Sigma_e$$

$$\gamma_{ij}(l) = (i,j)^{th} \text{ element of } \Gamma_y(l) = Cov(Y_{i,t}, Y_{i,t+l})$$

So, my gamma y L or is nothing but the autocovariance matrix takes this form, of course. So, expectation of y t minus mu into y t plus L minus mu and transpose. Now, assuming delta is 0, which implies mu is 0, right? So, these two mu's become 0 here in the above expression. So, in other words, my gamma y L reduces to expectation of y t into y t plus L transpose, all right.

Which can be written down or broken down further as phi 1 expectation of Yt, Yt plus L minus 1 transpose plus phi 2 and then expectation of Yt and then Yt plus L minus 2 transpose up to that point. So, phi p and then expectation of Yt into Yt plus L minus p transpose, and the last expression here is expectation of Et into Yt plus L transpose. Now, the whole thing I am doing here is that I am taking yt from here and then yt plus l from here, right? And then each yt and yt plus l has their own expressions from the last slide if you see, right? From here essentially, okay?

Because we are dealing with a vector autoregressive process, isn't it? So, I can sort of write down a var structure for yt and a var structure for yt plus l and then sort of multiply in every term, okay? But for this reason, it can be simplified further as something like this. So, the first expression you see here is nothing but phi 1 into gamma y L minus 1 because lag is nothing but L minus 1. So, the gap between these two subscripts is nothing but L minus 1.

So, in other words, this expression is nothing but phi 1 into the auto covariance function at lag L minus 1. Similarly, if you go a step forward, how much is the lag here? So, here the lag is L minus 2. So, this the middle expression reduces to phi t or rather phi 2 into gamma y at a lag L minus 2. Similarly, the last thing can be written down as phi p into gamma y L minus p and the last expression is sigma e. So, sigma e denotes the variance covariance matrix for my error vector which is E t. And hence I can talk about let us say gamma y 0.

So gamma y 0 would be above expression, but taking the value of L to be 0. So, gamma 0 is nothing but if you consider L to be 0. So, if you blindly plug in L to be 0 in the above expression, what will you have? So, you will have phi 1 into gamma y of minus 1 plus phi 2 into gamma y minus 2 and so on and so forth up to phi p and then gamma y minus p plus sigma e or as discussed in some previous session in this week, that gamma y minus 1 is exactly equal to gamma y 1 transpose, okay? or as discussed in some previous session in this week, that gamma y minus 1 is exactly equal to gamma y 1 transpose, okay?

So, gamma y minus 1 is nothing but gamma y 1 transpose. Similarly, gamma y minus 2 is gamma y 2 transpose. And similarly, gamma y minus p is gamma y p transpose, and then plus you have this sigma e, okay? Thus, I can again sort of summarize all the values. So, gamma y 0 takes this form, which is again given in the last slide, gamma y 1, which is again given in the last slide, gamma y 1.

So, gamma y 1 means what? So, L equals 1, is it not? So, if you plug in the value of L to be 1, the expression that you get is this. So, phi 1 gamma y 0 plus phi 2 gamma y minus 1, so on and so forth up to phi p gamma y minus of p minus 1. And in short, the last thing would be gamma y p in general, which is given by that expression.

So, hopefully this is clear, and then I think here I am missing or probably not because you are sort of evaluating this at lag 1. So, there will not be any sigma E component there. But in general, my small gamma IGL. So, what does that denote? So, my small gamma IGL is the IJth element of my gamma YL matrix, which is nothing but given by the covariance between YIT and YIT plus L.

So, covariance between YIT and YIT plus L. So, the same series, but at two different lags. So, T and T plus L. So, these are some sort of extension that one can gain insights on once you define some multivariate processes. So, be it VARMA or VMA with certain order or VAR with certain order and so on and so forth. So, can we sort of put forward some condition of stationarity is the first thing and can we sort of interchange the equation to take the other form. For example, VMA becomes VAR with certain order of course.

VAR becomes VMA with certain order right. And then the last thing would be to put forward expressions for autocorrelance matrices and obviously autocorrelation matrix. So, lastly the correlation matrix. takes this form. So, the individual correlation coefficients are given by this, right? So, rho ij L is nothing but the covariance divided by square root of the variances.

$$\rho_{ij}(l) = \frac{\gamma_{ij}(l)}{\sqrt{\gamma_{ii}(0)\gamma_{jj}(0)}}, \ \forall\, i, j = 1,2, \dots, k$$

Autocorrelation matrices: $\rho_y(l) = V^{-1/2}\Gamma_y(l)V^{-1/2}$

$$V^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{\gamma_{11}(0)}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sqrt{\gamma_{kk}(0)}} \end{bmatrix}$$

So, gamma ij L divided by under root gamma ii 0 into gamma jj 0, of course, for all the indices. So, for all ij between 1, 2, 3, 4 up to k. Or in general, my autocorrelation matrix, which is given by rho y L, takes this form, right? So, one thing to note here is that the expressions, the formulas stay the same, right? So, the expression for finding out the autocorrelation matrix is nothing but V to the power minus half into L. gamma YL into V to the power minus half.

But then here, what does this V to the power minus half matrix become? It takes this form. So, you have a matrix which is K cross K, of course, and then all the diagonal entries are 1 divided by under root of gamma 1 1 0, then gamma 2 2 0 up to gamma K K 0, right? And all the off-diagonal entries are 0s. So, if you plug in this V to the power minus half matrix in the above expression along with my gamma Y L, which is nothing but the autocorrelation matrix, and sort of evaluate, then you will get hold of the autocorrelation matrix given by rho Y L. So, hopefully, the ideas laid down particularly in this week are, or rather should be, clear if not all the intricate notation and equations. So, the idea is that you are slowly transitioning from a single series to multiple series.

So, the moment you transition from a single series even to two series, then the interdependencies between the two series come into the picture, isn't it? So, then we have to talk about autocorrelation matrices, autocorrelation matrices, cross-covariances, cross-correlations, etc., okay. Alright, so just to wrap up this week, or rather the theoretical part of that, we will sort of discuss or put forward some application areas just to sort of complement whatever theory we have discussed so far. So, the first application could be seen in macroeconomics. So, the first idea could be, let us say, something like this.

So, monetary policy analysis. So, a VARMA model can help in analyzing the effects of, let us say, interest rate changes or inflation and other monetary policies on variables like GDP, unemployment, and exchange rates. Okay. So here again, the whole idea is that can you sort of combine more than one time series processes or more than one sort of structures or features. So, for example, here we are talking about a combination of interest rate changes, inflation, and let us say other monetary policies and their effect on variables like GDP, unemployment, exchange rates, and so on.

So, here one can actually adopt a VARMA process. So, a simple ARMA process won't do the job because you don't have a single series, but you want to study the interrelationships between two or three series or two or three features on two or three extra features. Or, something like a second example would come from economic forecasting. So, let's say governments and financial institutions use a VARMA process to forecast key macroeconomic indicators such as GDP growth, inflation, or unemployment, considering the interplay between these variables. So, for example, how is GDP growth related to or affects inflation, or how is inflation related to employment or unemployment, etc.

And then, moving forward, the second application area could be in supply chain and operations. So, the first example could be inventory management. So, in industries with complex supply chains, a VARMA model helps forecast demand for multiple products, especially those with interdependencies, improving inventory management and reducing stockouts or excess inventory. So, stockouts means what? Stockouts means that the inventory or the warehouse does not have enough products.

So, the stock is sort of out. So, in other words, stockouts or excess inventory. So again, stockouts and excess inventory are on both extremes. So, we want to avoid both. We do not want excess inventory.

We neither want excess inventory nor do we want stockouts. So one can actually apply the VRMA model in inventory management, or let's say logistics and shipping. So shipping companies use VRMA to forecast shipping demand and lead times for different routes and services, taking into account interrelated demand patterns across regions or product types. So one example here from a logistics and shipping point of view could be, let's say, If you want to analyze the interdependencies or the interrelationships between the demand patterns across regions, let's say India and China or India and, let's say, Sri Lanka, something like that.

So, let's say if you have more than one region or more than one location, and then for each region you have the individual demand logistics data or shipping data or inventory data, whatever, and then since you have two time series processes, this is an example of a bivariate sort of situation. Another area could be in financial markets. So, let's say asset pricing. So, VRMA models are applied to study the co-movements between various asset classes, such as stocks, bonds, and commodities, helping in portfolio optimization and risk management. So, again here, one could study the interdependencies between, let's say, how a stock is performing vis-à-vis, let's say, a government bond or commodity, let's say crude oil or something like that.

And then the next example could be in volatility forecasting. So, multivariate time series models like VARMA are used in predicting the volatility of multiple assets simultaneously and their covariance or co-volatility structure, which is critical in risk management and derivative pricing. So, again, these are some examples coming from a financial sort of framework. The next thing could be the energy market. So, let's say electricity load forecasting.

So, a VARMA model could be used to forecast electricity demand where multiple time series such as temperature, humidity and energy prices are interdependent and affect the energy consumption patterns right or let us say oil and gas pricing. So, in energy markets a VARMA model help in forecasting the interrelated prices of let us say crude oil or natural gas and other energy sources ok. And probably the last one could come from, let us say, healthcare. So, epidemiology. So, VRMA model are employed in studying the spread of infectious diseases where the number of cases in different regions may be interdependent, helping in resource allocation and public health planning.

And another example could be, let us say, hospital resource planning. So, hospital use a multivariate time series model to predict patient arrivals, demand for critical care beds and other resources, considering the correlation between different types of admissions. And now the very last one could come from something like climate science background. So temperature and weather forecasting. So, in climate studies, a VARMA model helps in forecasting weather patterns by considering the interaction or interdependencies between various climate variables like temperature, humidity and wind speed across let us say multiple locations.

And second one could be let us say air quality monitoring. So, VRMA models are used to predict the levels of pollutants like CO_2 or nitrogen dioxide and particulate matter which are often interdependent in atmospheric conditions. So, again just to summarize very quickly in maybe a minute or so is that whenever you want to model and forecast multiple series. So, be it climate science, finance, energy, supply chain, logistics, wherever, right. Regardless of whichever area I want to implement in.

But the whole idea is that if you want to model and forecast more than one time series or more than one process. Then one has to apply some multivariate time series process structures or bivariate time series process structures. Now the next session will be a practical session. So I think it is a strong suggestion that you should again rework or review all the sessions so far this week. So as to understand the application idea of, let us say, applying a VARMA process or a VAR process or a VMA process to some practical data much more confidently.

Thank you.