Indian Institute of Science
Bangalore

NP-TEL
National Programme on
Technology Enhanced Learning

Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 07

FE Modelling of Planar structures

By

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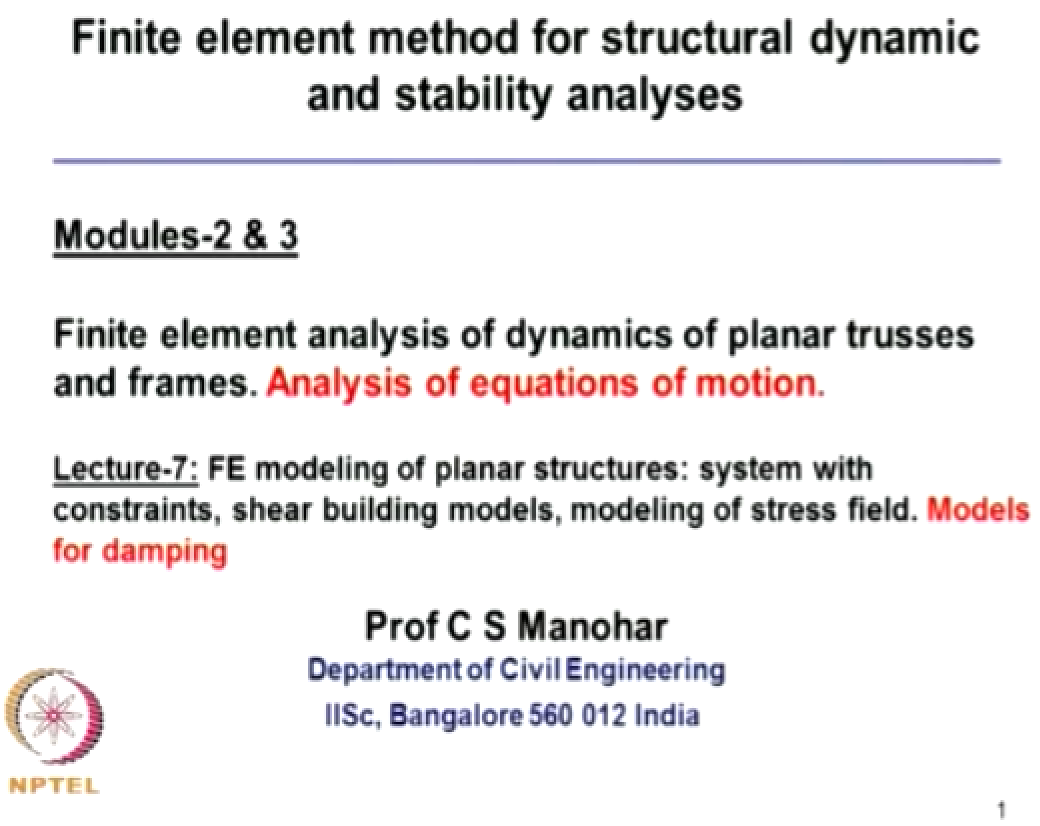

Finite element method for structural dynamic and stability analyses

Modules-2 & 3

Finite element analysis of dynamics of planar trusses and frames. Analysis of equations of motion.

Lecture-7: FE modeling of planar structures: system with constraints, shear building models, modeling of stress field. Models for damping

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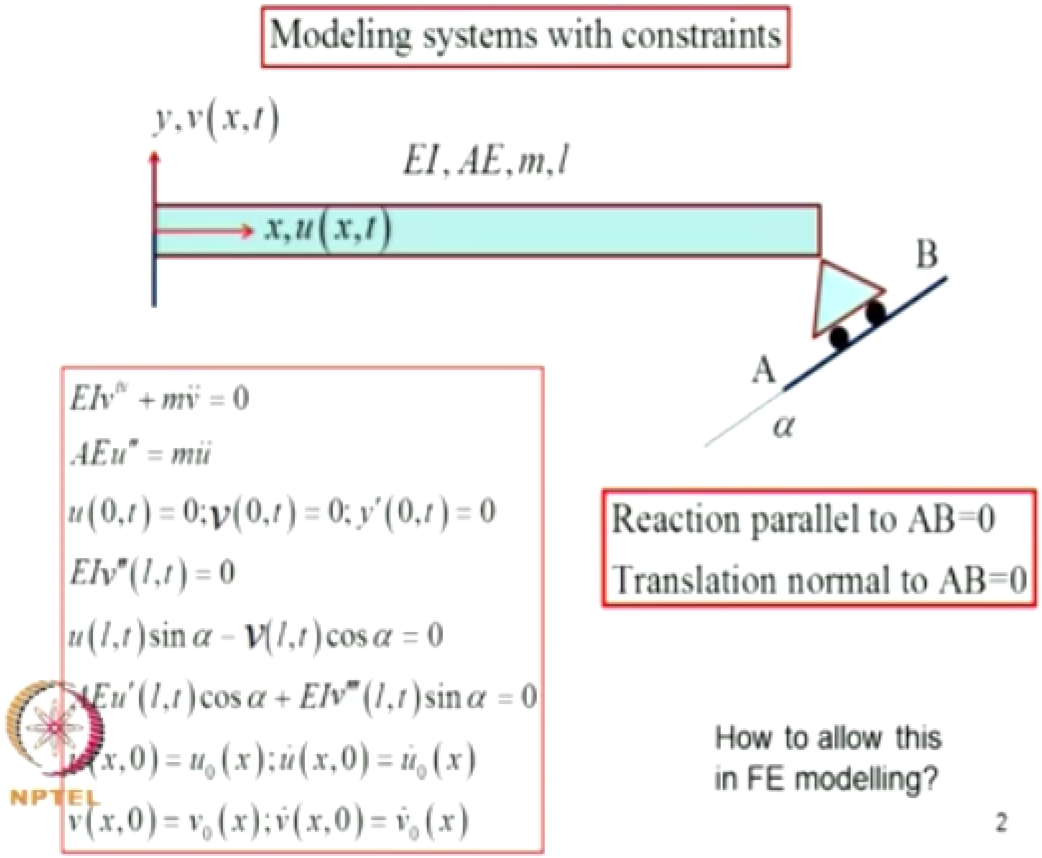
We have been discussing finite element analysis of dynamics of planar structures, we will continue with that discussion and in this lecture we will also begin talking about analysis of equations of motion which constitutes the topic for the next module.



Modeling systems with constraints

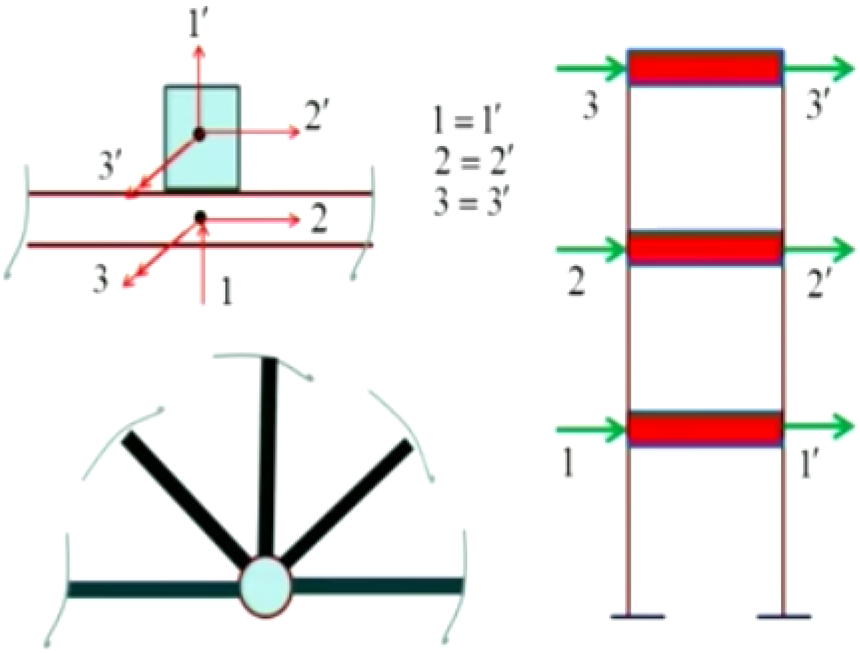
$$EIv^{iv} + m\ddot{v} = 0$$
$$AEu'' = m\ddot{u}$$
$$u(0,t) = 0; v(0,t) = 0; y'(0,t) = 0$$
$$EIv''(l,t) = 0$$
$$u(l,t)\sin\alpha - v(l,t)\cos\alpha = 0$$
$$AEu'(l,t)\cos\alpha + EIv'''(l,t)\sin\alpha = 0$$
$$u(x,0) = u_0(x); \dot{u}(x,0) = \dot{u}_0(x)$$
$$v(x,0) = v_0(x); \dot{v}(x,0) = \dot{v}_0(x)$$

Reaction parallel to AB=0
Translation normal to AB=0

How to allow this
in FE modelling?

In the previous lecture we tackle few numerical examples towards the very end I also talked about modeling systems with constraints, so we will address that specific aspect to start with, so if we consider a say cantilever beam which is now supported on a roller which is inclined to the horizontal through an angle alpha, so how do we formulate this problem, suppose we want to formulate this problem within the framework of axial vibration of bars and Euler Bernoulli beam theory using the partial differential equation approach we can write the equilibrium equation for the dependent variable V which is the transverse displacement, using Euler Bernoulli beam theory which reads this and similarly for the axial vibration we will be able to write the second order equation for the axial displacement U.

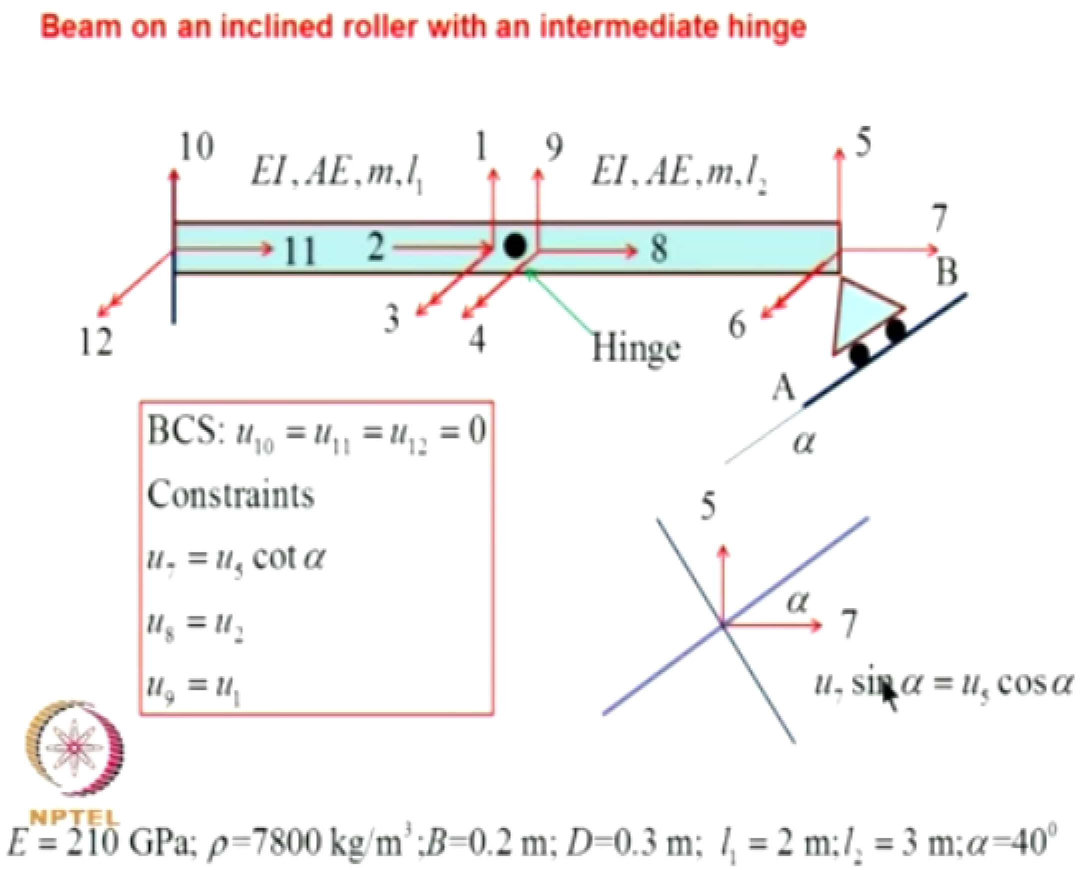
Now the boundary conditions at X = 0, U is 0, and V and V prime are 0, similarly at X = L the bending moment is 0, when we formulate the equation for the conditions at this end on displacements and reactions we need to honor 2 facts, namely reactions parallel to AB is 0 because the roller is in this direction, the reaction parallel to AB would be 0, similarly translation normal to AB would be 0, so if we use that for example if that translation normal to AB is 0, I will get the equation U sin alpha - y cos alpha = 0, similarly the shear force the reaction normal to AB is 0, will lead to this equation, so these are the initial conditions so we would see that the axial deformation and transverse deformation due to bending are coupled through the boundary conditions. So how do we allow for this in the finite element formulation, so this type of situations are also encountered in a few other context, many other context I am just outlining a few, for example in this figure we show say a plate, a slab on which we have



Ball and socket joint

placed a beam, so the centroidal axis for the slab and the mid plane of the slab and a centroidal axis of the beam are different, but we would like the deformations 1, 2, 3, the displacements 1, 2, 3 must be equal to 1 prime, 2 prime, 3 prime, so we will have to locate the nodes at 2 different places and then impose this as additional conditions.

Similarly in a ball and socket joint, at a joint if several members meet they share the common translations in, let’s say for example in this direction and in the Y direction, but each member can rotate independent of the others, this is not a rigid joint, so they can rotate independent of each other. This is an example of a so called shear building frame where we assume that the slabs are infinitely rigid in their own plane, therefore the actual displacement in, the displacement along this direction at this point and at this point are the same, so similarly 2 is 2 prime, 3 is 3 prime, so we have to impose this type of condition, so the question is how do we tackle this in finite element modeling, so to what I will do is I will illustrate this with a typical



Beam on an inclined roller with an intermediate hinge

BCS: $u_{10} = u_{11} = u_{12} = 0$

Constraints

$u_7 = u_5 \cot\alpha$

$u_8 = u_2$

$u_9 = u_1$

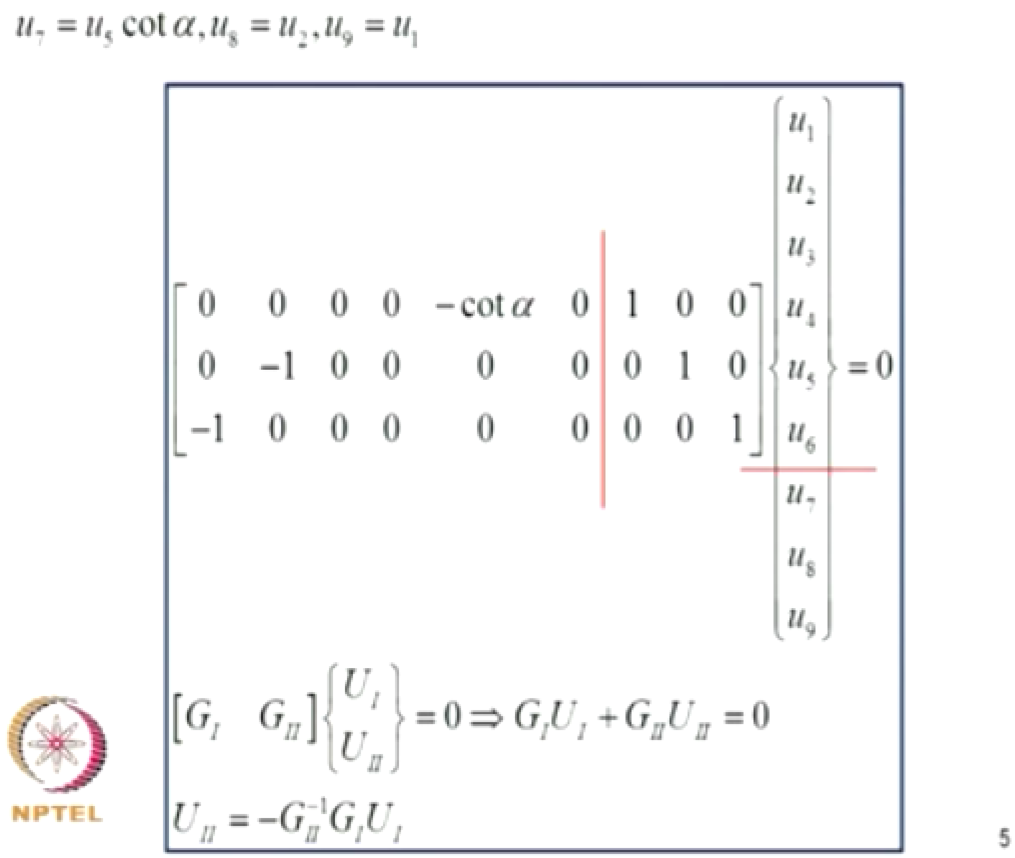
$u_7 \sin\alpha = u_5 \cos\alpha$

$E = 210$ GPa; ρ=7800 kg/m^3;B=0.2 m; D=0.3 m; $l_1 = 2$ m;$l_2 = 3$ m;α=40^0

example which contains some of the features that the previous 3 examples included, so let us consider again the cantilever beam on an inclined roller so for sake of discussion we will also introduce a hinge at the middle at a some distance L1.

Now so here at X = L, the roller is on an inclined plane alpha and here X= L1, I have shown the degrees of freedom at two distinct points but we should understand that there at the same geometric location, now the boundary conditions are U10 is 0, U11 is 0, U12 is 0, so what I have done I have made two elements here and the degrees of freedom are labeled as shown here, the constraint equations are now the translation normal to line AB is 0 therefore if you pay attention here so I will have U5 cos alpha must be equal to U7 sin alpha, then if this condition is satisfied in a translation normal to AB would be 0, that would mean U7 must be equal to U5 cot alpha.

Now U8 must be equal to U2, these 2 translations must be the same and U9 must be equal to U1, that is U9 must be equal to U1, but U3 need not be equal to U4, because this is a in, so now these equations will now appear as set of constraints that we need to explicitly handle, so I am going to do a numerical example and the data for that is listed here, so we will come to this as we go along.



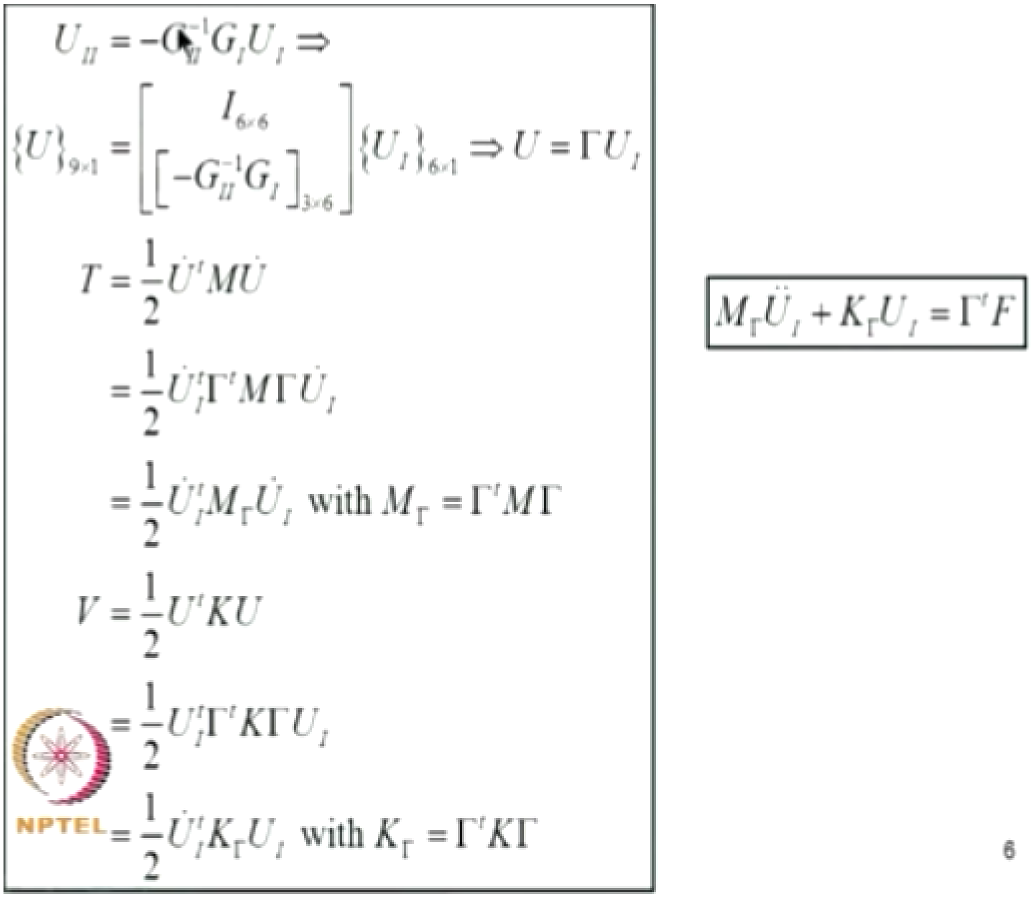
$$u_7 = u_5 \cot\alpha, u_8 = u_2, u_9 = u_1$$

$$\left[\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & -\cot\alpha & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \hline u_7 \\ u_8 \\ u_9 \end{array}\right\} = 0$$

$$\begin{bmatrix} G_I & G_{II} \end{bmatrix} \begin{Bmatrix} U_I \\ U_{II} \end{Bmatrix} = 0 \Rightarrow G_I U_I + G_{II} U_{II} = 0$$

$$U_{II} = -G_{II}^{-1} G_I U_I$$

Now the constraint equation U7 = U5 cot alpha, U8 = U2, U9 = U1 can be written in a matrix form as shown here, so the degrees of freedom here are since 10, 11, 12 are 0, the degrees of freedom are 1, 2, 3, 4, 5, 6, 7, 8, 9, so they are listed here, now these 9 degrees of freedom are constrained by these 3 constraint equations, so these 3 constraint equations are written here. For example 1, 2, 3, 4, 5 - U5 cot alpha is U7, U7 okay I think, yeah the first equation this requires some correction I will do that, then U2 - U2 + U8 = 0, so that is fine and similarly U1 - U1 +U9 must be equal to 0, so this can be put in this matrix form, so we can right now partition now this displacement vector into U1 and U2, where U1 are this 1, 2, 3, 4, 5, 6 which I will retain as degrees of freedom and I will eliminate 7, 8, 9 using these 3 equations in terms of U1 to U6 through this operation.



$$U_{II} = -G_{II}^{-1} G_I U_I \Rightarrow$$

$$\{U\}_{9\times1} = \begin{bmatrix} I_{6\times6} \\ \left[-G_{II}^{-1}G_I\right]_{3\times6} \end{bmatrix} \{U_I\}_{6\times1} \Rightarrow U = \Gamma U_I$$

$$T = \frac{1}{2}\dot{U}^t M \dot{U}$$

$$= \frac{1}{2}\dot{U}_I^t \Gamma^t M \Gamma \dot{U}_I$$

$$= \frac{1}{2}\dot{U}_I^t M_\Gamma \dot{U}_I \text{ with } M_\Gamma = \Gamma^t M \Gamma$$

$$V = \frac{1}{2}U^t K U$$

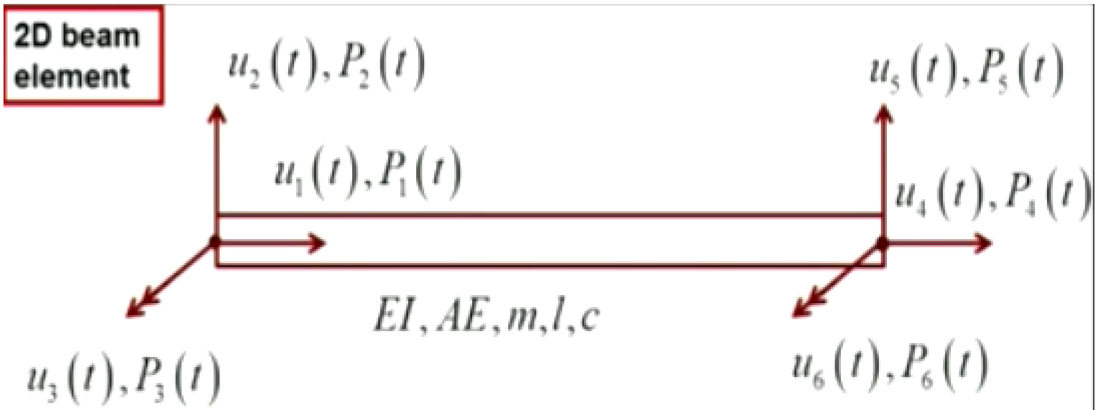
$$= \frac{1}{2}U_I^t \Gamma^t K \Gamma U_I$$

$$= \frac{1}{2}\dot{U}_I^t K_\Gamma U_I \text{ with } K_\Gamma = \Gamma^t K \Gamma$$

$$\boxed{M_\Gamma \ddot{U}_I + K_\Gamma U_I = \Gamma^t F}$$

Now U2 therefore is given by - G2 inverse G1 UI, so U9, the U9 cross 1 can be written as in terms of this transformation as first 6 degrees of freedom remain the same, the next 3 degrees of freedom are related to the first 3 through this relation, so I get U is gamma UI. Now the kinetic energy in the system is half U dot transpose MU dot, now for U I will make this substitution, so I will get this, so U dot transpose will be U1 transpose U1 dot transpose gamma transpose M gamma U1 dot, so this gamma transpose M gamma I will call it as M gamma which is now the matrix after imposing the constraints, so the kinetic energy becomes this similarly the potential energy would become something like this and the equation of motion can be written in this form, so it is this equation of motion we need to handle.

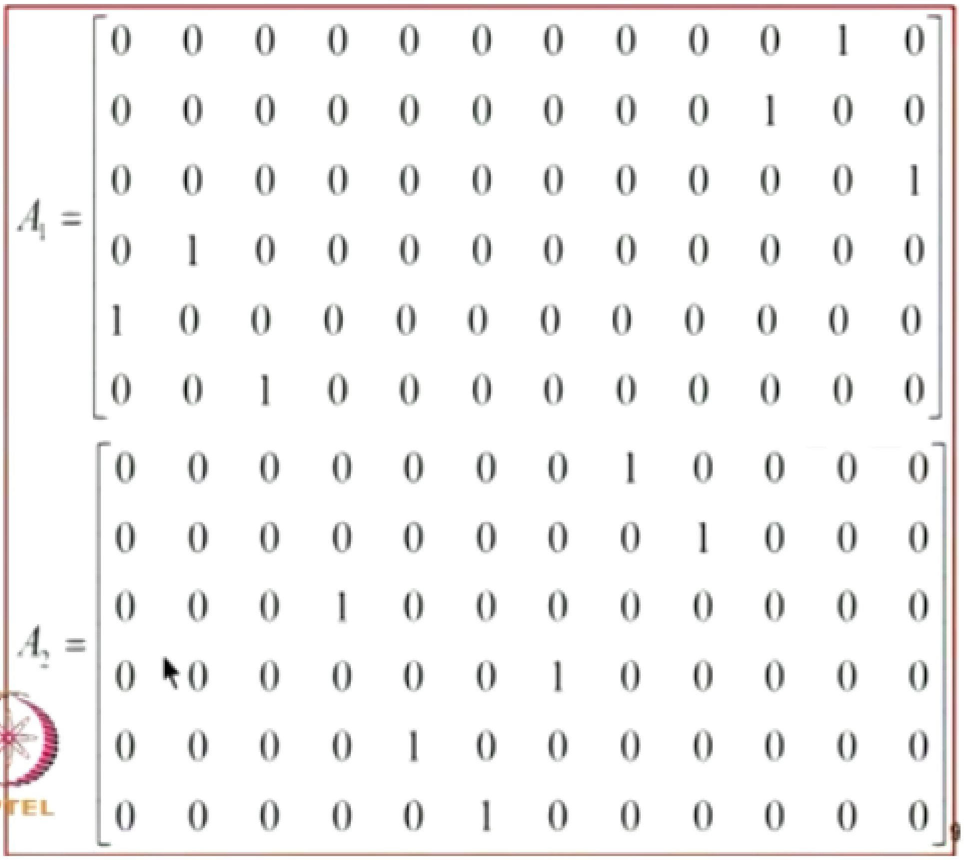
2D beam element



$$M = \frac{mL}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22l & 0 & 54 & -13l \\ 0 & 22l & 4l^2 & 0 & 13l & -3l^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13l & 0 & 156 & -22l \\ 0 & -13l & -3l^2 & 0 & -22l & 4l^2 \end{bmatrix}$$

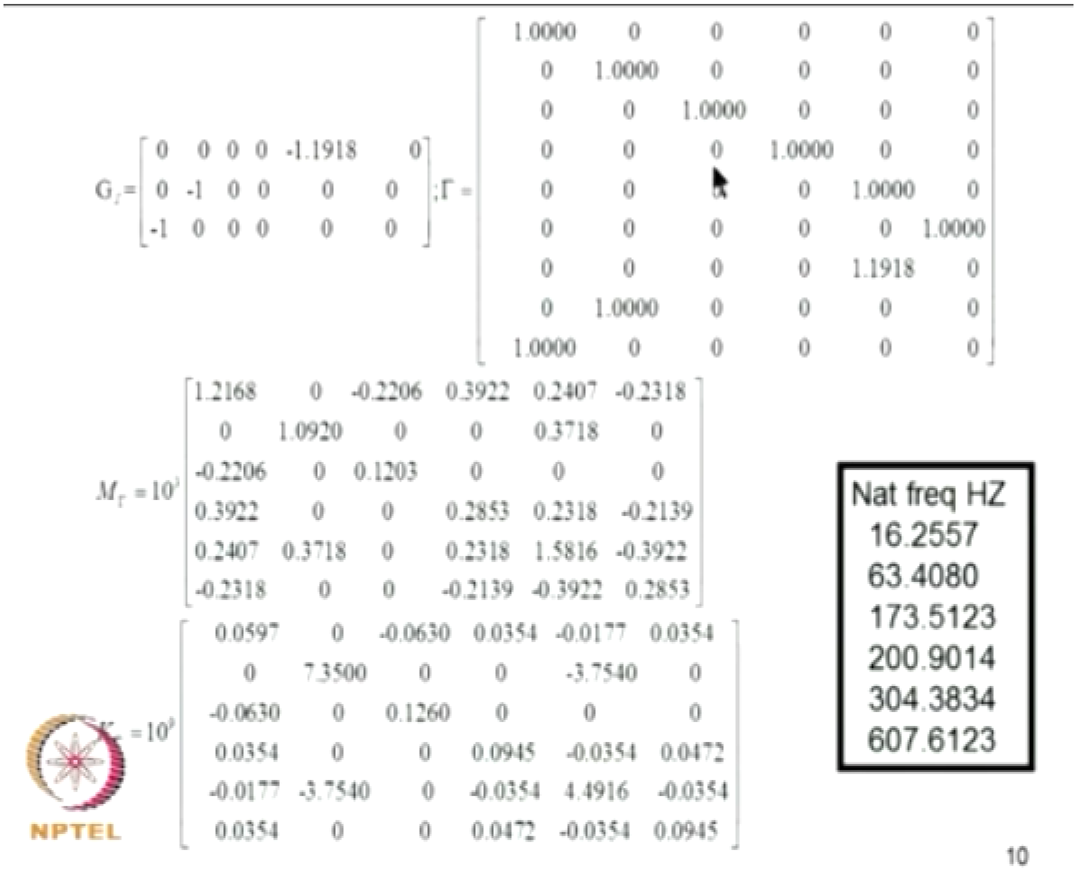
$$K = \frac{EI}{l^3}\begin{bmatrix} \left(\frac{l}{r}\right)^2 & 0 & 0 & -\left(\frac{l}{r}\right)^2 & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\left(\frac{l}{r}\right)^2 & 0 & 0 & \left(\frac{l}{r}\right)^2 & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix}; r = \sqrt{\frac{I}{A}}$$

So coming back to the numerical example we have this beam element, this is the mass matrix, this is the stiffness matrix which we have done in the previous class, so we will now assemble so there are two elements here 1 and 2 so if I will construct the A1, A2 matrices which I have shown here,



$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



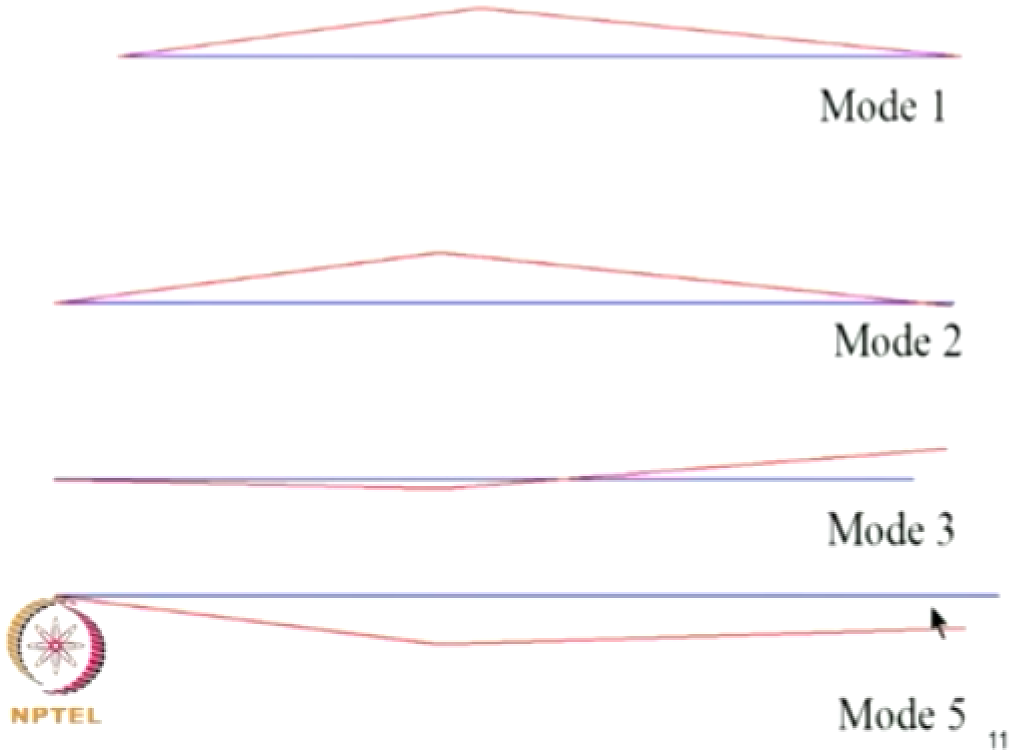
$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1.1918 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \Gamma = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0 & 1.1918 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{\Gamma} = 10^3 \begin{bmatrix} 1.2168 & 0 & -0.2206 & 0.3922 & 0.2407 & -0.2318 \\ 0 & 1.0920 & 0 & 0 & 0.3718 & 0 \\ -0.2206 & 0 & 0.1203 & 0 & 0 & 0 \\ 0.3922 & 0 & 0 & 0.2853 & 0.2318 & -0.2139 \\ 0.2407 & 0.3718 & 0 & 0.2318 & 1.5816 & -0.3922 \\ -0.2318 & 0 & 0 & -0.2139 & -0.3922 & 0.2853 \end{bmatrix}$$

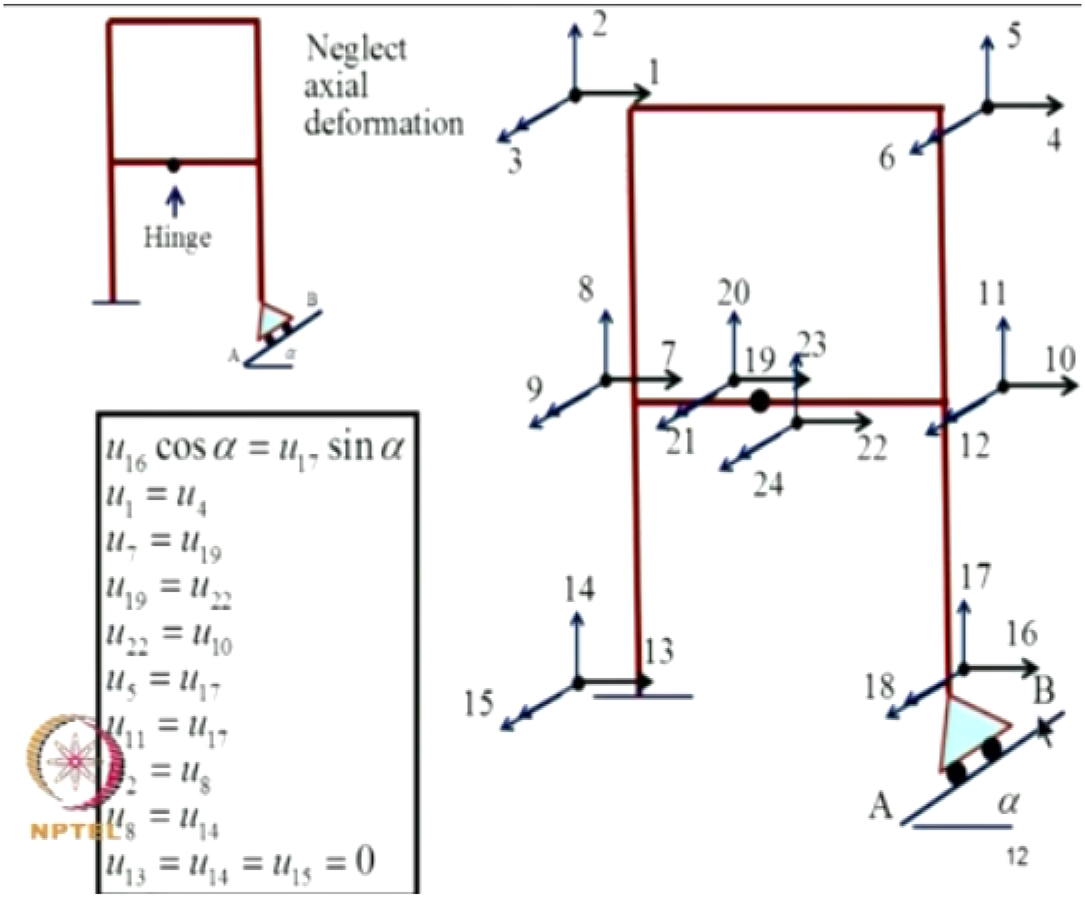
$$K = 10^3 \begin{bmatrix} 0.0597 & 0 & -0.0630 & 0.0354 & -0.0177 & 0.0354 \\ 0 & 7.3500 & 0 & 0 & -3.7540 & 0 \\ -0.0630 & 0 & 0.1260 & 0 & 0 & 0 \\ 0.0354 & 0 & 0 & 0.0945 & -0.0354 & 0.0472 \\ -0.0177 & -3.7540 & 0 & -0.0354 & 4.4916 & -0.0354 \\ 0.0354 & 0 & 0 & 0.0472 & -0.0354 & 0.0945 \end{bmatrix}$$

Nat freq HZ
16.2557
63.4080
173.5123
200.9014
304.3834
607.6123

then G1 matrix is this, gamma matrix is this, so if you now do gamma transpose M gamma you will get M gamma which is this matrix, this is KM, so now if we do the eigenvalue analysis on this problem, these are the natural frequencies that I get.



Now we can draw the mode shapes just for illustration I have shown the first few modes, the first mode shape will be in this form you can see here there is a you know lack of differentiability at this point, although the mode shapes are continuous because of the hinge here since the two rotations can be different there is a lack of differentiation on that, so this is mode 2, mode 3, if you see that this line if you draw the motion of this free end will be along the inclined plane on which the roller moves, so that is seen here.



$u_{16} \cos\alpha = u_{17} \sin\alpha$

$u_1 = u_4$

$u_7 = u_{19}$

$u_{19} = u_{22}$

$u_{22} = u_{10}$

$u_5 = u_{17}$

$u_{11} = u_{17}$

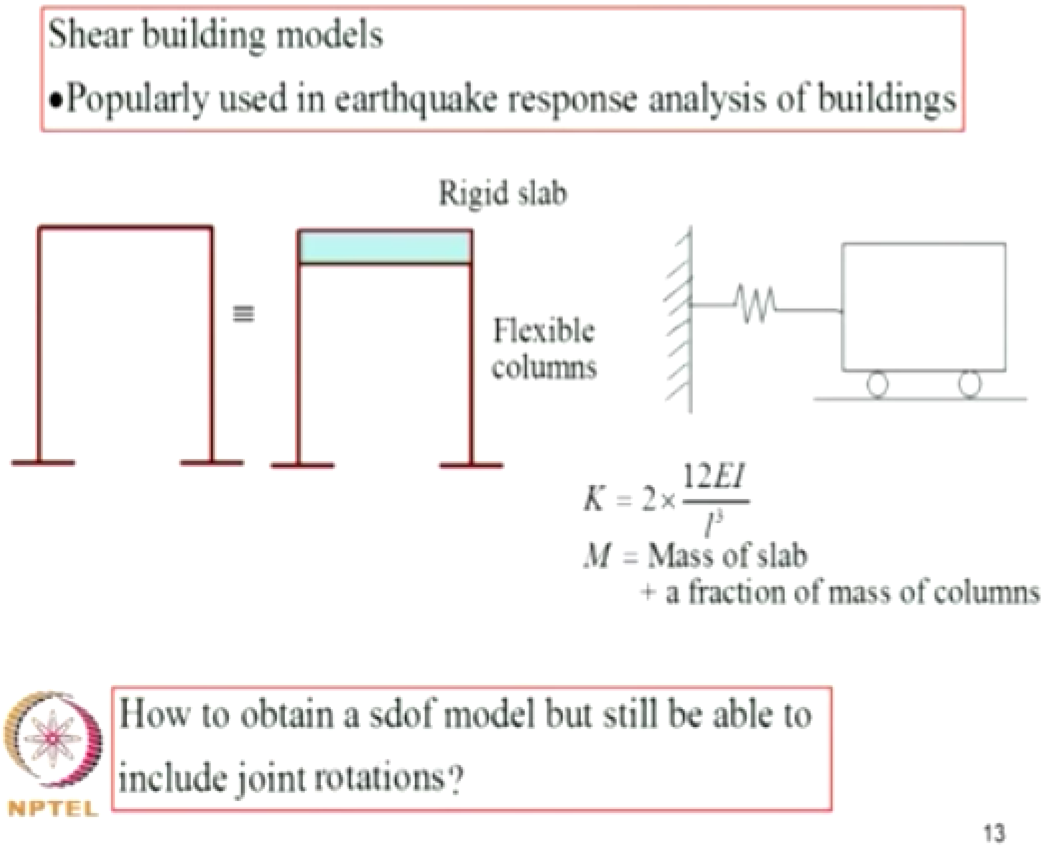
$_{2} = u_8$

$u_8 = u_{14}$

$u_{13} = u_{14} = u_{15} = 0$

Now a similar example in the analysis of a frame for example, suppose we consider a 2 story frame again for our illustration we'll have a roller on an inclined plane and a hinge, now additionally what I will say is we can ignore axial deformation, so what will be the constraint equations, what are the constraint equation again here I have U16 cos alpha is U17 sin alpha, in the numbering schemes are here, U1 is U4 that means this translation is same as U4 because we are ignoring axial deformation, similarly U7 is U10, U7 is U19 because there is a hinge here I have formed two elements again they are geometrically on the same point although they are shown at a slightly different place, so U7 is U19, U22 is U10, then U5 is U17, U5, U5 is U17, then U11 is U17, U17 incidentally is node 0 because this point moves, then U2 is U8, U2 is U8, U8 is U14 and we also have the boundary conditions here U13, 14, 15 = 0.

So now we can write this all these equations as a constraint equations and we can eliminate the constraints and apply the boundary conditions and we will be able to analyze this problem.



Shear building models

- Popularly used in earthquake response analysis of buildings

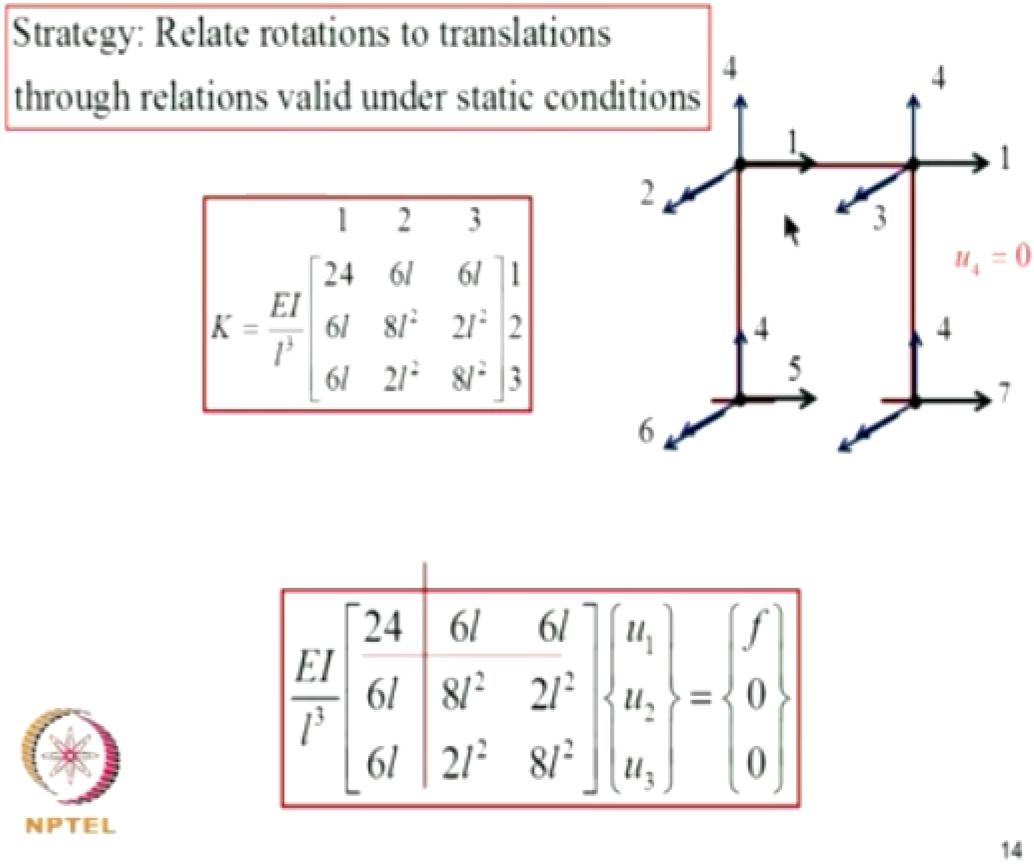
$$K = 2\times\frac{12EI}{l^3}$$

M = Mass of slab
+ a fraction of mass of columns

How to obtain a sdof model but still be able to include joint rotations?

Now a few remarks on what are known as shear building models which are commonly used in earthquake response analysis of structures, suppose if you have a one-story portal frame like this typically what we assume is this is, this represents a slab and slab is rigid, infinitely rigid in relation to the column in this direction, that is as far as horizontal motion is concerned the slab can be taken as infinitely rigid in its own plane, so the stiffness for motion in this direction is comes basically through the flexibility of the columns, the stiffness of the columns, so slabs are infinitely rigid, columns are flexible, so we can make a simple one degree freedom approximation where we lump the stiffness of the columns through a spring, and lump the mass of the slab and maybe a part of mass of the columns as a point mass, so typically we take the mass to be mass of the slab and a fraction of mass of the columns, the stiffness is 2 into 12EI / L cube, because we take that in the contribution of, to the horizontal stiffness of the frame due to each column can be added and we get this.

Now this is an heuristic simple model, this ignores the possible rotations at the joints. Now if I ask the question I would like to arrive at a single degree freedom model just the way I have done here, but I want to now make allowance for possible rotation of the joints, so the question is therefore how to obtain an SDOF model but still be able to include joint rotations, see we have seen in previous examples that by including joint rotation the system will have 3 degrees



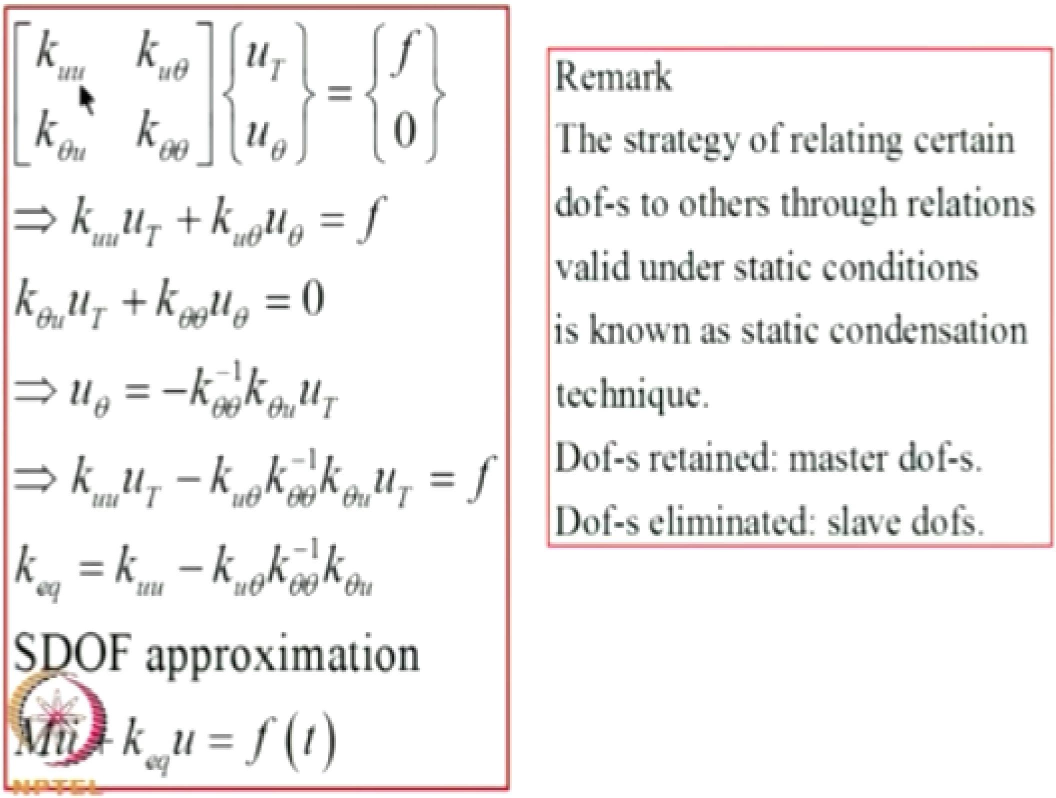
Strategy: Relate rotations to translations through relations valid under static conditions

$$K=\frac{EI}{l^3}\begin{array}{cccc} 1 & 2 & 3 & \\ 24 & 6l & 6l & 1 \\ 6l & 8l^2 & 2l^2 & 2 \\ 6l & 2l^2 & 8l^2 & 3 \end{array}$$

$$\frac{EI}{l^3}\left[\begin{array}{c|cc} 24 & 6l & 6l \\ \hline 6l & 8l^2 & 2l^2 \\ 6l & 2l^2 & 8l^2 \end{array}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} f \\ 0 \\ 0 \end{Bmatrix}$$

of freedom, for example 1, 2, and 3, now 2 and 3 were put to 0 in the previous model, only one was taken as a degree of freedom. Now if we now take 1, 2, 3 as degrees of freedom we have constructed this 3 by 3 stiffness matrix in earlier one of the earlier examples, and this is the stiffness matrix we get.

Now one of the strategy that we adopt, we can adopt to arrive at simplified models which do not ignore joint rotations is to relate rotations to translations through relations which are valid under static conditions, so suppose if you now consider static equilibrium of this system under some horizontal force F as shown here, the equilibrium equations will be this. Now what we do is U1 is the translation, U2, U3 are the rotations, so what we do is we partition the translation and the displacement and the force vector into a translation and set of rotations, that imposes a the partitioning of stiffness matrix as well, we can write it in general form as K UU K U theta,



$$\begin{bmatrix} k_{uu} & k_{u\theta} \\ k_{\theta u} & k_{\theta\theta} \end{bmatrix} \begin{Bmatrix} u_T \\ u_\theta \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$

$$\Rightarrow k_{uu}u_T + k_{u\theta}u_\theta = f$$

$$k_{\theta u}u_T + k_{\theta\theta}u_\theta = 0$$

$$\Rightarrow u_\theta = -k_{\theta\theta}^{-1}k_{\theta u}u_T$$

$$\Rightarrow k_{uu}u_T - k_{u\theta}k_{\theta\theta}^{-1}k_{\theta u}u_T = f$$

$$k_{eq} = k_{uu} - k_{u\theta}k_{\theta\theta}^{-1}k_{\theta u}$$

SDOF approximation

$$M\ddot{u} + k_{eq}u = f(t)$$

Remark

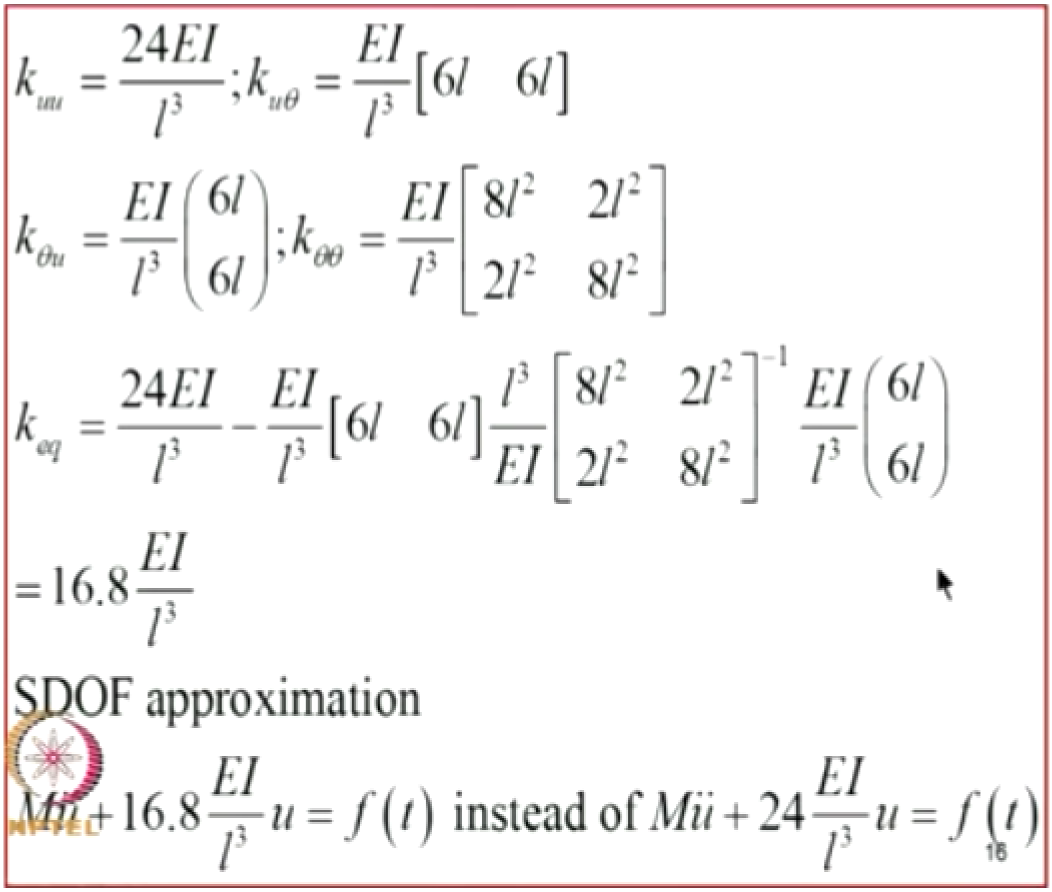
The strategy of relating certain dof-s to others through relations valid under static conditions is known as static condensation technique.

Dof-s retained: master dof-s.

Dof-s eliminated: slave dofs.

K theta, K theta theta, UT for translation, U theta for rotation equal to 0. Now the first of this equation will be K UU U T + K U theta U theta = F, the second of these equations will be K theta U, UT + K theta theta U theta = 0.

Now you look at the second equation now I can eliminate U theta or relate U theta to UT through this relation, again let me emphasize that this relation is valid only under static conditions, but what we do now is we extrapolate that and say that even under dynamic situation translations are related, rotations are related to translation through this equation, so consequently what happens if you now substitute for U theta in the first equation we get this equation and we can see that the equivalent stiffness will be now K UU - this extra term which we had not included earlier, we're simply taken K UU as the stiffness term in an earlier calculation, so the single degree of freedom approximation will be there for MU double dot plus K equivalent U = F(t), where M is still the mass of the slab plus part of mass of the columns, so we can observe that the strategy of relating certain degrees of freedoms to other through relations valid under static conditions is known as static conditions condensation technique, so the degrees of freedom that are retained are called master degrees of freedom, and the degrees of freedom which are eliminated are called slave degrees of freedom.



$$k_{uu} = \frac{24EI}{l^3}; k_{u\theta} = \frac{EI}{l^3}\begin{bmatrix} 6l & 6l \end{bmatrix}$$

$$k_{\theta u} = \frac{EI}{l^3}\begin{pmatrix} 6l \\ 6l \end{pmatrix}; k_{\theta\theta} = \frac{EI}{l^3}\begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}$$

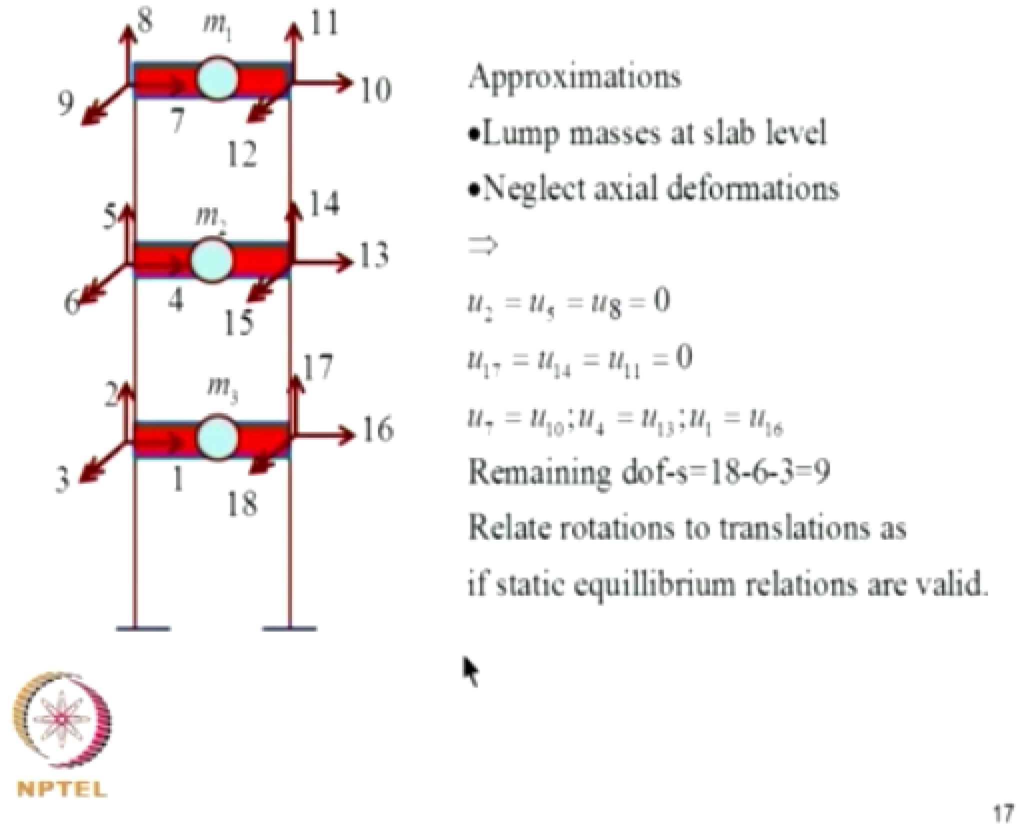
$$k_{eq} = \frac{24EI}{l^3} - \frac{EI}{l^3}\begin{bmatrix} 6l & 6l \end{bmatrix}\frac{l^3}{EI}\begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix}^{-1}\frac{EI}{l^3}\begin{pmatrix} 6l \\ 6l \end{pmatrix}$$

$$= 16.8\frac{EI}{l^3}$$

SDOF approximation

$$M\ddot{u} + 16.8\frac{EI}{l^3}u = f(t) \text{ instead of } M\ddot{u} + 24\frac{EI}{l^3}u = f(t)$$

Now if we apply this strategy to the problem on hand following the partitioning that is shown here and using the notation that is outlined here we get K UU as this, K U theta as this and K theta U is this, K theta theta is this, now K equivalent now can be computed through this relation and if you do that we get 16.8 EI / L cube, so in this single degree of freedom approximation the governing equation is MU double dot 16.8 DI by L cube U = F(t), so this model includes joint rotation therefore this has lesser stiffness than the one where the joint rotations were ignored in which case we got MU double dot 24 EI / L cube U = F(t).



Approximations

- Lump masses at slab level
- Neglect axial deformations

$\Rightarrow$

$u_2 = u_5 = u_8 = 0$

$u_{17} = u_{14} = u_{11} = 0$

$u_7 = u_{10}; u_4 = u_{13}; u_1 = u_{16}$

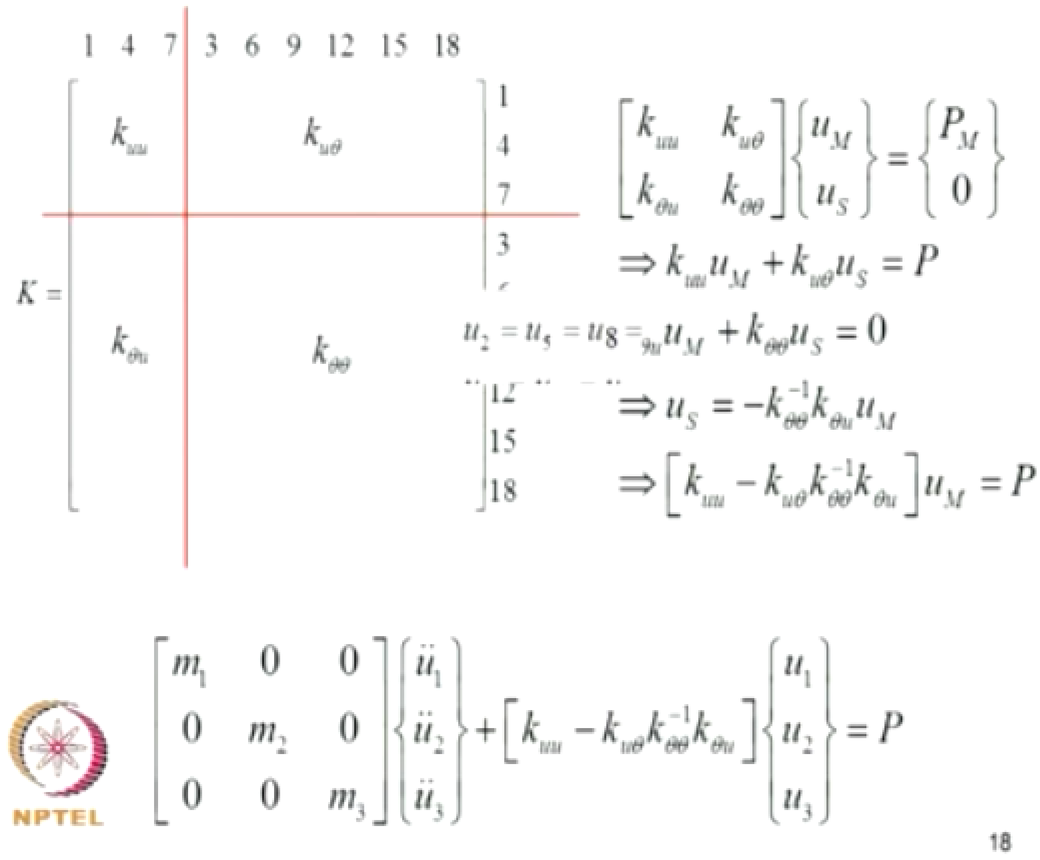
Remaining dof-s=18-6-3=9

Relate rotations to translations as

if static equillibrium relations are valid.

Now the same logic if we now extend to say 3-story frame, so there are 3 slabs, I lumped the mass of the slabs and some of the column mass at these places, and the 3 degrees of freedom that I wish to retain are the translation that is U16, U13, and U10, so the idealization that we are adopting is lump masses at slab level, neglect axial deformation, so if I do that U2 = U5 = this is U8 = 0 then U17, U14, U11 = 0, then U7 is U10, U4 is U13, U1 is U16, so now what are the remaining degrees of freedom, there are 18 degrees of freedom in this so 6 degrees of freedom 18 – 6 – 3 if we eliminate there are 9 degrees of freedom.

Now in the remaining 9 degrees of freedom which are basically the translations here and rotations at these joints, we want to now eliminate rotations and relate them to translation through relations which are valid only for static conditions, so that is a static condensation



$$K = \begin{array}{ccc|c} 1\ 4\ 7 & 3\ 6\ 9\ 12\ 15\ 18 & \\ k_{uu} & k_{u\theta} & 1,4,7 \\ \hline k_{\theta u} & k_{\theta\theta} & 3,\ ,\ ,12,15,18 \end{array}$$

$$\begin{bmatrix} k_{uu} & k_{u\theta} \\ k_{\theta u} & k_{\theta\theta} \end{bmatrix} \begin{Bmatrix} u_M \\ u_S \end{Bmatrix} = \begin{Bmatrix} P_M \\ 0 \end{Bmatrix}$$

$$\Rightarrow k_{uu}u_M + k_{u\theta}u_S = P$$

$$u_2 = u_5 = u_8 = {}_{\theta u}u_M + k_{\theta\theta}u_S = 0$$

$$\Rightarrow u_S = -k_{\theta\theta}^{-1}k_{\theta u}u_M$$

$$\Rightarrow \left[k_{uu} - k_{u\theta}k_{\theta\theta}^{-1}k_{\theta u}\right]u_M = P$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \left[k_{uu} - k_{u\theta}k_{\theta\theta}^{-1}k_{\theta u}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = P$$

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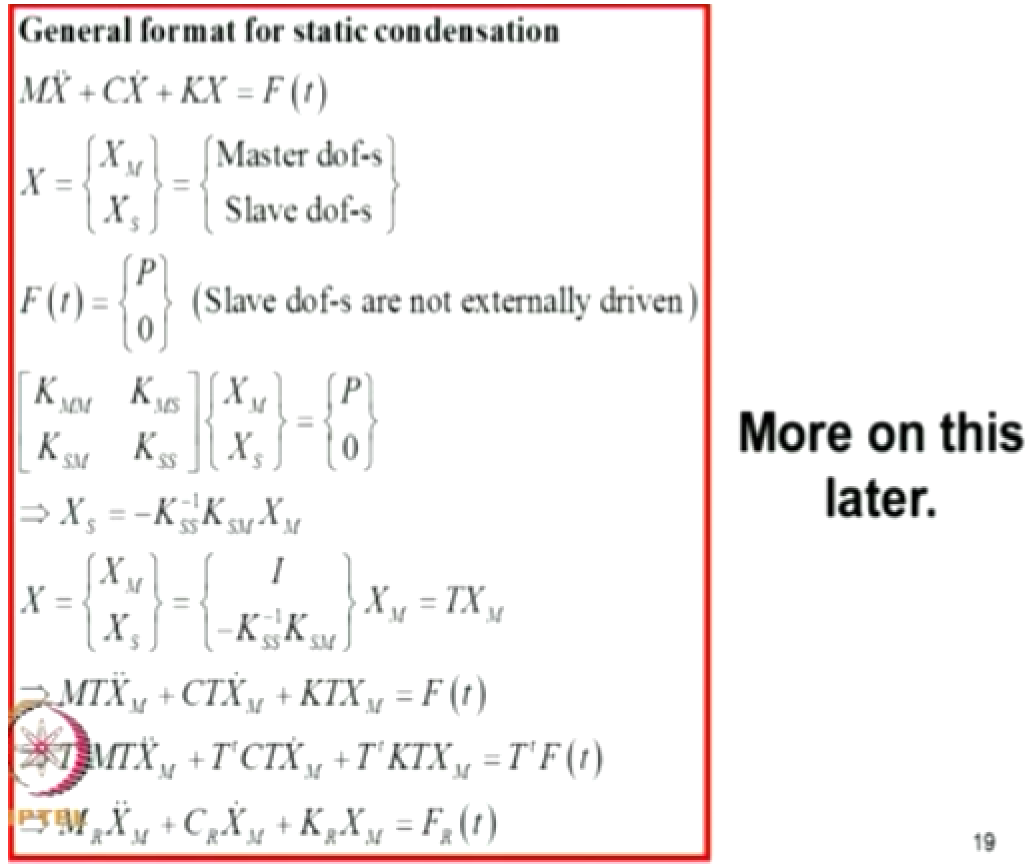
strategy if I do that now 1, 4, 7, that is 1, 4, 7 are the degrees of freedom the so called master degrees of freedom which I wish to retain, and 3, 6, 9, 12, 15, 18, that is 3, 6, 9, 12, 15, 18 which are joint rotations I want to eliminate their slaves, so I partition like this. So K UU is now 3 by 3, this is 3 by 6, this is 6 by 3, and this is 6 by 6.

Now I will write the equilibrium equation for the frame under static conditions, and I get this equation, so the first of the equation as usual leads to this, and the second one here the subscripts M and S are master and slaves, the second equation can be used to eliminate slaves in terms of masters and if you put back this relation into the original equation I will get this as the governing equation, so the stiffness matrix for the system which is 3 by 3 is actually given by this, the mass matrix is diagonal M1, M2, M3 and all entries are 0, these are the 3 horizontal accelerations this is the equivalent stiffness matrix which takes into account, joint rotations okay, so I get this as 3 by 3 matrix, in the traditional shear building model where joint rotations are ignored this correction would be absent, okay, so we get this equation. So that can be further analyzed to get the properties of this structure.



General format for static condensation

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$X = \begin{Bmatrix} X_M \\ X_S \end{Bmatrix} = \begin{Bmatrix} \text{Master dof-s} \\ \text{Slave dof-s} \end{Bmatrix}$$

$$F(t) = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \text{ (Slave dof-s are not externally driven)}$$

$$\begin{bmatrix} K_{MM} & K_{MS} \\ K_{SM} & K_{SS} \end{bmatrix} \begin{Bmatrix} X_M \\ X_S \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

$$\Rightarrow X_S = -K_{SS}^{-1} K_{SM} X_M$$

$$X = \begin{Bmatrix} X_M \\ X_S \end{Bmatrix} = \begin{Bmatrix} I \\ -K_{SS}^{-1} K_{SM} \end{Bmatrix} X_M = TX_M$$

$$\Rightarrow MT\ddot{X}_M + CT\dot{X}_M + KTX_M = F(t)$$

$$\Rightarrow T^t MT\ddot{X}_M + T^t CT\dot{X}_M + T^t KTX_M = T^t F(t)$$

$$\Rightarrow M_R\ddot{X}_M + C_R\dot{X}_M + K_R X_M = F_R(t)$$

More on this later.

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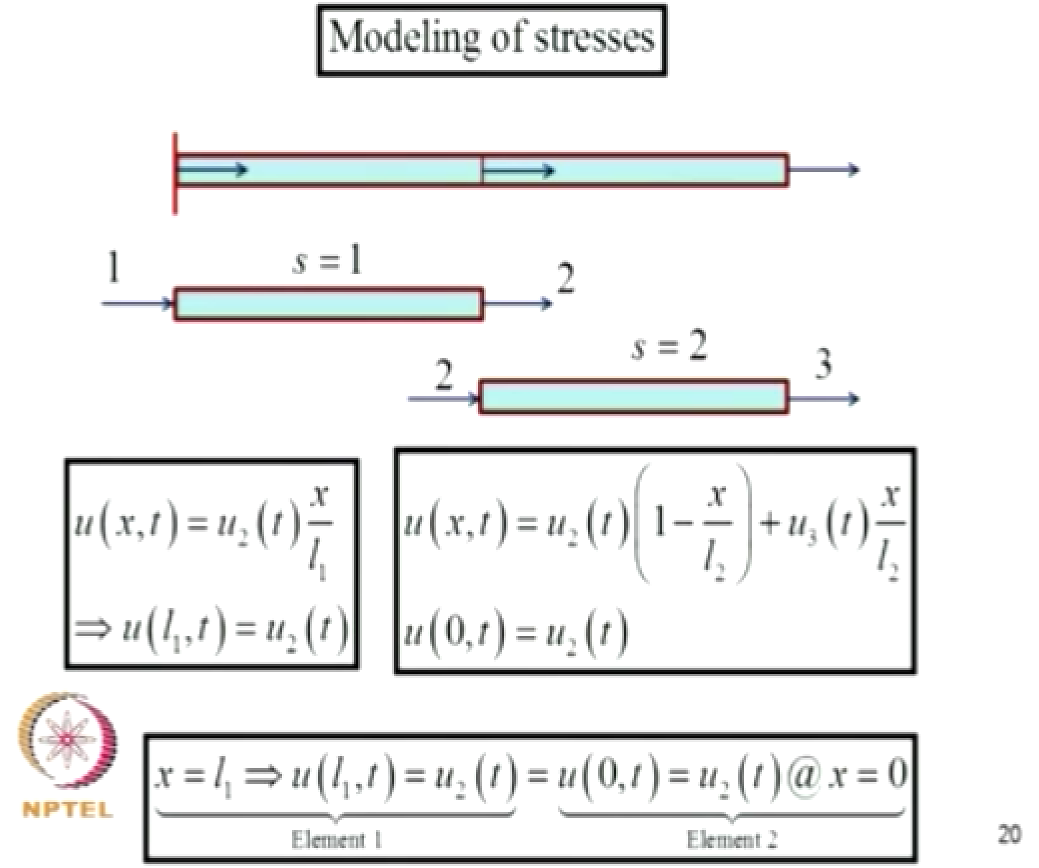
Now the general format for the strategy that I am talking about the so-called static condensation in the context of dynamical systems can be written as, can be explained as follows, suppose if you consider equation of motion for N degree freedom system as shown here we partition the degrees of freedom into masters and slaves, and the partitioning should ensure that the slave degrees of freedom do not receive any external force, there is only the masters which are driven, so then we write the static equilibrium equation and relate the slave degrees of freedom to the master degrees of freedom through relations which are strictly valid only under static conditions, that enables us to transform the state with, the system's displacement vector X through a transformation T, which is T matrix is I and this. Now this is a T matrix so now this transformation can be substituted into this, so X is TXM so slaves are eliminated, and I write this equation and pre multiplying by T transpose I get this equation and therefore the reduced mass matrix MR is T transpose MT, reduced damping matrix is T transpose CT, reduced stiffness matrices C transpose KT and the reduced force vector is T transpose F, so this is a general strategy which goes by the name static condensation.

We will return to this later when I talk about model reduction techniques in which at that time we will consider what are the limitations of this method and why it is needed in the first place in the modern context of modeling, and how we can improve upon this static condensation.



Modeling of stresses

$$u(x,t)=u_2(t)\frac{x}{l_1}$$
$$\Rightarrow u(l_1,t)=u_2(t)$$

$$u(x,t)=u_2(t)\left(1-\frac{x}{l_2}\right)+u_3(t)\frac{x}{l_2}$$
$$u(0,t)=u_2(t)$$

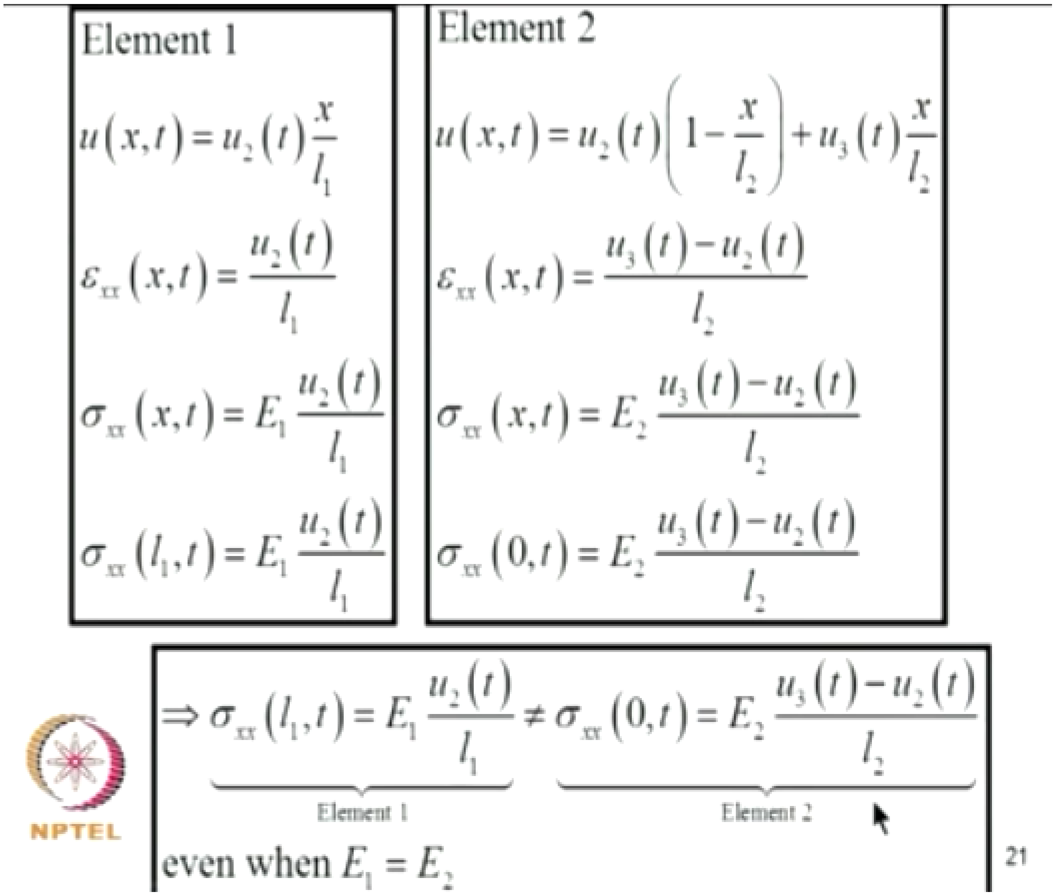
$$\underbrace{x=l_1 \Rightarrow u(l_1,t)=u_2(t)}_{\text{Element 1}}=\underbrace{u(0,t)=u_2(t)@x=0}_{\text{Element 2}}$$

Now let's talk about another aspect of modeling which we are not touched upon till now, we have now been able to formulate the problem in terms of nodal displacements and within an element we know how to interpolate the displacement field using the values at the nodes and the interpolation functions, in our analysis we are not only interested in displacements we're also interested in strains and stresses, so that issue is what I would like to now discuss, so to make the issues clear let us consider an axially vibrating rod which has been discretized into 2 elements, element number 1, element number 2.

Now within the element 1 this 1 is 0, U1 is 0 because it is a boundary condition, so U(x,t) within the first element is given by U2(t) into X / L1, so therefore at X = L1 which is the length of the first element the displacement is U2(t) which is what we have prescribed.

Now for the second element the displacement field is U2(t) into the first shape function + U3(t) into the second shape function, so again at X = 0 which is a common point for both these elements if you now compute the displacement I get U(0,t) is U2(t), so as far as displacement field is concerned across these two elements it is continuous, whether you view this point from point of view of this element or point of view of this element you get same value for the displacement.

Now that means at X = L1, U(L1,t) which is viewed from element 1 is U2(t) and viewed from element 2 which would be U(0,t) which is also U2(t), so there is no problem as far as displacement is concerned.



Element 1

$$u(x,t) = u_2(t)\frac{x}{l_1}$$

$$\varepsilon_{xx}(x,t) = \frac{u_2(t)}{l_1}$$

$$\sigma_{xx}(x,t) = E_1\frac{u_2(t)}{l_1}$$

$$\sigma_{xx}(l_1,t) = E_1\frac{u_2(t)}{l_1}$$

Element 2

$$u(x,t) = u_2(t)\left(1-\frac{x}{l_2}\right) + u_3(t)\frac{x}{l_2}$$

$$\varepsilon_{xx}(x,t) = \frac{u_3(t)-u_2(t)}{l_2}$$

$$\sigma_{xx}(x,t) = E_2\frac{u_3(t)-u_2(t)}{l_2}$$

$$\sigma_{xx}(0,t) = E_2\frac{u_3(t)-u_2(t)}{l_2}$$

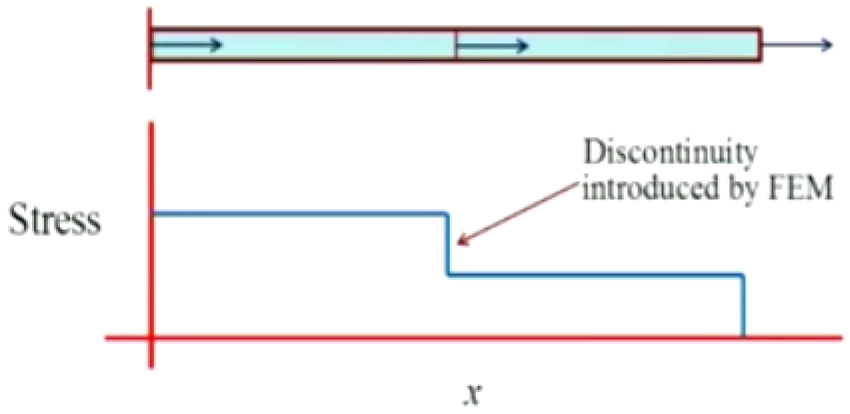
$$\Rightarrow \underbrace{\sigma_{xx}(l_1,t) = E_1\frac{u_2(t)}{l_1}}_{\text{Element 1}} \neq \underbrace{\sigma_{xx}(0,t) = E_2\frac{u_3(t)-u_2(t)}{l_2}}_{\text{Element 2}}$$

even when $E_1 = E_2$

Now how about the gradients? We know that strains are a special derivative of the displacement for example epsilon XX if you differentiate this with respect to X, I will get U2(t) / L1, stress is young's modulus into U2(t) / L1, let us assume the two elements have Young's modulus of E1 and E2, stress is therefore at X =L1 that is at the right end, the stress will be this.

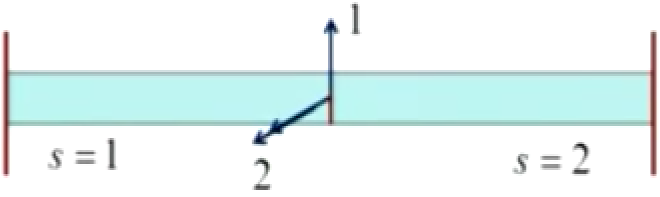
Now let's analyze element 2, this is a displacement field, this is a strain field, you differentiate with respect to X we get this. Now I take a similarly stress will be E2 into epsilon XX which is what we get here. Now at X = 0 which is a left end of the element I get now stress as E2 into U3 - U2 by L2. Now if you compare these two at the common point viewed from element 1 the stress is given by this, and viewed from element 2 the stress is given by this.

Now let us assume that the two Young's modulus are equal, even in that case the stress viewed from element 1 is different from stress at the same point viewed from element 2, so it is clear that this problem also is associated with strains, in the true solution strains must be continuous even though Young's modulus are different, strains must be the same, but even when Young's modulus are equal stresses are different, and strains also would be different, so that means if I now plot the stress as a function of X, we see that even for the system with same Young's



Displcement based FEM introduces discontinuities in spatial variation of quantities which are truly continuous.

modulus suppose both bars have the same Young's modulus I get stressed to process a discontinuity at the boundary of the element, so this is one of the limitations of finite element method that the approach that we have used the displacement based finite element method actually introduces discontinuity in special variation of quantities which are truly continuous, so the exact solution in this case could be something cover like this with no discontinuity here, this may be the point and there will be error here and there will be an error here also, so this is a limitation.



Recall

$$v(x,t)=\sum_{i=1}^{4}u_i(t)\phi_i(x)$$

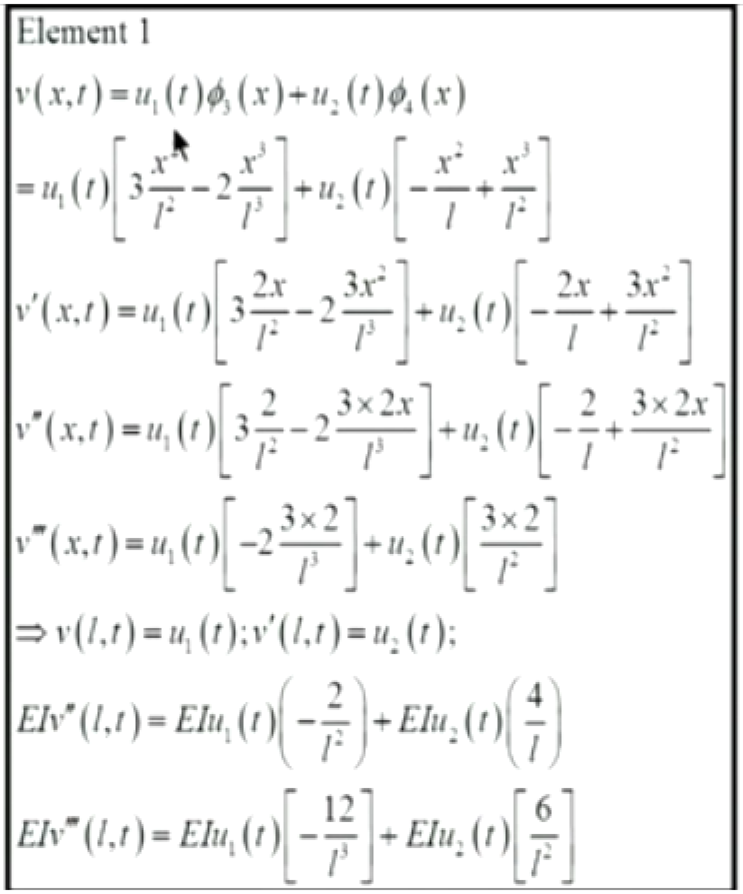
$$\phi_1(x)=1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}$$

$$\phi_2(x)=x-2\frac{x^2}{l}+\frac{x^3}{l^2};$$

$$\phi_3(x)=3\frac{x^2}{l^2}-2\frac{x^3}{l^3};$$

$$\phi_4(x)=-\frac{x^2}{l}+\frac{x^3}{l^2}$$

The same you know kind of features can be seen even for beam element, suppose if I now take a fixed beam clamped at the two ends and discretize using 2 elements S1, S2, and this is an Euler Bernoulli beam, therefore this system will have 2 degrees of freedom, 1 translation and 1 rotation, so the displacement field within each element is given by this approximation and Phi 1 Phi 2, Phi 3, Phi 4 are the cubic polynomials which we have used for interpolation.



Element 1

$$v(x,t)=u_1(t)\phi_3(x)+u_2(t)\phi_4(x)$$

$$=u_1(t)\left[3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right]+u_2(t)\left[-\frac{x^2}{l}+\frac{x^3}{l^2}\right]$$

$$v'(x,t)=u_1(t)\left[3\frac{2x}{l^2}-2\frac{3x^2}{l^3}\right]+u_2(t)\left[-\frac{2x}{l}+\frac{3x^2}{l^2}\right]$$

$$v''(x,t)=u_1(t)\left[3\frac{2}{l^2}-2\frac{3\times 2x}{l^3}\right]+u_2(t)\left[-\frac{2}{l}+\frac{3\times 2x}{l^2}\right]$$

$$v'''(x,t)=u_1(t)\left[-2\frac{3\times 2}{l^3}\right]+u_2(t)\left[\frac{3\times 2}{l^2}\right]$$

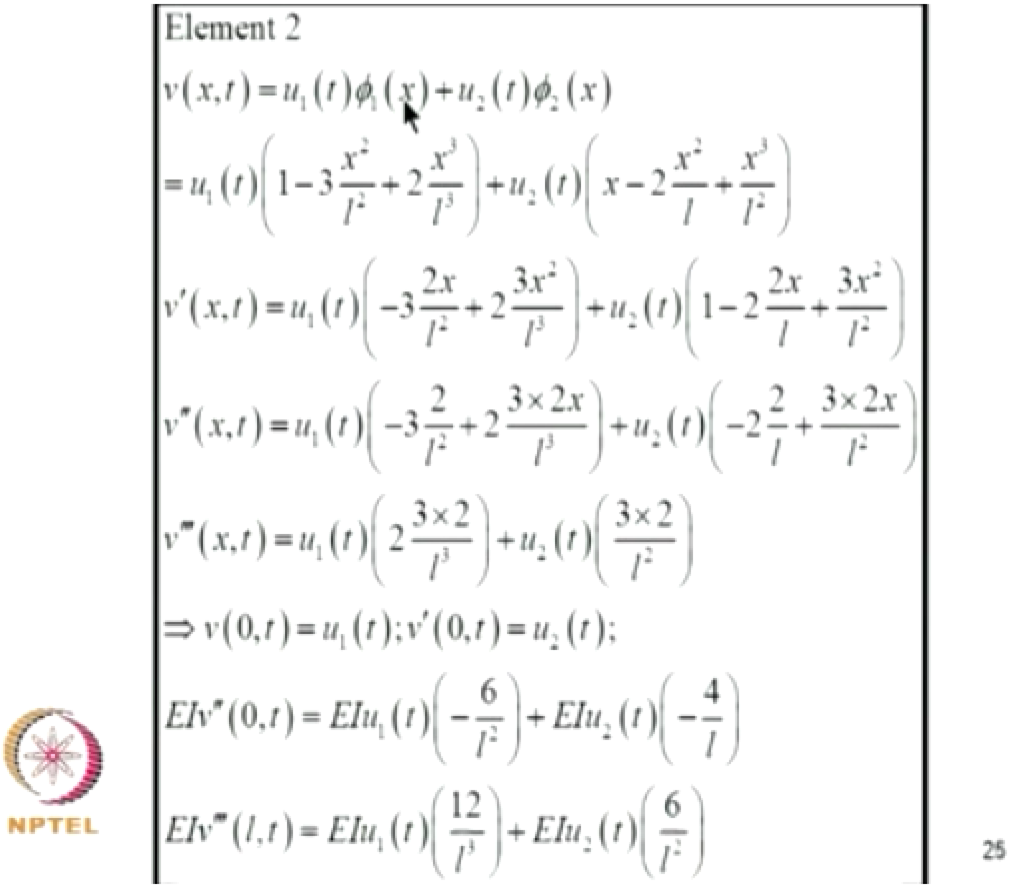
$$\Rightarrow v(l,t)=u_1(t);v'(l,t)=u_2(t);$$

$$EIv''(l,t)=EIu_1(t)\left(-\frac{2}{l^2}\right)+EIu_2(t)\left(\frac{4}{l}\right)$$

$$EIv'''(l,t)=EIu_1(t)\left[-\frac{12}{l^3}\right]+EIu_2(t)\left[\frac{6}{l^2}\right]$$

Now for the element 1 we see that these 2 degrees of freedom that is about displacement here, the translation here and rotations here are 0, so I get V(x,t) as U1 Phi 3 + U2 Phi 4, U1 and U2 please remember are these 2. Now you can substitute for Phi 3 and Phi 4 we can go ahead and find the slope Dou E by Dou X, this will be the expression we can find out V double Prime and multiplied by EI will give the bending moment, and V triple prime this is this, and multiplied by EI will give the shear force.

Now let’s look at X = L, the displacement will be U1 slope will be U2, this is a bending moment, this is a shear force, this is when I am viewing this element as I mean at this point as a member of this element.



Element 2

$$v(x,t)=u_1(t)\phi_1(x)+u_2(t)\phi_2(x)$$

$$=u_1(t)\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right)+u_2(t)\left(x-2\frac{x^2}{l}+\frac{x^3}{l^2}\right)$$

$$v'(x,t)=u_1(t)\left(-3\frac{2x}{l^2}+2\frac{3x^2}{l^3}\right)+u_2(t)\left(1-2\frac{2x}{l}+\frac{3x^2}{l^2}\right)$$

$$v''(x,t)=u_1(t)\left(-3\frac{2}{l^2}+2\frac{3\times 2x}{l^3}\right)+u_2(t)\left(-2\frac{2}{l}+\frac{3\times 2x}{l^2}\right)$$

$$v'''(x,t)=u_1(t)\left(2\frac{3\times 2}{l^3}\right)+u_2(t)\left(\frac{3\times 2}{l^2}\right)$$

$$\Rightarrow v(0,t)=u_1(t);v'(0,t)=u_2(t);$$

$$EIv''(0,t)=EIu_1(t)\left(-\frac{6}{l^2}\right)+EIu_2(t)\left(-\frac{4}{l}\right)$$

$$EIv'''(l,t)=EIu_1(t)\left(\frac{12}{l^3}\right)+EIu_2(t)\left(\frac{6}{l^2}\right)$$

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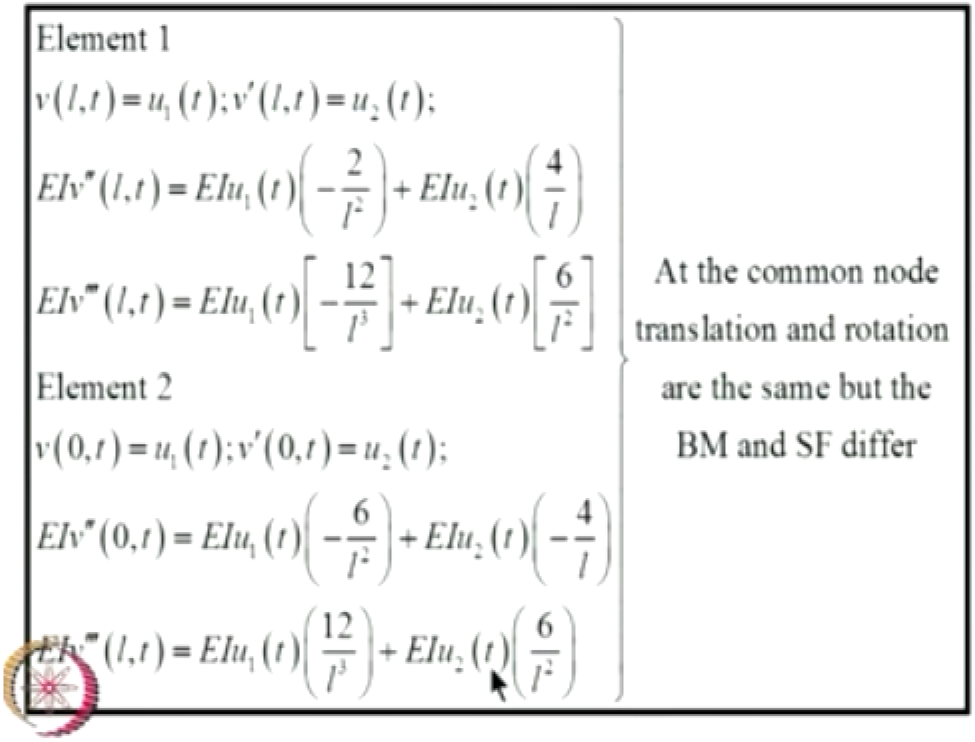
Now let us look at element 2, in element to what happens these 2 degrees of freedom are 0 the vertical translation is 0, rotation here is 0, so I get U1 Phi 1 + U2 Phi 2, so we can repeat this calculation I can find V prime, V double prime, V triple prime by simple differentiation they are polynomials and then look at what is V at X = 0, so this common point for this element this point is at X = 0 in the local coordinate system, so if I look at that I get the translation to be U1, the slope to be U2 which is fine, because I got the same result when I viewed the common point as an member of the first element, the same values I am also getting when I view that point as a member of the second element, but how about bending moment? Bending moment is this, shear force is this, now these two are not same as what we got from element 1, so in summary I have



Element 1

$v(l,t)=u_1(t); v'(l,t)=u_2(t);$

$EIv''(l,t)=EIu_1(t)\left(-\frac{2}{l^2}\right)+EIu_2(t)\left(\frac{4}{l}\right)$

$EIv'''(l,t)=EIu_1(t)\left[-\frac{12}{l^3}\right]+EIu_2(t)\left[\frac{6}{l^2}\right]$

Element 2

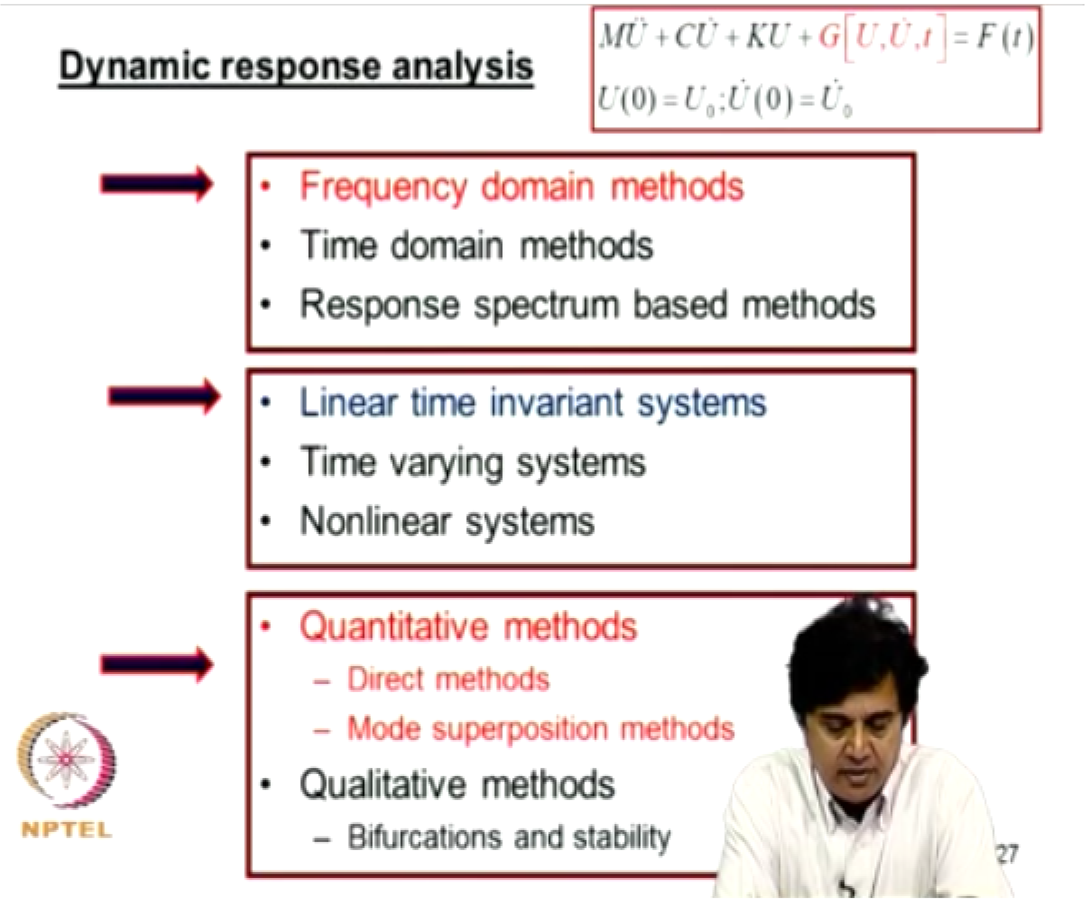
$v(0,t)=u_1(t); v'(0,t)=u_2(t);$

$EIv''(0,t)=EIu_1(t)\left(-\frac{6}{l^2}\right)+EIu_2(t)\left(-\frac{4}{l}\right)$

$EIv'''(l,t)=EIu_1(t)\left(\frac{12}{l^3}\right)+EIu_2(t)\left(\frac{6}{l^2}\right)$

At the common node translation and rotation are the same but the BM and SF differ

properties of element 1 listed here at X = L, and properties of the system states at X = 0 for element 2, so we see that V(l,t) is same as V(0,t), V prime (l,t) is V prime (0,t) that is U2(t) there is no problem with display the translation and the slope, but bending moment we get one expression here which is different from this expression shear force is again different, so at the nodes at the common node translation and rotation are the same but the bending moment and shear force differ, so if you compute now bending stress and shear stress you will get different answers at the common node depending on your how you are viewing that point either as a member of element 1, or member of element 2, this again is points towards one of the limitations of finite element method the kind of approach that we are taking displacement based finite element method which introduces discontinuities in quantities which are in reality or not discontinuous, okay.



Dynamic response analysis

$$M\ddot{U}+C\dot{U}+KU+G\left[U,\dot{U},t\right]=F(t)$$
$$U(0)=U_0;\dot{U}(0)=\dot{U}_0$$

- Frequency domain methods
- Time domain methods
- Response spectrum based methods

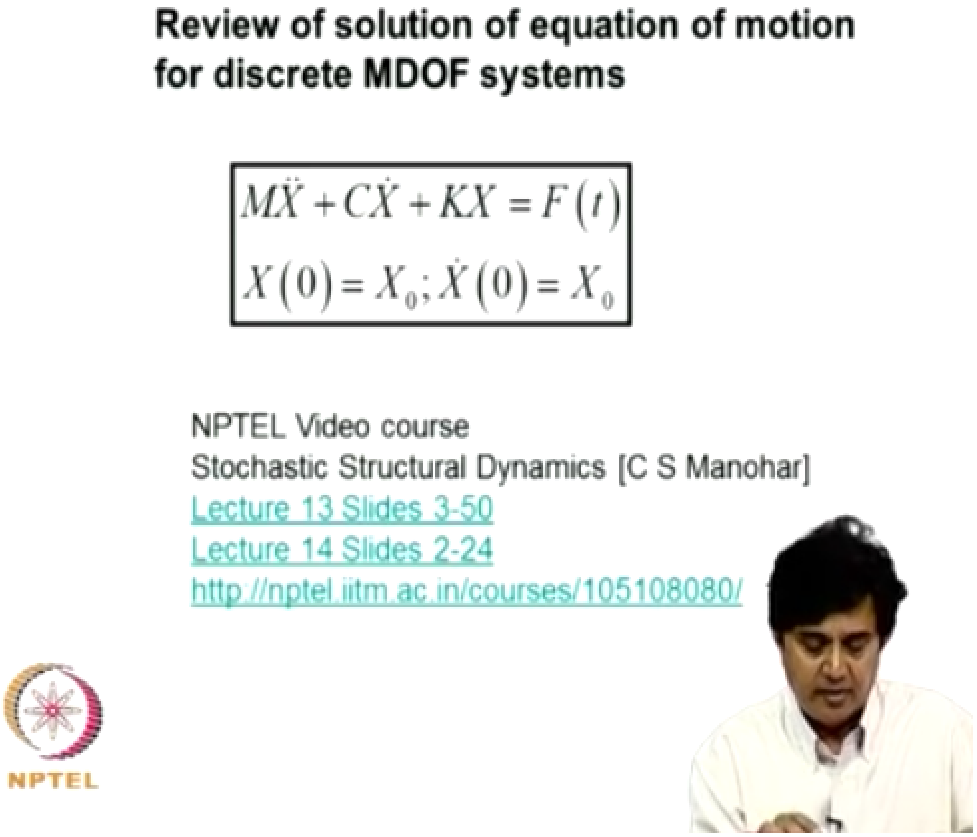
- Linear time invariant systems
- Time varying systems
- Nonlinear systems

- Quantitative methods
  - Direct methods
  - Mode superposition methods
- Qualitative methods
  - Bifurcations and stability

Now we can now start talking about the next step in our development of the subject how to analyze the response, we have to now look into certain details, so at the end of finite element modeling the equilibrium equation that we got was MU double dot + CU dot + KU is F(t), but if the problem were to be nonlinear which I have not really discussed but you can see that if you include non-linearity there will be additional vector here G(U) U dot, T, we will come to this later how to model this but right now for sake of generality I have included this, so there are initial conditions on displacement and velocity so this constitute a set of second order ordinary differential equations which constitute initial value problems the time variable here is still continuous.

Now I am now interested in discussing how to analyze this set of equations, what are the different frameworks available and what are the issues. Now to look at this problem we can start with suppose if the problem is linear G will be 0, then the problem can be tackled either in frequency domain or time domain, so depending on how we tackle the problem we can adopt a frequency domain method or a time domain method there is yet another way of analyzing the problem known as the response spectrum based methods which is commonly used in earthquake engineering where the part of the problem is solved in time domain, and the remaining part is solved using in the model domain that is the like frequency domain, so it is a kind of a hybrid strategy and I will come to the details in due course, the idea here in response spectrum based method is part of the problem is solved in time domain and that solution is made available to the analyst in some form the way the external actions are modeled is in terms of a set of response spectrum that would be the starting point and by that time the time domain analysis would have been completed, so the user of this method need to do a free vibration analysis and construct the force response, so this classification is based on how you analyze the problem either in frequency or time or kind of a combined time and modal domains.

Now the response analysis procedure can also be viewed from the point of view of whether the system is linear time in variant that is MCK are independent of time or the systems could be time varying where either MCK or this non-linearity function could be time dependent or the system could be non-linear where G, this function G is nonzero so the methods of analysis also depends on this classification of the systems yet another way of looking at the analysis of this equation of motion is we can classify the methods as being quantitative or qualitative, in quantitative methods we can further classify the methods of direct methods or more superposition methods, in direct methods we tackle this equation without any further transformation on U, whereas in mode superposition methods we attempt to transform U into a new coordinate space so that some of the representations become simpler and then analyze the problem, then qualitative methods we are not so much interested in details of time histories of response that we are not interested in how U varies as a function of time or U dot varies as a function of time etcetera, but on the other hand we are interested in fixed points and their stability so maybe at some point later in the course we will address qualitative methods also, for the time being to initiate the discussion we'll start with frequency domain methods for linear time invariant systems and we will focus on quantitative analysis, we will be coming, we will be getting to other details in due course.

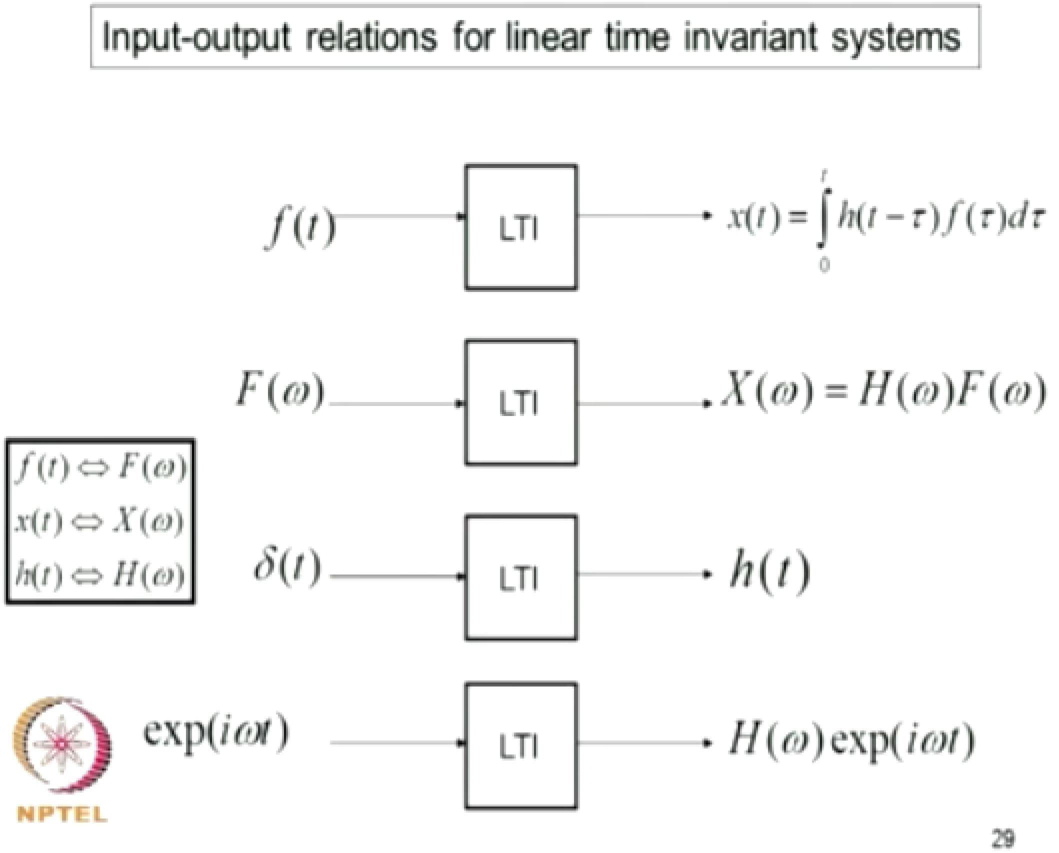


Review of solution of equation of motion for discrete MDOF systems

$$\boxed{\begin{aligned} &M\ddot{X} + C\dot{X} + KX = F(t) \\ &X(0) = X_0; \dot{X}(0) = X_0 \end{aligned}}$$

NPTEL Video course
Stochastic Structural Dynamics [C S Manohar]
Lecture 13 Slides 3-50
Lecture 14 Slides 2-24
http://nptel.iitm.ac.in/courses/105108080/

Now again at this point I would like to draw your attention to certain discussions that are already available in another video course at developed by me, and again I have earlier given this reference I would like to reiterate that the same set of lectures it will be useful if we are not already gone through these lectures to visit these lectures and go through the review, so I will assume in my discussion that you would have done this, so I will not repeat what is covered in these lectures, I may just touch upon some of the issues to retain a kind of a continuity, but I will not be repeating all the details. A general framework for representing the input-output relations for linear time-invariant systems displayed in this view graph suppose the forcing function F(t) is specified in time domain and it is the input to a linear time invariant system, LTI stands for linear time invariant system, the output will be through a convolution integral where H is an impulse response function. On the other hand you specify the input in the Fourier domain, you give the Fourier transform of F(t), the output can be expressed in terms of the Fourier transform of input and in terms of the system frequency response function known as H(omega) this is convolution integral is an integral, a fairly complicated integral at that whereas the in frequency domain it is algebraic operation, so this operation is far simpler than doing convolution, so that is one reason why we go into frequency domain.



Input-output relations for linear time invariant systems

$f(t) \to$ LTI $\to x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$

$F(\omega) \to$ LTI $\to X(\omega) = H(\omega) F(\omega)$

$\delta(t) \to$ LTI $\to h(t)$

$\exp(i\omega t) \to$ LTI $\to H(\omega)\exp(i\omega t)$

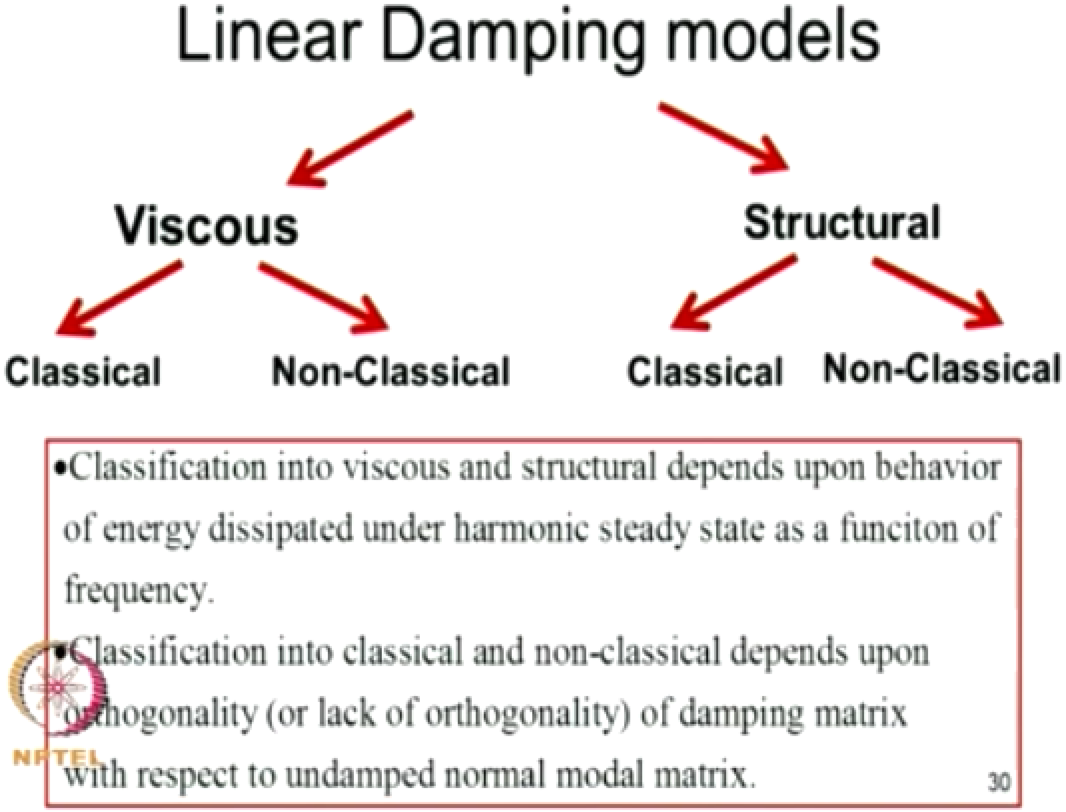
$f(t) \Leftrightarrow F(\omega)$

$x(t) \Leftrightarrow X(\omega)$

$h(t) \Leftrightarrow H(\omega)$

The impulse response function that is used in time domain representation is nothing but the response of the system given unit impulse, the frequency response function H(omega) is defined through this graph you apply an unit harmonic excitation and the response we get is H(omega) raise to I omega T, so it can also be shown that F(t) F(omega) form Fourier transform pair this notation means F(t) and F(omega) are Fourier transform pairs, this X(t) and X(omega) are Fourier transform pairs and not only that the system function, the system impulse response H(t) and the system frequency response function H(omega) also form Fourier transform pair.

Now this would form the basis for our discussion this idea is applicable to not only single degree freedom systems but to any linear time-invariant even multi degree freedom systems.



Linear Damping models

Viscous

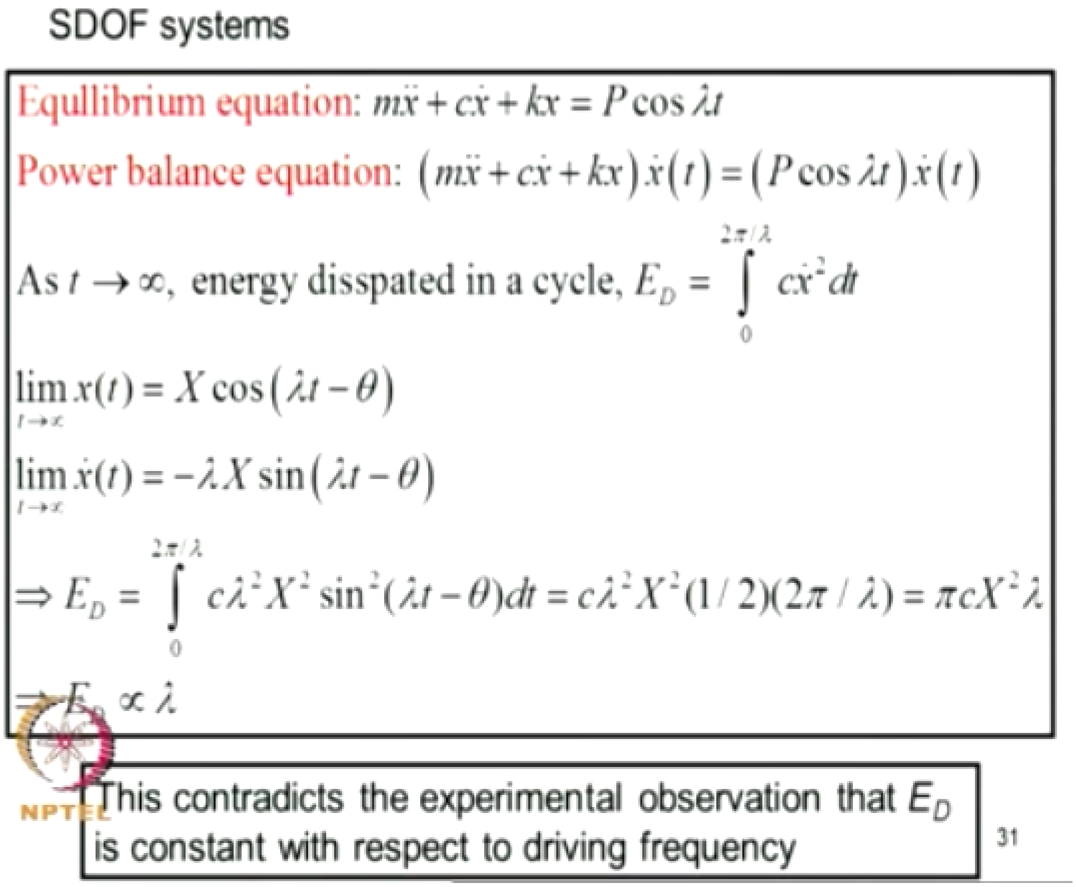
- **Classical**
- **Non-Classical**

Structural

- **Classical**
- **Non-Classical**

- Classification into viscous and structural depends upon behavior of energy dissipated under harmonic steady state as a funciton of frequency.
- Classification into classical and non-classical depends upon orthogonality (or lack of orthogonality) of damping matrix with respect to undamped normal modal matrix.

Now one of the issues that we need to discuss before we can analyze response of the system is questions on modeling damping, the methods of analysis that we use for analyzing the response is closely linked to how we model damping, so these two questions need to be considered concurrently. Now a broad classification of linear damping models would be to say that damping is viscous or structural, I will explain in a short while what is the difference and each of this could be classical or non-classical that means we can have a viscous classical damping, viscous non-classical damping, structural classical, or structural non-classical, the classification into viscous and structural damping is something to do with how the energy dissipated per cycle behaves as a function of frequency, the classification into classical and non-classical is something to do with the structure of the damping matrix, we say that the damping is classical if the damping matrix is such that they un-damped normal modes diagonalize the damping matrix, otherwise we call it as non-classical, so this classification is applicable to structural damping also.



SDOF systems

Equllibrium equation: $m\ddot{x}+c\dot{x}+kx=P\cos\lambda t$

Power balance equation: $\left(m\ddot{x}+c\dot{x}+kx\right)\dot{x}(t)=\left(P\cos\lambda t\right)\dot{x}(t)$

As $t\to\infty$, energy disspated in a cycle, $E_D=\int_0^{2\pi/\lambda}c\dot{x}^2dt$

$$\lim_{t\to\infty}x(t)=X\cos\left(\lambda t-\theta\right)$$

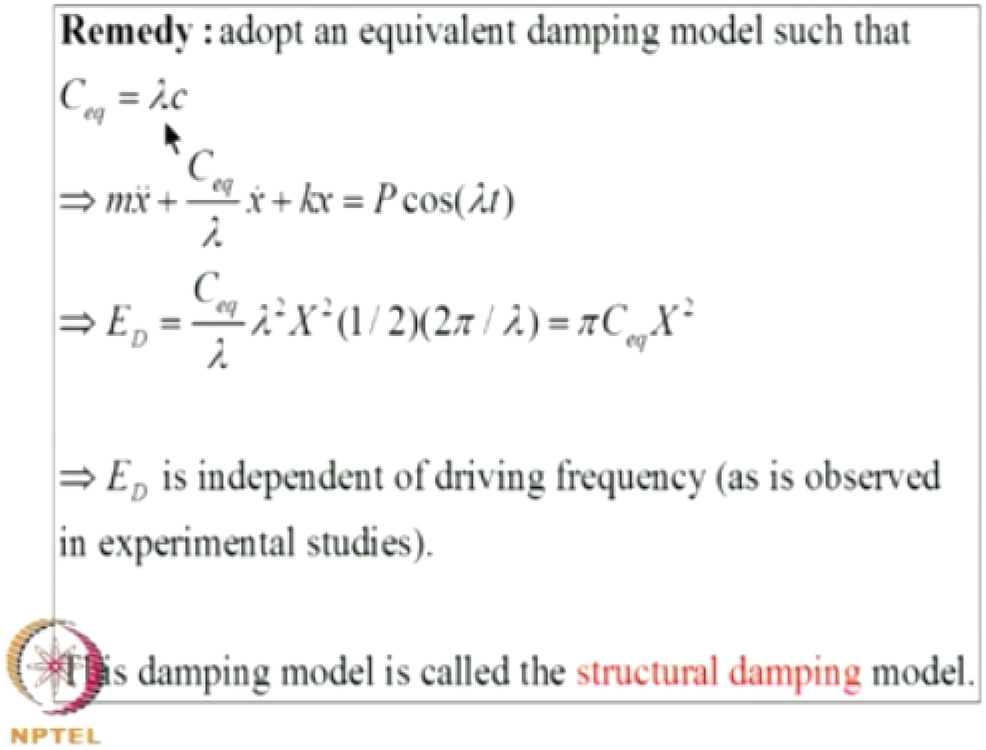
$$\lim_{t\to\infty}\dot{x}(t)=-\lambda X\sin\left(\lambda t-\theta\right)$$

$$\Rightarrow E_D=\int_0^{2\pi/\lambda}c\lambda^2X^2\sin^2(\lambda t-\theta)dt=c\lambda^2X^2(1/2)(2\pi/\lambda)=\pi cX^2\lambda$$

$$\Rightarrow E_D\propto\lambda$$

This contradicts the experimental observation that E_D is constant with respect to driving frequency

So we will begin by asking outlining what is difference between viscous and structural damping, so that can be done with the help of a single degree freedom system suppose if I have a single degree freedom system damped single degree freedom system driven harmonically the equation of motion is given here and I would be interested in the steady state behavior, the harmonic steady state behavior this is a equilibrium equation if I want to write the power balance equation that is if I multi, this is a force and if I multiply by displacement I get work done and if I multiply by velocity I get work done per unit time which is the power, so if I by multiplying both sides by velocity I get this is external power input and this is the work done by the system forces on the velocity, so as T tends to infinity the energy dissipated per cycle here CX dot square if you integrate from 0 to 2 Phi by lambda we get this.



Remedy : adopt an equivalent damping model such that

$$C_{eq} = \lambda c$$

$$\Rightarrow m\ddot{x} + \frac{C_{eq}}{\lambda}\dot{x} + kx = P\cos(\lambda t)$$

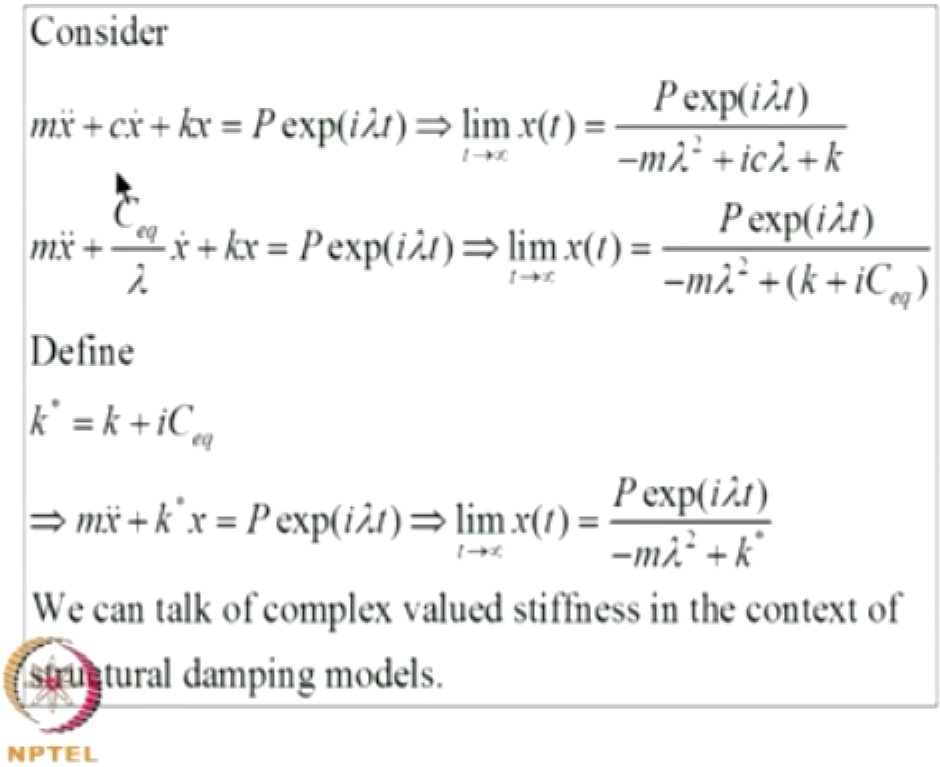
$$\Rightarrow E_D = \frac{C_{eq}}{\lambda}\lambda^2 X^2 (1/2)(2\pi/\lambda) = \pi C_{eq} X^2$$

$\Rightarrow E_D$ is independent of driving frequency (as is observed in experimental studies).

This damping model is called the structural damping model.

Now in steady state we know that the response will be of the form X cos lambda T – theta, now therefore X dot(t) is given by this and if I substitute into the expression for energy dissipated in a cycle I get, if I carry out this integration I get the value as Phi CX square lambda, so from this we observe that the energy dissipated in a cycle is proportional to the driving frequency, according to the viscous damping model but in experimental work it is observed that this is not true, the energy dissipated per cycle remains constant with respect to the driving frequency so viscous damping model therefore does not capture what is experimentally observed in an acceptable manner, so the remedy for that would be to introduce an equivalent damping C equivalent, that is this C into this lambda I call it as C equivalent, and I write the equation of motion as C equivalent by lambda for C I am writing C equivalent by lambda X dot KX = P cos lambda T, now energy dissipated per cycle if I now substitute into the original equation if this is a damping model I get as Phi C EQ X square, this is fine this is independent of frequency so this is fine.

Now this damping model is known as structural damping model, now you have to pay attention to this equation this lambda is same as this lambda, so this is not a differential equation in the traditional sense, for example there is nothing like a free vibration here, when you put right hand side as 0 what lambda you have to put here and that is not defined so it is defined only when there is a harmonic excitation and only with this model works only in steady state, so you have to bear that in mind.



Consider

$$m\ddot{x}+c\dot{x}+kx=P\exp(i\lambda t)\Rightarrow\lim_{t\to\infty}x(t)=\frac{P\exp(i\lambda t)}{-m\lambda^2+ic\lambda+k}$$

$$m\ddot{x}+\frac{C_{eq}}{\lambda}\dot{x}+kx=P\exp(i\lambda t)\Rightarrow\lim_{t\to\infty}x(t)=\frac{P\exp(i\lambda t)}{-m\lambda^2+(k+iC_{eq})}$$

Define

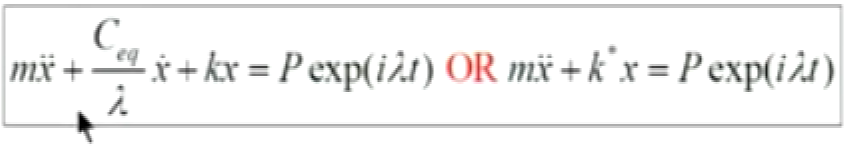
$$k^*=k+iC_{eq}$$

$$\Rightarrow m\ddot{x}+k^*x=P\exp(i\lambda t)\Rightarrow\lim_{t\to\infty}x(t)=\frac{P\exp(i\lambda t)}{-m\lambda^2+k^*}$$

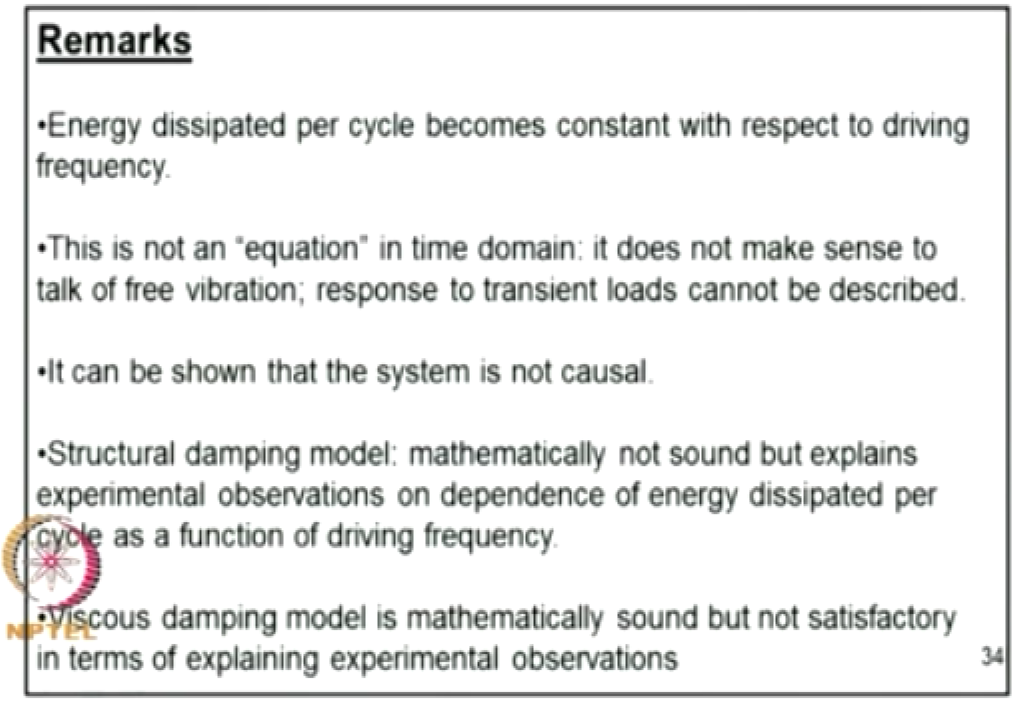
We can talk of complex valued stiffness in the context of structural damping models.

So now if we now consider let us look at the nature of the frequency response function suppose if I consider the viscously damped single degree freedom system in steady state I know that response is given by this, for viscously a structurally damped system in steady state I get the frequency response function is steady state to be given by this, the steady state response to be given by this.

Now you can see here in the denominator I am having K + I CEQ, so I can define that as K star and call it as a complex spring constant, the real part is elastic stiffness, the imaginary part is associated with structural damping, so structurally damped system therefore can be viewed as a system with complex valued stiffness, where real part is the elastic stiffness imaginary part is structural damping coefficient, so that is often done even in finite element codes so you need to be aware of this terminology, so I can write now for viscously dam system the equation either



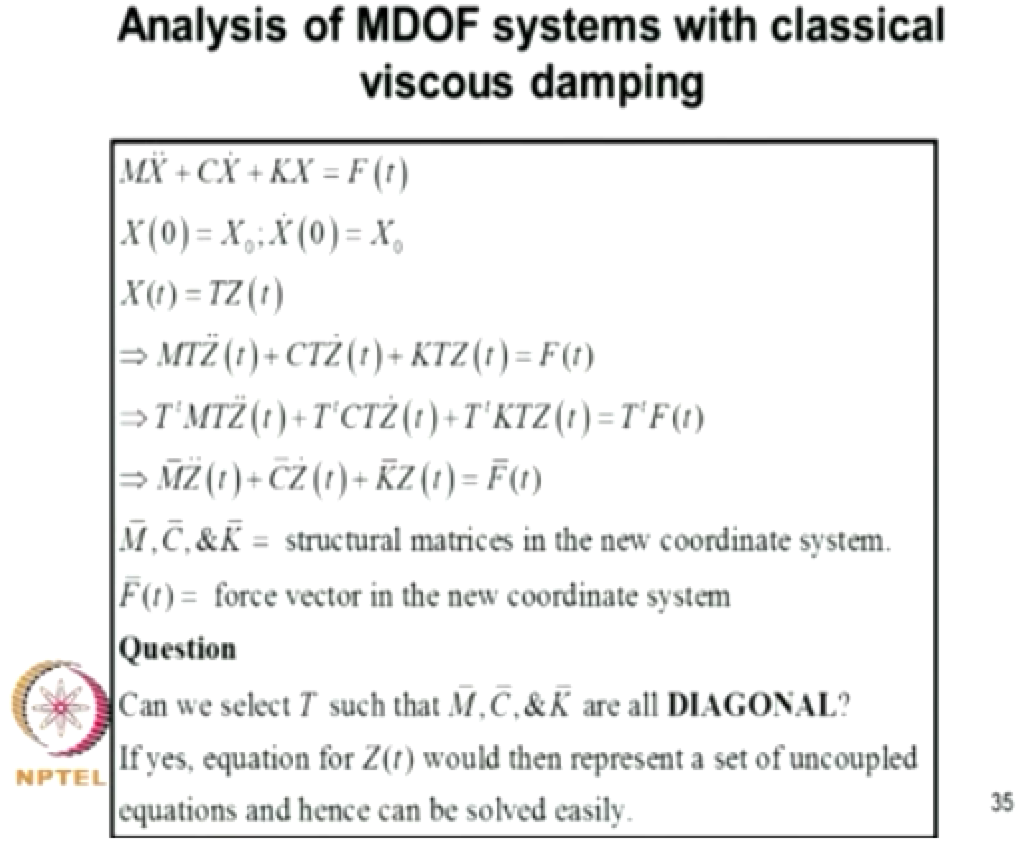
$$m\ddot{x}+\frac{C_{eq}}{\lambda}\dot{x}+kx=P\exp(i\lambda t)\ \text{OR}\ m\ddot{x}+k^{*}x=P\exp(i\lambda t)$$



Remarks

- Energy dissipated per cycle becomes constant with respect to driving frequency.
- This is not an "equation" in time domain: it does not make sense to talk of free vibration; response to transient loads cannot be described.
- It can be shown that the system is not causal.
- Structural damping model: mathematically not sound but explains experimental observations on dependence of energy dissipated per cycle as a function of driving frequency.
- Viscous damping model is mathematically sound but not satisfactory in terms of explaining experimental observations

in this form or in terms of complex stiffness in this form, so both the system will have the same transfer function. Now we can make few observations the energy dissipated per cycle becomes constant with respect to the driving frequency, for the structural damping model this is not an equation that is what I am saying this refers to this is not an equation in time domain it does not make sense to talk of free vibration response to transient loads therefore cannot be described, so if you now look at this equation you may get a impression that this is a valid differential equation, that is not true, okay, it is not a you cannot do any free vibration analysis for a transient domain analysis with this type of damping, we can show in fact that this system is not causal so it violates one of the physical requirements or a mathematical requirement that impulse response function for this system is not 0 for time less than 0, so we can summarize now structural damping model is mathematically not sound, but explains experimental observations on dependence of energy dissipated per cycle as a function of driving frequency correctly, this description is captured correctly whereas viscous damping model is mathematically sound but not satisfactory in terms of explaining experimental observation, so we have to live with this limitations.



Analysis of MDOF systems with classical viscous damping

$$M\ddot{X} + C\dot{X} + KX = F(t)$$
$$X(0) = X_0; \dot{X}(0) = X_0$$
$$X(t) = TZ(t)$$
$$\Rightarrow MT\ddot{Z}(t) + CT\dot{Z}(t) + KTZ(t) = F(t)$$
$$\Rightarrow T^t MT\ddot{Z}(t) + T^t CT\dot{Z}(t) + T^t KTZ(t) = T^t F(t)$$
$$\Rightarrow \bar{M}\ddot{Z}(t) + \bar{C}\dot{Z}(t) + \bar{K}Z(t) = \bar{F}(t)$$

$\bar{M}, \bar{C}, \& \bar{K} =$ structural matrices in the new coordinate system.

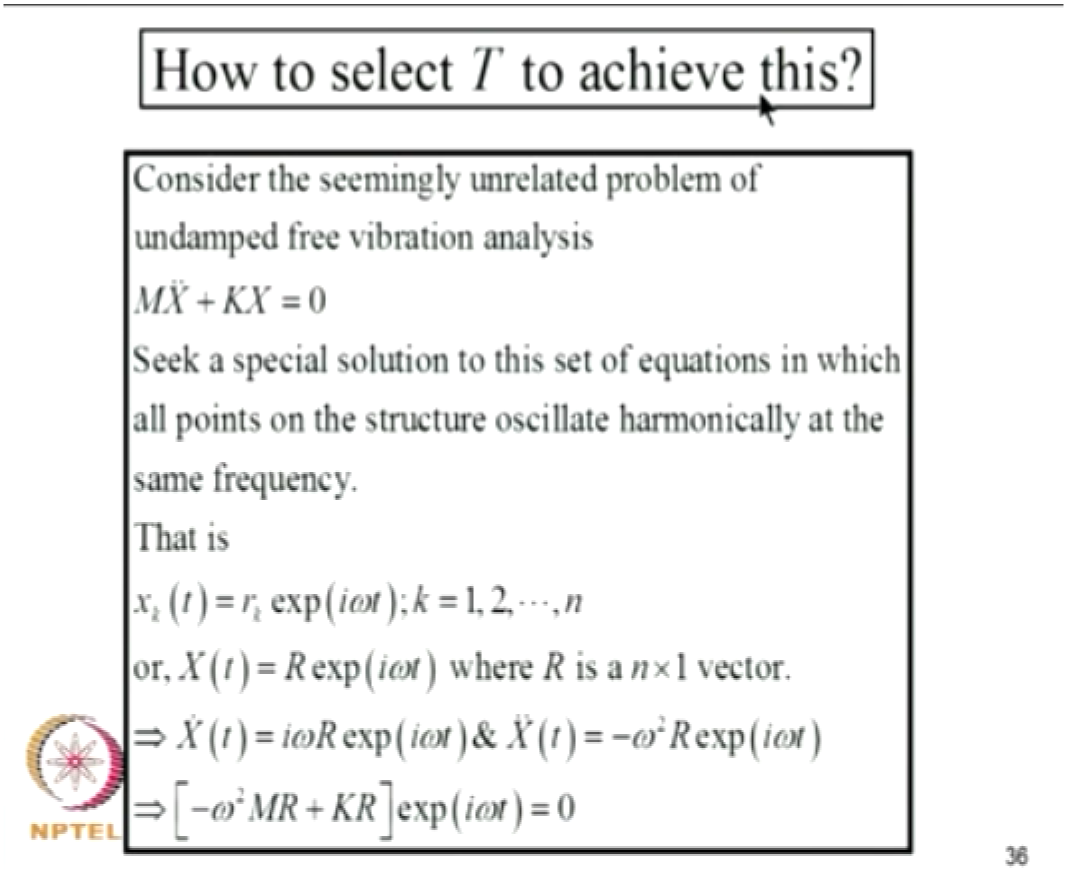
$\bar{F}(t) =$ force vector in the new coordinate system

Question

Can we select T such that $\bar{M}, \bar{C}, \& \bar{K}$ are all **DIAGONAL**?

If yes, equation for $Z(t)$ would then represent a set of uncoupled equations and hence can be solved easily.

Now we will start by analyzing the response of multi degree freedom systems with classical viscous damping, okay now the word classical need to be explained so I will quickly run through these arguments so this is the governing equation where I am having a viscous damping term CX dot F(t) is the forcing function and these are the initial conditions, what I wish to do is I would like to find a transformation X(t) equal to TZ, so let us propose this transformation and if you substitute this into the governing equation and pre multiplied by T transpose I get this equation, now in the transform coordinate system the mass matrix is M bar which is T transpose MT, the damping matrix is C bar which is T transpose CT, K bar is the transform stiffness matrix which is a T transpose KT, and F bar is FT transpose F, why are we interested in doing this transformation, if we can select a T, so that this M bar, C bar and K bar can become diagonal that would mean as a consequence of this transformation we would have uncoupled the governing equations of motion, so if we can find that transformation it will simplify our solution procedure enormously, so this leads to the concept of natural coordinates normal modes and natural frequencies, so we'll quickly recapitulate what they are how to select T to



How to select T to achieve this?

Consider the seemingly unrelated problem of undamped free vibration analysis

$M\ddot{X} + KX = 0$

Seek a special solution to this set of equations in which all points on the structure oscillate harmonically at the same frequency.

That is

$x_k(t) = r_k \exp(i\omega t); k = 1, 2, \cdots, n$

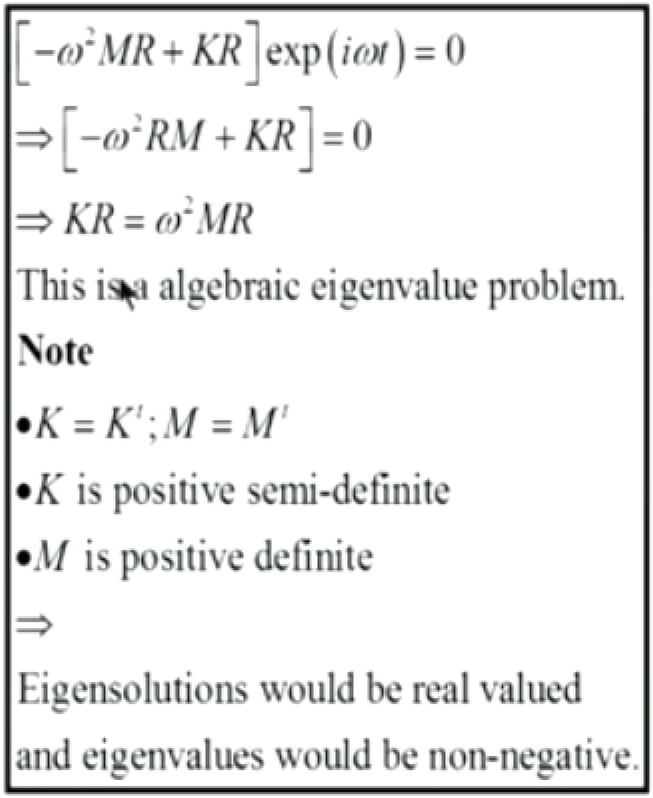
or, $X(t) = R\exp(i\omega t)$ where R is a $n \times 1$ vector.

$\Rightarrow \dot{X}(t) = i\omega R\exp(i\omega t) \,\&\, \ddot{X}(t) = -\omega^2 R\exp(i\omega t)$

$\Rightarrow \left[-\omega^2 MR + KR\right]\exp(i\omega t) = 0$

achieve this, so what we do is we start by considering seemingly unrelated problem of undamped free vibration, and we look for special free vibration solutions where all points on the structure vibrate harmonically at the same frequency that means if you take ratio of displacements at 2 different places that ratio will be independent of time, so that means the motion will be synchronous all the points reach their respective maximum, minimum simultaneously, so any 2 points on the structure consequently would be either in perfectly in phase or perfectly out of phase, there is no intermediate phase difference is possible.

Now let us substitute this into the assume solution into the governing equation, we are assuming in vector form X is RE raise to omega T, X dot is this, and I want to know now for what should be the condition R and omega which are the unknowns for this type of solution to exist, so you substitute into the governing equation I get this equation E raise to I omega T cannot be 0 for all T therefore the term inside the bracket would be 0 and that leads to a general



$$\left[-\omega^2 MR + KR\right]\exp(i\omega t) = 0$$

$$\Rightarrow \left[-\omega^2 RM + KR\right] = 0$$

$$\Rightarrow KR = \omega^2 MR$$

This is a algebraic eigenvalue problem.

Note

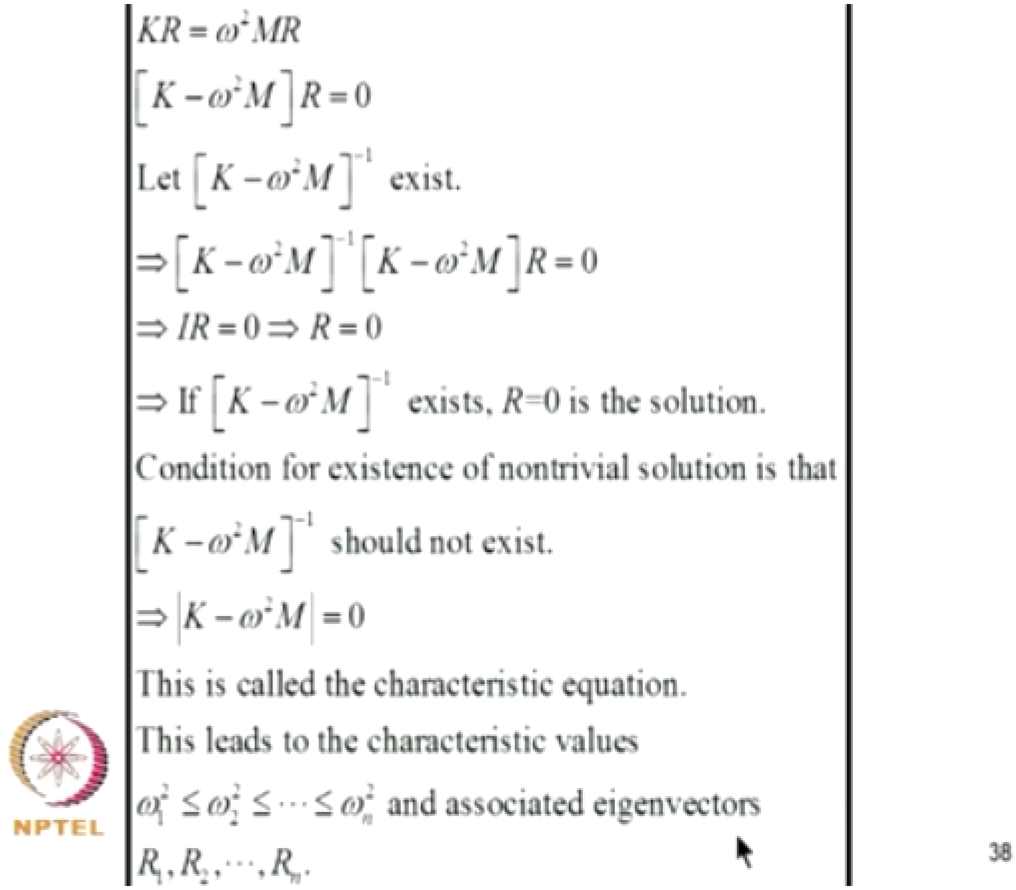
- $K = K^t; M = M^t$
- K is positive semi-definite
- M is positive definite

$\Rightarrow$

Eigensolutions would be real valued and eigenvalues would be non-negative.

algebraic eigenvalue problem, involving stiffness matrix and mass matrix we know that stiffness matrix is symmetric, mass matrix is symmetric, and stiffness matrix is positive semi definite, and mass matrix is positive definite, the semi definiteness comes because of possible rigid body motions that means structure can move as a rigid body without change in the potential energy therefore K can be semi definite, whereas kinetic energy is always positive definite.

Now because K and M satisfy these properties we can show that Eigen solutions would be real valued, and eigenvalues would be non-negative so that would mean we can rank order the eigenvalues, so to get the eigenvalues we go through the usual argument this is eigenvalue



$KR = \omega^2 MR$

$\left[K - \omega^2 M\right] R = 0$

Let $\left[K - \omega^2 M\right]^{-1}$ exist.

$\Rightarrow \left[K - \omega^2 M\right]^{-1}\left[K - \omega^2 M\right] R = 0$

$\Rightarrow IR = 0 \Rightarrow R = 0$

$\Rightarrow$ If $\left[K - \omega^2 M\right]^{-1}$ exists, R=0 is the solution.

Condition for existence of nontrivial solution is that

$\left[K - \omega^2 M\right]^{-1}$ should not exist.

$\Rightarrow \left|K - \omega^2 M\right| = 0$

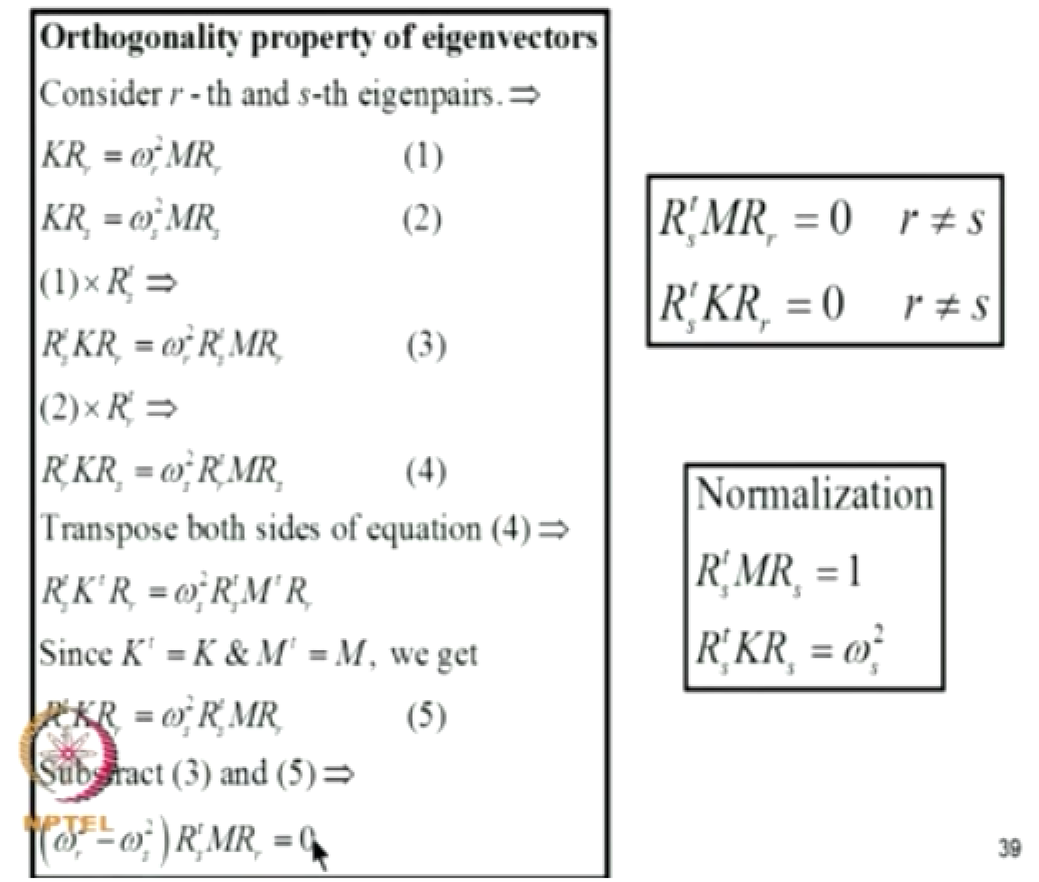
This is called the characteristic equation.

This leads to the characteristic values

$\omega_1^2 \le \omega_2^2 \le \cdots \le \omega_n^2$ and associated eigenvectors

$R_1, R_2, \cdots, R_n$.

problem which can be rewritten in this form if inverse of this matrix K - omega square M exist then it means that only R = 0 is a solution which is trivial solution in which we are not interested therefore for existence of non-trivial solution the inverse of this matrix must not exist and that would mean the matrix should be singular and that would mean the determinant of this must be 0, and this is the so-called characteristic equation, and as I already said the eigenvalues omega square are real and non-negative therefore they can be ordered as omega 1, omega 2, omega N square as shown here, and associated with each of these eigenvalues I will have these Eigen vectors.



Orthogonality property of eigenvectors

Consider r - th and s-th eigenpairs. ⇒

$$KR_r = \omega_r^2 MR_r \qquad (1)$$

$$KR_s = \omega_s^2 MR_s \qquad (2)$$

$$(1)\times R_s^t \Rightarrow$$

$$R_s^t KR_r = \omega_r^2 R_s^t MR_r \qquad (3)$$

$$(2)\times R_r^t \Rightarrow$$

$$R_r^t KR_s = \omega_s^2 R_r^t MR_s \qquad (4)$$

Transpose both sides of equation (4) ⇒

$$R_s^t K^t R_r = \omega_s^2 R_s^t M^t R_r$$

Since $K^t = K$ & $M^t = M$, we get

$$R_s^t KR_r = \omega_s^2 R_s^t MR_r \qquad (5)$$

Subtract (3) and (5) ⇒

$$\left(\omega_r^2 - \omega_s^2\right) R_s^t MR_r = 0$$

$$R_s^t MR_r = 0 \quad r \neq s$$

$$R_s^t KR_r = 0 \quad r \neq s$$

Normalization

$$R_s^t MR_s = 1$$

$$R_s^t KR_s = \omega_s^2$$

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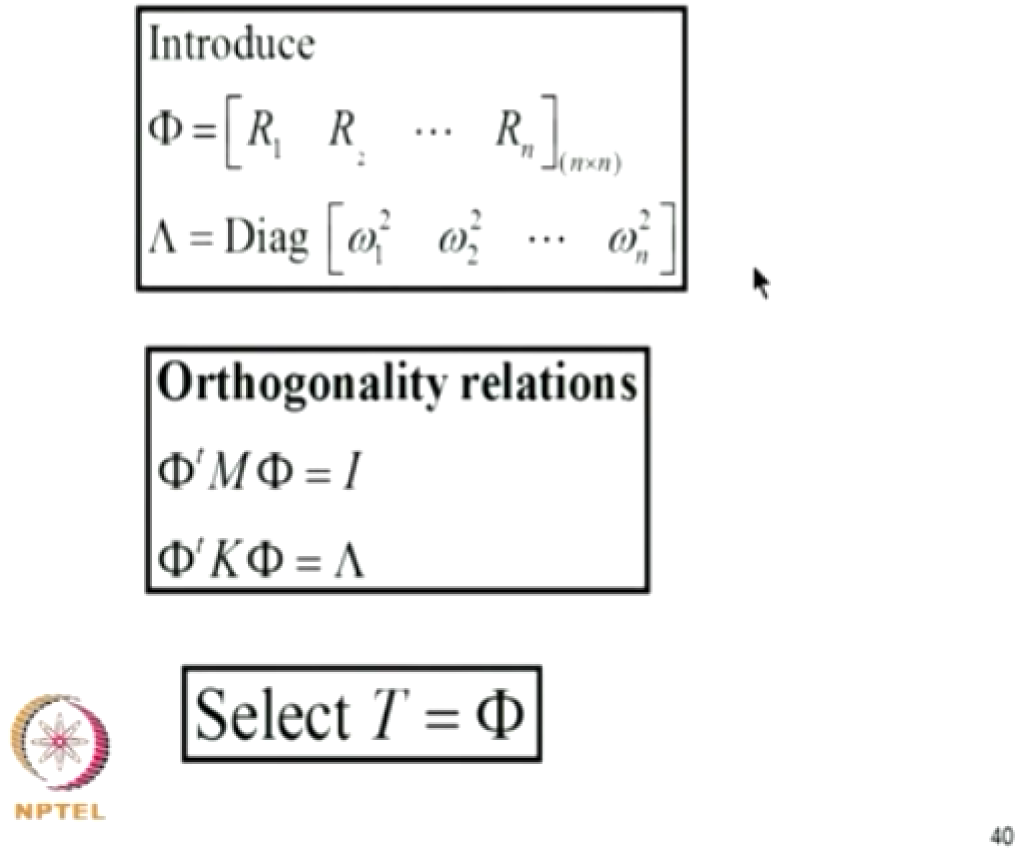
Now the interesting property of this Eigen vectors is that they have the satisfy what is known as orthogonality property, so what does that mean suppose now if you consider two Eigen pairs the R-th and S-th Eigen pairs the equations are KRR = omega R square MRR, KRS is omega S square MRS, so let us label these equations as 1 and 2. Now I will pre multiply the first equation by RS transpose, so I get this equation and pre multiply the second equation by RR transpose I get this, so label them as 3 and 4. Now if we now transpose both sides of equation 4, I get this equation so here we notice that AB transpose is B transpose A transpose, that relation is used, now since K is symmetric, K transpose is same as K, and M transpose is same as M therefore equation 4 after transposing both sides reads as shown here.

Now if I subtract 3 and 5, I get this equation. Now if R is not equal to S, we can reduce from this equation that RS transpose MRR is 0 for R not equal to S and similarly RS transpose KRR would be 0 for R not equal to S, so these are okay, these are very evident if omega R is not equal to omega S, but even if there are repeated roots omega R = Omega S we still insist that these R be selected to satisfy these requirements which is possible.

Now the Eigen vectors are not unique a constant multiple of Eigen vector is also a valid Eigen vector, so we eliminate that arbitrariness by demanding that the Eigen vectors are normalized following the requirement that RS transpose MRS is 1, this would mean if you now go back to this RS transpose KRS will be Omega SS square, so the Eigen vectors we satisfy these requirements are known as normalized Eigen vectors, and this condition we say that the Eigen vectors are orthogonal with respect to mass and stiffness matrix.



Introduce

$$\Phi = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}_{(n\times n)}$$

$$\Lambda = \text{Diag}\begin{bmatrix} \omega_1^2 & \omega_2^2 & \cdots & \omega_n^2 \end{bmatrix}$$

Orthogonality relations

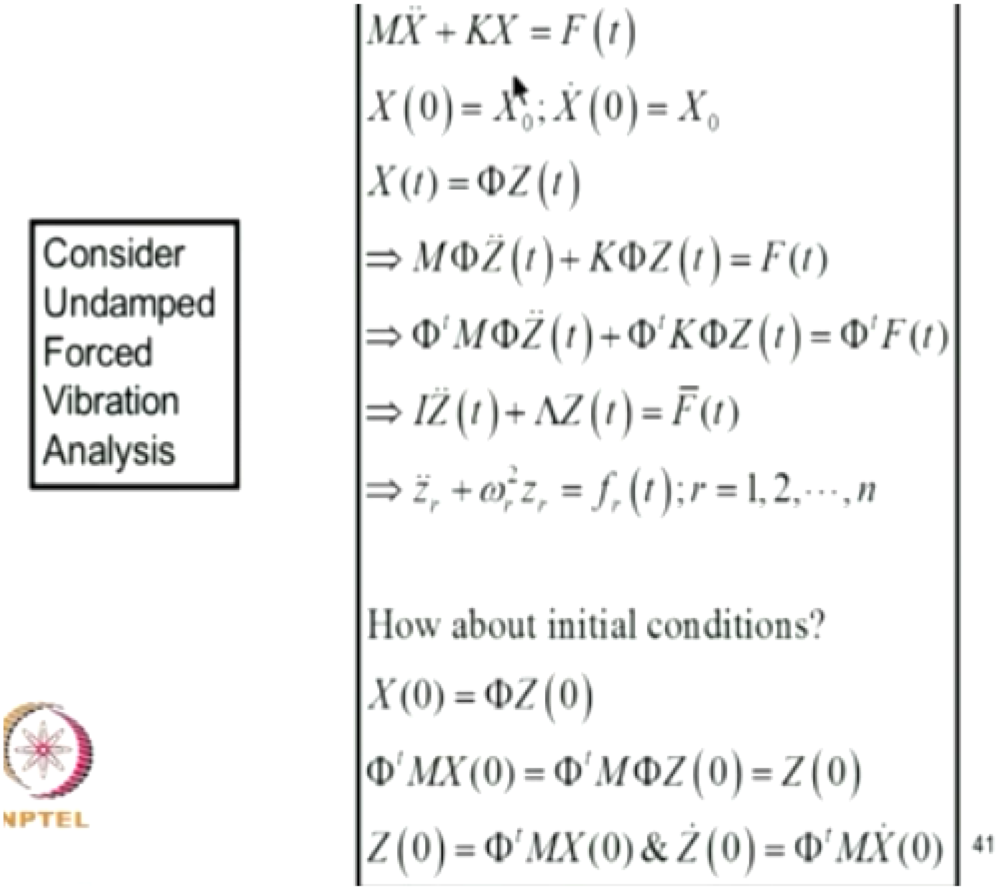
$$\Phi^t M \Phi = I$$

$$\Phi^t K \Phi = \Lambda$$

Select $T = \Phi$

So now we pull all the normalized Eigen vectors into one matrix call it as modal matrix Phi, and I define another diagonal matrix lambda which is along the diagonal I have square of the natural frequencies which are the eigenvalues, the eigenvalues on the diagonal.

The orthogonality relation in terms of the modal matrix can be written as Phi transpose M Phi I, and Phi transpose K Phi is lambda. Now I will now select the transformation T that I was originally looking for to be Phi, so that upon this transformation therefore the transformed mass matrix and stiffness will be diagonal, so if we now consider say un-damped force vibration analysis damping will introduce shortly with certain initial conditions,



Consider Undamped Forced Vibration Analysis

$$M\ddot{X} + KX = F(t)$$

$$X(0) = X_0; \dot{X}(0) = X_0$$

$$X(t) = \Phi Z(t)$$

$$\Rightarrow M\Phi\ddot{Z}(t) + K\Phi Z(t) = F(t)$$

$$\Rightarrow \Phi^t M\Phi\ddot{Z}(t) + \Phi^t K\Phi Z(t) = \Phi^t F(t)$$

$$\Rightarrow I\ddot{Z}(t) + \Lambda Z(t) = \bar{F}(t)$$

$$\Rightarrow \ddot{z}_r + \omega_r^2 z_r = f_r(t); r = 1, 2, \cdots, n$$

How about initial conditions?

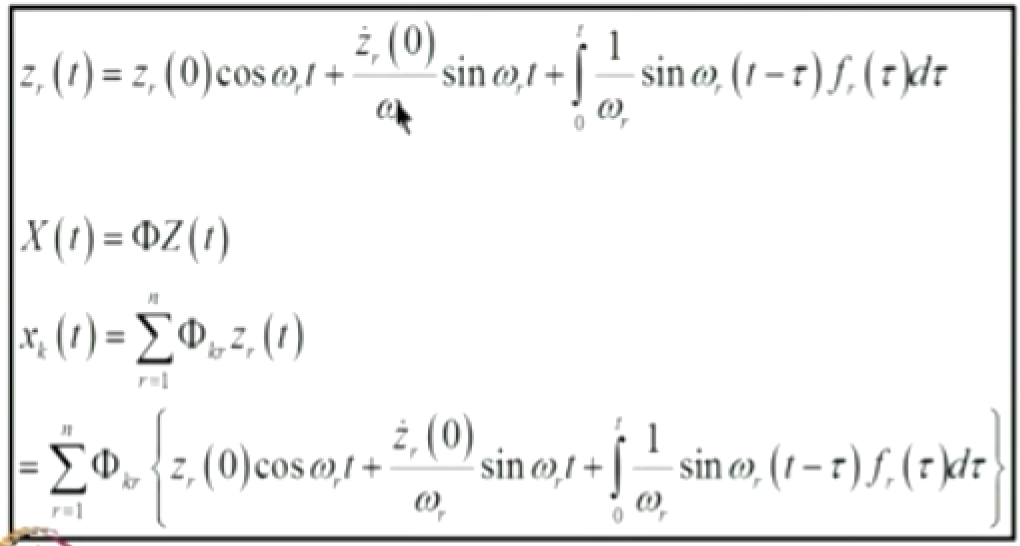
$$X(0) = \Phi Z(0)$$

$$\Phi^t MX(0) = \Phi^t M\Phi Z(0) = Z(0)$$

$$Z(0) = \Phi^t MX(0) \& \dot{Z}(0) = \Phi^t M\dot{X}(0)$$

so this is original equation I make the transformation X = Phi Z, and go through substitute this into the original equation I get this and pre multiplied by Phi transpose I get this.

Now Phi transpose M Phi, my analysis is an identity matrix which is IZ double dot, and Phi transpose K Phi is another diagonal matrix which is capital Lambda, so I get these equations, these equations therefore are equivalent to set of uncoupled second order differential equations as shown here. Now how do we get initial conditions? I have initial condition in X space to get in Z space I use this equation you can get Z(0) by inverting Phi but that is not the way we wish to do it so we will use orthogonality relations again I will pre multiply by Phi transpose M both sides and I get this equation and used our orthogonality relation Phi transpose M Phi is I, therefore I get Z(0) as Phi transpose MX(0), Z dot(0) is Phi transpose MX dot.



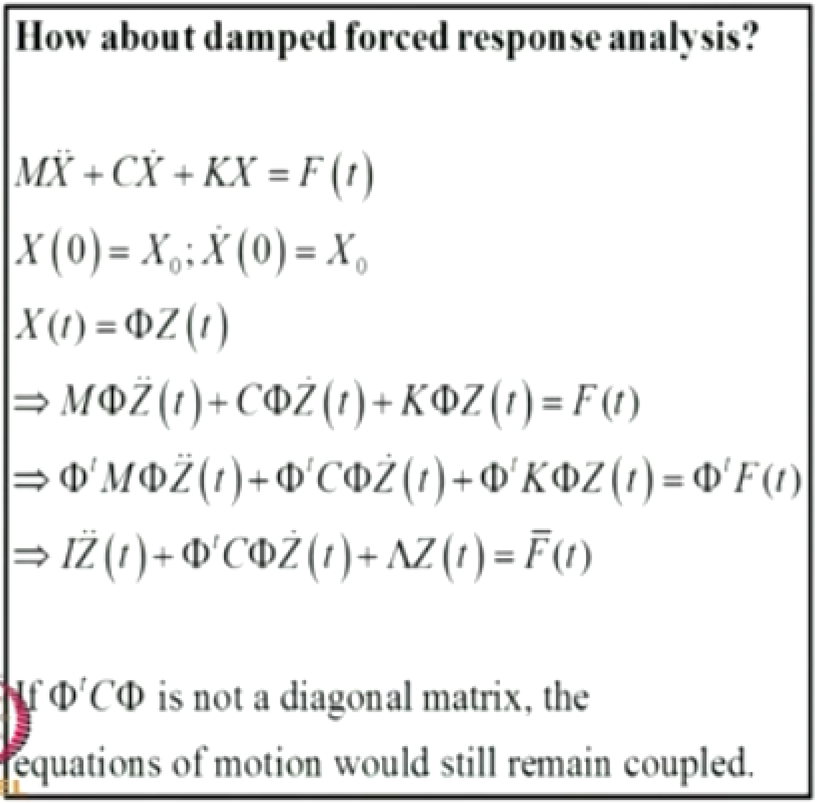
$$z_r(t) = z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)f_r(\tau)d\tau$$

$$X(t) = \Phi Z(t)$$

$$x_k(t) = \sum_{r=1}^{n}\Phi_{kr}z_r(t)$$

$$= \sum_{r=1}^{n}\Phi_{kr}\left\{z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)f_r(\tau)d\tau\right\}$$

Now I have now got the uncoupled equations and the initial conditions so I can write the complete solutions in terms of the prescribed initial conditions and then Duhamel integral, that is the impulse response function and this I will do for all R, R = 1 to N and moment I find all the Z's I will go back to my X and get any XK(t) T by this summation, so this is the final answer. So what we have done is we have uncoupled the equation of motion and solve a family of single degree freedom problems and then constructed back the final solution.



How about damped forced response analysis?

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$X(0) = X_0; \dot{X}(0) = X_0$$

$$X(t) = \Phi Z(t)$$

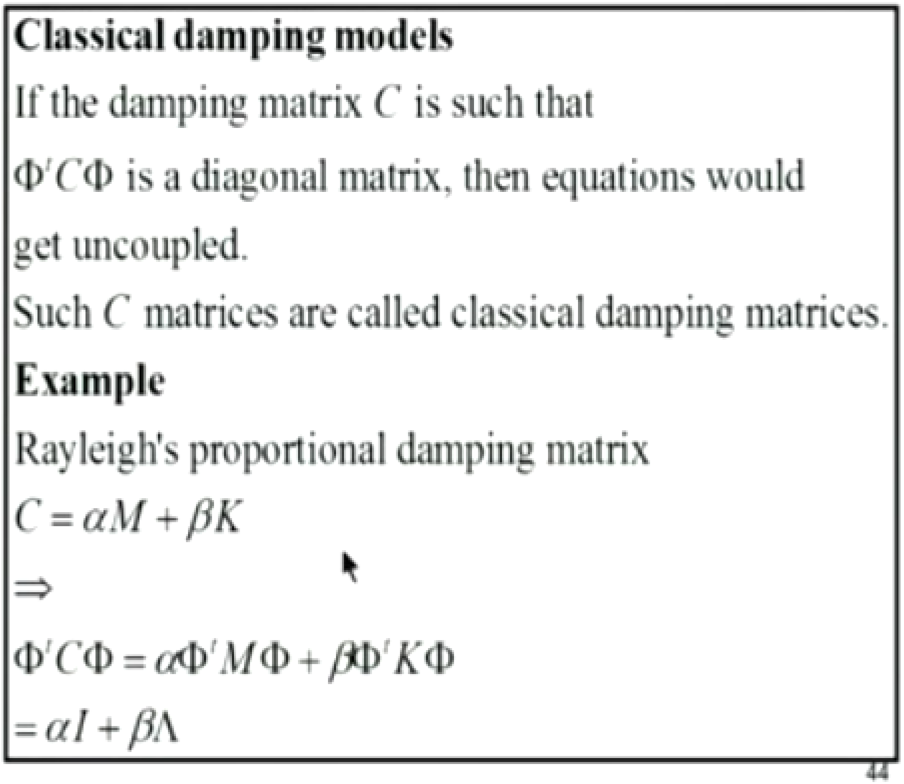
$$\Rightarrow M\Phi\ddot{Z}(t) + C\Phi\dot{Z}(t) + K\Phi Z(t) = F(t)$$

$$\Rightarrow \Phi^t M\Phi\ddot{Z}(t) + \Phi^t C\Phi\dot{Z}(t) + \Phi^t K\Phi Z(t) = \Phi^t F(t)$$

$$\Rightarrow I\ddot{Z}(t) + \Phi^t C\Phi\dot{Z}(t) + \Lambda Z(t) = \bar{F}(t)$$

If $\Phi^t C\Phi$ is not a diagonal matrix, the equations of motion would still remain coupled.

Now what happens if there is damping, now if we now namely proceed I will start with the equation and make the transformation where Phi is the un-damped normalized modal vector so I will get here you substitute this assume solution here I get this equation pre multiplied by Phi transpose I get this equation and Phi transpose M Phi is I, Phi transpose K Phi is capital Lambda, but I am still left with Phi transpose C Phi, this Phi transpose C Phi there is no reason to believe that it is a diagonal matrix, it will be a diagonal matrix it could be, but there is no reason to expect apriori that it is bound to be a diagonal matrix, so if C is such that Phi transpose C Phi is not diagonal this governing equation still remain coupled and we have only succeeded in diagonalizing mass and stiffness matrix and all the troubles are still written through the damping matrix, trouble means couplings.



Classical damping models

If the damping matrix C is such that

$\Phi' C \Phi$ is a diagonal matrix, then equations would get uncoupled.

Such C matrices are called classical damping matrices.

Example

Rayleigh's proportional damping matrix

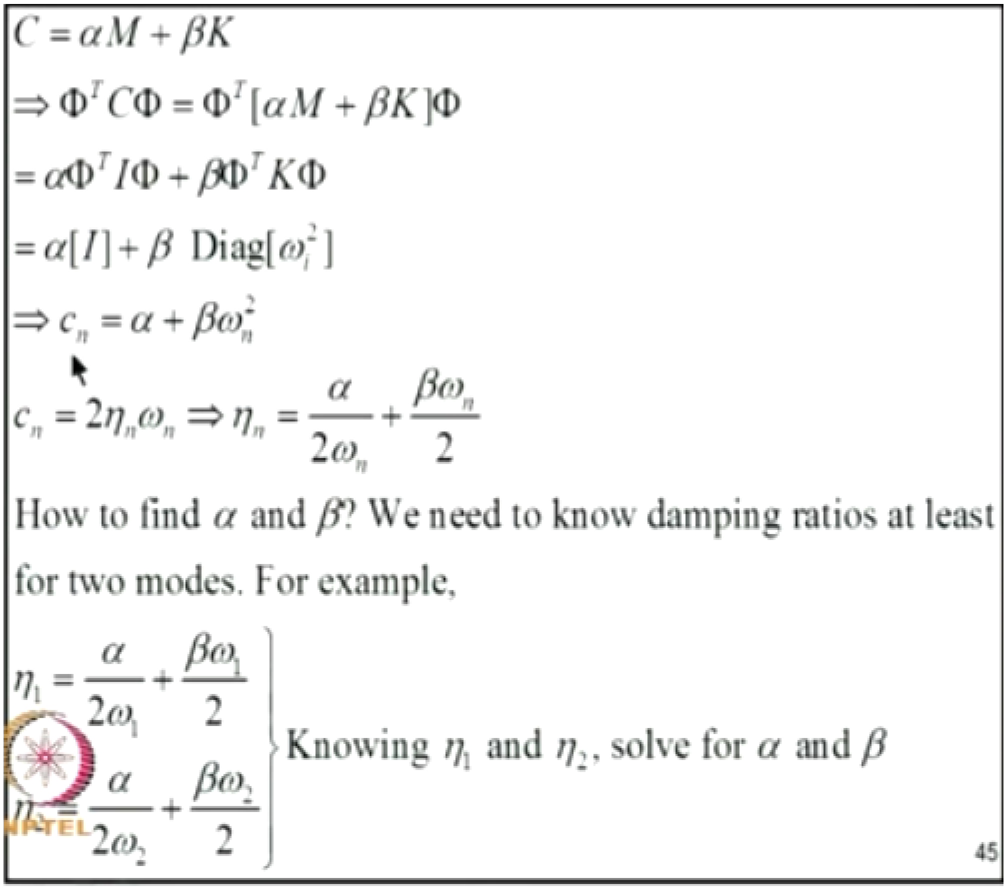
$$C = \alpha M + \beta K$$

$$\Rightarrow$$

$$\Phi' C \Phi = \alpha \Phi' M \Phi + \beta \Phi' K \Phi$$

$$= \alpha I + \beta \Lambda$$

Now it is here that we introduce the notion of classical damping, if we select only those class of C matrices which satisfy the requirement that Phi transpose C Phi is the diagonal matrix then the equations would get uncoupled so such C matrices are called classical damping matrices. One example for that is the so-called Rayleigh's proportional damping matrix, if we now take C as alpha M + beta K, okay that means damping matrix is linearly dependent on mass and stiffness matrix this is from a physical point of view you must understand that there is no justification for this it is only for sake of mathematical expediency that we are using this model. Now you construct now Phi transpose C Phi, I will get alpha into Phi transpose M Phi + beta into Phi transpose K Phi and the first matrix is simply alpha into I, second Matrix is beta into capital lambda therefore Phi transpose C Phi is also diagonal, okay, so if we are happy using only this type of damping matrix model then the un-damped normal modes will still uncouple the equations of motion.



$$C = \alpha M + \beta K$$

$$\Rightarrow \Phi^T C \Phi = \Phi^T [\alpha M + \beta K] \Phi$$

$$= \alpha \Phi^T I \Phi + \beta \Phi^T K \Phi$$

$$= \alpha [I] + \beta \ \text{Diag}[\omega_i^2]$$

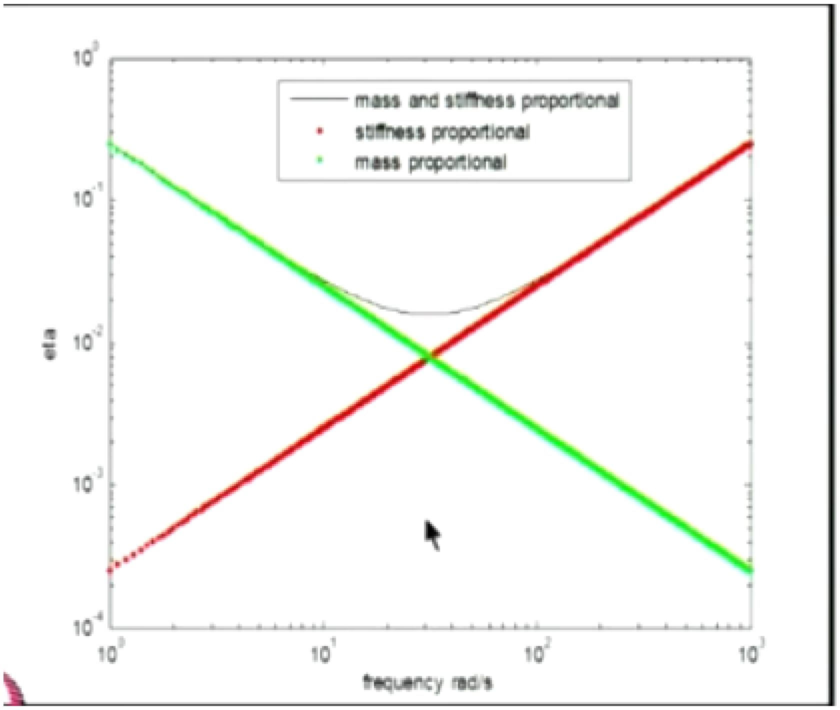
$$\Rightarrow c_n = \alpha + \beta \omega_n^2$$

$$c_n = 2\eta_n \omega_n \Rightarrow \eta_n = \frac{\alpha}{2\omega_n} + \frac{\beta \omega_n}{2}$$

How to find α and β? We need to know damping ratios at least for two modes. For example,

$$\left.\begin{aligned} \eta_1 &= \frac{\alpha}{2\omega_1} + \frac{\beta \omega_1}{2} \\ \eta_2 &= \frac{\alpha}{2\omega_2} + \frac{\beta \omega_2}{2} \end{aligned}\right\} \text{Knowing } \eta_1 \text{ and } \eta_2 \text{, solve for } \alpha \text{ and } \beta$$

So if this is the case then the damping in the nth degree of freedom will be alpha + beta omega N square, so if I now use the notation 2 ETA and omega n from this we can get ETA N as alpha by 2 Omega N square + beta omega N by 2, this damping matrix model has 2 parameters alpha and beta, how do you get alpha and beta? You should measure damping at least in 2 modes suppose you measure damping in first 2 modes ETA 1 and ETA 2 you write this equation for N = 1 and N = 2 we get two equations, solve for this and get alpha and beta, and moment alpha and beta are found what happens to ETA 3, ETA 4, ETA 5 etcetera they will be emerge, they will emerge from this relation,



so if I plot it, it will look like this, for this red line is only with this is with alpha equal to 0 and this is with beta equal to 0, so this is mass proportional, this is stiffness proportional, and the black line that you see here is when both alpha and beta are nonzero, so in conclusion we can notice that the Rayleigh's proportional damping model imposes certain variations on damping ratios as a function of frequency, it has 2 free model parameters with those 2 free model parameters you can adjust damping in 2 modes, this need not be the first 2 modes, it could be any 2 modes for which you have measured the damping.

So what I will do is I will close this lecture at this point in the next lecture we will continue with this discussion and see how to modify the Rayleigh's damping model, so this lecture will conclude here.

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