Indian Institute of Science
Bangalore

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Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 08
FRF-s and damping models - 1

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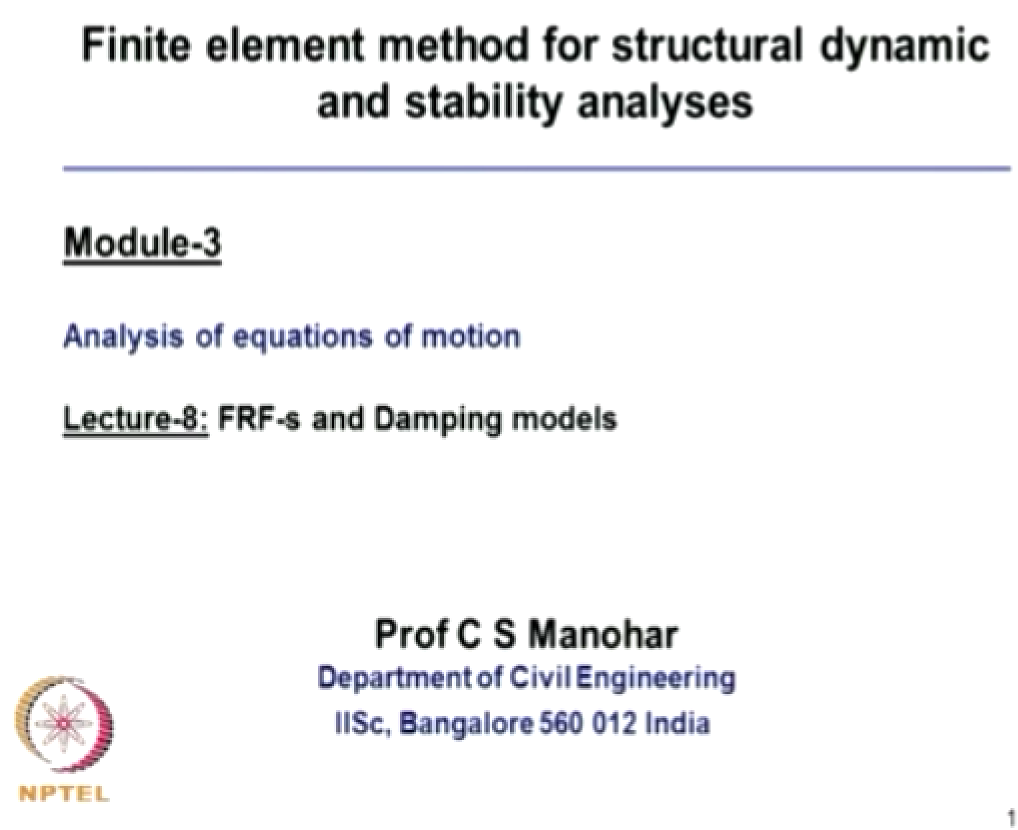
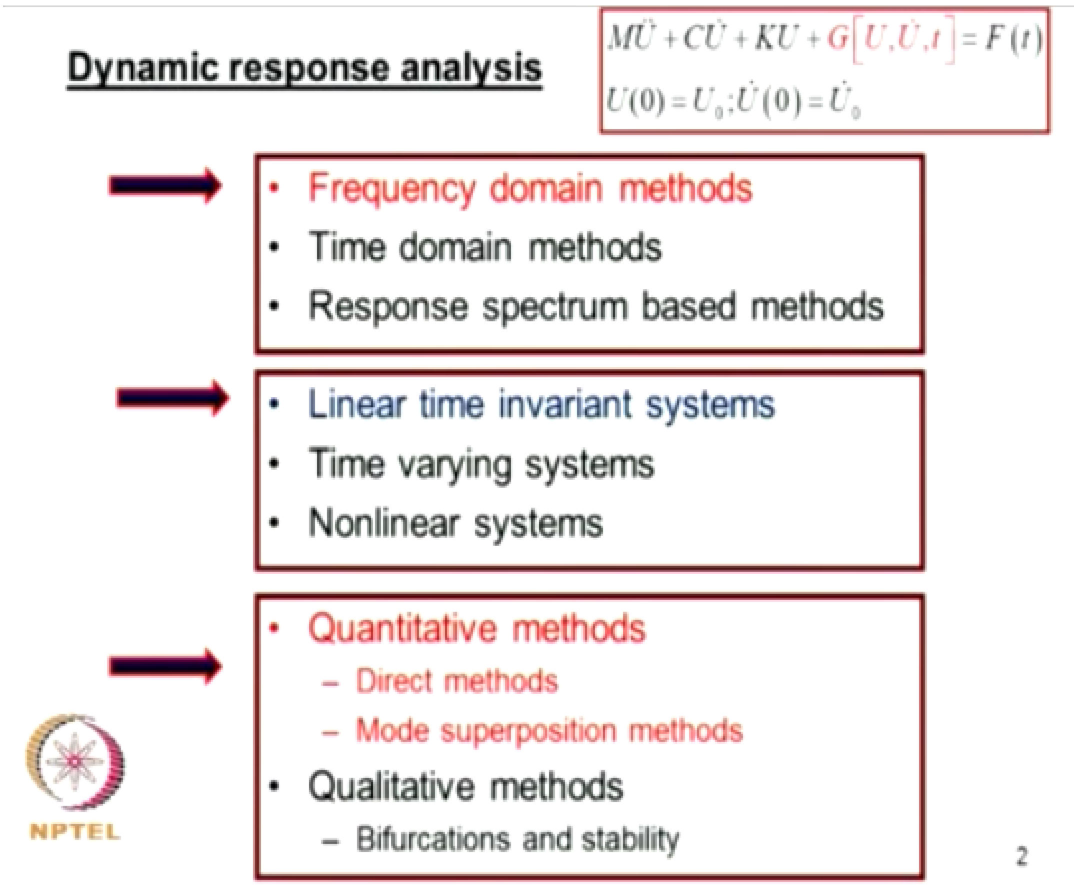
We have been discussing issues related to analysis of equations of motion, and in the previous lecture we started discussing about damping models, the methods used for analyzing equations of motion and the models that we adopt for damping are interrelated especially if one wants to

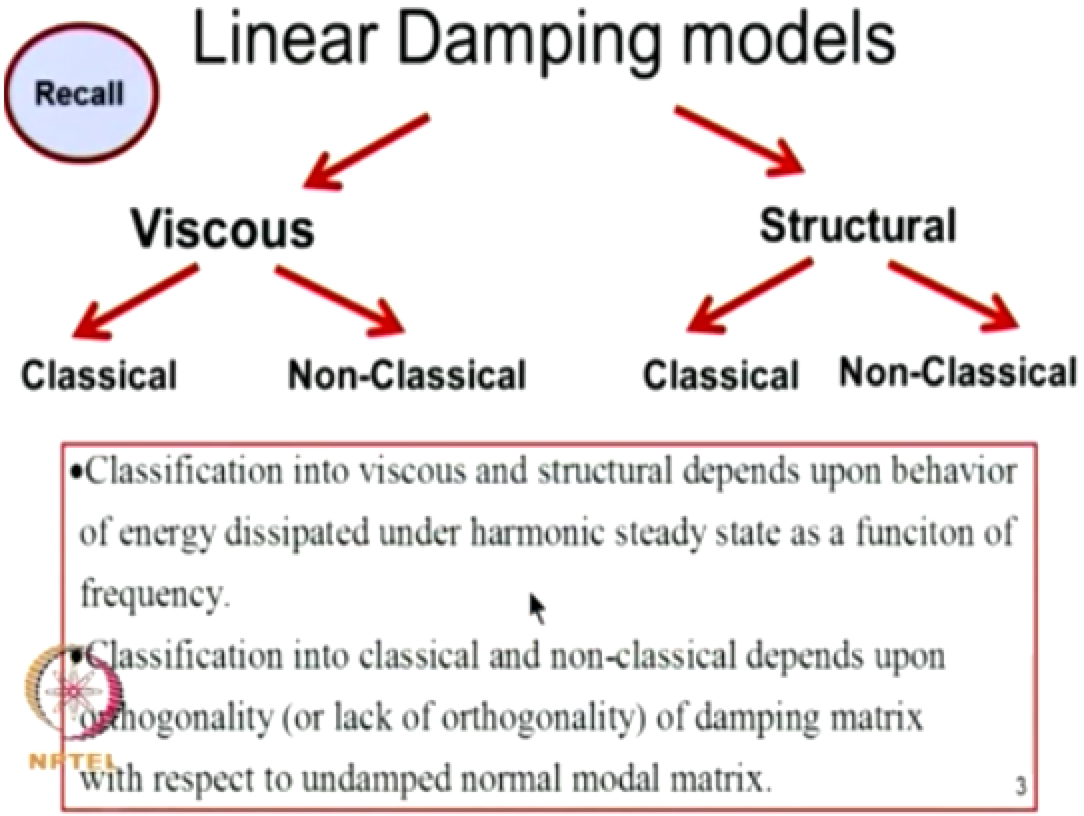


use mode superposition method to analyze the response, so we will continue with that discussion.

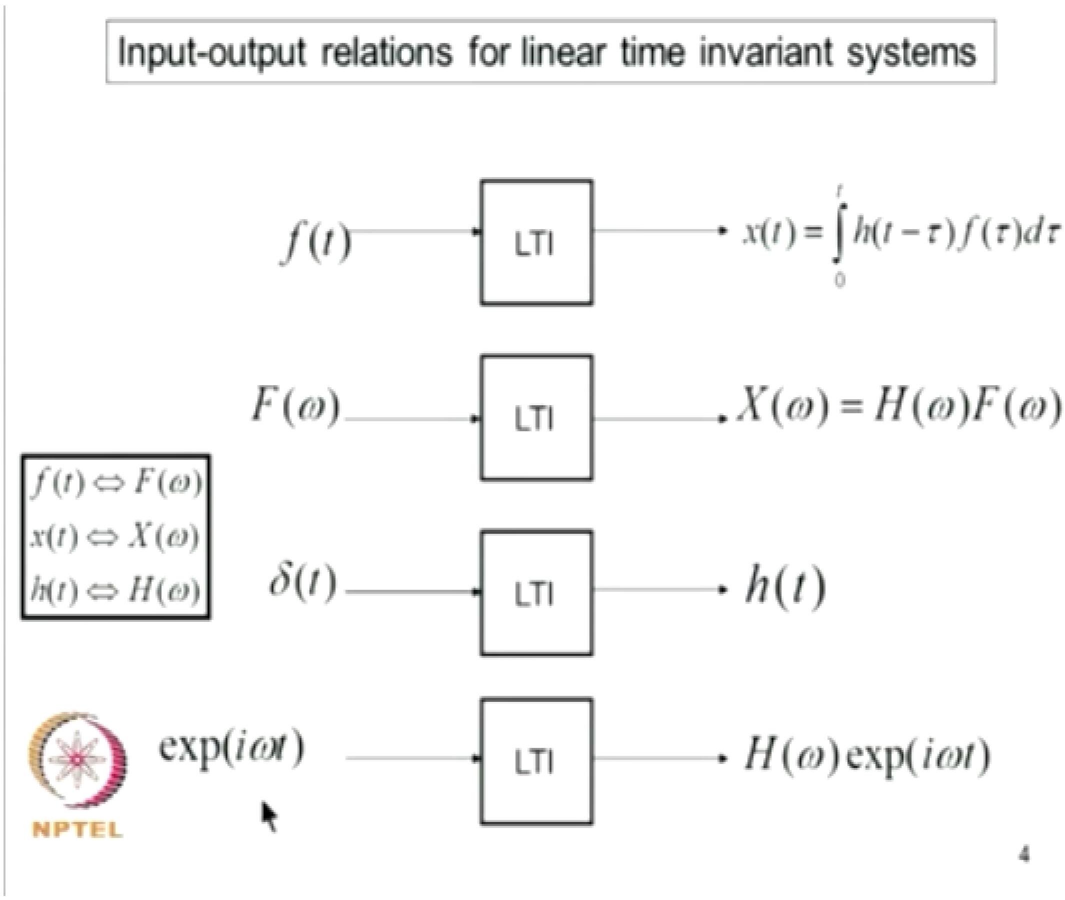


So we are looking at equations which are generally of this form, we can look at methods of analysis through three different perspective, one is whether we do frequency or time domain analysis or we use response spectrum based methods. The other perspective is on nature of the system which could be linear time invariant system or time varying systems or nonlinear systems. And another, yet another perspective is whether we are doing a quantitative analysis or qualitative analysis.

Right now we are focusing on linear time invariant systems we are trying to develop frequency domain methods which falls into the category of quantitative methods, so we are going to discuss both direct methods and mode superposition methods.



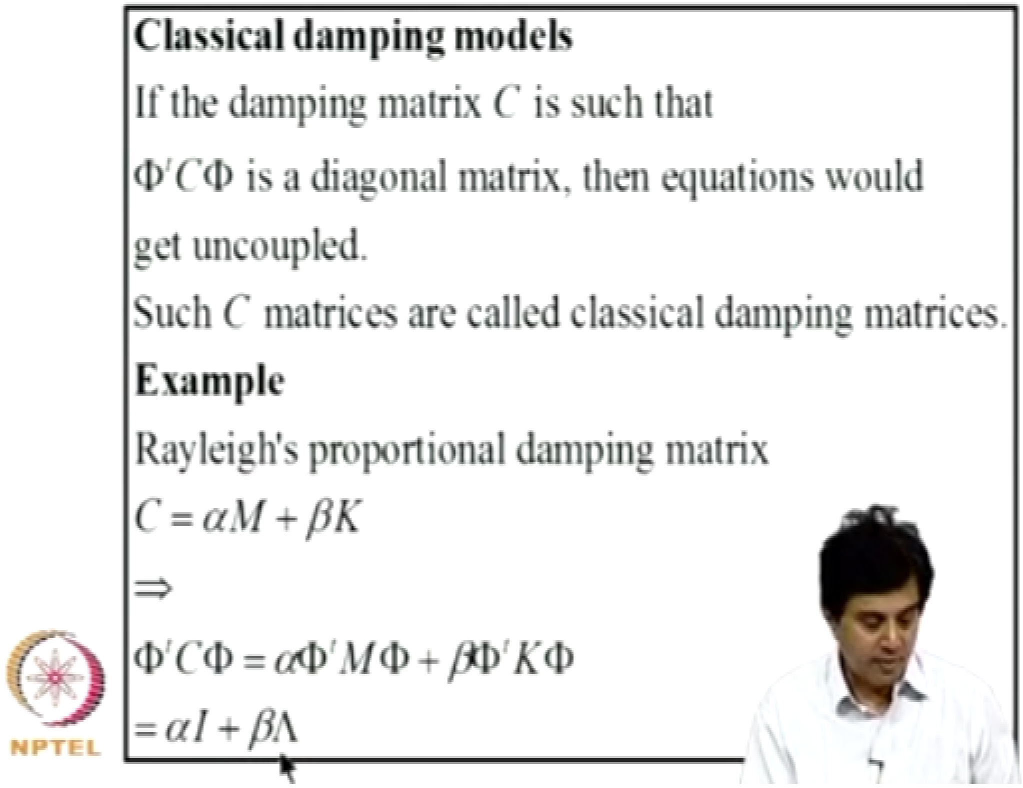
Now in our discussion on damping models we saw that there are two alternative strategies to model damping, one is risk within the framework of linear system modeling, one is viscous damping, other one is structural damping, the classification into viscous and structural depends upon behavior of energy dissipated per cycle as a function of frequency, we also classify the damping models as being classical or non-classical and that classification is on, is related to issue of whether un-damped normal modes are orthogonal to the damping matrix or not, so if damping matrix is orthogonal to the un-damped normal modes then we say that damping model is classical, otherwise it is non-classical, so we also saw that the input or generic form of input output relation for linear time invariant systems either is in time domain it is through a



convolution integral, in frequency domain it is through a frequency response function and the impulse response this H(t) is impulse response which is a response of the system to N unit impulse, and H(omega) is amplitude of the response to an external harmonic excitation.

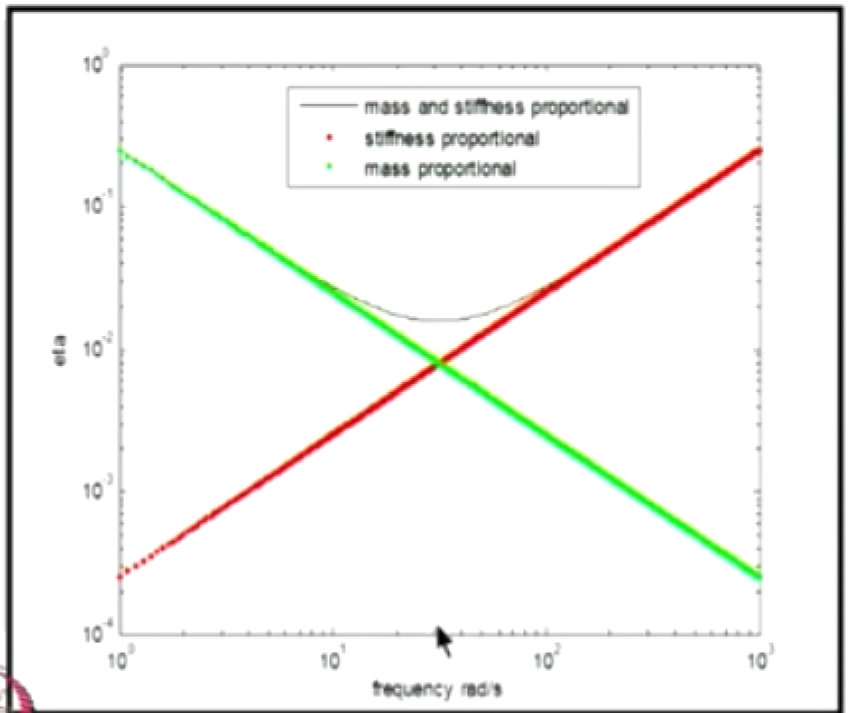
So we were discussing the questions about damped forced response analysis, so we consider an equation of this form with given initial conditions and we started with the un-damped normal modes which is MU Phi is the modal matrix and we made this transformation Phi into Z(t) Z(t) is a new coordinate system, so upon substituting that into the governing equation I get this equation, and pre multiplying this equation by Phi transpose I get this equation and here Phi transpose M Phi is the new mass matrix in the new coordinate system Z(t) and that is an identity matrix that is how we have normalized the normal modes and Phi transpose C Phi remains as it is, this is what we need to discuss now. And Phi transpose K Phi is a diagonal matrix with the eigenvalues on the diagonal F bar(t) is the generalized force which is Phi transpose F(t).

Now the question is what happens to this Phi transpose C Phi, if Phi transpose C Phi is a non-diagonal matrix then equation of motion would still remain coupled and all our exercise in finding Phi and making this transformation will not be fruitful, so what we do in classical



damping models we restrict the class of damping models that is the damping matrix C to only those matrices where Phi transpose C Phi is a diagonal matrix such damping matrices are called classical damping matrices, and in such cases the un-damped normal mode that is Phi transpose C Phi will be a diagonal matrix that means the un-damped normal mode diagonalizes C matrix also.

So an example for this we saw is so-called Rayleigh's proportional damping matrix alpha M + beta K, and if I now find Phi transpose C Phi I will get alpha into Phi transpose M phi beta into Phi transpose K Phi and this is alpha I + beta into capital Lambda.



Now using this we showed that the damping ratio as a function of frequency has typically this type of behavior it has 2 open parameters and we determine the 2 open parameter, we write

$$C = \alpha M + \beta K$$

$$\Rightarrow \Phi^T C \Phi = \Phi^T [\alpha M + \beta K] \Phi$$

$$= \alpha \Phi^T I \Phi + \beta \Phi^T K \Phi$$

$$= \alpha [I] + \beta \ \text{Diag}[\omega_i^2]$$

$$\Rightarrow c_n = \alpha + \beta \omega_n^2$$

$$c_n = 2\eta_n \omega_n \Rightarrow \eta_n = \frac{\alpha}{2\omega_n} + \frac{\beta \omega_n}{2}$$

How to find α and β? We need to know damping ratios at least for two modes. For example,

$$\left.\begin{aligned} \eta_1 &= \frac{\alpha}{2\omega_1} + \frac{\beta\omega_1}{2} \\ \eta_2 &= \frac{\alpha}{2\omega_2} + \frac{\beta\omega_2}{2} \end{aligned}\right\} \text{Knowing } \eta_1 \text{ and } \eta_2\text{, solve for } \alpha \text{ and } \beta$$

question for say for example for mode 1 and mode 2 we know the damping, we will write these 2 equations and will solve for alpha and beta, and moment I find alpha and beta the damping for all other modes are evaluated from this equation.

$$I\ddot{Z}(t)+\Phi^t C\Phi\dot{Z}(t)+\Lambda Z(t)=\bar{F}(t)$$

$$Z(0)=\Phi^t MX(0)\ \&\ \dot{Z}(0)=\Phi^t M\dot{X}(0)$$

$$\Rightarrow$$

$$\ddot{z}_r+2\eta_r\omega_r\dot{z}_r+\omega_r^2 z_r=f_r(t);r=1,2,\cdots,n$$

with $z_r(0)\ \&\ \dot{z}_r(0)$ specified.

$$\Rightarrow$$

$$z_r(t)=\exp(-\eta_r\omega_r t)\left[a_r\cos\omega_{dr}t+b_r\sin\omega_{dr}t\right]+$$

$$\int_0^t\frac{1}{\omega_{dr}}\exp\left[-\eta_r\omega_r(t-\tau)\right]f_r(\tau)d\tau$$

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Now let us assume that Phi transpose C Phi is diagonal and let's take it the analysis to each logical, and so I will get initial conditions through this relation and the equation for the family of single degree freedom system which are the generalized coordinates is given by this, and consequently I can solve this family of single degree freedom systems in terms of the impulse response function and integration constant they are in BR they can be related to ZR(0) and ZR dot (0), and if we do that I get X(t) after solving this I can go back to X(t) using this relation

$$z_r(t) = \exp(-\eta_r\omega_r t)\left[a_r \cos\omega_{dr} t + b_r \sin\omega_{dr} t\right] + \int_0^t \frac{1}{\omega_{dr}} \exp\left[-\eta_r\omega_r (t-\tau)\right] f_r(\tau)\, d\tau$$

$$X(t) = \Phi Z(t)$$

$$x_k(t) = \sum_{r=1}^{n} \Phi_{kr} z_r(t)$$

$$= \sum_{r=1}^{n} \Phi_{kr} \left\{ \exp(-\eta_r\omega_r t)\left[a_r \cos\omega_{dr} t + b_r \sin\omega_{dr} t\right] + \int_0^t \frac{1}{\omega_{dr}} \exp\left[-\eta_r\omega_r (t-\tau)\right] f_r(\tau)\, d\tau \right\}$$

$$k = 1, 2, \cdots, n$$

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because decisions have to be made in X(t) space, Z(t) is an abstract coordinate system in which engineering decisions are not possible, so we have to go back to X(t) to be able to make you know useful engineering decision, so the K-th element of X(t) if I write in summation form it will be given by this, so this K runs from 1 to N, so this completes the solution.

Remarks

- The success of $C = \alpha M + \beta K$ in making $\Phi^t C \Phi$ diagonal depends upon the two orthogonality relations: $\Phi^t M \Phi = I$ and $\Phi^t K \Phi = \Lambda$.
- The fact that the model $C = \alpha M + \beta K$ admits two open parameters, α and β, is related to the fact that we have two orthogonality relations.
- Having only two model parameters permits us to to capture damping in only two modes of oscillations. This is restrictive, since, once α and β are chosen, the variation of η w.r.t. frequency is also artifically fixed.
- How to make the model less inflexible?
  - Introduce more model parameters.
  - Do we have newer orthogonality relations?

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Now let's examine now in some detail the model C = alpha M + beta K, the success of this model say alpha M + beta K in making Phi transpose C Phi diagonal depends on two orthogonality relations Phi transpose M Phi is diagonal, and Phi transpose K Phi is diagonal. Now there are two model parameters here alpha and beta, the fact that it has two model parameters is very much related to the fact that there are two orthogonality relation that we are using, that means the fact that the model alpha M + beta K admits two open parameters that is alpha and beta is related to the fact that we have two orthogonality relations, however in a damping model having only two model parameters permits us to capture damping only in two modes of oscillations, this is somewhat restrictive since once alpha and beta are determined using two known damping ratios, the damping ratios for all other modes will get automatically fixed by the nature of the values of alpha and beta, suppose if we know damping for more than two modes we have no way of accommodating that in this model, and if the resulting damping model is the dim, not physically acceptable the way the damping ratio varies with frequency is deemed not acceptable then there is no way to improve the model within the framework of this proportional damping model, so therefore it appears that now this model for damping is somewhat inflexible, so questions that we could ask at this stage is how to make the model less inflexible or more flexible so the idea is to introduce more model parameters.

Now if I introduce more model parameters instead of alpha beta I have say alpha 1, alpha 2, alpha N, and beta 1, beta 2, beta N, how do I select them? And how do I ensure by that the choice that I make on alpha and beta, ensures that Phi transpose C Phi is still diagonal, so this leads us to the question do we have newer orthogonality relations or the two that we are talking about are the only two orthogonality relations.

The two infinite families of orthogonality relations

1st family

Consider $K\Phi = M\Phi\Lambda$ with $\Phi' M \Phi = I$ & $\Phi' K \Phi = \Lambda$

$K\Phi = M\Phi\Lambda \Rightarrow M^{-1}K\Phi = \Phi\Lambda$

$\Rightarrow \Phi' K M^{-1} K \Phi = \Phi' K \Phi \Lambda = \Lambda^2$ (a diagonal matrix)

$\Rightarrow \Phi$ is orthogonal w.r.t. $KM^{-1}K$

$M^{-1}K\Phi = \Phi\Lambda \Rightarrow$

$\Phi' K M^{-1} K M^{-1} K \Phi = \Phi' K M^{-1} K \Phi \Lambda = \Lambda^3$ (a diagonal matrix)

$\Rightarrow \Phi$ is orthogonal w.r.t. $KM^{-1}KM^{-1}K$

$\Rightarrow \Phi' K M^{-1} K M^{-1} K M^{-1} K \Phi = \Phi' K M^{-1} K M^{-1} K \Phi \Lambda = \Lambda^4$

(a diagonal matrix)

Φ is orthogonal w.r.t. $KM^{-1}KM^{-1}KM^{-1}K$

:

Now it transpires that there are actually two infinite families of orthogonality relations, so the first family we can construct by considering the eigenvalue problem K phi is M Phi lambda with Phi Transpose M Phi is I, Phi Transpose K Phi is Lambda. Now I will rewrite this as, I will pre multiply this by M Inverse and I can write it as M inverse K Phi is equal to Phi into Lambda.

Now I will pre multiply this by Phi transpose K so I will get this relation. On the right hand side you will see that I am getting Phi transpose K Phi which is a diagonal matrix, so this becomes lambda square, so lambda square is also a diagonal matrix, so what it means? It means that Phi is orthogonal with respect to KM, M inverse K matrix. Now we can continue with this argument now I will consider M inverse K Phi = Phi Lambda, and pre multiply by Phi Transpose KM inverse K, so I will again on the right hand side I get Phi Transpose KM inverse K Phi Lambda = Lambda, and this Phi transpose KM inverse K we have just now seen that it is orthogonal, Phi is orthogonal to this matrix therefore it is again lambda square, if I put lambda square here it becomes Lambda cube, so this is again a diagonal matrix, so we can continue this operation and we get an infinite family of orthogonality relation.

2^{nd} family

Consider $K\Phi = M\Phi\Lambda$ with $\Phi' M\Phi = I$ & $\Phi' K\Phi = \Lambda$

$\Rightarrow K^{-1}M\Phi = \Phi\Lambda^{-1}$

$\Phi' MK^{-1}M\Phi = \Phi' M\Phi\Lambda^{-1} = \Lambda^{-1}$ (a diagonal matrix)

$\Rightarrow \Phi$ is orthogonal w.r.t. $MK^{-1}M$

$K^{-1}M\Phi = \Phi\Lambda^{-1}$

$\Rightarrow \Phi' MK^{-1}MK^{-1}M\Phi = \Phi' MK^{-1}M\Phi\Lambda^{-1} = \Lambda^{-2}$ (a diagonal matrix)

$\Rightarrow \Phi$ is orthogonal w.r.t. $MK^{-1}MK^{-1}M$

$\Rightarrow \Phi' MK^{-1}MK^{-1}MK^{-1}M\Phi = \Phi' MK^{-1}MK^{-1}M\Phi\Lambda^{-1} = \Lambda^{-3}$

(a diagonal matrix)

$\Rightarrow \Phi$ is orthogonal w.r.t. $MK^{-1}MK^{-1}MK^{-1}M$

There is another family to obtain that what we do is we again consider K Phi = M Phi lambda and now pre multiplied by K inverse, so I get KM inverse M Phi and also I will take lambda to the other side call it as lambda inverse, now I will now pre multiply by Phi transpose M, so I get Phi transpose MK inverse M Phi is Phi transpose M Phi, Phi transpose M Phi we know is identity matrix, so this becomes Lambda – 1, now that means Phi is orthogonal to MK inverse M, now I will pre multiply by now what I am showing in the red here, Phi transpose MK inverse M and use the fact that this orthogonality relation is valid, I will get this as lambda to the power of – 2, so if I repeat this I get another family, infinite family of orthogonality relation.

Generalized classical damping model

$$C = \alpha_1 M + \alpha_2 MK^{-1}M + \alpha_3 MK^{-1}MK^{-1}M + \alpha_4 MK^{-1}MK^{-1}MK^{-1}M + \cdots +$$
$$\beta_1 K + \beta_2 KM^{-1}K + \beta_3 KM^{-1}KM^{-1}K + \beta_4 KM^{-1}KM^{-1}KM^{-1}K + \cdots$$

$$\Rightarrow \Phi^t C \Phi = \alpha_1 \Lambda^0 + \alpha_2 \Lambda^{-1} + \alpha_3 \Lambda^{-2} + \alpha_4 \Lambda^{-3} + \cdots$$
$$+ \beta_1 \Lambda + \beta_2 \Lambda^2 + \beta_3 \Lambda^3 + \beta_4 \Lambda^4 + \cdots$$

$\Rightarrow \Phi^t C \Phi$ is diagonal.

$\Rightarrow \Phi$ is orthogonal to C.

$$2\eta_n \omega_n = \alpha_1 + \frac{\alpha_2}{\omega_n^2} + \frac{\alpha_2}{\omega_n^4} + \cdots + \beta_1 \omega_n^2 + \beta_2 \omega_n^4 + \cdots$$

$$\eta_n = \frac{\alpha_1}{2\omega_n} + \frac{\alpha_2}{2\omega_n^3} + \frac{\alpha_2}{2\omega_n^5} + \cdots + \frac{\beta_1 \omega_n}{2} + \frac{\beta_2 \omega_n^3}{2} + \cdots$$

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So this would mean now that I can develop now a generalized classical damping model beyond alpha M plus beta K by using the additional orthogonality relation that we have developed, so I can write to C as linear superposition of various combinations of K and M matrices as shown here. So if we use this expansion we can show that Phi transpose C Phi remains diagonal and it has this form.

Now that means Phi is orthogonal to C, if C is of this form. Now so I have now as many model parameters as I deem desirable, so if I now you know use the orthogonality relations and write the diagonal entry in the Nth row as 2 ETA N Omega N, I can write the right hand side in this form, so this is the way ETA N varies with Omega N, so earlier we had only this term and this term now we have more newly introduced terms which will lend flexibility to my modeling of damping, Generalized classical damping model can also be written as

$$C = M\sum_{n=-\infty}^{\infty} a_n\left[M^{-1}K\right]^n$$

Example

$$C = M\sum_{n=-2}^{2} a_n\left[M^{-1}K\right]^n$$

$$= M\left(a_{-2}K^{-1}MK^{-1}M + a_{-1}K^{-1}M + a_0 + a_1M^{-1}K + a_2M^{-1}KM^{-1}K\right)$$

$$= a_{-2}MK^{-1}MK^{-1}M + a_{-1}MK^{-1}M + a_0M + a_1K + a_2KM^{-1}K$$

Remarks

- Clearly

$$\Phi^t C\Phi = a_{-2}\Phi^t MK^{-1}MK^{-1}M\Phi + a_{-1}\Phi^t MK^{-1}M\Phi + a_0\Phi^t M\Phi + a_1\Phi^t K\Phi + a_2\Phi^t KM^{-1}K\Phi$$

is diagonal.

- With $a_{-2}=0, a_{-1}=0, a_2=0,$ we get $C = a_0M + a_1K,$ which is the Rayleigh's proportional damping model.

so I can in fact write the 2 infinite family of orthogonality relations in a single equation and I can write C in this form M into summation N minus infinity + infinity, and AN into M inverse K to the power of N. For example if I now take 5 terms in this, from N = - 2 to + two if I expand this out I will get this, and if I now write Phi transpose C Phi I will get this equation and we know that all these matrices are diagonal matrices by virtue of the orthogonality relation that we have and consequently Phi transpose C Phi is also diagonal.

If we now take in this summation only from 0 to 1 this summation N = 0, and N = 1, we will get the familiar Rayleigh damping model which is A naught M plus A1 K, so this model embeds within itself the Rayleigh damping model, but it is more general.

$$\eta_n = \frac{\alpha_1}{2\omega_n} + \frac{\alpha_2}{2\omega_n^3} + \frac{\alpha_2}{2\omega_n^5} + \cdots + \frac{\beta_1\omega_n}{2} + \frac{\beta_2\omega_n^3}{2} + \cdots$$

Remarks

- We can introduce as many model parameters as the number of modes for which we have damping ratios known.
- Damping ratios for modes can be arbitrarily fixed (for example, the damping ratio can be the same for all the particpating modes) with an understanding that we can arrive at a classical damping model based on the above model.
- Notice that $\Phi^t C \Phi = [2\eta_n\omega_n] \Rightarrow C = [\Phi^t]^{-1} [2\eta_n\omega_n] \Phi^{-1}$

This approach would not be practicable since we seldom determine Φ for all possible modes of the system.

- Caution: for those modes for which damping ratios are not explicitly specified, but are infered from model for C, care must be taken to ensure that that ratios are physically meaningful.

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So the damping ratio as a function of frequency has now this form, so we can make few observations on this. Now we can introduce as many model parameters as a number of modes for which we have damping ratios known, then damping ratios for modes can be arbitrarily fixed, for example the damping ratio can be the same for all participating modes typically we take say damping is 5% for all modes or say analyzing RCC structure under an earthquake, right, so that is not inconsistent with the expectation that C matrix is classical, we can arrive at a valid classical damping model based on that assumption.

Now of course I can find Phi transpose C Phi if I write it as a diagonal matrix of 2 ETA and Omega N, from this I can evaluate Phi as shown here, but this approach is not going to be very helpful because we seldom actually find the full modal matrix in a practical analysis, suppose if you have a thousand degree of freedom model we would not really evaluate all thousand normal modes, we will evaluate perhaps about 50 or 100 normal modes and the Phi matrix will often be rectangular, it's not square matrix in calculation, so we cannot use this approach to find C.

Now a word of caution if you are using the models of this kind, for those modes for which damping ratios are not explicitly specified and you bank on this relation to reduce them, you should be careful that the value that you get for ETA are physically admissible, they should not become negative and if they become more than 1, you should be sure that you want to accept that, ETA more than 1 mean system is over damped and we should really consider whether that is admissible in a given problem.

FRF nomenclature and modal representations

SDOF SYSTEMS

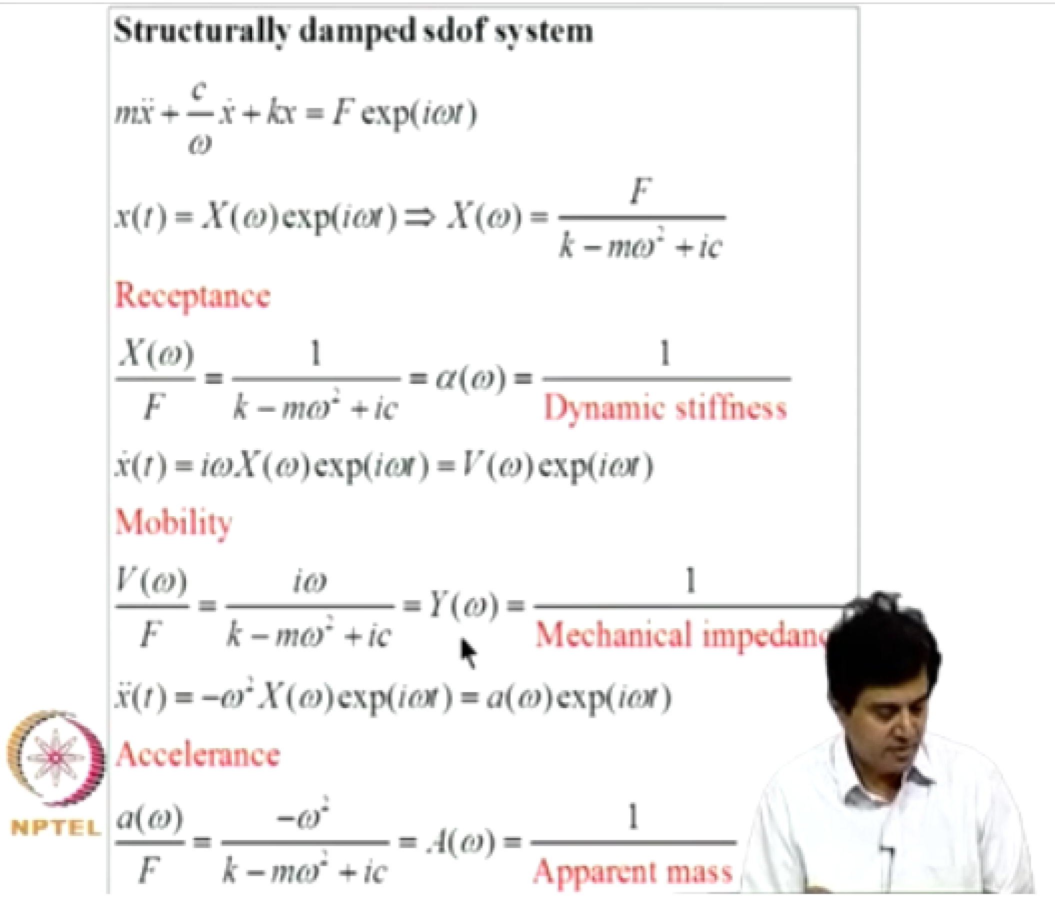
- Receptance, mobility, and accelerance
- Dynamic stiffness, Mechanical impedance, apparent mass
- FRFs for viscously and structurally damped systems
- Asymptotes

MDOF SYSTEMS

- Modal representation of FRFs for proportional and non-proportional damping models
- Direct and cross FRFs
- Resonance and anti resonance

NPTEL

Now we will now move to the next topic in our discussion that is calculation of frequency response functions for multi degree freedom systems, so before we get into the details we will introduce a few nomenclature that is found in describing frequency response function of linear time-invariant systems, so we will start with single degree freedom systems and then move to multi degree freedom system there are various terms like receptance, mobility, accelerance, dynamic stiffness, mechanical impedance, apparent mass etc which are used, so it is useful to have a clear definition for all these terms in one place, so that if you are reading any special papers or books you would not feel uncomfortable if you come across these terms,



so let us start with that, so let us start with a viscously damped single degree freedom system which is driven harmonically FE raise to I Omega T, I am discussing only steady-state response here, so in steady state X(t) will, since system is linear and it is being driven harmonically the response in steady state would also be harmonic at the driving frequency with an amplitude given by this quantity, F divided by this.

Now I define a quantity known as receptance, which is the displacement per unit force and that ratio is known as receptance and I denoted by alpha(omega), we also introduced a term known as dynamic stiffness which is reciprocal of the receptance, so dynamic stiffness is K - M omega square plus I Omega C, so this nomenclature is fairly clear if there is no dynamics that means Omega = 0, stiffness is K which is the static stiffness. Now this stiffness, dynamic stiffness includes inertial effects, dissipation effects, and the driving frequency, that's why the word dynamic stiffness.

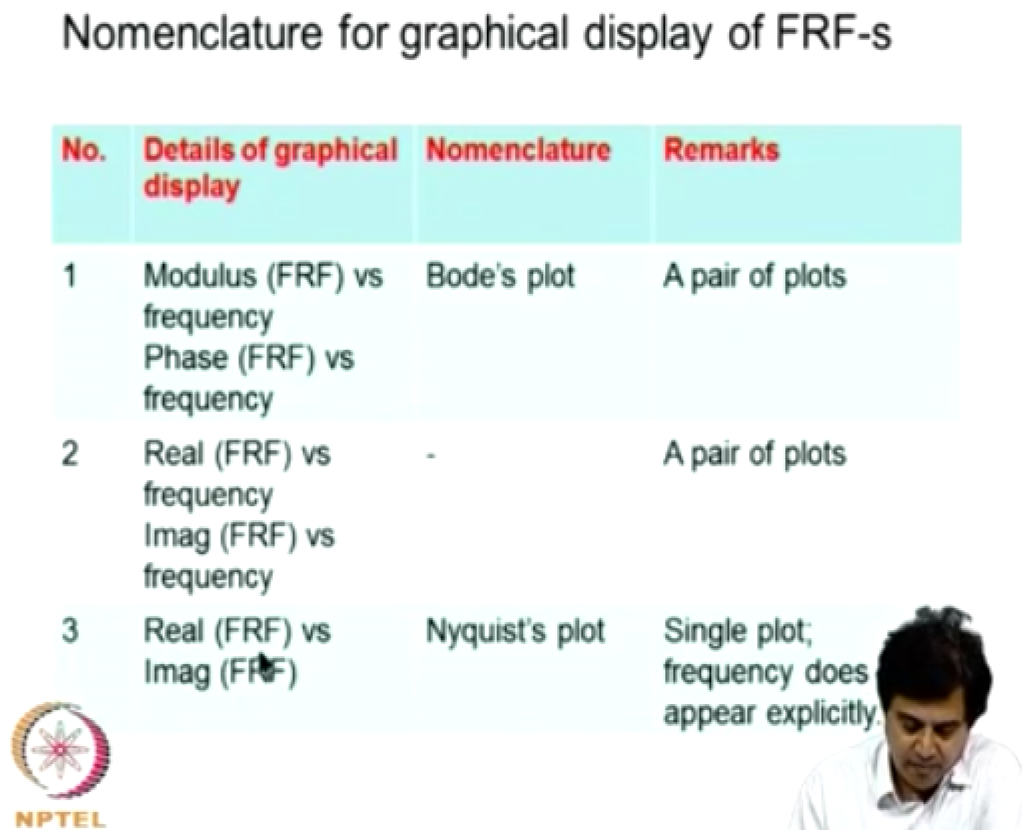
Now having derived the displacement I can get the velocity, so I differentiate this with respect to T, I get X dot as I Omega X(Omega), I call I Omega X(Omega) is V(Omega) which is a Fourier transform and amplitude of velocity, and they define velocity per unit force as a quantity known as mobility, and that is given by this and that is denoted by capital Y(Omega), the reciprocal of mobility that is force per unit velocity is known as mechanical impedance, okay so that is given by 1 / Y(Omega), similarly I can now derive the acceleration I get this expression and A(Omega) is denotes - Omega square, X(Omega) and the acceleration per unit force I call it as accelerance, and the reciprocal of that is known as apparent mass, so these are the terminology that is often used.

Similarly for structurally damped systems the damping matrix, the damping term will be C / Omega X dot + KX, so either we can use a complex stiffness or write expression in this form, so we can quickly reduce the expression for receptance which is given here, mobility which is given here, and excellence which is given here, and that automatically defined dynamic stiffness, mechanical impedance, and apparent mass, so derivation of this is straightforward, if you have followed the derivation for the viscously damped system.

Nomenclature for FRF

Response Quantity (R)	$\frac{R}{F}$	$\frac{F}{R}$
Displacement	Receptance Admittance Dynamic compliance Dynamic flexibility	Dynamic stiffness
Velocity	Mobility	Mechanical impedance
Acceleration	Accelerance	Apparent mass

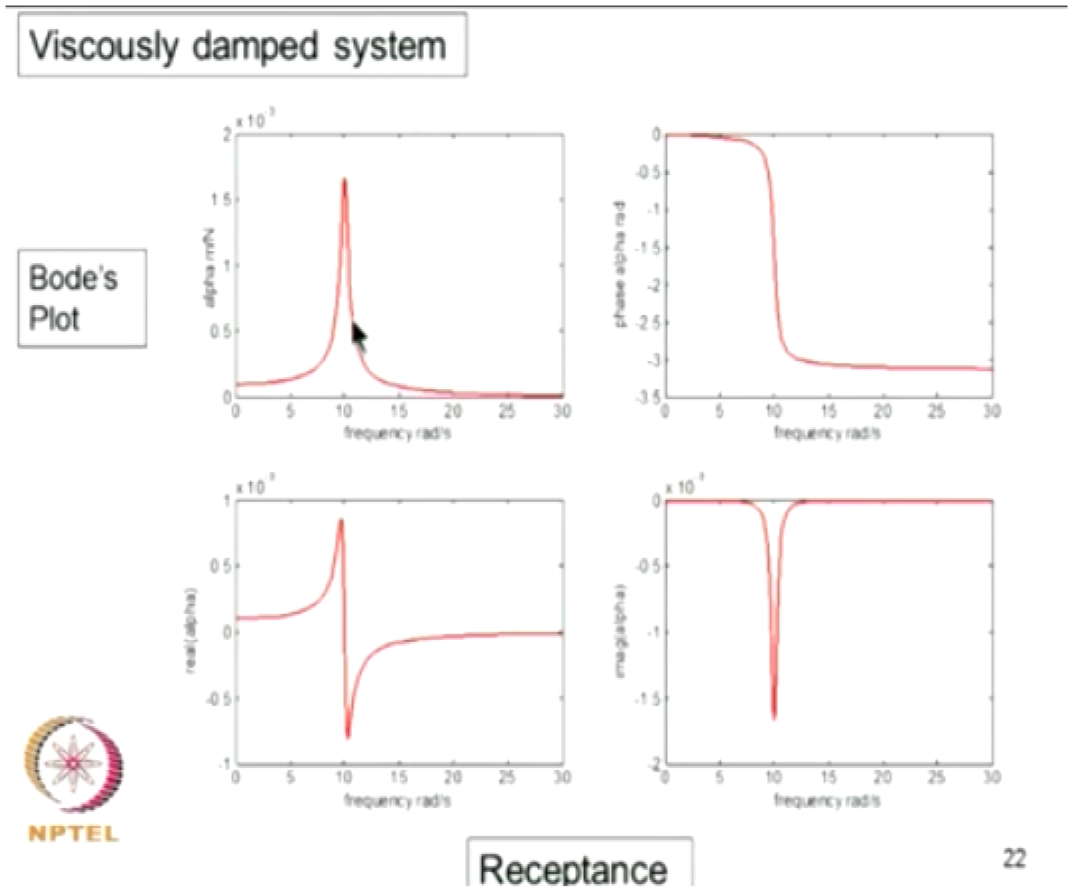
Now in the existing literature the word receptance is also sometimes, receptance is also described by admittance, dynamic compliance or dynamic flexibility that means these four terms receptance, admittance, dynamic compliance, and dynamic flexibility are all synonymous. So I have tabulated here all the terminologies this is a response per unit force, this is force per unit response, so these are the terminologies, so this is displacement, velocity and acceleration.



Nomenclature for graphical display of FRF-s

No.	Details of graphical display	Nomenclature	Remarks
1	Modulus (FRF) vs frequency Phase (FRF) vs frequency	Bode's plot	A pair of plots
2	Real (FRF) vs frequency Imag (FRF) vs frequency	-	A pair of plots
3	Real (FRF) vs Imag (FRF)	Nyquist's plot	Single plot; frequency does appear explicitly

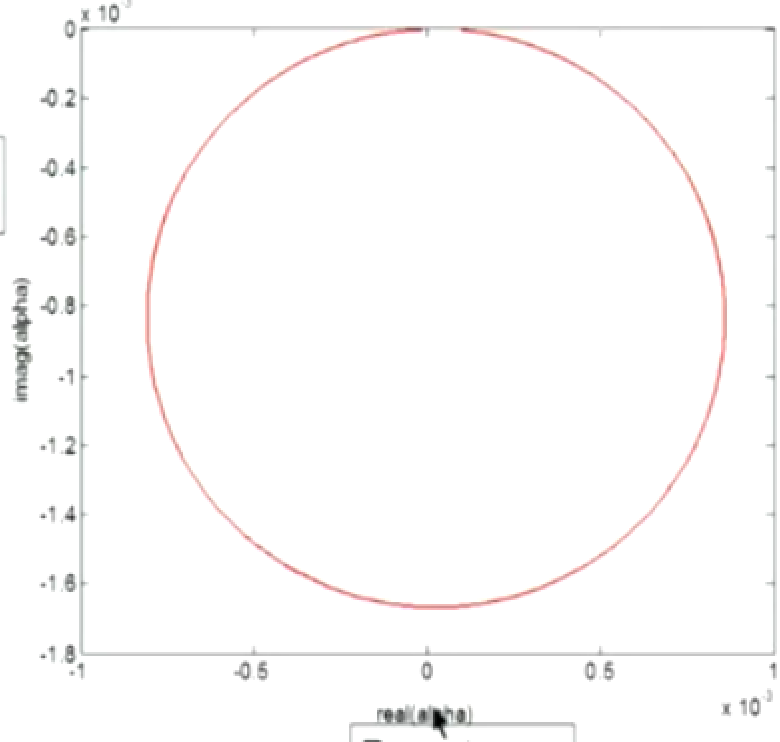
Now there are different practices that we follow in displaying frequency response function, right, so if you plot now, frequency response function collectively denotes either receptance, mobility, or accelerance, so the modulus of frequency response function versus the frequency and the phase of a frequency response function which is frequency is known as Bode's plot, so Bode's plot actually is a pair of plots where we display amplitude and phase spectrum, a word spectrum is used to denote any plot where our X axis we have a frequency. Now we can also plot the real part and imaginary part versus frequency, there is no specific name for this, on the other hand we can plot real part of the frequency response function versus the imaginary part of the frequency response function, this is known as a Nyquist plot. This is a single plot where frequency does not appear explicitly, so we can see how these graphs look like for simple systems, for example for a viscously damped system the Bode's plot that is amplitude and



phase of receptance look like this, and this is a real part and imaginary part, so we will soon discuss what is the nature of this variation. The phase is changing here, the amplitude is growing here so on and so forth, so we need to discuss why they are happening and where they happen and what these amplitude means etc. The Nyquist plot where I plot real part of receptance versus imaginary part looks like a circle as shown here.

Viscously damped system

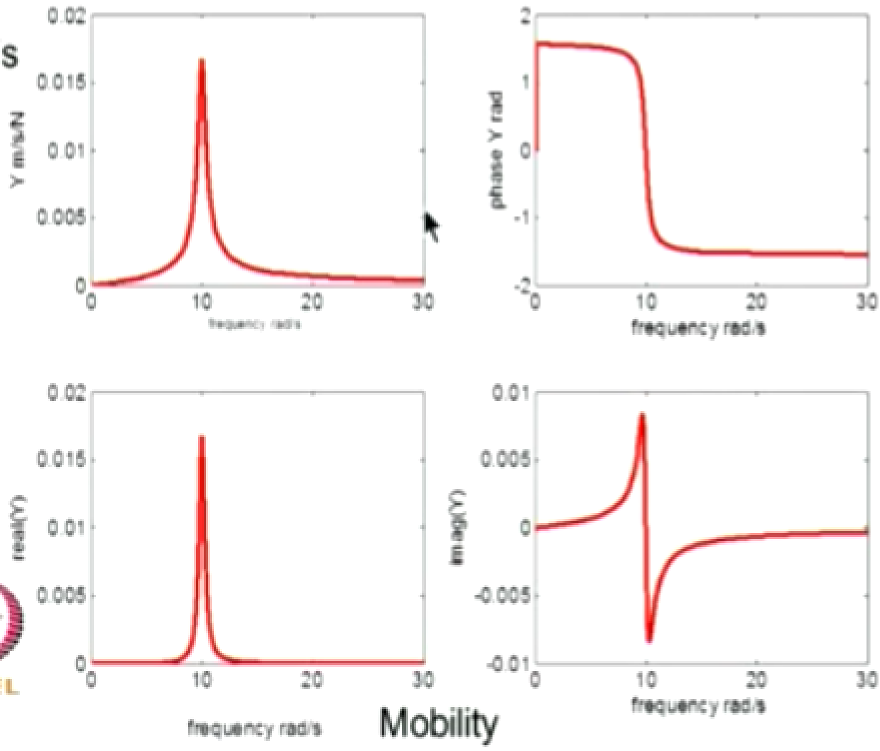
Nyquist's Plot



Receptance

Viscously damped system

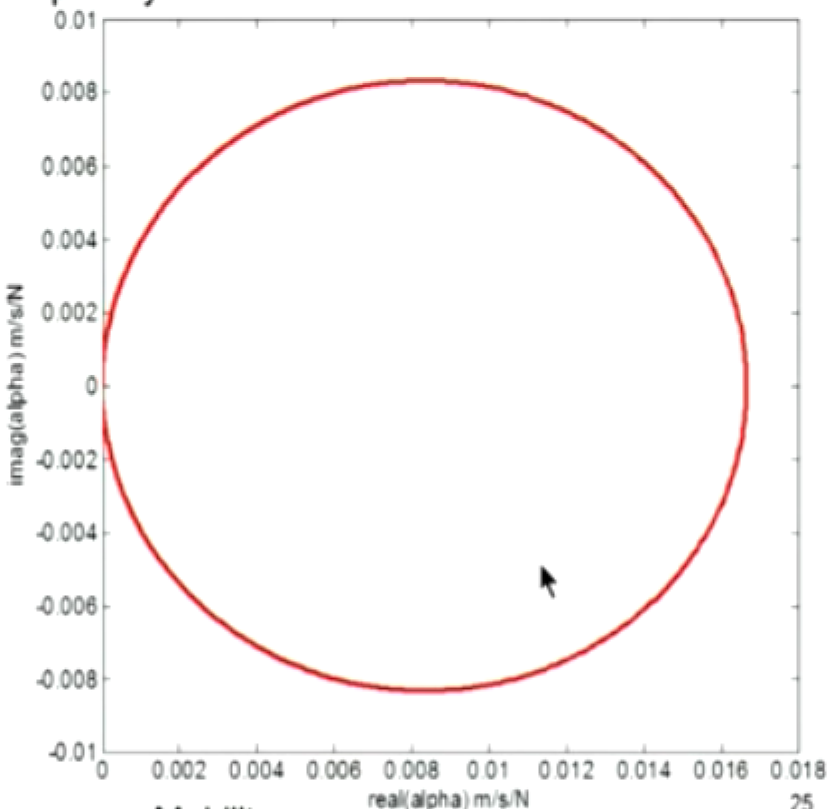
Bode's plot



Mobility

Viscously damped system

Nyquist's plot



Mobility

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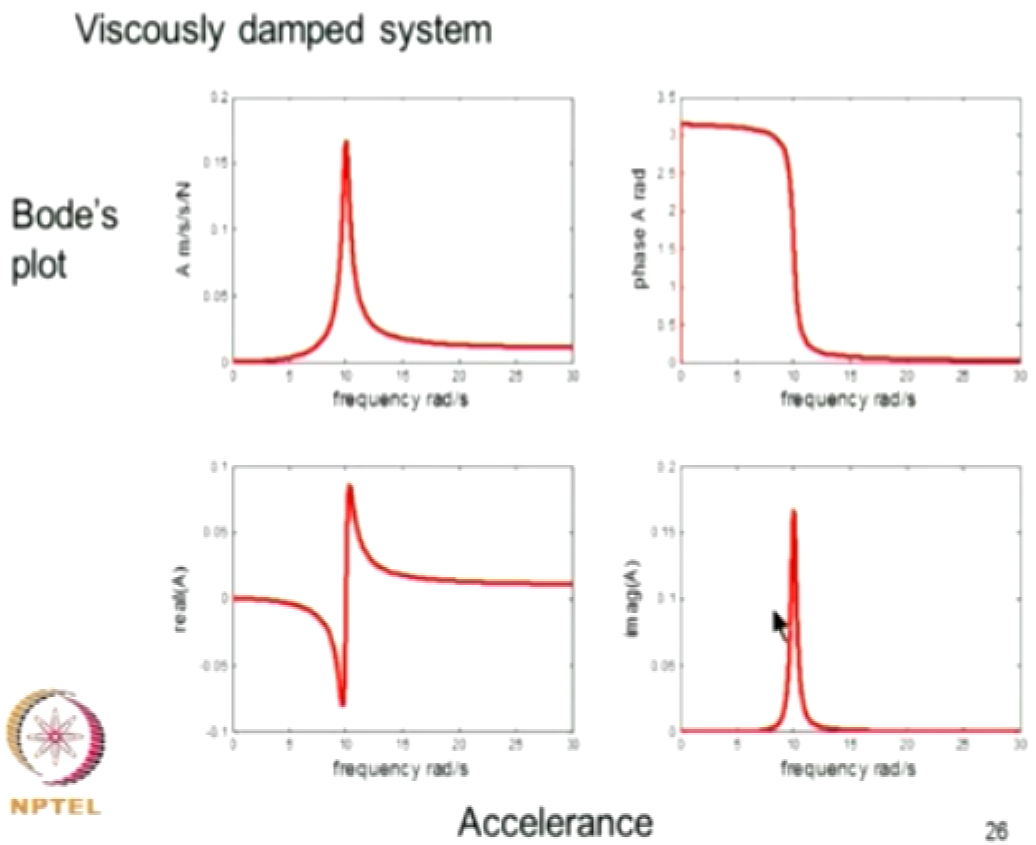
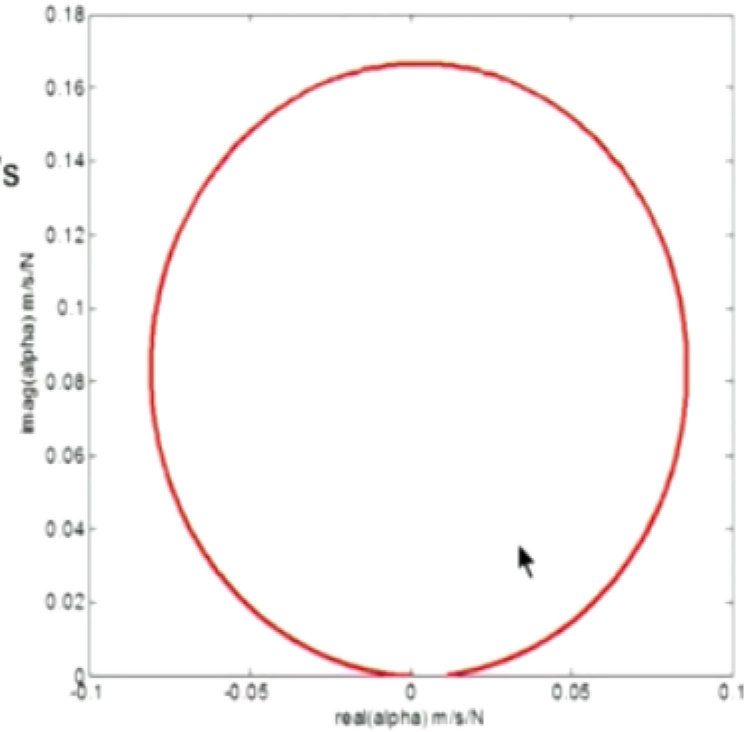
Similarly if you plot now mobility we get these graphs, Nyquist plot for mobility, Bode's plot for accelerance, this is real and imaginary part of accelerance versus frequency, this is Nyquist



plot for excellence, this is for viscously damped system, we can look at structurally damped

Viscously damped system

Nyquist's plot



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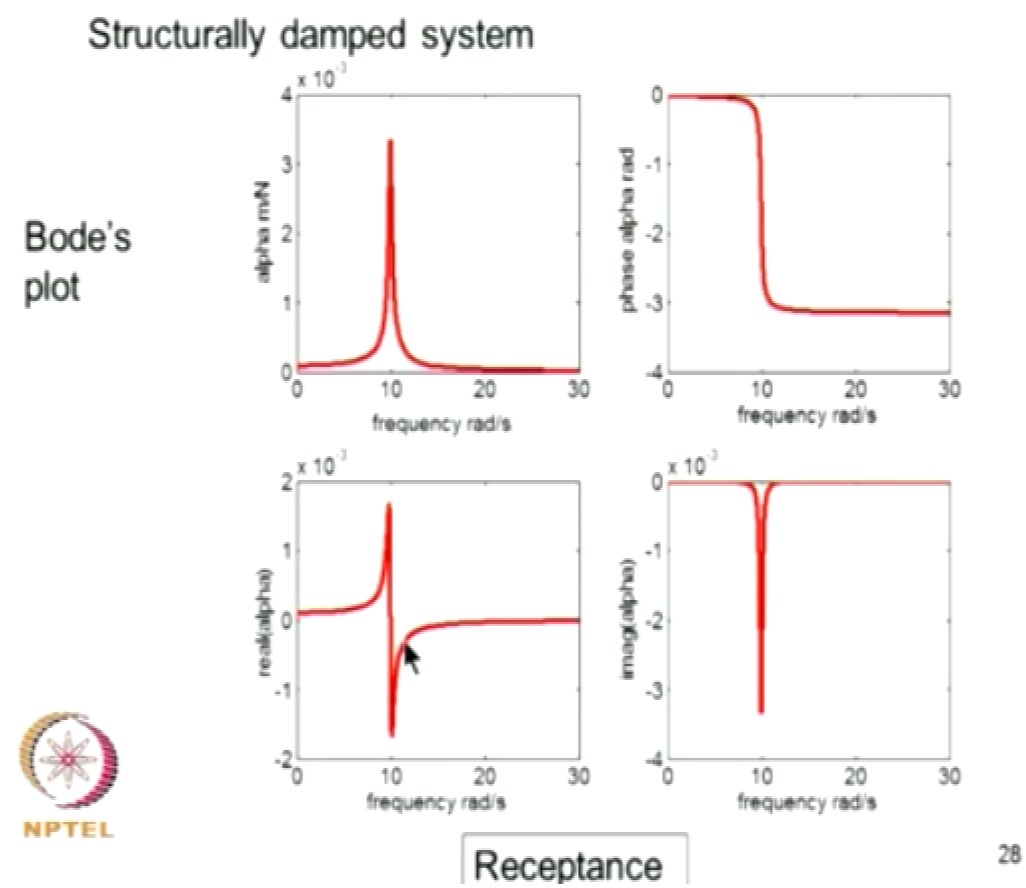
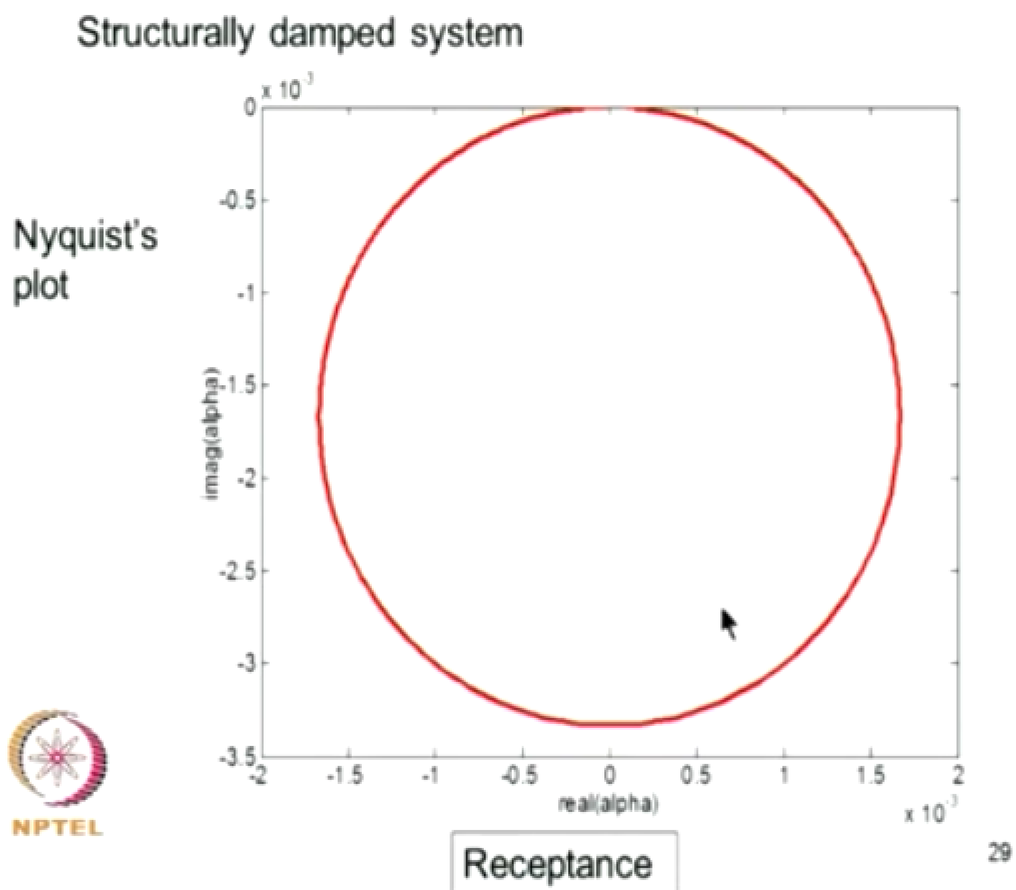
Accelerance



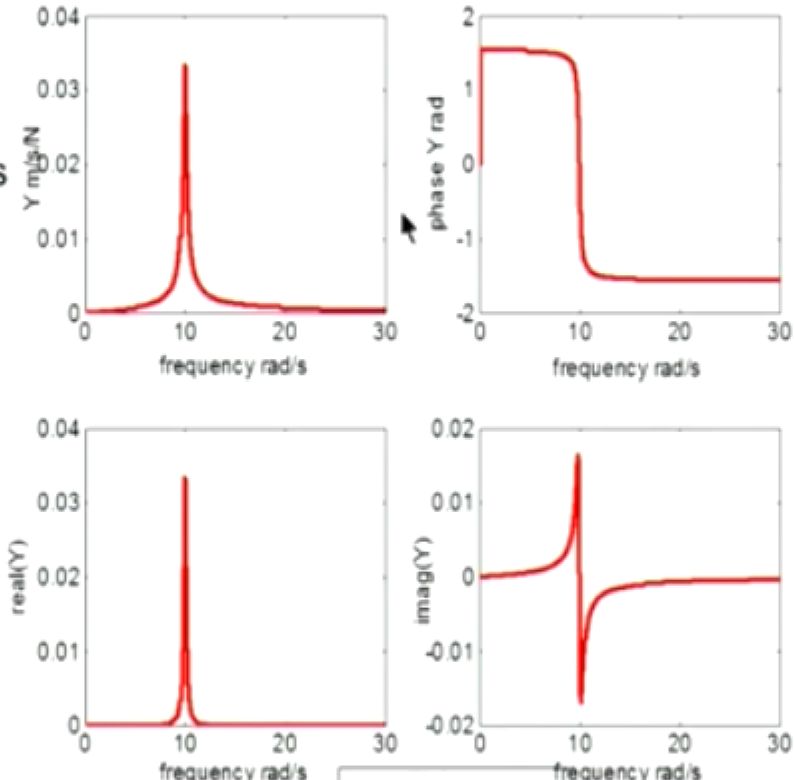
system, we have again Bode’s plot for receptance, the real and imaginary parts of receptance,



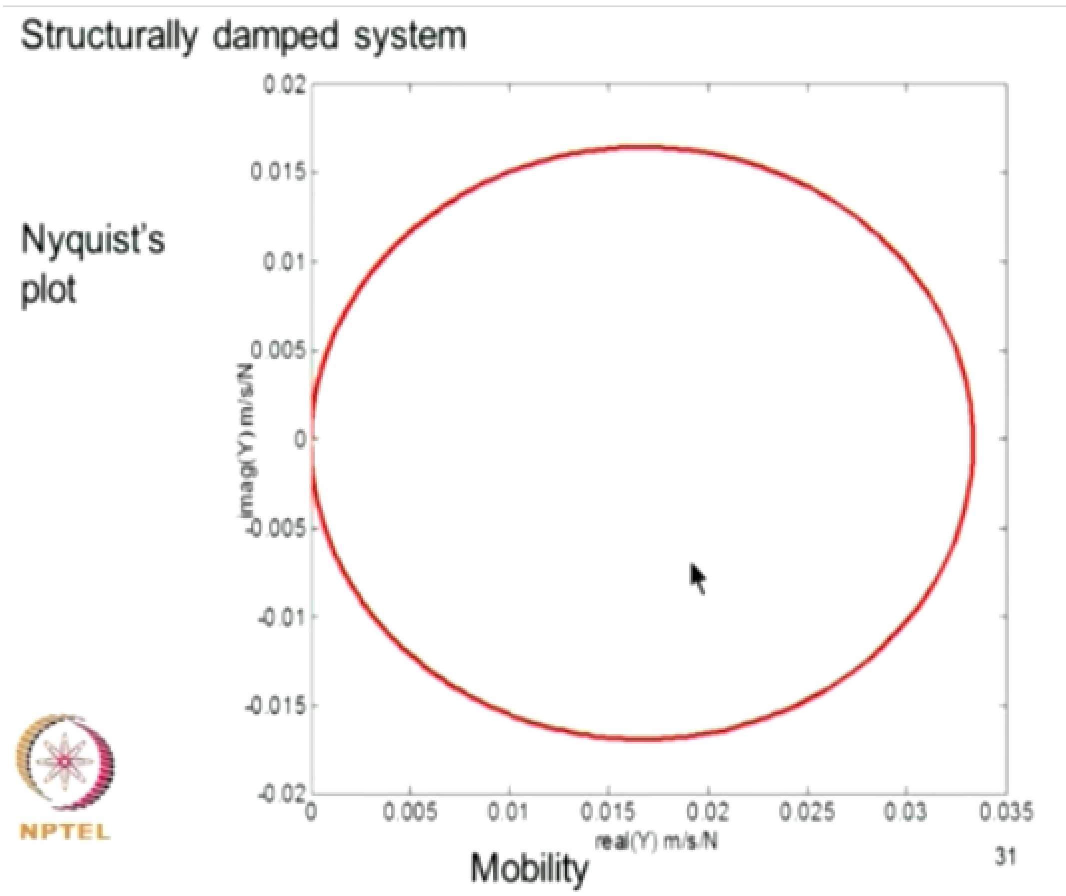
this is a Nyquist plot for receptance, this is a Bode's for mobility,

Structurally damped system

Bode's plot



Mobility



real and imaginary part of the mobility and so on and so forth, this is a Nyquist plot, so we can see that they have some characteristic features which we need to understand.

Asymptotic properties of FRF - s for viscously damped system

$$\alpha(\omega) = \frac{1}{-m\omega^2 + ic\omega + k} \quad \lim_{\omega\to\infty} \alpha(\omega) \to \frac{1}{-m\omega^2} \quad \lim_{\omega\to 0} \alpha(\omega) \to \frac{1}{k}$$

$$Y(\omega) = \frac{i\omega}{-m\omega^2 + ic\omega + k} \quad \lim_{\omega\to\infty} Y(\omega) \to \frac{i}{-m\omega} \quad \lim_{\omega\to 0} Y(\omega) \to \frac{i\omega}{k}$$

$$A(\omega) = \frac{-\omega^2}{-m\omega^2 + ic\omega + k} \quad \lim_{\omega\to\infty} A(\omega) \to \frac{1}{m} \quad \lim_{\omega\to 0} A(\omega) \to \frac{-\omega^2}{k}$$

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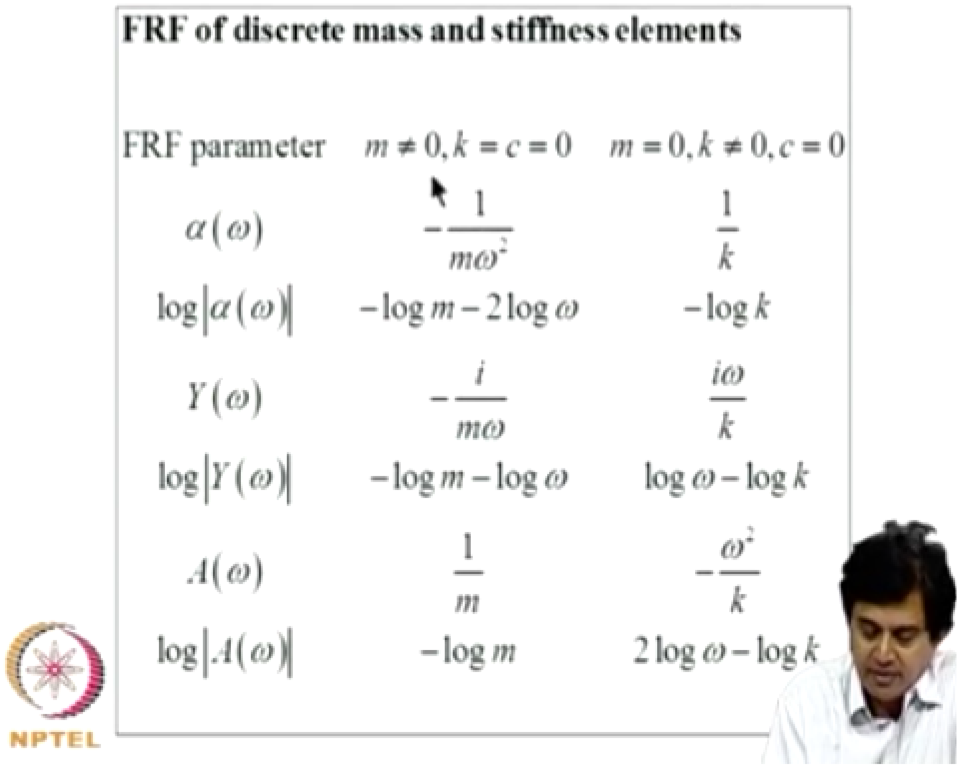
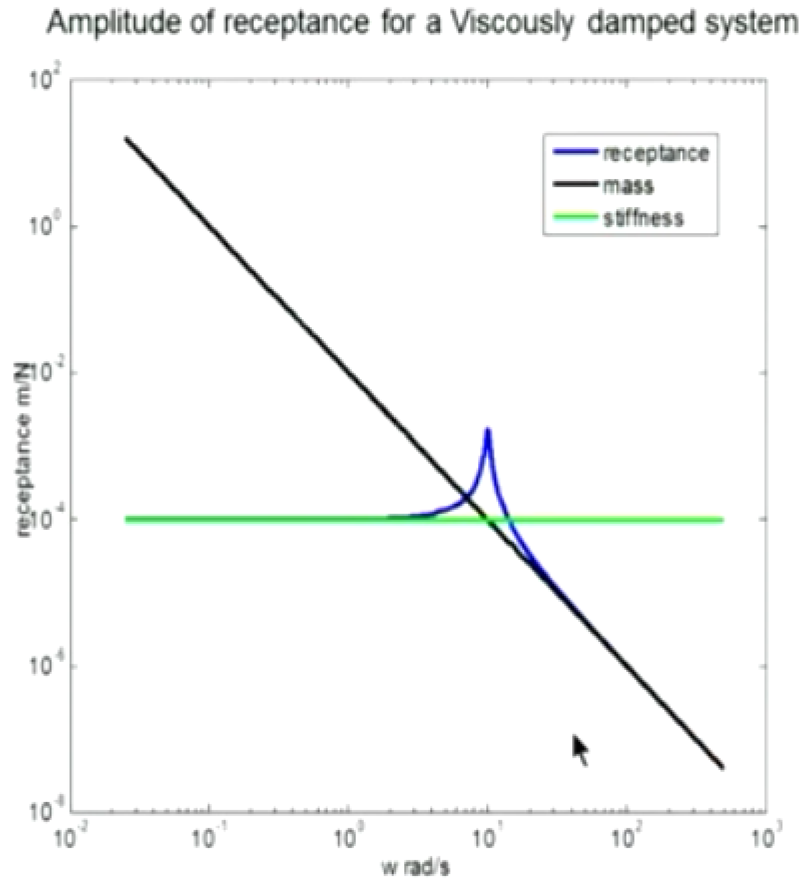
So now to understand this what we can do is we can consider the limiting behavior of these functions as omega goes to 0, and omega goes to infinity, for example let us focus on viscously damped system, suppose this is the expression for the receptance, so as omega becomes very large you can see that in the denominator the first term would dominate and I get this as 1 by - M omega square, as omega goes to 0, the first 2 terms will vanish and K will dominate, this will become 1 / K. So similarly mobility as omega tends to infinity will be I / - M Omega, and this is mobility, this will be I omega / K, and similarly accelerance has this limiting behavior.

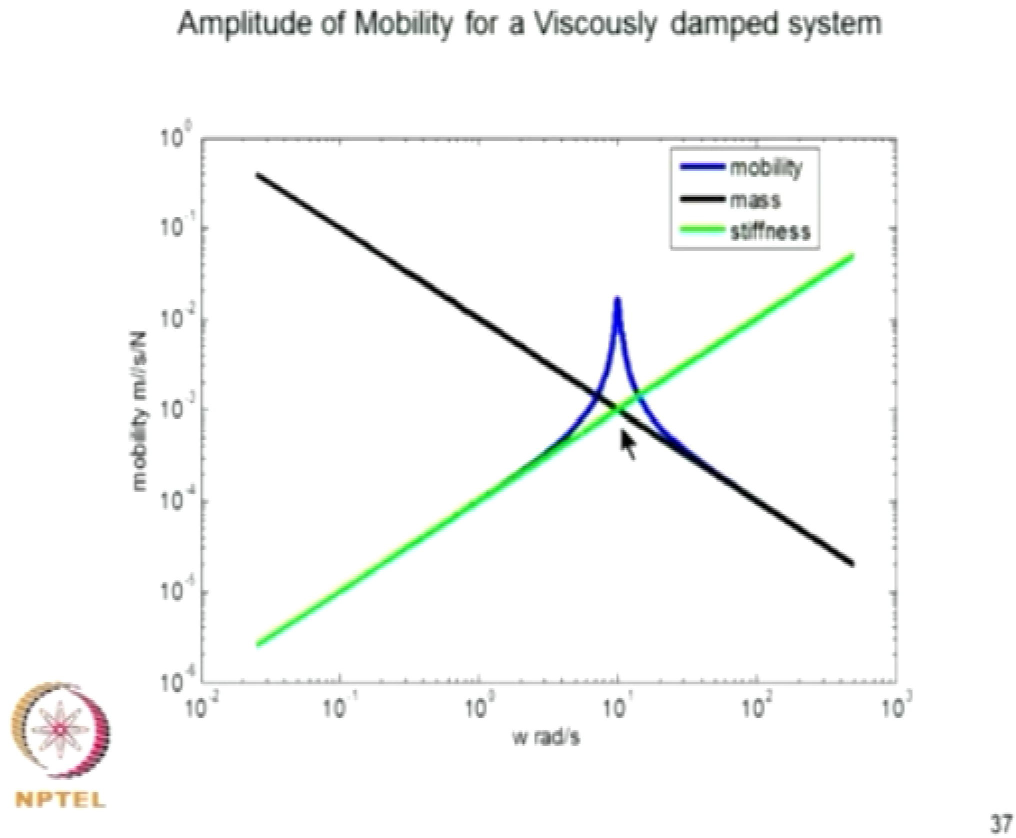


Now we can now consider if we look at receptance for a mass element that means I will consider a mass spring dashpot system and put K and C to be 0, so the receptance will be - 1 / M omega square, and if you take now only stiffness it is 1 / K, so if you now take logarithm of the amplitude I get this value, similarly I can get derived these similar quantities for mobility and accelerance.

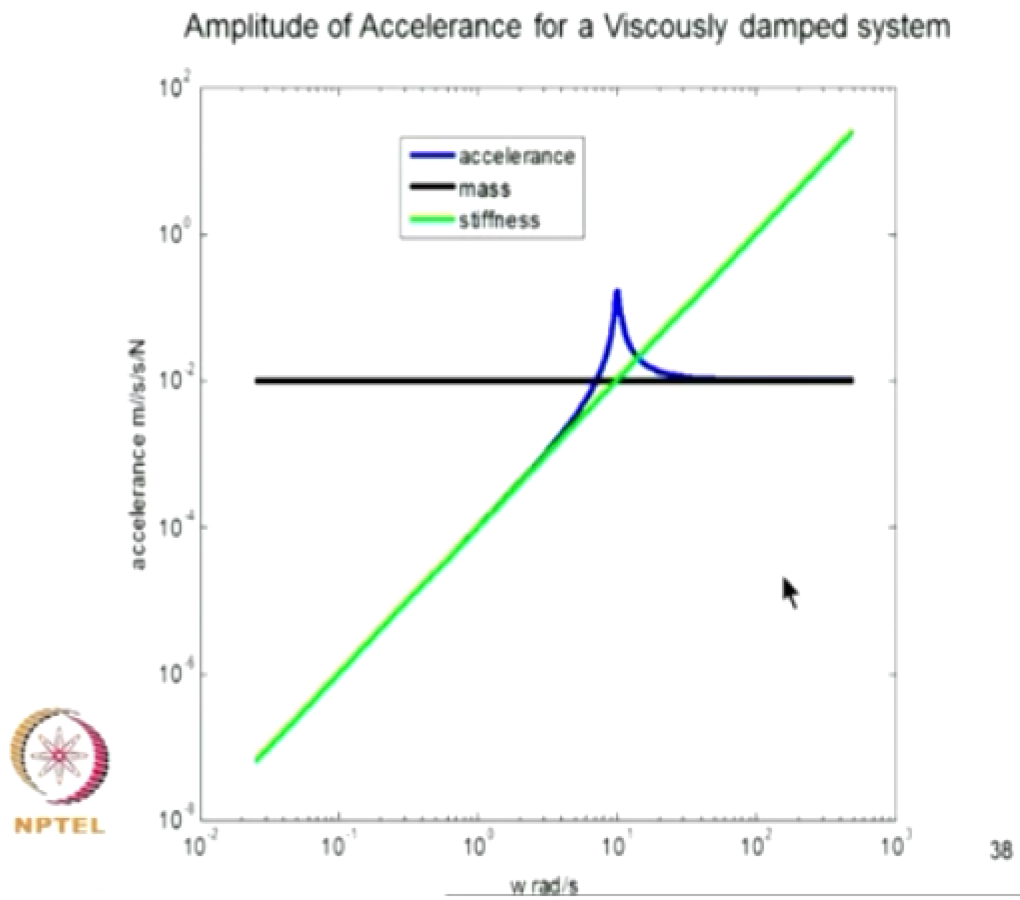
So now if I plot the frequency response functions in the log, log plot, that means Y axis is also logarithm and X axis is also logarithm you will see that these lines are all straight lines, so how do they look? For example for receptance of a viscously damped system the blue line is the



actual receptance function, please note that X axis is now on log scale, Y is also on log scale, so these asymptotes the stiffness, you know you can see as low frequency the FRF goes to the behavior of a stiffness, so it's the stiffness dominated behavior. As omega becomes large the FRF becomes asymptotic to the mass element, so high frequency behavior is dominated by inertial forces, so here of course all the three quantities mass, damping, and spring constant control the behavior, so this is for mobility, the green line is the asymptotes for stiffness, the



black line is for asymptotes for mass, and blue line is the actual mobility curve, and as you can see for a small omega the FRF is asymptotic to stiffness, and for large frequency it is asymptotic to mass, so this is similar plot for accelerance and we again make the same



observation, so it gives a clue on how the FRF behave for small frequencies and very large frequencies, this has important bearing on mode superposition method especially on questions on truncation of modes, so we will see what the implications are shortly.

Nyquist's plot for mobility of a viscously damped system is a circle with radius *1/2c* and centre at *(1/2c,0)*

$$Y(\omega) = \frac{i\omega}{k - m\omega^2 + i\omega c}$$

$$\mathrm{Re}\left[Y(\omega)\right] = \frac{\omega^2 c^2}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\mathrm{Im}\left[Y(\omega)\right] = \frac{\omega(k - m\omega^2)}{(k - m\omega^2)^2 + (\omega c)^2}$$

Define

$$U = \mathrm{Re}\left[Y(\omega)\right] - \frac{1}{2c} \quad \& \quad V = \mathrm{Im}\left[Y(\omega)\right]$$

$$\Rightarrow U^2 + V^2 = \left(\frac{1}{2c}\right)^2$$

NPTEL

Now one more feature that we observed is that these FRF's tend to, seem to appear like circles so indeed you can show that for a viscously damped system the mobility if you consider and look at its Nyquist plot, suppose if I introduce U as real part of mobility and I shift it by 1 / 2 C and V as this you can show that U square + V square is a circle, so that means the Nyquist plot for mobility for a viscously damped system is a circle with radius 1 / 2C and its center is located here, so this helps, this is one way of measuring damping, if you are doing an experiment you plot the mobility and then as a Nyquist plot and measure its location and radius you will get an idea about the damping.

Nyquist's plot for receptance of a structurally damped system is a circle with radius *1/2h* and centre at *(0,-1/2h)*

$$\alpha(\omega) = \frac{1}{(k - m\omega^2) + ih}$$

$$\mathrm{Re}\left[\alpha(\omega)\right] = \frac{k - m\omega^2}{(k - m\omega^2)^2 + h^2}$$

$$\mathrm{Im}\left[\alpha(\omega)\right] = \frac{-h}{(k - m\omega^2)^2 + h^2}$$

Define

$$U = \mathrm{Re}\left[\alpha(\omega)\right] \quad \& \quad V = \mathrm{Im}\left[\alpha(\omega)\right] + \frac{1}{2h}$$

$$\Rightarrow U^2 + V^2 = \left\{\frac{1}{2h}\right\}^2$$

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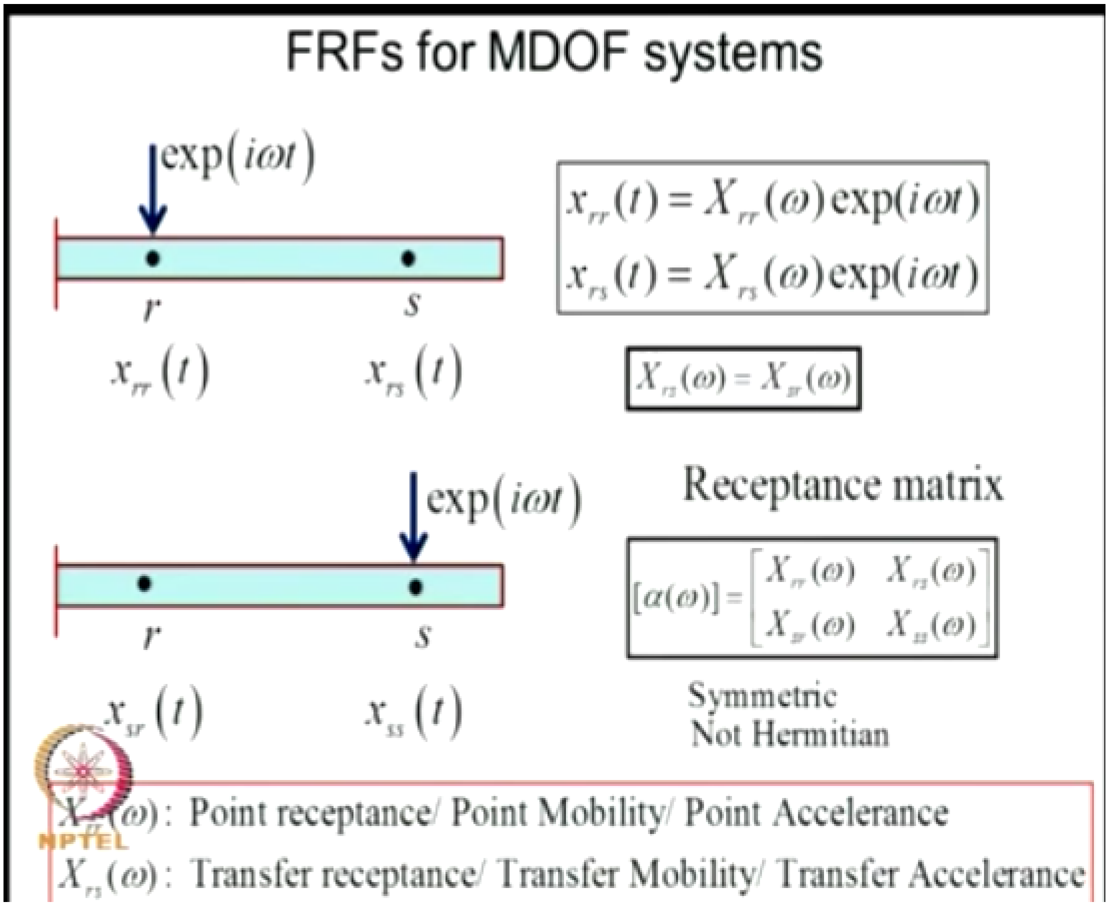
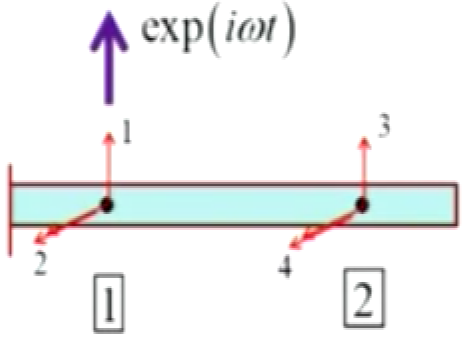
Similarly for a structurally damped system you can show that receptance will be a circle with this as the ratio, where H is the complex part of the stiffness, the imaginary part of the stiffness associated with energy dissipation.



Now I talked about single degree freedom systems, now if you come to multi degree freedom systems there is yet another you know basket of terminologies which we should feel comfortable using, so suppose if I consider a cantilever beam and there are two stations, say R and S, so R is a drive point, S is a measurement point. Now I Drive at R the system with an harmonic excitation E raise to I Omega T, XRR(t) is the drive point response XRS(t) is the you know transfer response at some other station, so I can as well drive at S and measure at R, so I have XRR, XRS, XSR, XSS. Now we will assume that system has reached steady state and is a linear time-invariant system, and as time becomes large the response is harmonic at a driving frequency and capital XRR and XRS are the amplitudes of response.

Now I will assemble these amplitudes into a matrix and call it as a receptance matrix, so XRR(Omega) XRS(Omega), XSR(Omega) XSS(Omega), you can see that this is a symmetric matrix it is a complex valued matrix it is symmetric, but not Hermitian, it is actually symmetric.

Now we call XRR(Omega) as point receptance or point mobility or point accelerance depending on whether you are talking about displacement, velocity or acceleration, XRS(Omega) we call it as transfer receptance, or transfer mobility, or transfer accelerance, so the word point and transfer refers to drive point and measure point, measurement points, it may



$$\delta_1(t) = \Delta_{11}(\omega)\exp(i\omega t)$$
$$\delta_2(t) = \Delta_{12}(\omega)\exp(i\omega t)$$
$$\delta_3(t) = \Delta_{13}(\omega)\exp(i\omega t)$$
$$\delta_4(t) = \Delta_{14}(\omega)\exp(i\omega t)$$

$\Delta_{11}(\omega) =$ Direct point receptance (mobility/accelerance)

$\Delta_{12}(\omega) =$ Cross point receptance (mobility/accelerance)

$\Delta_{13}(\omega) =$ Direct transfer receptance (mobility/accelerance)

$\Delta_{14}(\omega) =$ Cross transfer receptance (mobility/accelerance)

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so happen that I may apply a force E raise to I omega T and measure translation and rotations, okay suppose at station 1 and 2 now at each station there are two degrees of freedom say 1, 2, 3, 4, so Delta 1(t) can be written as Delta 1 1 of omega E raised to I Omega T and so on and so forth, so this Delta 1 1 of Omega, we call it as direct point receptance, we use the word direct and cross to indicate that I am applying a force and measuring a rotation that is cross, if you apply force and measure translation it is direct so if the drive point and measurement point coincide it is point, if not it is transfer, so we have various combinations of these terminologies. So if you are reading, as I said already if you're reading research paper you may come across these terms so it is useful to know what these things mean.

$$\{\text{Displ}\} = [\alpha(\omega)]\{\text{Force}\}; \quad [\alpha(\omega)] = \text{Receptance}$$

$$\{\text{Vel}\} = [Y(\omega)]\{\text{Force}\}; \quad [Y(\omega)] = \text{Mobility}$$

$$\{\text{Accl}\} = [A(\omega)]\{\text{Force}\}; \quad [A(\omega)] = \text{Accelerance}$$

$$[\alpha(\omega)]^{-1}\{\text{Displ}\} = \{\text{Force}\}; \quad \Delta(\omega) = \alpha^{-1}(\omega) = \text{Dynamic stiffness matrix}$$

$$[Y(\omega)]^{-1}\{\text{Vel}\} = \{\text{Force}\}; \quad \Gamma(\omega) = Y^{-1}(\omega) = \text{Mechanical impedance matrix}$$

$$[A(\omega)]^{-1}\{\text{Accl}\} = \{\text{Force}\}; \quad \Xi(\omega) = A^{-1}(\omega) = \text{Apparent mass matrix}$$

Remarks

- We often measure accelerance
- Obviously receptance, mobility and accelerance are easier to measure than Dynamic stiffness, mechanical impedance and apparent mass.

NPTEL

Caution

$$[\alpha]^{-1} = [\Delta] \text{ does not imply } \Delta_{ij}(\omega) = \frac{1}{\alpha_{ij}(\omega)}$$

So the displacement vector is written as receptance matrix into the force, the velocity vector is written as mobility into force, acceleration is accelerance into force, so what was scalar quantities for single degree freedom systems becomes now matrices, so the inverse of receptance is a dynamic stiffness matrix, the inverse of mobility is a mechanical impedance matrix, the inverse of accelerance is apparent mass matrix. So if you are doing experimental work we often measure acceleration and therefore we will be measuring accelerance.

Now obviously receptance, mobility and accelerance are easier to measure than dynamic stiffness, mechanical impedance and apparent mass, I’ll leave it as an exercise for you to think. One more word of caution is although alpha inverse implies delta it does not mean that individual terms in dynamic stiffness matrix are reciprocals of individual terms in receptance matrix, so this is a common error that people seem to make, so you should be very cautious about this, okay.

Viscously damped MDOF system with s-th dof driven by an unit harmonic force

$$M\ddot{X} + C\dot{X} + KX = F\exp(i\omega t)$$

$$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

s – th entry

$X_{rs}(t) =$ response of the r-th coordinate due to unit harmonic driving at s-th coordinate.

$$\lim_{t\to\infty} X_{rs}(t) = ?$$

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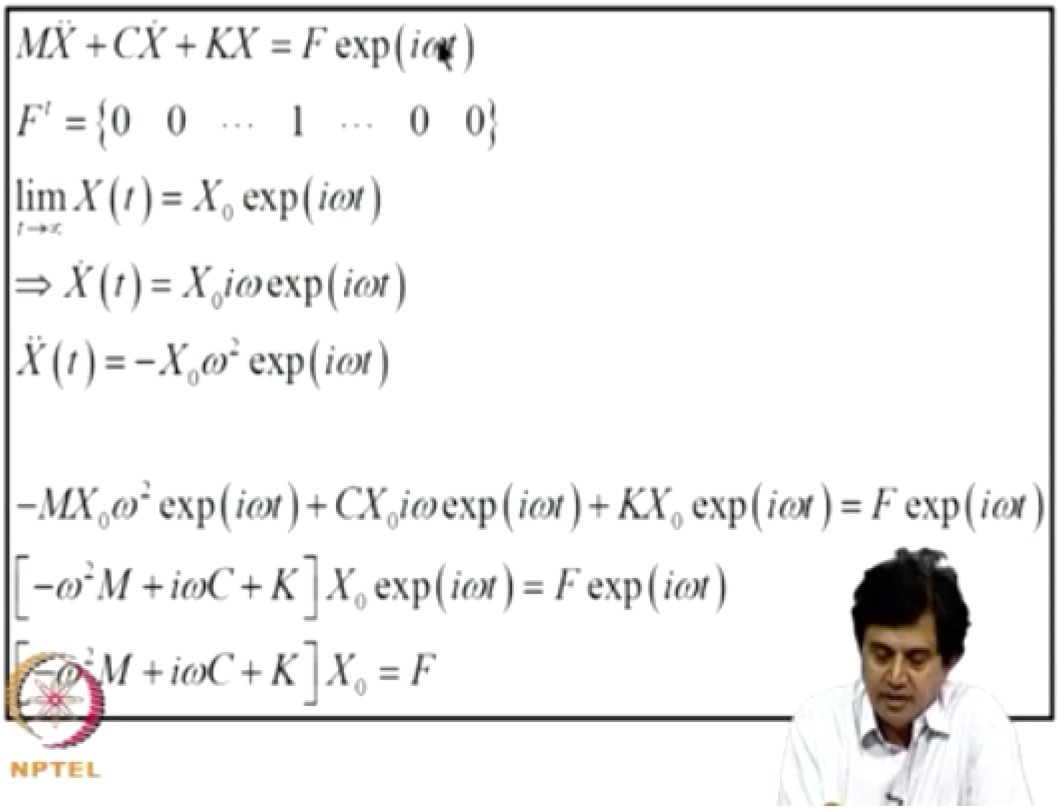
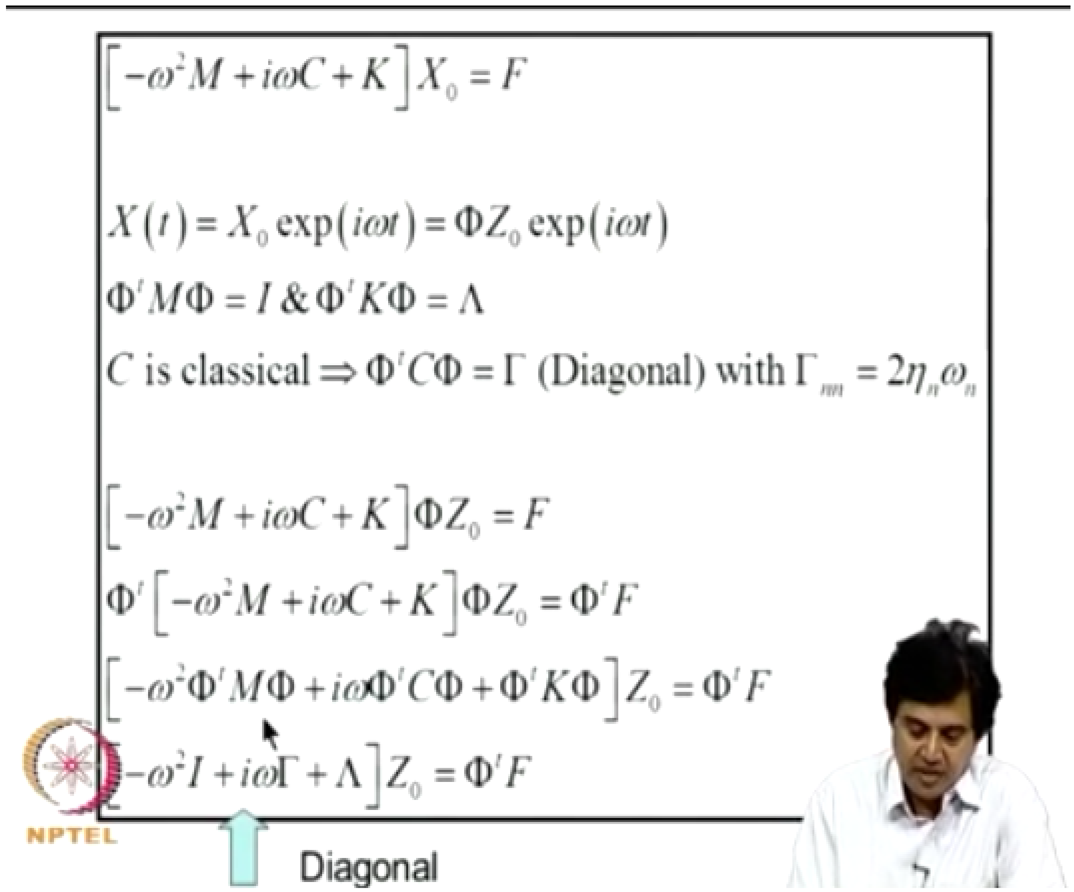
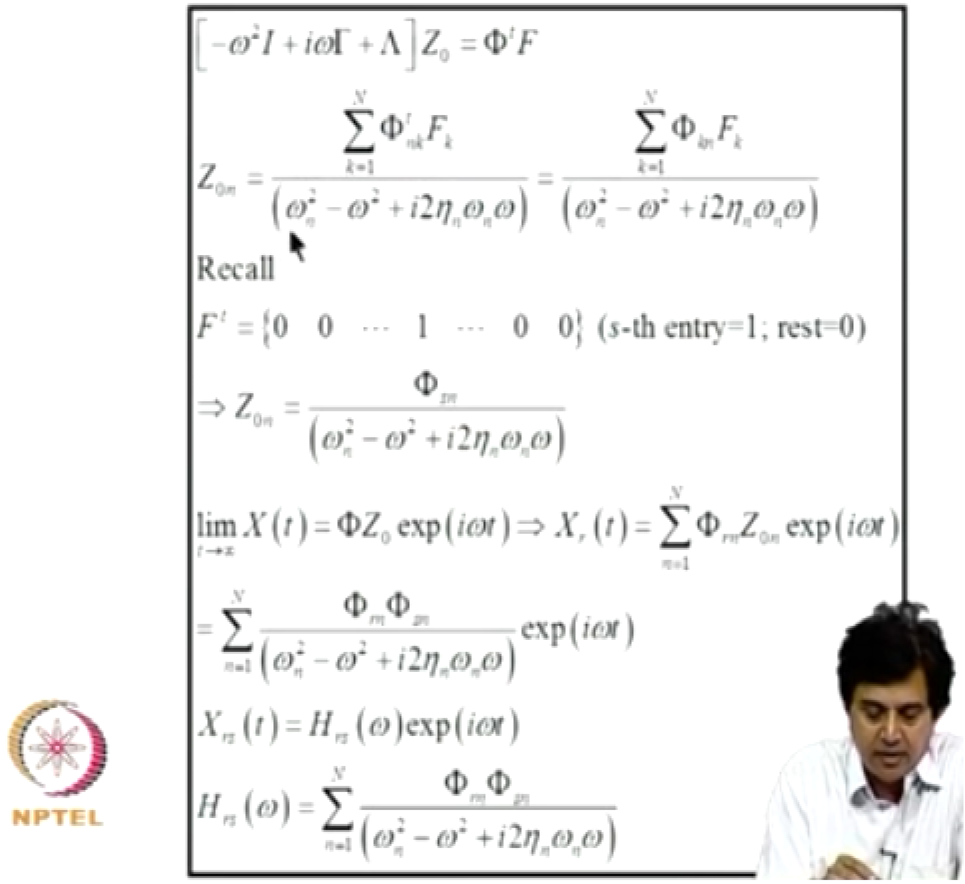
Now equipped with this terminologies we will now return to the problem of driving the frequency response function of a viscously damped multi degree freedom system, so we'll assume that damping is classical to start with and then we'll later on consider what happens if system is non-classical damped, so the equation of motion is MX double dot + CX dot + KX = F E raised to I Omega T, this F is a vector such that only the S-th entry is 1, that means I am driving the system by a concentrated harmonic load at the S-th degree of freedom, it could be either an bending moment or a shear force it depends on the model that you are using, XRS(t) is response of the R-th coordinate due to unit harmonic driving at S-th coordinate, so we are interested in knowing as T tends to infinity what happens to this, that is a question.



Now the system is linear it is time invariant and it is being driven harmonically therefore in steady state all response quantities are harmonic at the driving frequency with varying amplitudes, so I can assume therefore solution to be of the form X naught E raised to I Omega T, now I will differentiate this to get velocity and acceleration and put it back in the governing equation, I get the equilibrium equation in frequency domain as this, so E raised to I Omega T cannot be 0 for all T and this is the equilibrium equation in the frequency domain. So this matrix is a dynamic stiffness matrix and its inverse is the receptance matrix.



Now we have to evaluate this matrix, now we will now make the transformation X = Phi Z, so and Z we take it as Z naught E raised to I Omega T, so now if I make this substitution into this equation and use the fact that C is a classical, classically damp matrix and these orthogonality relations are true, that is phi transpose M Phi is I, Phi Transpose K Phi is a diagonal matrix and Phi Transpose C Phi is another diagonal matrix, if I now substitute here for X naught I will write Phi Z not, which is F, and pre multiplied by Phi Transpose I will get this. Now I will get subsequently - Omega square Phi transpose M Phi + I Omega Phi Transpose C Phi + Phi Transpose K Phi into Z naught is equal to this, so this matrix now Phi Transpose M Phi is I, Phi Transpose C Phi is a diagonal matrix and I have given the name capital Gamma for that where diagonal entries are 2 ETA and Omega N, and capital Lambda is a diagonal matrix of square of natural frequency, so this matrix is a diagonal matrix this is because of the orthogonality relation satisfied by Phi, Phi is now orthogonal to all the three structural matrices mass, stiffness, and damping, this Phi transpose F is the generalized force vector, so now I can write it in the scalar form, we can write in this form we get this equation.



Now right hand side is Phi Transpose F, so I am getting this, now since the forcing the elements of the forcing vector is 0 except for one entry this summation in the numerator actually collapses to a single term and that is given by this, so this is Z, Z0N, and now I want X0N so that is, I have to now transform this using the modal vector matrix Phi Z not, so based on that I will get for the R-th coordinate this is the response. So now substituting for this Z0N I get this expression and XRS(t) which is the quantity that I am looking for can be written in terms of an amplitude and E raised to I Omega T, and this amplitude is now obtained as a summation or normal modes, natural frequencies and normal modes.

$$X_{rs}(t) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)} \exp(i\omega t)$$

$$H_{rs}(\omega) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

Remarks

- $X_{rs}(t) = X_{sr}(t)$
- $H_{rs}(\omega) = H_{sr}(\omega)$
- $\left[H(\omega)\right] = \left[H_{rs}(\omega)\right]$
- $\left[H(\omega)\right]$ is symmetric but not Hermitian
- $\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1} = \left[\sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$

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So this is the expression for the frequency response function. We can see that XRS(t) is XSR(t) that would mean HRS of Omega is HSR(Omega), it is symmetric but not Hermitian, so this matrix can be written, this is the receptance matrix where the individual entries are this, okay, and it is actually equal to inverse of this matrix.

- $\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1}$
  - Conceptually simple
  - Computationally difficult to implement

- $\left[H(\omega)\right] = \left[\sum_{n=1}^{N^* \le N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$
  - Computationally easier to implement
  - Not all modes need to be included

    (Nor it is advisable to include all modes)

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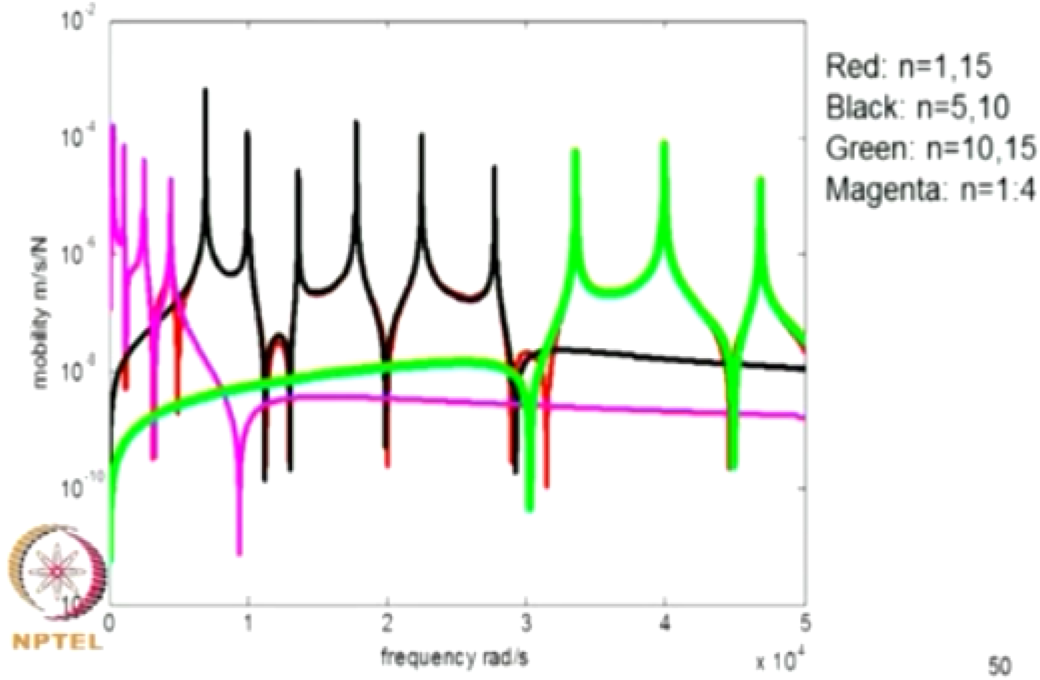
Now in the numerical work of course we can directly invert this matrix without doing the mode superposition, but then you should see that this I need to invert for every value of Omega in which I'm interested, suppose I am interested in producing a spectrum, a graph of various amplitude and phase of elements of H as a function of Omega then for all the Omega values in which you are interested in you should invert this matrix, so that is a computationally a demanding task, therefore what we do is this is replaced by any summation and that is the advantage of mode superposition, so this is a direct frequency response function calculation, this is frequency response function calculation using mode superposition, this is more revealing this is characterless this has certain features that we can understand by looking at the nature of these individual terms.

$$H_{jk}(\omega) = -\frac{1}{\omega^2 M_{jk}^R} + \sum_{r=m_1}^{m_2} \left[\frac{{}_r A_{jk}}{\omega_r^2 - \omega^2 + i2\eta_r\omega_r\omega} \right] + \frac{1}{K_{jk}^R}$$



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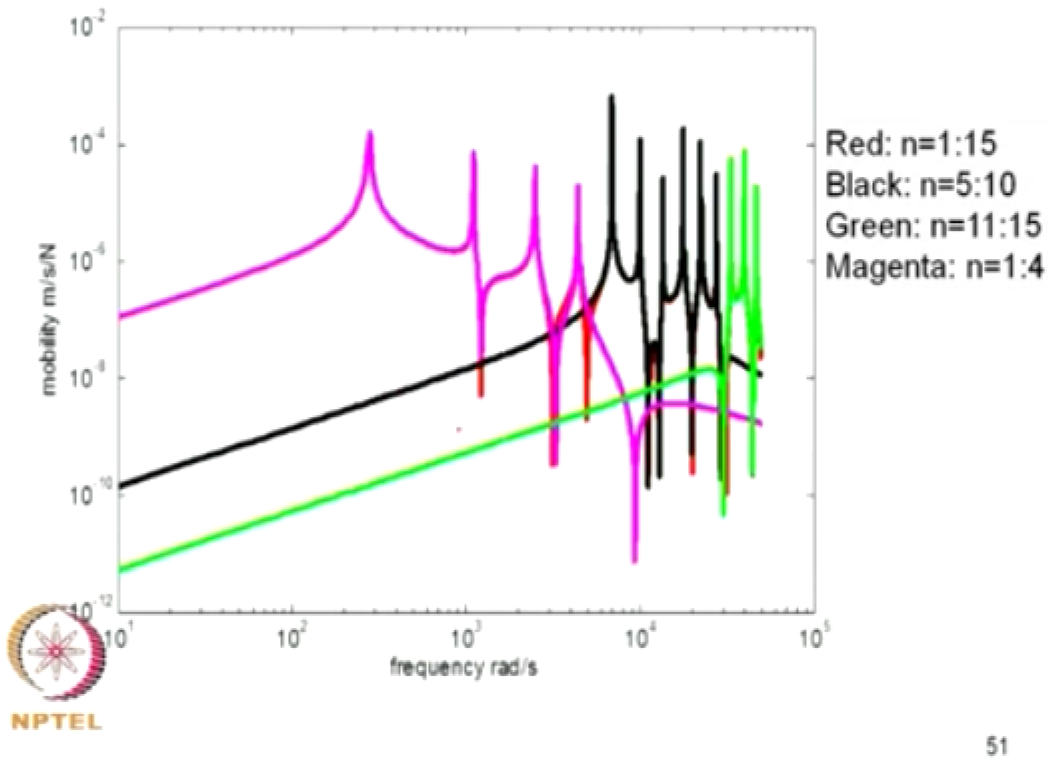
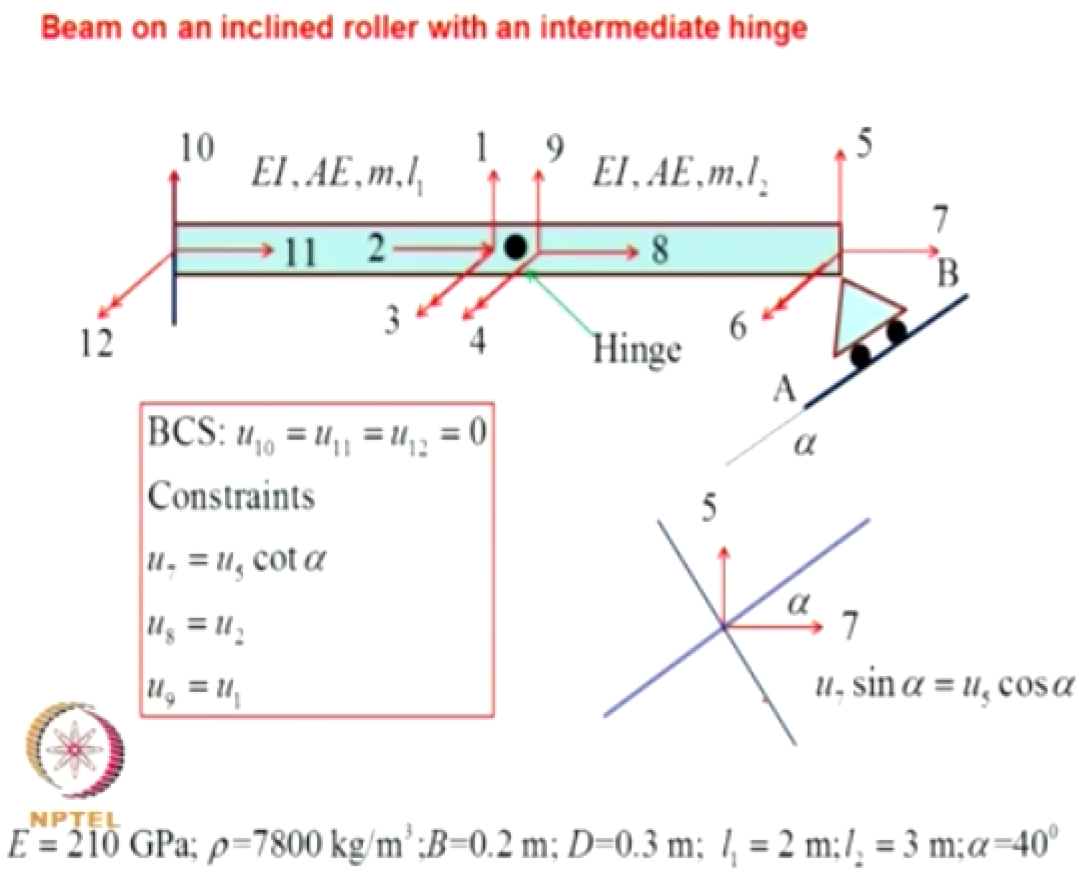
Now if we look at this summation in modeling although this summation runs from N equal to, capital N which are all the modes seldom we actually include all the modes, we truncate it, may be in 100 degree freedom system or may be 10 modes we will use and compute the answer, how many modes should be included in a given dynamic analysis is a question that we will address in due course, but it is suffice at this stage to appreciate the fact that although a system with N degrees of freedom admits N normal modes in response representation we will not use the N modes. So now here suppose you are computing the frequency response function at a given value of omega and you're retaining modes from M1 to M2 that means modes from 0 to M1 - 1 are ignored, and modes from M2 to capital N are ignored.

Now these errors, this is an error because you are not included the first few modes, this is an error that has happened because you are not included few higher modes. Now let us look at the implication of this, now the red line that you see here it is buried here is the frequency response function for a dynamical system where all modes have been included, okay some 15 modes have been included, the black line that you see here is when the modes from 5 to 10 are included, so here 0 to 4 is not included, and 11 to 15 is not included. Now just to see what is not being included I have plotted here a green one you see here in this region there is no contribution from the initial few modes so this is captured, this is the contribution from 11, 12 and 15 modes, so the magenta line that you see here initially is when I find this response function including only the first four modes. So now suppose if you are interested in frequency response between say fifth to the tenth mode that is a black line, the magenta curve is in the low frequency regime of the black curve or the corrections to this curve is in the mass control region of this system, right, so this is the mass asymptotic which has not been included in my calculation, so that is a correction that we do for the first few modes which are not included, they are actually the inertial correction for the lower modes, the higher modes that we have not included that is this green one, the corrections are in the stiffness region of this region, so that is this correction, okay, so later on when we have to evaluate number of modes to be retained and how to correct for modes that are not being included in an analysis we'll revisit this description but at this stage when we are looking at the expression for the frequency response function it is useful to notice these facts.



Now the same graph is shown in log, log scale, so that you can see the asymptotes are all straight lines, so this you can you know relate to what I said just now the only difference between this plot and the previous plot as I said the X-axis is now on log scale.



Now let's quickly work out the frequency response function for a multi degree freedom system, you may recall that we have studied this problem in the previous class so we have derived,

$$G_J = \begin{bmatrix} 0 & 0 & 0 & 0 & -1.1918 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \Gamma = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0 & 1.1918 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_T = 10^3 \begin{bmatrix} 1.2168 & 0 & -0.2206 & 0.3922 & 0.2407 & -0.2318 \\ 0 & 1.0920 & 0 & 0 & 0.3718 & 0 \\ -0.2206 & 0 & 0.1203 & 0 & 0 & 0 \\ 0.3922 & 0 & 0 & 0.2853 & 0.2318 & -0.2139 \\ 0.2407 & 0.3718 & 0 & 0.2318 & 1.5816 & -0.3922 \\ -0.2318 & 0 & 0 & -0.2139 & -0.3922 & 0.2853 \end{bmatrix}$$

$$K_T = 10^3 \begin{bmatrix} 0.0597 & 0 & -0.0630 & 0.0354 & -0.0177 & 0.0354 \\ 0 & 7.3500 & 0 & 0 & -3.7540 & 0 \\ -0.0630 & 0 & 0.1260 & 0 & 0 & 0 \\ 0.0354 & 0 & 0 & 0.0945 & -0.0354 & 0.0472 \\ -0.0177 & -3.7540 & 0 & -0.0354 & 4.4916 & -0.0354 \\ 0.0354 & 0 & 0 & 0.0472 & -0.0354 & 0.0945 \end{bmatrix}$$

Nat freq HZ
16.2557
63.4080
173.5123
200.9014
304.3834
607.6123

this mode system is modeled as a 6 degree of freedom system and this was my mass matrix, and this was my stiffness matrix.

Damping: η=0.001 for all modes

$$\Phi = \begin{bmatrix} 0.0307 & 0.0262 & 0.0040 & -0.0206 & 0.0453 & -0.0058 \\ 0.0000 & -0.0006 & -0.0112 & -0.0103 & -0.0059 & -0.0279 \\ 0.0149 & 0.0069 & 0.0467 & -0.0904 & 0.1073 & -0.0113 \\ -0.0051 & -0.0435 & 0.0295 & -0.0015 & -0.1331 & 0.0276 \\ 0.0000 & -0.0012 & -0.0162 & -0.0133 & -0.0038 & 0.0261 \\ -0.0099 & 0.0340 & 0.0129 & -0.0465 & -0.0799 & 0.0531 \end{bmatrix}$$

$$C = 10^3 \begin{bmatrix} 0.3099 & 0.0136 & -0.1284 & 0.0843 & -0.0349 & 0.0145 \\ 0.0136 & 4.9322 & -0.0012 & 0.0135 & -0.9014 & 0.0401 \\ -0.1284 & -0.0012 & 0.2075 & 0.0425 & 0.0031 & -0.0046 \\ 0.0843 & 0.0135 & 0.0425 & 0.2020 & 0.0026 & -0.0415 \\ -0.0349 & -0.9014 & 0.0031 & 0.0026 & 4.0355 & -0.2980 \\ 0.0145 & 0.0401 & -0.0046 & -0.0415 & -0.2980 & 0.2410 \end{bmatrix}$$

$$\Phi' C \Phi = \begin{bmatrix} 0.2043 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7968 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 2.1804 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0 & 0.0000 & 2.5246 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 3.8250 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 7.6355 \end{bmatrix}$$

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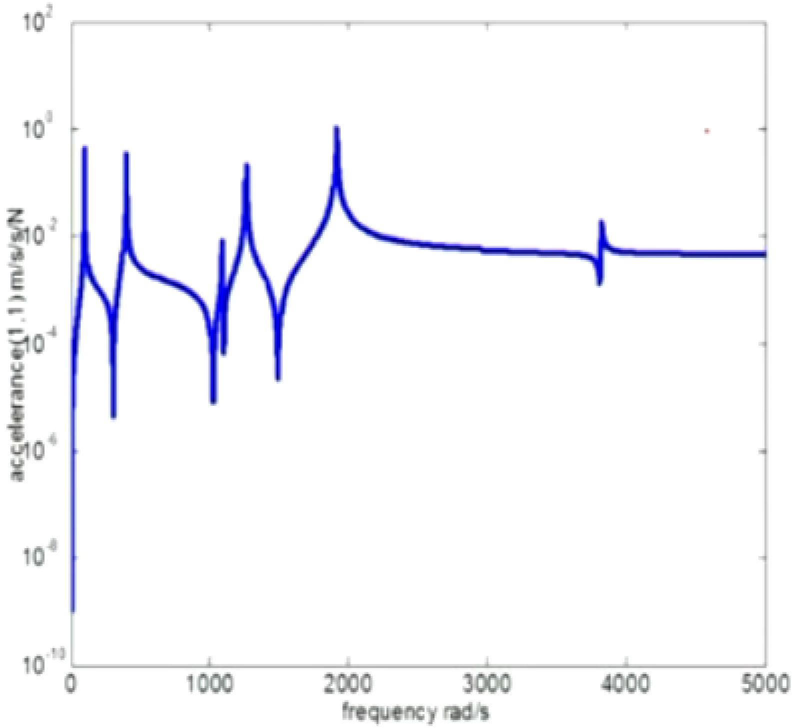
So now if I assume that all modes have damping of say .001, that is very lowly damped we can construct a C matrix which is shown here and just to make sure that Phi Transpose C Phi is diagonal I have displayed that here,



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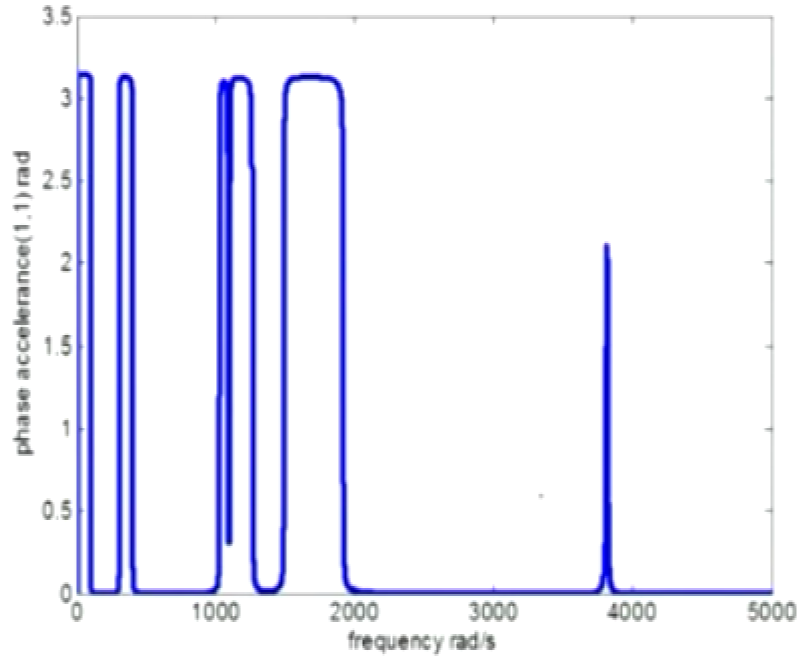
so this is a classical damping matrix and Phi is the modal matrix.

Now this is the accelerance at 1, 1 that means I am applying harmonic load at degree of freedom 1 and I am measuring acceleration there, so this is across the frequency range of interest. Now what you are seeing here is there are characteristic behavior of peak and a dip, okay so 1, 2, 3, 4, 5, 6, peaks that you are seeing are the 6 natural frequencies of the system. Now we need to explain what happens between two peaks that needs explanation, now before we do that we can look at this a phase angle for the same graph and we are seeing if you see



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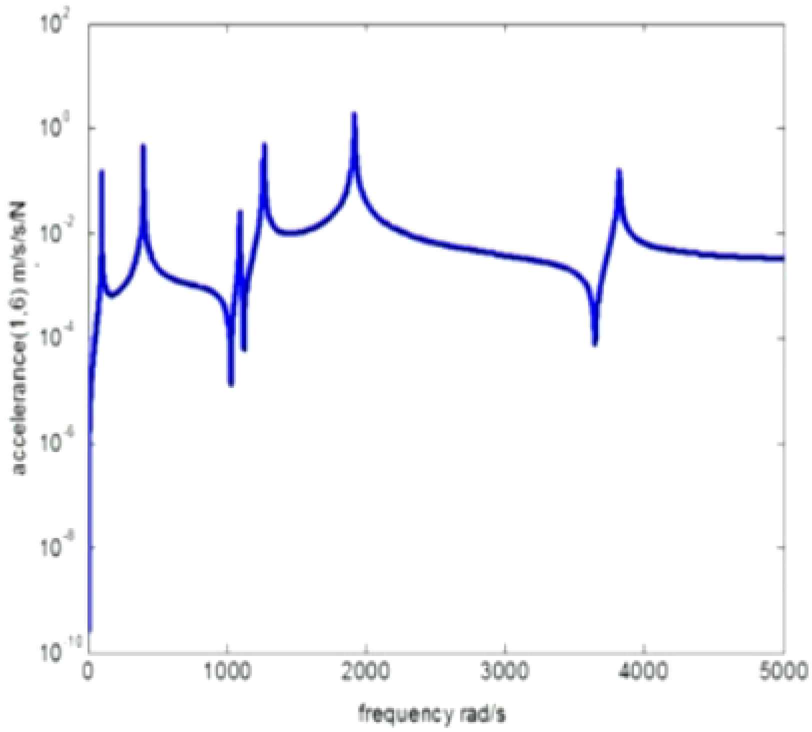
carefully you are seeing that the phase is undergoing rapid changes whenever there is a peak or a dip in that FRF, okay, so the peak and the dip here are accompanied by rapid changes in the phase angle. Now this is for accelerance 1, 6 that means you are driving at 1 and measuring the



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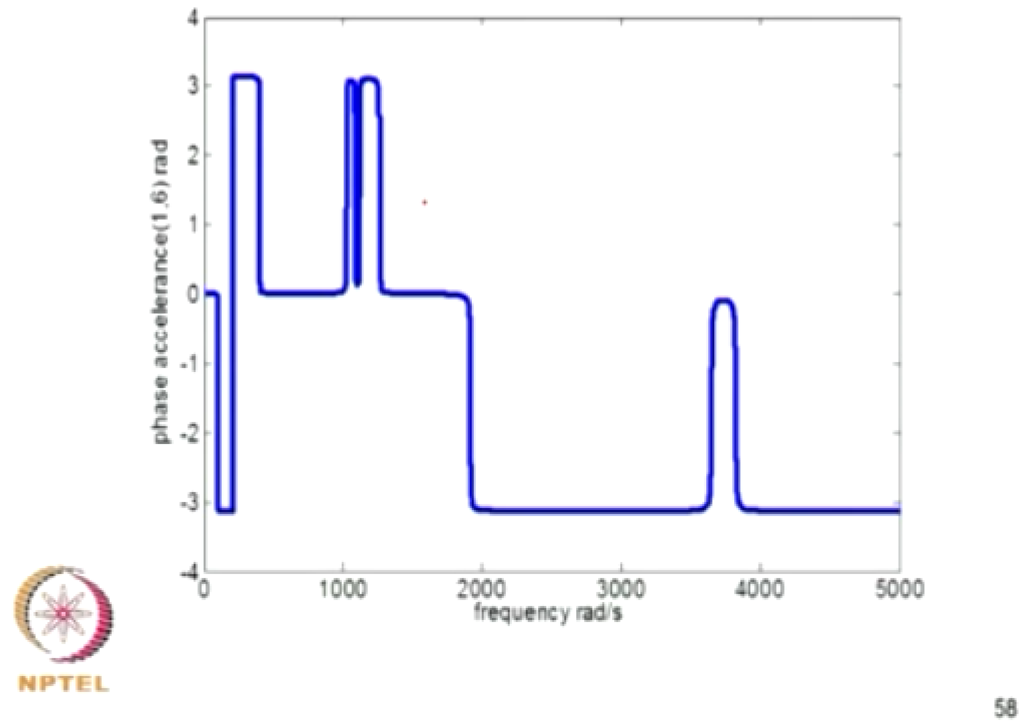
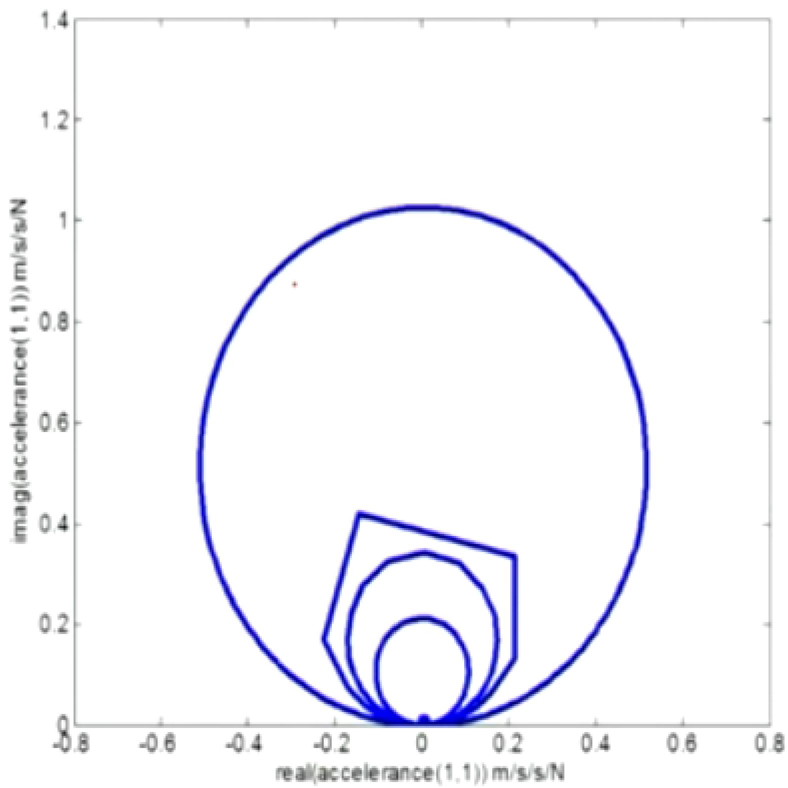
6 degree of freedom. Here again we are getting peaks at the system natural frequencies but between 2 peaks there seem to be a different type of behavior here, the different type of behavior here, a different type of behavior here, so what are these? Okay, we will look at this so



this is the phase angle for the system again it is going under, it is undergoing rapid changes at resonance.



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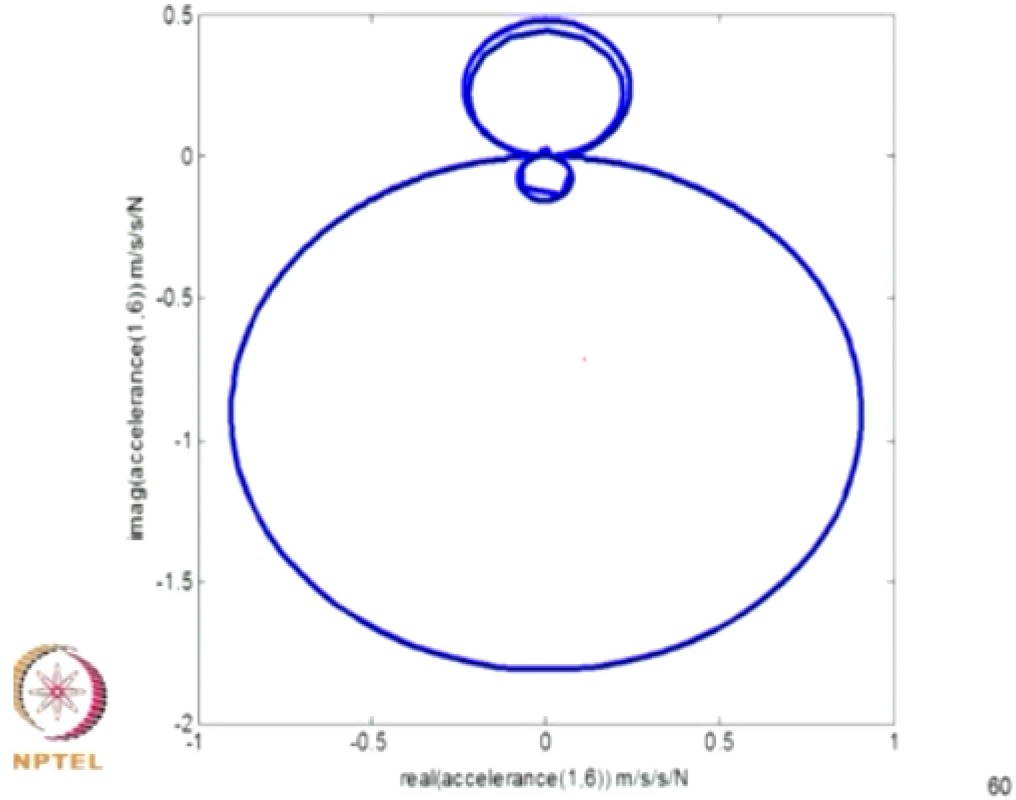
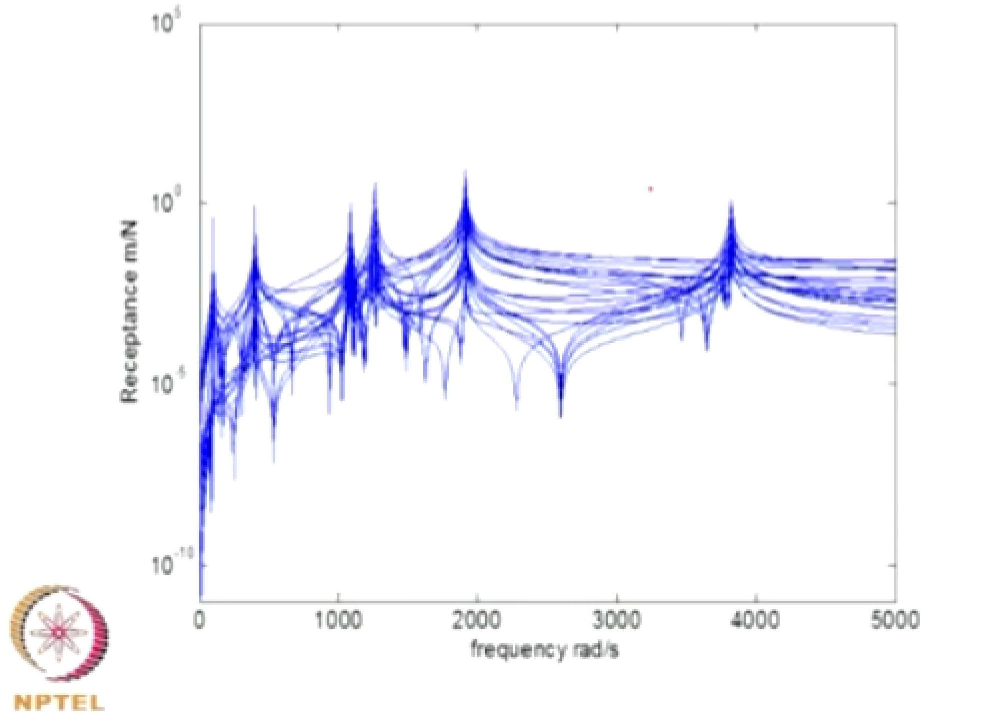
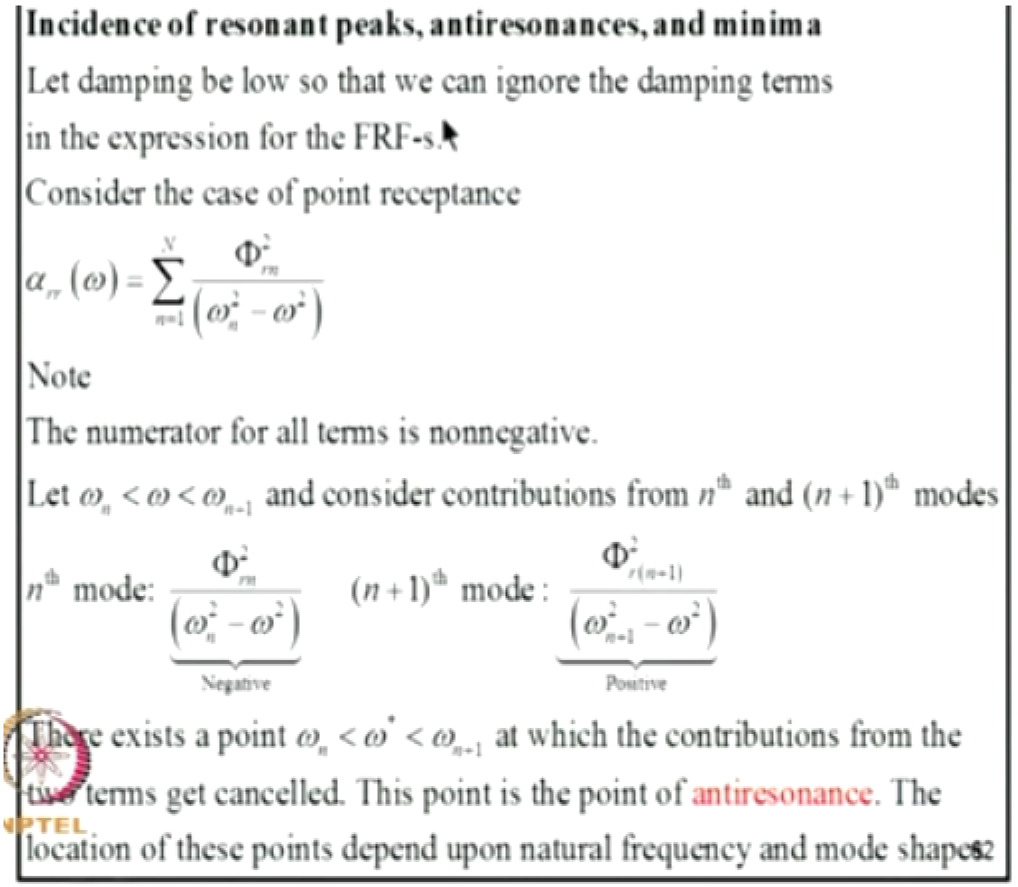
Now this is a Nyquist plot for these graphs and you see that each loop corresponds to one of the modes, for a single degree freedom system it’s a one closed-loop so there are now say 6 loops if you carefully see you will be able to see that and this is for 1, 6 Nyquist plot the loops will flip



on the other side and appears like this. Now all FRF's for receptance have been superposed in



this graph, now what you need to appreciate here is, this is the first natural frequency, this is the first natural frequency, the second, this third, four, five, six, so all the FRF's are peaking at the natural frequencies, but you look at the dips what happens between the two peaks there is a great deal of variability, there are no characteristic frequencies at which all FRF's are reaching a Minima or a dip like this, so we need to, at this say we can notice it and we will shortly offer an explanation.



Now so incidence of resonant peaks, anti-resonances and minima, the word anti resonances I need to explain, so let's consider FRF and let's assume that damping is fairly low so that we need not worry about it when we discussed this, so we will ignore those terms. Now let us consider the case of a point receptance, this is the sub mode super position expression for the FRF. Now first thing we can observe is the numerator is always, it's a square of a number and it cannot be negative, it is not negative. Now let us consider a frequency which lies between Nth and N + 1 natural frequency, okay. Now we will consider at this value of Omega how does Nth mode contribute and N + 1 mode contribute so the contribution from Nth mode will be this term but N + 1 mode this will be this one.

Now you see here the response is made up of some multiplied, this plus this, okay numerator in either case is non-negative, the first denominator since Omega is greater than Omega N this term will be negative, but on the other hand Omega is less than Omega N + 1 this will be positive, so that would mean this contribution will be negative, this contribution will be positive, so for a certain Omega in between Omega N and Omega N + 1 this positive and negative contributions would cancel and you get almost a 0 response and that point is known as anti-resonance, so you are seeing this is an anti-resonance, this is an anti-resonance, this is a ninety resonance.

Now consider the case of cross receptance

$$\alpha_{rs}(\omega)=\sum_{n=1}^{N}\frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2\right)}$$

Let $\omega_n < \omega < \omega_{n+1}$ and consider contributions from n^{th} and $(n+1)^{th}$ modes

n^{th} mode: $\dfrac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2\right)}$ $\quad (n+1)^{th}$ mode : $\dfrac{\Phi_{r(n+1)}\Phi_{s(n+1)}}{\left(\omega_{n+1}^2-\omega^2\right)}$

Num-1	Num-2	Term-1	Term-2	Remark
<0	<0	>0	<0	Antiresonance
<0	>0	>0	>0	Minimum
>0	<0	<0	<0	Minimum
[illegible]	>0	<0	>0	Antiresonance

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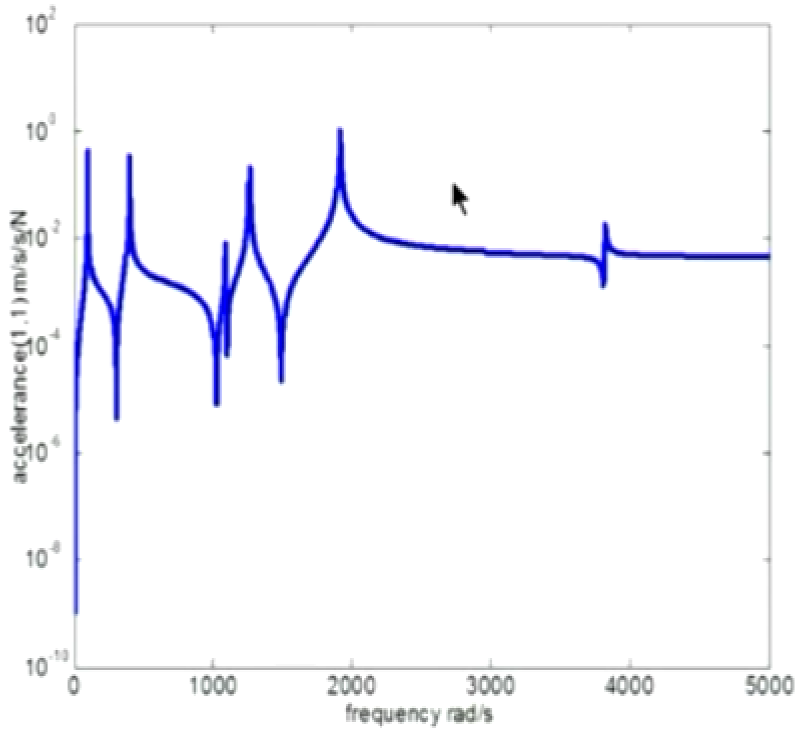
Now let us now consider cross receptance that means drive point and measurement points are different so the mode superposition method this is the expression for the FRF. Again let us consider Omega lying between Omega N and Omega N + 1, so the contribution from nth mode is this N + 1 mode is this. Now you look at the numerator, there is no guarantee that it is non-negative, it can be negative or positive, denominator this will be negative, this will be positive, so we can consider different combinations now, numerator 1, numerator 2,denominator this is negative, this is positive, that is known, suppose numerator 1 is negative, and numerator 2 is also negative the first term will be negative by, negative which will be positive and this will be negative by positive which will be negative, so between these two contributing terms or between these two frequencies now there will be an anti-resonance, that depends on sign of the mode shapes.

On the other hand if mode shapes are such that this numerator is negative, but this numerator is positive so then what happens the first term negative by negative positive, this positive by positive it is positive, so it contributes to a minimum there won't be anti-resonance now, whereas similarly we have other situations where we get a minimum or anti- resonance. So if you now go back to this graph, okay,



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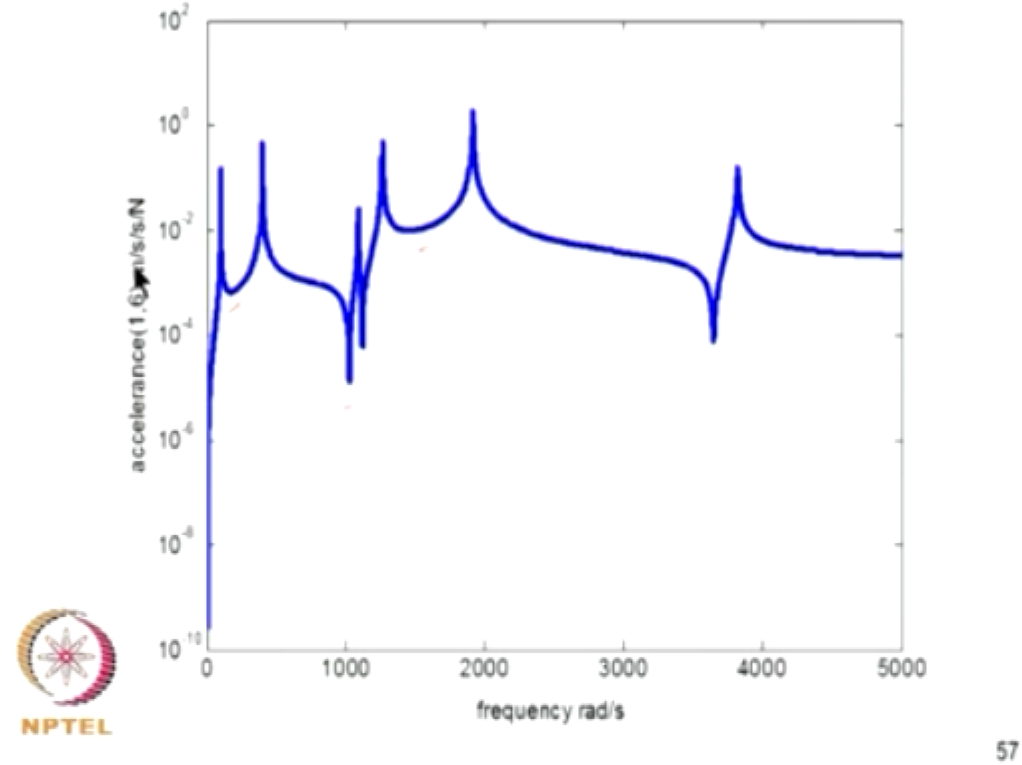
this is a drive point so between two resonant peaks there invariably intelligence so anti-resonance and resonant points alternate, but if you come to a say accelerance 1, 6 that mean driving point and measurement points are different there are different possibilities, for example



between peak 1 and peak 2 there is no anti-resonance point but there is only a minima, 2 small positive numbers add up to produce another small number. Whereas now you look at behavior between this peak and this peak, the two terms one is positive, another is negative and they are cancelling out and producing a very small number, okay of course in this system damping is not zero so it is showing a nonzero value otherwise it will dip to minus infinity, I mean it to zero, sorry, so what we can observe with, make few observation is all frequency response functions

Remarks

- All FRF-s peak at the same frequencies (natural frequencies)

This is not true for antiresonace and points of minima.

- For a point FRF, between every two resonances, an antiresonance occurs without exception.
- Trasnfer FRF-s show a mixture of antiresonance and minima.
- Resonant peaks are accompanied by large responses and rapid changes in phase angle. Antiresonance points are accompanied by low responses and rapid changes in phase angle.
- Presence of damping could make identitfication of resonance, antiresonance, and minima difficult.

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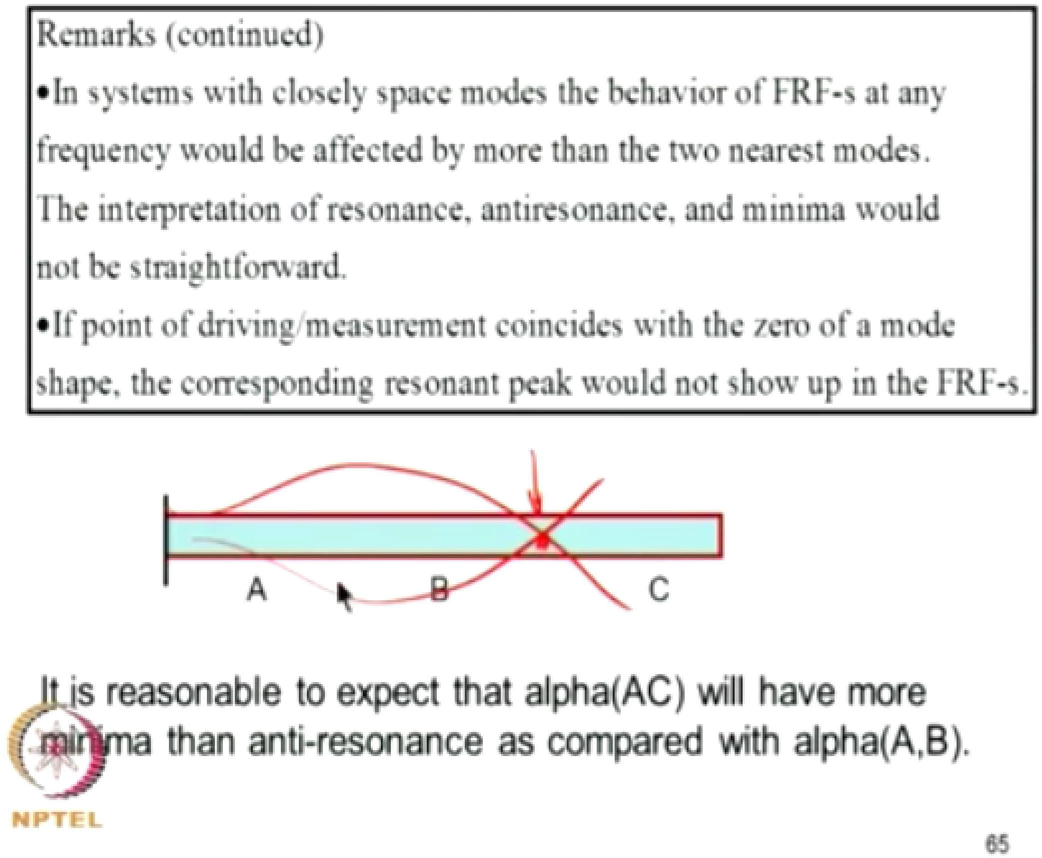
peak at the same frequencies which are natural frequencies but this is not true for anti-resonance points and minima because the location of anti-resonance and minima depends on mode shapes also in addition to the natural frequencies. For a point frequency response function that means drive point and measurement points are the same between every two resonances and anti-resonance occurs without an exception, whereas for transfer FRF's they show a mixture of anti-resonance and minima, resonant peaks are accompanied by large responses and rapid changes in phase angle, anti-resonance points are accompanied by low responses and rapid changes in phase angle. Presence of damping could make identification of resonance, anti-resonance and minima difficult, so when damping is absent they are very clearly pronounced, but if damping is large the so-called modal bandwidth will increase and contribution from neighboring modes also will increase, we will discuss that sometime later, so we may not get clear you know peaks and anti-resonance and minima points.



Similarly if a system has closely space modes I am assuming that at any given frequency in my analysis so far that it is only the two nearest modes that are contributing, but if modes are closely spaced that is not necessarily true, the other modes on the right and left can contribute, so again then the nature of peaks you know valleys and minima etcetera would change, the description that I have given or for well-spaced modes.

Now if point out driving or measurement coincides with the zero of a mode shape that means that we call zero of a mode shape as a node and if you are driving it point where node is occurring then the corresponding resonant peak would not show up in the FRF, okay, for example in a cantilever beam there can be a zero somewhere here for the second mode, somewhere here the mode shape could be like this, like this, so there will be a zero, suppose if you drive here the second mode frequency, at that frequency there will not be any peak in the FRF, okay that also you should bear in mind.

Now I said if at the drive point the resonant peaks and anti-resonant points alternate, so if your drive point is close to the measurement point that is measurement point is close to A, but it is not exactly A, then we would expect that there will be more anti-resonances than minima, because all the modes are likely to be in phase unless between A and this point there is a change in sign of the mode shape, there won't be a minima, okay, so but on the other hand if you move too far away stations the propensity for occurrence of anti-resonance comes down, so it is reasonable to expect that the receptance AC will have more minima than anti-resonance as compared with alpha AB. There's another feature that you can see, if you carefully analyze the frequency response functions.

MDOF system with s-th dof driven by an unit impulse force

$$M\ddot{X} + C\dot{X} + KX = F\delta(t)$$

$$X(0) = 0; \dot{X}(0) = 0$$

$$F' = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

s – th entry

$X_{rs}(t) =$ response of the r-th coordinate due to unit impulse driving at s-th coordinate.

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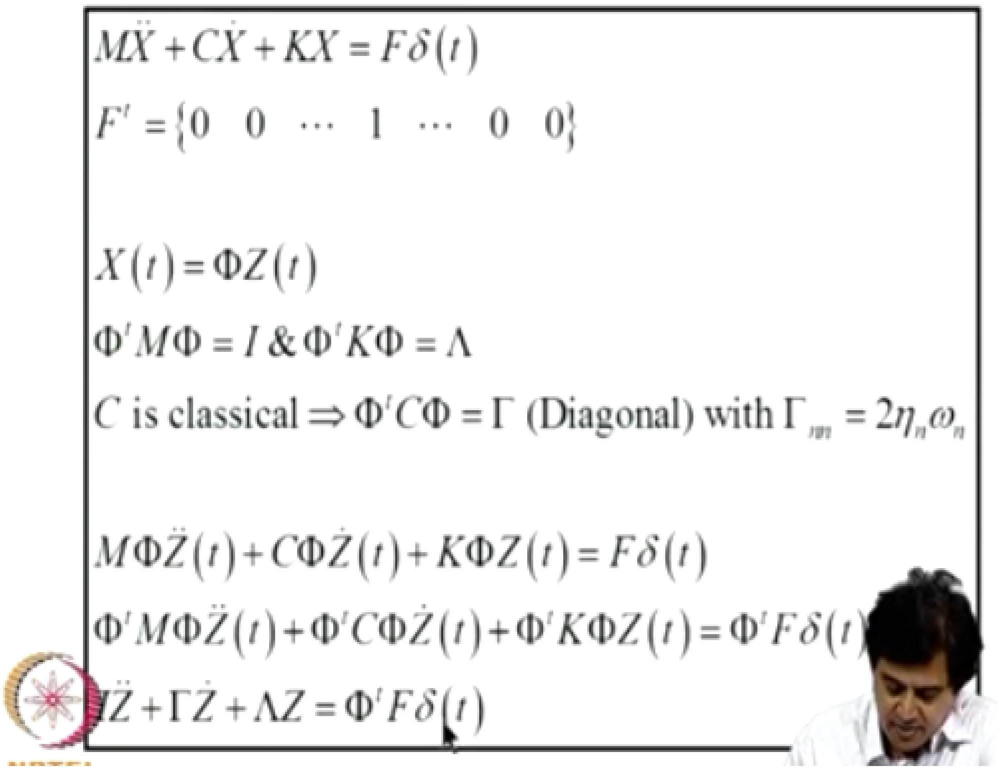
Now we have analyzed now the response of a classically damped viscous, viscously damped system with classical damping matrix under harmonic load, so we can now switch over to time domain and carry out a similar analysis this is straightforward, so if I have the equilibrium equation where I am now driving in the system impulsively at some S-th coordinate, so F is again 0 except for one coordinate, so how do I get the response at say R-th location, okay, so this again can be formulated we write the equation and make the model transformation where



Phi transpose M Phi is diagonal Phi Transpose K Phi is diagonal, and C is taken to be classical where Phi Transpose C Phi is gamma, so we make the substitution and multiplied by Phi Transpose I get this equation, so this is a set of single degree freedom system equations with on the right hand side I have impulsive forces, so I can write this as ZN double dot + 2 ETA N

$$I\ddot{Z} + \Gamma\dot{Z} + \Lambda Z = \Phi^t F\delta(t)$$

$$\ddot{z}_n + 2\eta_n\omega_n\dot{z}_n + \omega_n^2 z_n = \sum_{j=1}^{N}\Phi_{jn}F_j\delta(t) = \Phi_{sn}\delta(t)$$

$$z_n(0) = 0; \dot{z}_n(0) = 0$$

$$\Rightarrow$$

$$z_n(t) = \Phi_{sn}h_n(t) = \frac{\Phi_{sn}}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

$$X = \Phi Z \Rightarrow$$

$$X_r(t) = \sum_{n=1}^{N}\Phi_{rn}z_n(t)$$

$$h_{rs}(t) = \sum_{n=1}^{N}\Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

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Omega N ZN dot + Omega N square ZN and this summation will collapse to a single term because elements of F are 0 except for one number, so this is what I get. So initial conditions can be reduced if you assume the system starts from rest, even for Z coordinates the initial conditions would be 0, so I can get ZN(t), see the nth degree of freedom system is now driven by impulsive force with a magnitude of Phi SN, so if HN(t) is the unit impulse response for the nth degree of freedom system, the response will be Phi SN into HN(t) so that is given by this. Now we go back to X coordinate system write it as Phi Z, so XR(t) is therefore summation over all the modes ZN(t), so this is my element of the so-called impulse response function matrix, HRS(t) is given in terms of normal modes and natural frequencies and modal damping ratios as shown here. So this is a mode superposition based approach for evaluating elements of impulse response function matrix.

$$X_r(t) = h_{rs}(t) = \sum_{n=1}^{N} \Phi_{rn}\Phi_{sn} \frac{1}{\omega_{dn}} \exp(-\eta_n \omega_n t) \sin \omega_{dn} t$$

Remarks

- $h_{rs}(t) = h_{sr}(t)$
- $\left[h(t)\right] = \left[h_{rs}(t)\right]$ = Matrix of impulse response functions
- $\left[h(t)\right] = \left[h(t)\right]^t$
- Not all modes need to be included in the summation
- If an arbitrary load $f_s(\tau)$ is applied at the s-th dof (instead of unit impulse excitation)

$$X_{rs}(t) = \int_0^t h_{rs}(t-\tau) f_s(\tau) d\tau$$

$$= \int_0^t f_s(\tau) \left\{ \sum_{n=1}^{N} \Phi_{rn}\Phi_{sn} \frac{1}{\omega_{dn}} \exp\left[-\eta_n \omega_n (t-\tau)\right] \sin \omega_{dn}(t-\tau) \right\} d\tau$$

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So again we can observe that this HRS(t) is HSR(t) and if you assemble them in a matrix is a matrix of impulse response functions and this matrix is symmetric. Again we need not include all the modes, I in fact we should not include all the modes especially if you are doing a finite element type of model because higher modes are less accurate than lower modes, so all modes need not be included and this summation can go over only those modes which have been evaluated with acceptable accuracy. We can write now the response for RS(t) in terms of the impulse response function which is given by this.

So what we have done in today's class is we have derived the frequency response function and impulse response function for a viscously damped system in which the matrix is classical, damping matrix is classical. So in the next lecture we will revisit these problems and ask the question how to proceed if the C matrix is non-classical, so we will close this lecture at this point.

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