Indian Institute of Science
Bangalore
NP-TEL
National Programme on
Technology Enhanced Learning

<u>Course Title</u>
Finite element method for structural dynamic
And stability analyses
Lecture – 27
Structural stability analysis – Introduction
(continued)
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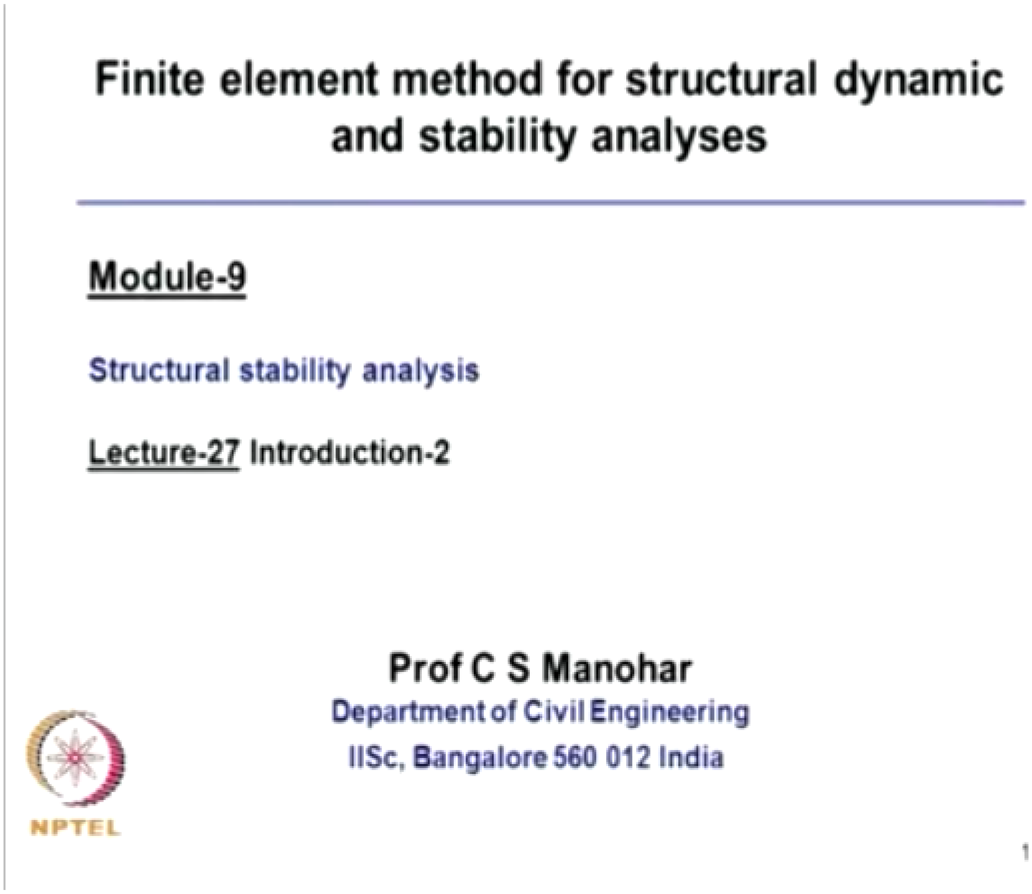
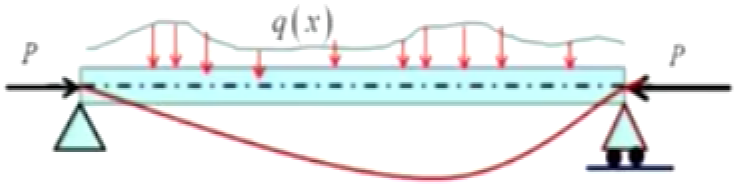


In the previous lecture we started discussing about problems of structural stability analysis, we will continue with that topic, and this lecture also will be by and large introductory in nature, so we'll begin by quickly recalling what we discussed in the previous lecture.

Beam columns

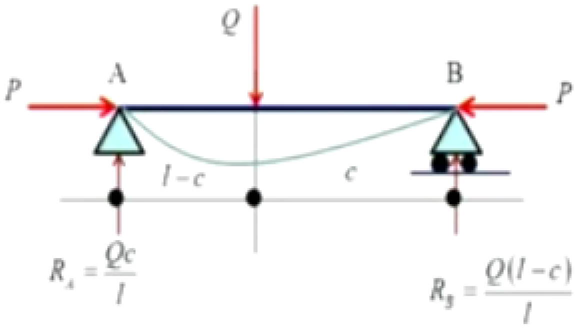
Recall



$$y(x) = y_0(x, P=0)\phi(x, P)$$

$\phi(x, P)$ is not well behaved for some values of P

So one of the problem we considered was that of a beam column, that is a beam element which carries both transverse and axial loads, so in the absence of axial loads we know the how to analyze the structure, the question that we are now considering is what is the effect of presence of axial loads on the transverse response of the beam. So we showed that response can be generically written as response with P = 0 multiplied by a modification function, and this modification is interesting because this function is not well behaved for some values of P, specifically it becomes unbounded that would mean the response becomes unbounded, so it is this possibility of structural behavior that we are interested in investigating.



Single concentrated load at midspan

Summary

$$\delta = \delta_0 \frac{3\{\tan u - u\}}{u^3} = \delta_0 \chi(u)$$

$$\theta(0) = \theta_0 \frac{2(1-\cos u)}{u^2 \cos u} = \theta_0 \varepsilon(u)$$

$$M\left(\frac{l}{2}\right) = M_0 \frac{\tan u}{u} = M_0 \xi(u)$$

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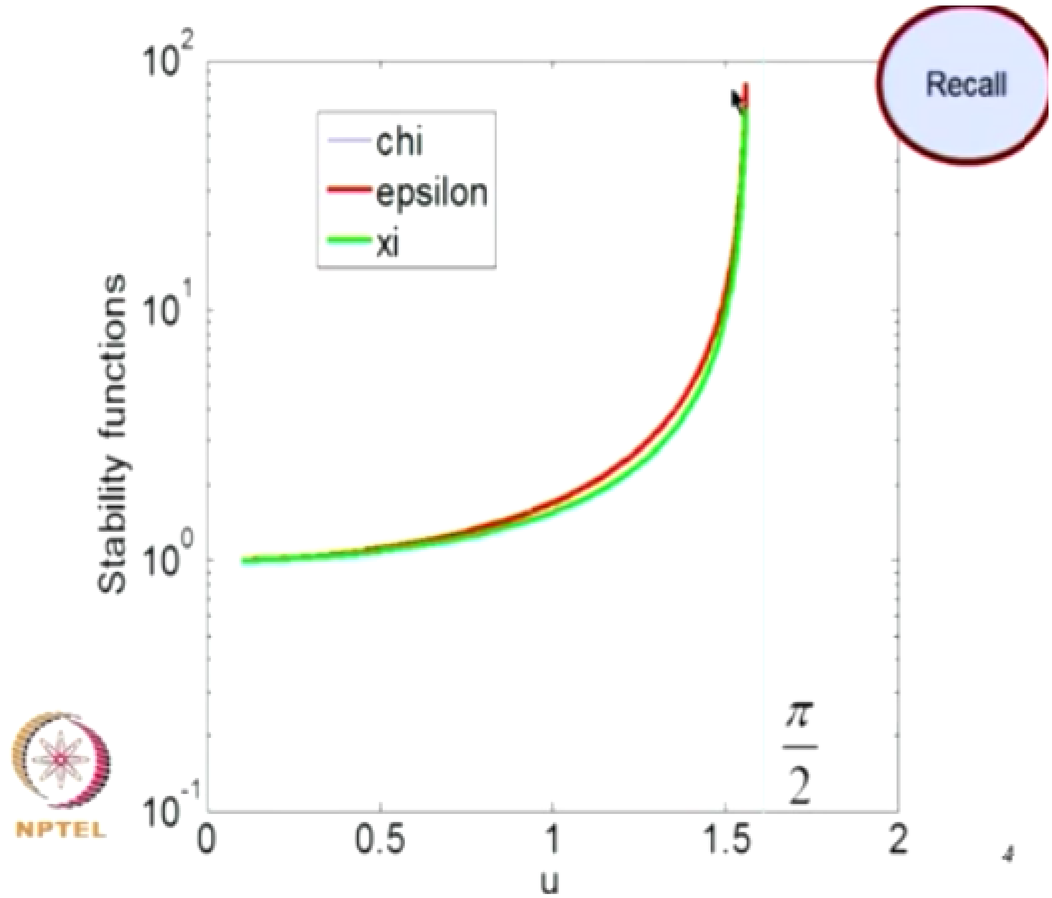
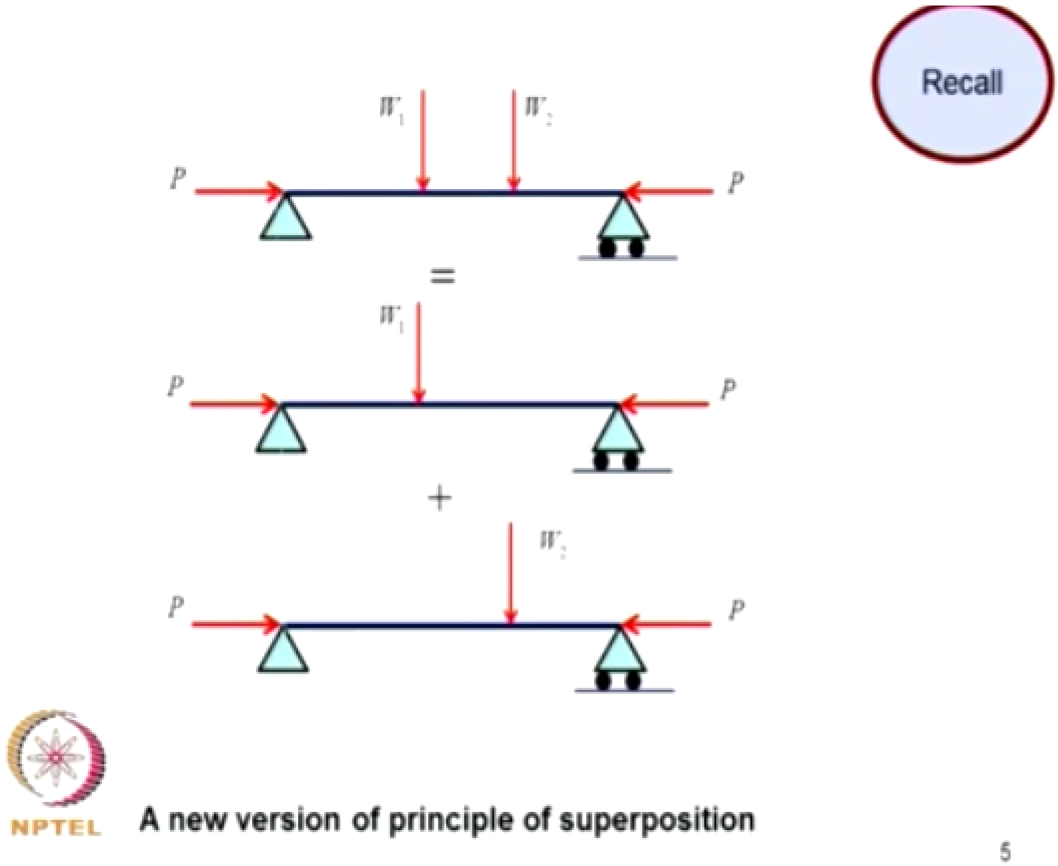
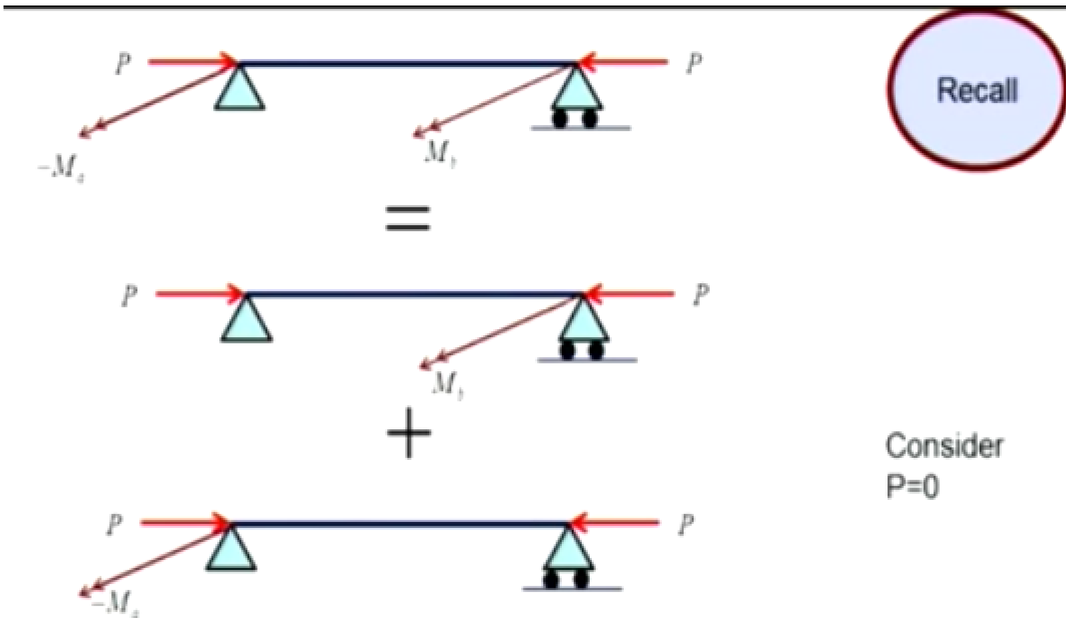
So we consider the problem of a beam, single span beam carrying a concentrated load at a distance C, and we analyze this problem and we showed that for a case of a single concentrated load at the mid span, the mid-span deflection was delta naught is a mid-span deflection when P is 0, and that is modified by a function called chi(u) which is 3 tan U – U/U cube, and similarly the slope at the support is slope when P = 0 multiplied by a modification function, and similarly maximum bending moment is modified by this function, the U is a load parameter is a function of P, and this function chi(u), epsilon(u) and XI(u) are known as stability functions, and their



behavior as a function of U is shown here, what is interesting is, as U approaches value of pi/2 this magnification factor becomes unbounded, so presence of an axial load can have dramatic effects in the neighborhood of this critical values, in this region when load is relatively small you can see that the value is very close to 1 and we could afford to ignore the effect of axial loads on transverse response, but if you are somewhere in this region the response becomes significantly amplified.



Now another important feature that we noticed was the principle of superposition is no longer valid in its traditional form it needs to be modified, for example if there is a beam carrying loads W1 and W2 and axial load P, if we analyze this beam problem under the action of W1 alone, W2 alone, and P alone and add the responses, we will not get the response to the system of loading. On the other hand a particular form of superposition is still possible where we will analyze the response under W1 and P, W2 and P, and if you add these 2 responses we will get response for this system, using this modified version of principle of superposition and treating this as a building block, the solution to this problem as a building block, we solved a few



$$\begin{Bmatrix}\theta_a\\ \theta_b\end{Bmatrix}=\begin{bmatrix}\frac{l}{3EI}\psi(u) & \frac{l}{6EI}\phi(u)\\ \frac{l}{6EI}\phi(u) & \frac{l}{3EI}\psi(u)\end{bmatrix}\begin{Bmatrix}M_a\\ M_b\end{Bmatrix}$$

$$\phi(u)=\frac{3}{u}\left[\frac{1}{\sin 2u}-\frac{1}{2u}\right]$$

$$\psi(u)=\frac{3}{2u}\left[\frac{1}{2u}-\frac{1}{\tan 2u}\right]$$

problems, specifically we solve the problem of a beam loaded by 2 end couples and we derived the flexibility matrix for this system which relates the end rotations to the applied moments, and this is the flexibility matrix and in the limit of P going to 0 this reverts back to the traditional flexibility matrix, and in the P in the neighborhood of P critical these elements of this matrix become unbounded and thereby indicating critical behavior that should be of engineering interest.

Recall

$$BM_{max} = -EI\frac{d^2y}{dx^2} = M_0 \sec u$$

$$\sigma_{max} = \frac{P}{A} + \frac{Pe}{I}\sec\left\{\frac{l}{2}\sqrt{\frac{P}{EI}}\right\} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left\{\frac{l}{2}\sqrt{\frac{P}{EI}}\right\}\right]$$

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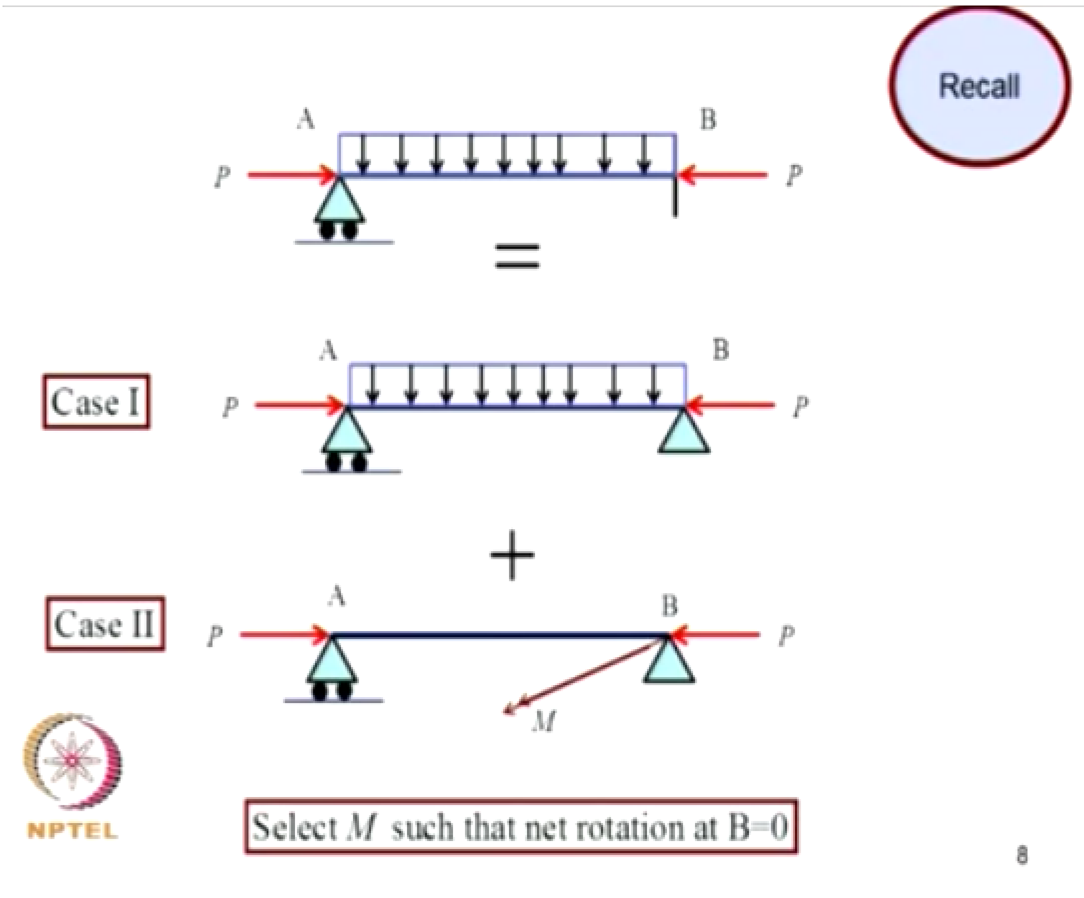
We also considered what would happen if axial load is applied eccentrically? So in real applications one cannot apply axial loads truly axially and one possible way of depicting that is the load is applied with an eccentricity E, we analyze this problem and we showed that the maximum bending moment in this case is given by M naught into secant U, if a moment is applied without the presence of axial load M naught will be the moment, but due to the application of simultaneous presence of axial load this is a modification factor, and the bending stress due to this we have computed and this is commonly used in design of steel structures.



Now we started discussing briefly about indeterminate structures, suppose we have a propped cantilever carrying UDL and an axial load P, so what we did was we release this fixity condition and the problem became statically determinate we have solved this problem, and we considered another problem where to this system we added an unknown moment M, now this moment M was selected such that the rotation at B is 0, when we add these 2 solutions the net rotation at B must be equal to 0.

References

- S P Timoshenko and J M Gere, 1963, Theory of elastic stability, McGraw-Hill, London.
- G J Simitses and D J Hodges, 2006, Fundamentals of structural stability, Elsevier, Amsterdam.

Today's lecture

- Indeterminate structures
- Imperfections and ideal situations
- Familiarization with few typical situations arising in structural stability analysis through a few examples

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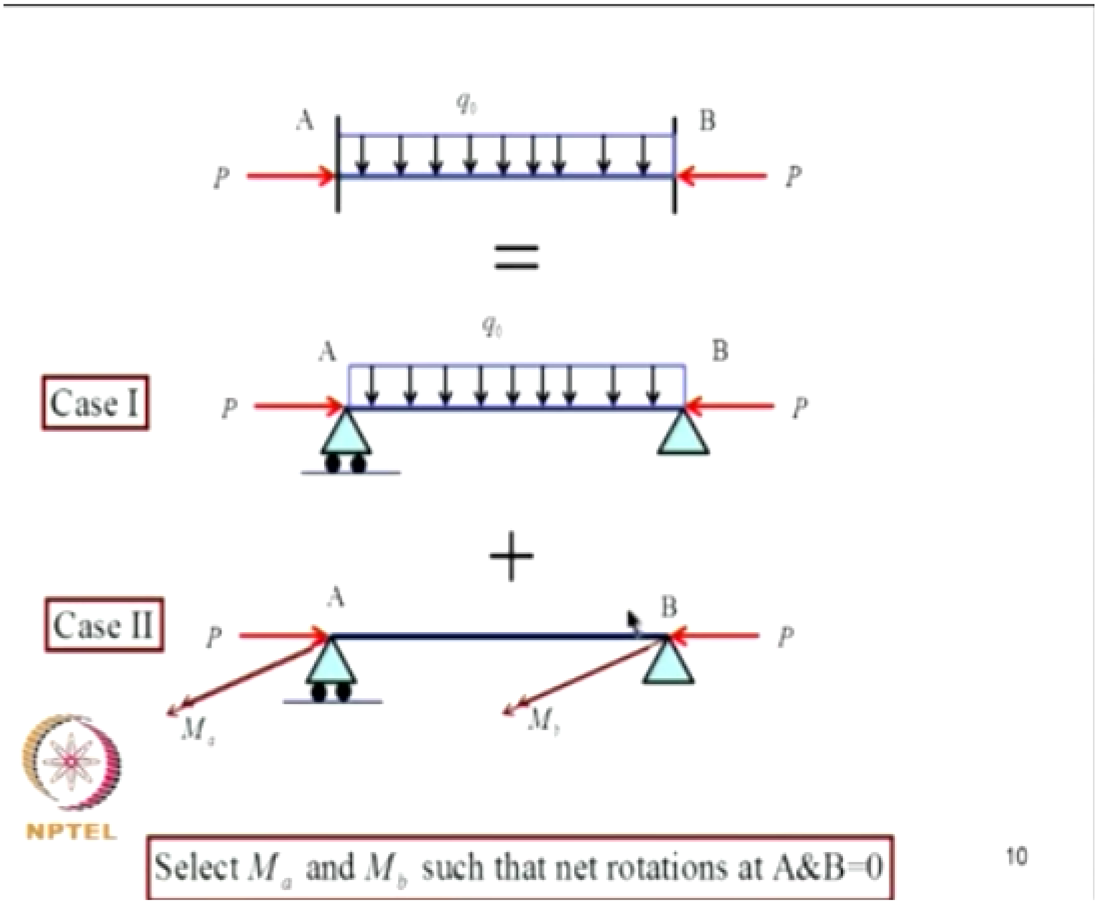
Now in today's lecture we will continue discussing behavior of indeterminate structures, and also we will discuss issues related to imperfections and ideal situations, and we will consider a few typical situation arising in structural stability analysis through a few examples so that we become familiar with the issues that we need to eventually address using computational tools, the material that I am going to present in today's lecture are covered in the books by Timoshenko and Gere and Simitses and Hodges, and some of these examples have been taken from these references.



So we will now consider early, in the previous example we considered propped cantilever beam, now we will consider a beam which is fixed at the 2 ends carrying a UDL, so what we will do is again we will release the 2 fixity conditions, and replace it by end conditions as shown here, and then we will solve another problem where the beam is loaded by 2 moments MA and MB, now because of this load Q naught there will be an end rotation here and because of these moment there will be another end rotation here, will this MA and MB are unknowns they need to be selected such that the rotations that are caused here, and rotations that are caused here when added becomes 0, because in the built-up structure there is no rotations at the end. The idea of splitting the problem into these 2 situations is we already solve these 2 problems and we are using principle of superposition in the modified way that we enunciated, so we select MA and MB such that net rotations at A and B are 0.

By symmetry, $M_A = M_B = M_0$

$$\frac{ql^3}{24EI}\chi(u) + \frac{M_0 l}{2EI}\frac{\tan u}{u} = 0 \quad \left(\text{with } \chi(u) = \frac{3\{\tan u - u\}}{u^3}\right)$$

$$M_0 = -\frac{ql^2}{12}\frac{u\chi(u)}{\tan u}$$

Excercise

Verify Maximum bending moment $\left(\text{at } x = \frac{l}{2}\right) = \frac{ql^2}{24}\frac{6(u - \sin u)}{u^2 \sin u}$

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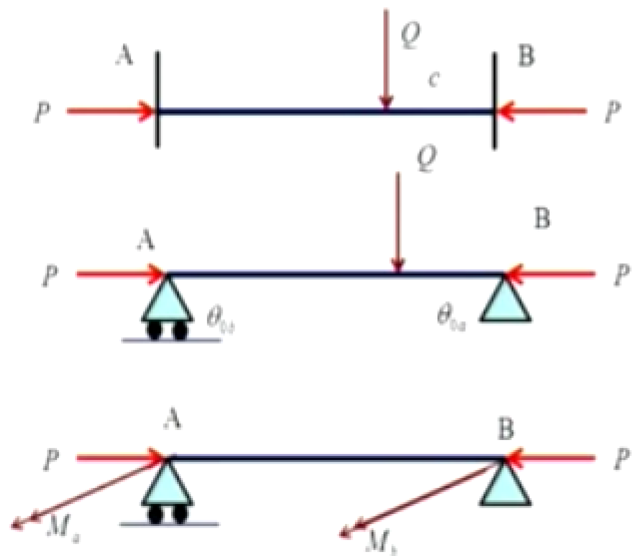
So by symmetry we can deduce that this MA must be equal to MB, because loading is symmetric, boundary condition is symmetric so this will be M naught, so the condition when translated into the solution that we already derived becomes this QL cube/24EI chi(u) + this must be equal to 0 and based on that we get M naught, so once M naught is determined I know the complete solution to this problem, and this problem, this problem is already solved, once M naught is determined this problem also gets solved and we can add the responses to construct response of the beam for this situation.

Now I leave it as an exercise for you to verify that the maximum bending moment in this case at X = L/2 is given by this, this is the maximum bending moment in the absence of axial load P, and this is a modification factor.



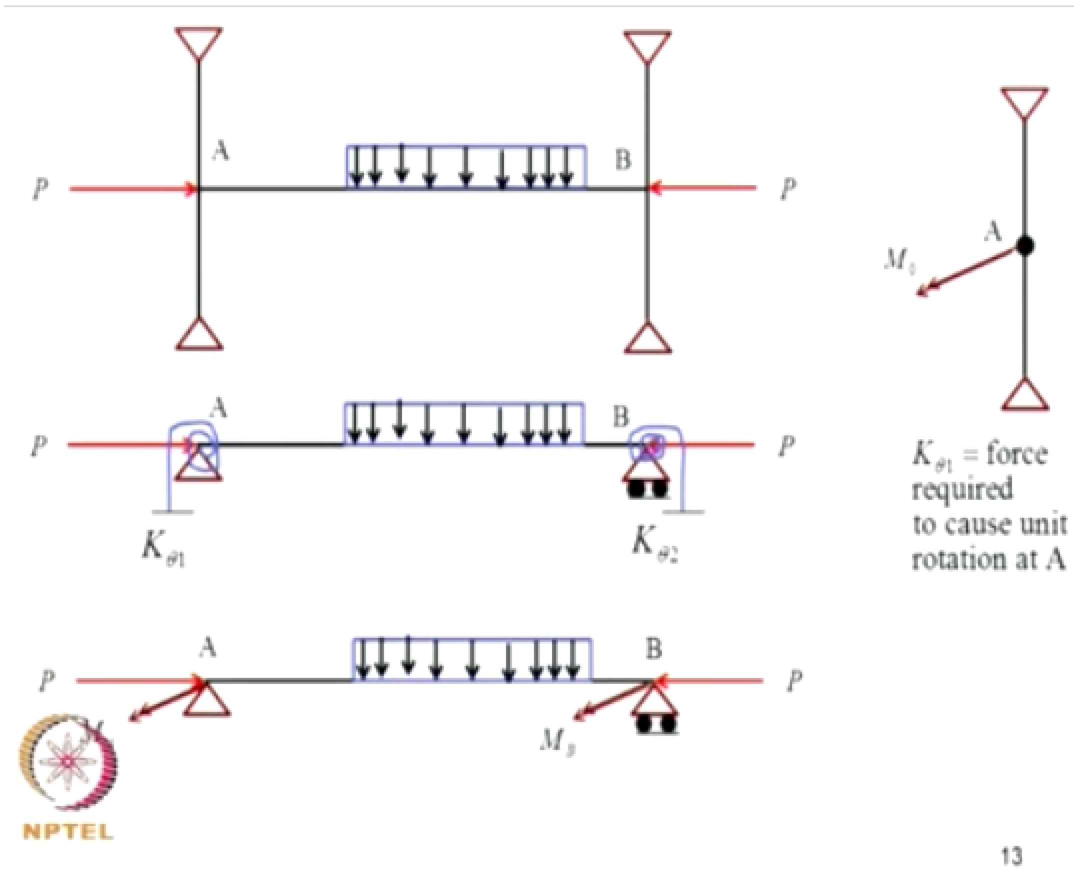
Select M_a and M_b so that $\theta_a = 0$ and $\theta_b = 0$

$$\theta_a = \theta_{0a} + \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u) = 0$$

$$\theta_b = \theta_{0b} + \frac{M_b l}{6EI}\phi(u) + \frac{M_a l}{6EI}\psi(u) = 0$$

This can be generalized to obtain the generalized three moment equations. Similarly we can get slope-deflection equations.

Now this is a special case where the load is now, concentrated load, we can again we release end loads and solve these 2 problems and again select MA and MB, so that theta A is 0 and theta B is 0, here there is no advantage of symmetry, so we have to find out MA and MB separately, so we get the 2 equations which we should be using to find MA and MB. Now this approach can be generalized to obtain, generalize 3 moment equations similarly we can get slope deflection equations you can do moment is to, moment distribution whatever you are traditionally doing, we could do this, so this is at a very classical set of tools, the things that we are trying to do is we are just investigating how all these well-known results can get modified because of, now our additional consideration of presence of axial loads.



Now I will now go on considering few situations as I said so which sensitizes has to the kind of issues that are of relevance in dealing with stability problems. Now let us consider a beam which is supported as shown here an idealization for this situation is will replace these 2 members by equivalent rotary springs as shown here, and how do we evaluate these rotary springs? We will consider these 2 members separately and find out what moment I should apply here to so as to produce unit rotation, so K theta 1 is the force required to cause unit rotation at A, that is why this K theta 1, so similarly K theta 2 will be the moment required to produce unit rotation at B and that is K theta 2, that K theta 1 and K theta 2 will be function of properties of this member.

Once this is done the problem is now how do I solve this problem? So we consider this problem where there are no N springs, we have solved this problem. Now again the solution to this problem now needs to be handle as follows, we select MA and MB, here again MA and MB are unknowns instead of solving this problem we solve this problem first, and the rotation here

Select M_a and M_b such that

$$-\frac{M_a}{K_{\theta_1}} = \theta_{0a} + \frac{M_a l}{3EI}\psi(u) + \frac{M_b l}{6EI}\phi(u)$$

$$-\frac{M_b}{K_{\theta_2}} = \theta_{0b} + \frac{M_b l}{6EI}\phi(u) + \frac{M_a l}{6EI}\psi(u)$$

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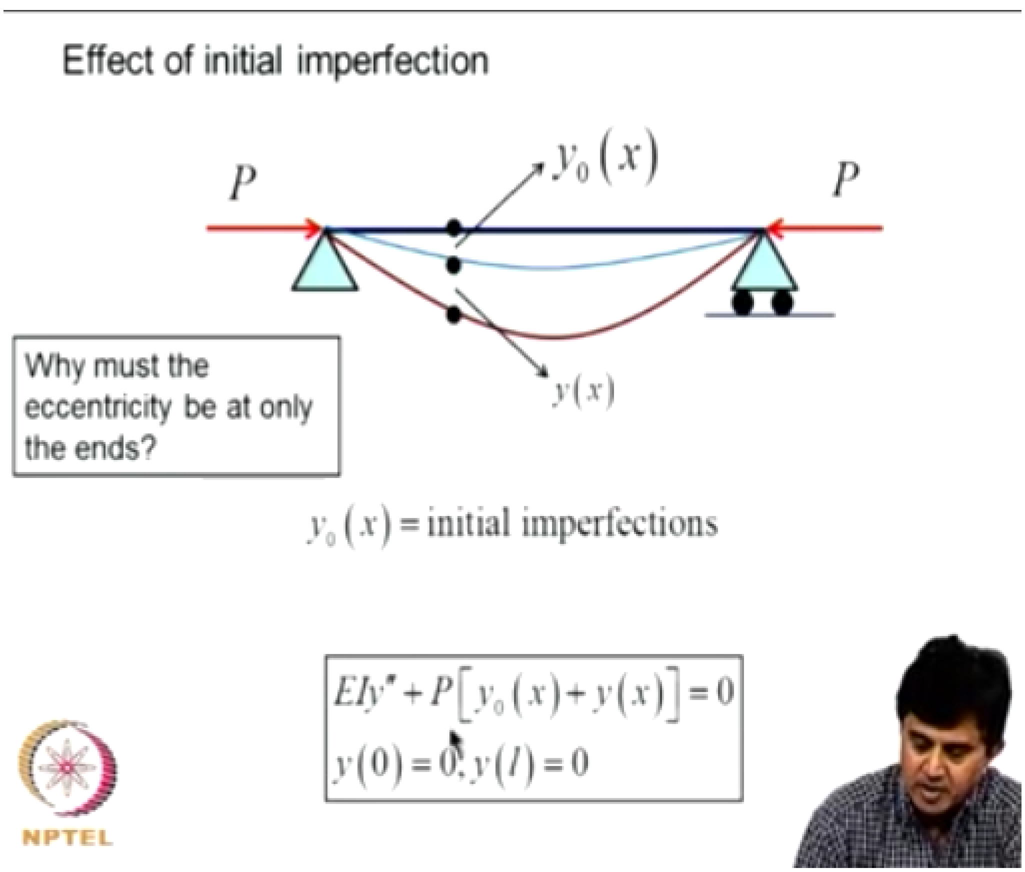
won't be 0 but it will be MA/K theta 1, so K theta 1 is already known so that must be equal to this, and MB similarly is given by this equation and by solving these 2 equations we will be able to find MA and MB, and subsequently we will get solution to the original problem as shown here.



Now in the previous lecture we considered the effect of eccentrically applied loads that is this axial load was not applied truly axially but at the support there was slight eccentricity. Now why must the eccentricity be limited only at the point of application of the load at the supports there can be imperfections which is distributed throughout the beam, suppose if we consider that situation, suppose the beam has the initial imperfection which is Y naught(x) and then we apply axial loads P, so the equation for equilibrium of this system will be EIY double prime plus the bending moment at this section will be P into initial imperfection plus the deflection due to Y(x), so this is given by this, so here again we must understand the reason why we are writing this equation in this form is that we are looking for equilibrium positions in the neighborhood of 0 displacement position, and this equation is valid only under the assumption that there exists this neighboring equilibrium state, so if it exists this will be the equilibrium position, so that point must be borne in mind.

$$EIy'' + P\left[y_0(x) + y(x)\right] = 0; y(0) = 0; y(l) = 0$$

$$EIy'' + Py(x) = -Py_0(x)$$

$\Rightarrow$ Inital imperfection here creates the same effect as that of an externally applied load.

$$y_0(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = a_1 \sin\frac{\pi x}{l} + \sum_{n=2}^{\infty} a_n \sin\frac{n\pi x}{l} \approx a_1 \sin\frac{\pi x}{l}$$

$$y'' + \lambda^2 y = -\lambda^2 y_0(x) = -a\lambda^2 \sin\frac{\pi x}{l}$$

$$y(x) = A\cos\lambda x + B\sin\lambda x - \frac{a\lambda^2}{\lambda^2 - \frac{\pi^2}{l^2}}\sin\frac{\pi x}{l}$$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow A = 0 \\ y(l) = 0 \Rightarrow B = 0 \end{array}\right\} \Rightarrow y(x) = -\frac{a\lambda^2}{\lambda^2 - \frac{\pi^2}{l^2}}\sin\frac{\pi x}{l}$$

So now we can deal with this equation, so this is the equation, now PY naught can be taken to the right hand side so this behaves like a distributed load now, and there is an axial load also they are not independent, they are related by the same multiplication factor P that means initial imperfection here creates the same effect as that of an externally applied load. Now to proceed further what we can do is we can expand Y naught(x) in Fourier series involving sine functions, and this if we decide to retain the first term in this approximate for sake of discussion, A1 sine pi X/L, so we are considering an imperfection which can be reasonably be approximated by a half sine wave. Now I'll substitute it here so I again divide by EI and call P by EI as lambda square, so I get Y double prime + lambda square Y as this, prime is differentiation with respect to X so this. Now we can, this solution to this equation will have a complementary part function and a particular integral.

Now this is a boundary value problem, so the boundary conditions are at X = 0 and X = L. so at X = 0, Y is 0, and at X = L, Y is 0, so if I impose these 2 conditions I will get Y(x) into be in this form, okay.

$$y(x)+y_0(x)=a\sin\frac{\pi x}{l}\left\{1-\frac{\lambda^2}{\lambda^2-\frac{\pi^2}{l^2}}\right\}$$

Final deflection= {Initial imperfection} {Modification factor}

$$\text{Modification factor}=\left\{1-\frac{\lambda^2}{\lambda^2-\frac{\pi^2}{l^2}}\right\}$$

This factor can be ill behaved when $\lambda^2-\frac{\pi^2}{l^2}\rightarrow 0$

$$\Rightarrow P_{cr}=\frac{\pi^2 EI}{l^2}$$

$\Rightarrow$ If $P = P_{cr}$, slightest imperfection would get dramatically modified.

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Now if I now consider the total displacement at the cross section it will be Y(x) + Y naught(x) and that I can write in this form, now that means final deflection is the initial imperfection multiplied by a modification fact, so this is the initial imperfection because of presence of axial loads an initial imperfection gets amplified through this factor, mind you there is no transverse load in this problem, it is only the axial load on a imperfect beam, so the modification factor here is given by this.

Now lambda square we know is square root P/EI and we can substitute that, and a critical condition will occur when lambda square goes to pi square/L square, so if this goes to 0 then this modification factor becomes unbounded and this condition will be satisfied when P is P critical which is the Euler buckling load for the beam, so if P = P critical slightest imperfection would get dramatically modified that means the whatever the slight deformation of the beam that already as an imperfection because of axial loads it can get dramatically modified provided the axial load is in the neighborhood of this critical load okay, so this axial loads reaching the critical value again turns out to be of engineering importance, so we can now simplify this

$$y(x)+y_0(x)=a\sin\frac{\pi x}{l}\left\{1-\frac{\lambda^2}{\lambda^2-\frac{\pi^2}{l^2}}\right\}$$

$$\text{Max BM}=P\left[y(x)+y_0(x)\right]_{x=\frac{l}{2}}$$

$$M_{max}=Pa\left\{1+\frac{1}{\frac{P_{cr}}{P}-1}\right\}$$

$$\sigma=\frac{P}{A}+Pa\left\{1+\frac{1}{\frac{P_{cr}}{P}-1}\right\}\frac{c}{r^2}\ \text{(Perry Robertson formula)}$$

c=distance from NA to the outermost fibre & $I = Ar^2$

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expression we have the total displacement, now maximum bending moment will be P into this, at X = L/2, so if I put that and substitute for lambda square in terms of P critical and P, I will get the maximum bending moment will be PA into this.

Now the axial stress if you find, due to axial load P there is a P/A + this is a contribution due to initial imperfection, and this is again like a secant formula that we had encountered earlier this is the Perry Robertson formula which again is commonly used in metal structure design.

Remark

$$y'' + \lambda^2 y = -a\lambda^2 \sin\frac{\pi x}{l}; y(0) = 0 \,\&\, y(l) = 0 \Rightarrow y(x) = -\frac{a\lambda^2}{\lambda^2 - \frac{\pi^2}{l^2}} \sin\frac{\pi x}{l}$$

As response grows, nonlinear terms kick in and amplitudes may be limited.

Compare this situation with problem of resonance in sdof systems:

$$\ddot{x} + \omega^2 x = P\cos\lambda t; x(0) = 0; \dot{x}(0) = 0 \Rightarrow x(t) = \frac{P}{\lambda^2 - \omega^2}\left[\cos\omega t - \cos\lambda t\right]$$

$$\lim_{\lambda\to\omega} x(t) = \lim_{\lambda\to\omega} \frac{P}{\lambda^2 - \omega^2}\left[\cos\omega t - \cos\lambda t\right] = \frac{Pt}{2\lambda}\sin\lambda t$$

$$\Rightarrow \lim_{\lambda\to\omega}\lim_{t\to\infty} x(t) \to \infty$$

Resonance response amplitudes are limited by damping.

Nonlinearity would also become important as response grows.

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Now we can make a remark here that if you examine the nature of this equation it is Y double prime + lambda square Y is - A lambda square sine pi X/R, and these are the boundary conditions, and this is a solution. Now as the response gross that means if you are in the neighborhood of P = P critical the non-linearity you know steps in, so material non-linearity can come in and so on and so forth and consequently the amplitudes may be limited by a possible nonlinear behavior of the system, although a linear theory depicts that amplitude becomes unbounded, moment and other nonlinearities are included because response amplitudes have become large, then the amplitudes may not really tend to infinity.

Now we can compare to gain insight into the behavior we can compare this situation with problem of resonance in signal degree freedom systems, see the governing equation here is Y double prime + lambda square Y in excited by a sinusoidal function, so suppose if I now consider a single degree freedom system undamped driven harmonically by harmonic load the equation will be X double dot + omega square X is P cos lambda T, so let’s assume that the system starts from rest, the main mathematically although the field equation has the same form here, the important difference is this is a boundary value problem whereas this an initial value problem, okay, excepting for that the structure of the differential equation is exactly the same, so the nature of complementary function in particular integral everything will be same, but once you start imposing the boundary conditions here or the initial conditions here the solution would differ, so here X(t) we know we have got it in this form, so at resonance a critical condition is rigid as lambda tends to be go to omega, just as P becomes P critical here, so in that case the solution now we know it becomes a sine lambda T which is modulated by a linearly varying function of time. Here as lambda goes to omega and T tends to infinity X(t) tends to infinity, so this is the response, this is how the time envelope growth and this is how, so this shoots to infinity as T tends to infinity.

Remark

$$y'' + \lambda^2 y = -a\lambda^2 \sin\frac{\pi x}{l};\ y(0)=0\ \&\ y(l)=0 \Rightarrow y(x) = -\frac{a\lambda^2}{\lambda^2 - \frac{\pi^2}{l^2}}\sin\frac{\pi x}{l}$$

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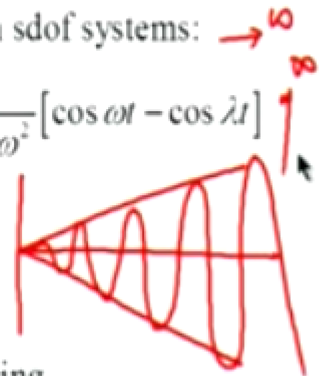
$$\lim_{\lambda\to\omega} x(t) = \lim_{\lambda\to\omega}\frac{P}{\lambda^2-\omega^2}\left[\cos\omega t - \cos\lambda t\right] = \frac{Pt}{2\lambda}\sin\lambda t$$

$$\Rightarrow \lim_{\lambda\to\omega}\lim_{t\to\infty} x(t) \to \infty$$



Resonance response amplitudes are limited by damping.

Nonlinearity would also become important as response grows.

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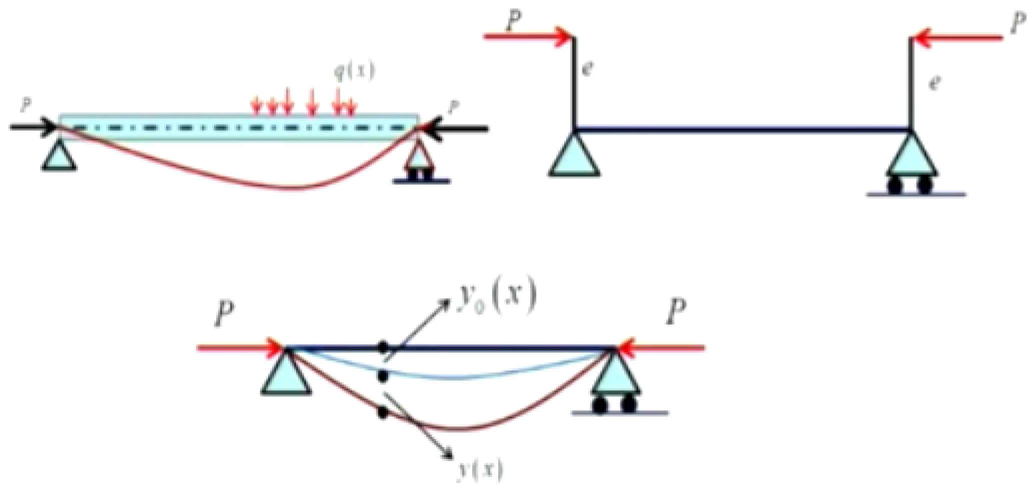
Now in this case what limits this amplitude, even in a linear system what limits the amplitude is presence of damping, okay, and of course if response becomes large here also non-linearity can come in, but even when non-linearity is not included in the response, the response here doesn't not become unbounded, because presence of damping ensures that at resonance in steady state the response amplitude is limited, so this difference must be I mean if you analyze the difference you will get further insight into how this structure is behaving.



These three problems are mathematically equivalent.
The
- transverse load
- eccentrically applied axial load
- initial imperfections

are manifestations of departures from an ideal situation.
How about the study of the ideal situation itself?

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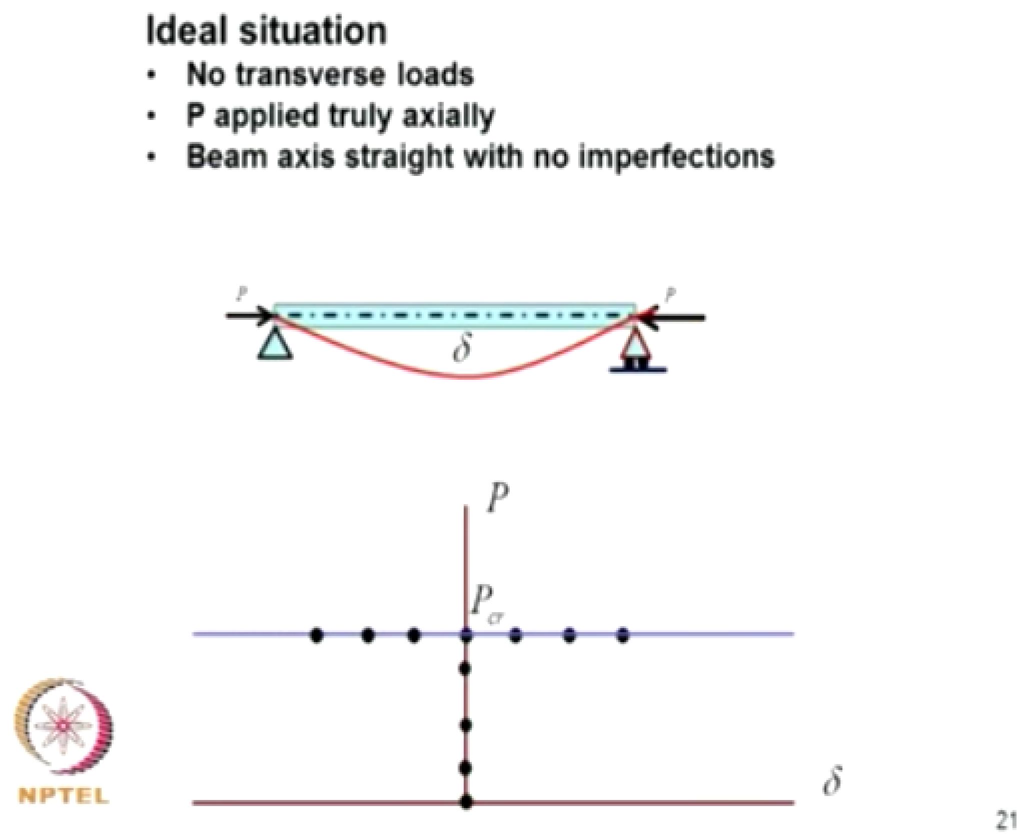
Now we have considered so far 3 types of problems, we considered axially loaded single-span beams in a way, presence of a transverse load or this load being applied eccentrically or they're being a presence of imperfection or a combination of all of them in a way depict situations which are far from ideal, an ideal situation will be there is a straight beam carrying no lateral loads, transverse loads and it is loaded actually by truly axial loads, there is no initial imperfection, there is no transverse load, there is no eccentricity in applying P, that is the ideal situation, the departure from ideal as we have seen can create amplification of the response especially when P is in the neighborhood of certain critical values. Now why this happens? So to understand this we can ask the question how about the study of the ideal situation itself, does



it tell us anything about why this system is behaving in this manner so to initiate that discussion we can consider the ideal situation where there are no transverse loads, P is applied truly axially, and beam axis is straight with no imperfections, so we can do a thought experiment, we consider beam loaded as shown here satisfying these specific requirements as spelt out, what we will do is we will increment P in steps of P, I will start with 0 delta P, 2 delta P, so on and so forth.

Now at every increment in load, at 0 of course this transverse displacement is 0 so what I am plotting here is a load deflection diagram here where load is plotted on Y axis, and delta is on the X-axis. When P = 0 of course response is 0 we start here, now P is increased to delta P, then what we do is we pluck the beam and allow it to vibrate, and because the beam will be damped so it may, there are 2 possibilities the beam may return to its original position or it may not return to original position, it so happens that it will return to the original position if P is small, okay, we will find explanations for all this, this is some kind of a preview. As we go on doing this for 2P, 3P, etcetera, we will soon reach a stage where when you pluck the beam it oscillates but it does not return to its original position but take some other position, so the final position that it takes depends on in which direction you have plucked and by how much you have plucked, so the load deflection diagram here has 2 branches, if you call this as a branch, this as a branch, they're 3 branches, okay, compare this with situation where in the ideal, linear, a transversely loaded beam, and if you are loading this by P and measuring delta this will be a straight line, okay, there is a single branch, it will go to infinity in a linear model, so I mean that itself, that may not be realistic but as far as mathematical model is concerned that is what is employed in assuming that system is linearly.

Now there is a always a single branch here, but here, for a given P is only one delta, so unique single branch for a load deflection diagram will be seen for this type of system, whereas here for P = P critical there is no unique response, the uniqueness is lost, okay, so what does all this

- Apply load P in increments as $0, \Delta P, 2\Delta P, \cdots$
- At each value of load P, pluck the beam from its equilibrium position and see if it returns to its original position.
- Up to $P = P_{cr}$, the beam returns to its original position. There would be a single load-deflection path.
- At $P = P_{cr}$, the beam does not return to its original position and occupies a neighbouring equilibrium state.
- The load-deflection path now bifurcates into two new paths.
- The load-deflection path ceases to be unique.

There seems to be three paths possible. (Are more possible?)

- How to characterize this mathematically?

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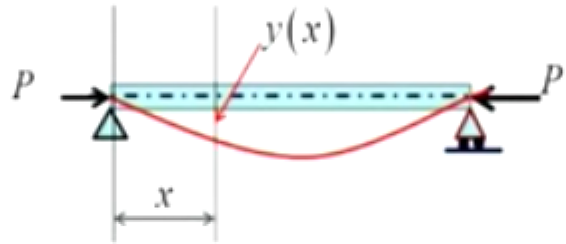
mean? So what we are doing is we are applying load P in increments as 0, delta P, 2 delta P, at each value of P we pluck the beam from its equilibrium position and see if it returns to its original position, up to P = P critical the beam returns to its original position there will be a single load deflection path, at P = P critical, the beam does not return to its original position and occupies a neighboring equilibrium state, we say that the load deflection path has now bifurcated into 2 new parts, that is as a consequence the load deflection path ceases to be unique, there seems to be 3 paths possible we do not know there may be more and this analysis we have to carry out the analysis to verify.

How to characterize this mathematical? I mean of course these observations are made based on from hindsight gained by doing a mathematical analysis, so we will see now how all these statements are actually made, what should be P such that the adjoining equilibrium position

What should be P such that a adjoining equillibrium position becomes possible?

Assume: an adjoining equillibrium position is indeed possible.



$$EI\frac{d^2y}{dx^2}+Py=0;\ y(0)=0;\ y(L)=0$$

This is an eigenvalue problem.

For what values of P does this equation admit a nontrivial solution?

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becomes possible is a question we will ask, okay, so we will assume that under the action of axial load P the beam can assume an adjoining equilibrium state, so this is the adjoining equilibrium state, and the deflection at a point X is Y(x) transverse deflection, so that means we are starting within an a priori assumption that an adjoining equilibrium position is indeed possible, under that assumption we can write now the equation EID square Y/DX square = - PY, where the PY is a bending moment at cross section X, and Y is at X = 0, 0 at X = L is again 0. Now the question we are asking is for Y = 0 is a solution for any value of P, are there any values of P for which non-trivial Y(x) is possible, that means it satisfies this equation and the boundary conditions, but Y is not 0, so again this is an Eigenvalue problem, but now it is associated with the differential operator second order linear differential operator, so the question is for what values of P does this equation admit a non-trivial solution, so we can solve

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0; \lambda^2 = \frac{P}{EI}$$

$$y(x) = A\cos\lambda x + B\sin\lambda x$$

$$y(0) = 0 \Rightarrow A = 0$$

$$y(L) = 0 \Rightarrow B\sin\lambda l = 0$$

For nontrivial solutions, $\sin\lambda l = 0 \Rightarrow \lambda l = n\pi, n = 1, 2, \cdots, \infty$

$\lambda_n = \frac{n\pi}{l} n = 1, 2, \cdots$ are the eigenvalues and $\phi_n(x) = \sin\frac{n\pi x}{l}$

are the associated eigenfunctions.

There exist infinitely many values of P given by $P_n = n^2\pi^2\frac{EI}{l}, n = 1, 2, \cdots, \infty$

at which adjoining equillibrium states become possible.

$P_{cr} = \pi^2\frac{EI}{l} = \min[P_n]$ is called the Euler's buckling load.

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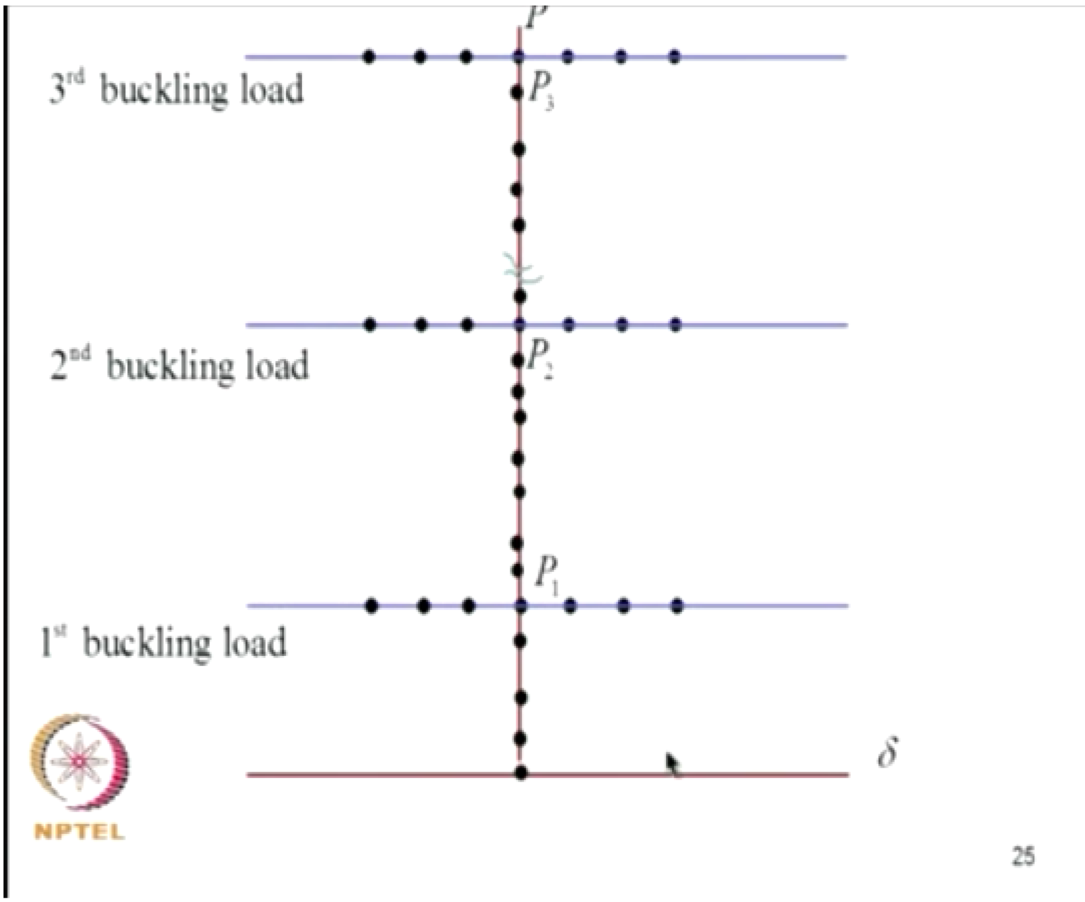
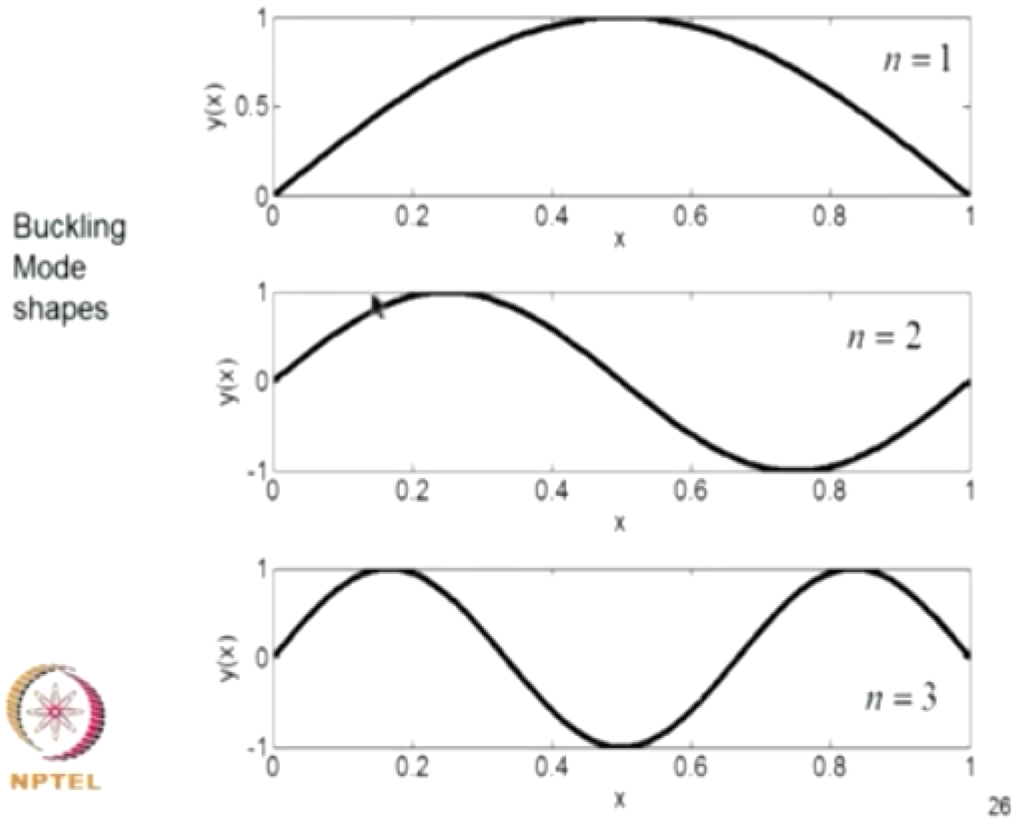
this equation I divide by EI and call P/EI as lambda square, and this solution is A cos lambda is + B sine lambda X, at X = 0, Y is 0, so A would become 0, at X = L, B sine lambda L is 0, now either be can be 0 or sine lambda L can be 0, if B is 0, Y(x) is 0 which is a trivial solution that is not what we are looking for, therefore for non-trivial solution sine lambda L must be equal to 0 or lambda L is N phi, so where N runs from 1 to infinity so at least we have answered one question now, whenever P is such that lambda L is N phi an adjoining equilibrium position is possible, and there are infinity of them, and the loads appear as I related to the eigenvalues of this problem, and the deflected shapes are given by the Eigen functions associated with these eigenvalues and in this case the Eigen functions are sine N phi X/L, there exist infinitely many values of P given by PN = N square phi square EI/L, N = 1 to infinity at which adjoining equilibrium states become possible. The lowest value of P for which such adjoining equilibrium states are possible is called the critical load and it is called the Euler's buckling load.



So now equipped with this analysis we can redraw the load deflection diagram now, so here if P is less than P critical our load deflection path will trace this branch, and as P becomes PB is equal to P1 as you pluck the beam it will occupy different positions and if you go on increasing the load there are other branches that are possible, so it turns out that there are infinity of P critical is possible and the load deflection diagram will have infinite number of branches, but of course from engineering point of view we are interested in the lowest value of P at which the structure would lose its stability and we would keep the load P, we will design the structure in such a way that the load P, critical value of load P is below this, the P critical is placed sufficiently high so that the axial loads experienced by the structure would not come close to the critical values.



The buckling mode shapes are sinusoidal functions, this is the first mode, this is second mode and this is a third more and so on and so forth.

Other boundary conditions: both ends clamped

$$EI\frac{d^4y}{dx^4}+P\frac{d^2y}{dx^2}=0; y(0)=0, y(l)=0; y'(0)=0, y'(l)=0;$$

$$y^{iv}+\lambda^2 y''=0$$

$$y(x)=A\cos\lambda x+B\sin\lambda x+Cx+D$$

$$y'(x)=-\lambda A\sin\lambda x+\lambda B\cos\lambda x+C$$

$$y(0)=0\Rightarrow A+D=0$$

$$y'(0)=0\Rightarrow \lambda B+C=0$$

$$y(l)=0\Rightarrow A\cos\lambda l+B\sin\lambda l+Cl+D=0$$

$$y'(l)=0\Rightarrow -\lambda A\sin\lambda l+\lambda B\cos\lambda l+C=0$$

$$\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & \lambda & 1 & 0\\ \cos\lambda l & \sin\lambda l & l & 1\\ -\lambda\sin\lambda l & \lambda\cos\lambda l & 1 & 0\end{bmatrix}\begin{Bmatrix}A\\B\\C\\D\end{Bmatrix}=0$$

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Now how about other boundary conditions? So suppose if you have a beam with 2 ends clamped, now a clarification can be made at this point when we write problems here, when I write P like this for a fixed beam it must be understood that what is being prevented here is the rotation about this axis, and the displacement about this action, the displacement in this direction is permitted, so this convention of showing this end as fixed by using this notation it should be understood that the beam is permitted to move in the U direction, this direction, otherwise the load P will not have any effect on the beam, so this issue must be understood in many of the books this is how it is represented, but unless the axial point at which the load acts the beam is permitted to move in the direction of the load all these discussions would not be relevant okay.

Okay, now for indeterminate structures like this we'll start with the fourth order equation relating EI D4Y/DX4 is equal to the applied loads, and in this we will include the effect of axial loads that is then the governing equation will be P D4Y/DX4 + PD square Y/DX square = 0, sometime later in the class this lecture we will derive this equation using Hamilton's principle but I am assuming that you are familiar with this equation, so in this revision where I am assuming that you know this, so again I divide by EI I get this equation, and the solution this is a fourth order equation there will be 4 integration constant A cos lambda X + B sine lambda X + CX + D is the complementary function. Now the boundary conditions are in terms of Y and Y prime, so Y prime is given by this and there are 4 constants and 4 boundary conditions, moment we impose those 4 conditions we get 4 equations and those 4 equations can be written in this form, so for non-trivial solution the determinant of this inverse of this matrix must not exist, and for that the determinant of this matrix must be equal to 0, and this is a characteristic

For nontrivial solutions

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \lambda & 1 & 0 \\ \cos\lambda l & \sin\lambda l & l & 1 \\ -\lambda\sin\lambda l & \lambda\cos\lambda l & 1 & 0 \end{vmatrix} = 0$$

$$\sin\frac{\lambda l}{2}\left(-\sin\frac{\lambda l}{2}+\frac{\lambda l}{2}\tan\frac{\lambda l}{2}\right)=0$$

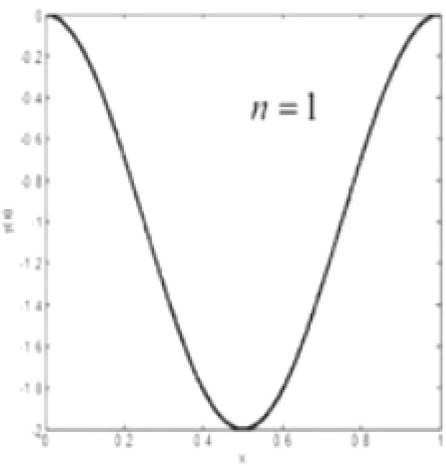
$$\Rightarrow \sin\frac{\lambda l}{2}=0 \text{ or } \tan\frac{\lambda l}{2}=\frac{\lambda l}{2}$$

$$\Rightarrow \frac{\lambda l}{2}=n\pi, n=1,2,\cdots$$

$$P_n=\frac{4n^2\pi^2}{l^2}EI \Rightarrow P_{cr}=\frac{4\pi^2}{l^2}EI=\frac{39.478}{l^2}EI$$

$$(x)=\cos\frac{2n\pi x}{l}-1, n=1,2,\cdots$$

$$\tan\frac{\lambda l}{2}=\frac{\lambda l}{2}\Rightarrow\frac{\lambda_1 l}{2}=4.493\Rightarrow P_{cr}^{*}=\frac{80.763}{l^2}EI$$



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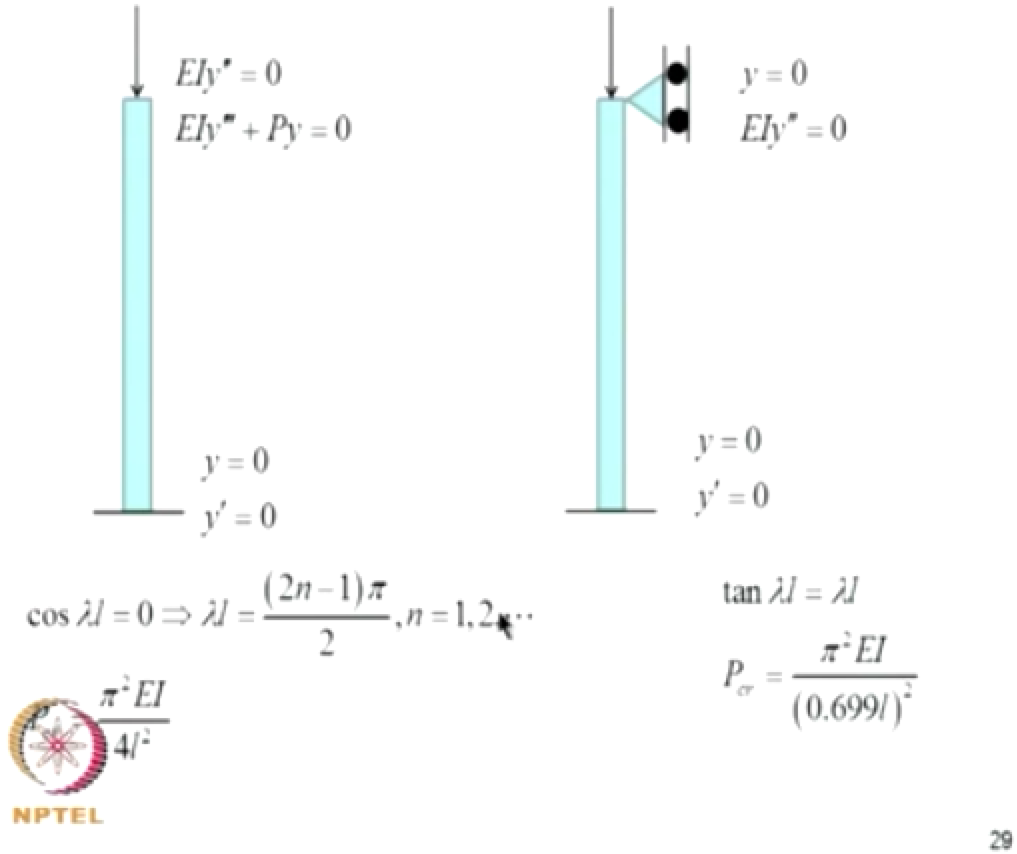
equation which governs lambda which lambda is the actual load parameter, so those values of P which satisfy this equation permit the existence of a adjoining equilibrium state, so if you simplify this there are 2 solutions that become possible. Sine lambda L = 0 or tan lambda L/2 = lambda L/2, this leads to one infinity of solutions, this leads to another infinity of solutions, we will see that this leads to anti-symmetric modes and this leads to symmetric modes, and the lowest value of critical load is given by this branch, and PN is given here, and P critical in this case turns out to be 39.478/L square into EI, so the buckling mode shape is given by this corresponding to this branch, and the first one is depicted here.

The second branch tan lambda L/2 = lambda L/2, if 1, 1 solves, the first root is given by 4.493 and the P critical value is a 80.763 EI/L square as you can see this is higher than this, therefore this is the critical value, this is the first root of this branch tan lambda L/2 = lambda L/2 and phi is the first root of this, and this is a first critical load from this branch, and this is a first critical load from this branch, among these 2 this is the lowest therefore this is the critical load for the beam. So using similar logic now we can consider other boundary conditions like a cantilever



beam, propped cantilever and I have given some broad hints on how this can be analyzed, you have to write the fourth order equation there will be 4 constants, 4 boundary conditions, use them right the characteristic equation, solve, and you will get the characteristic equation in this case as cos lambda L = 0, here tan lambda L = lambda L, and the critical loads for this case is given by this, and for this case it is given by this.

This boundary condition here at the fixed end Y and Y prime are 0, at this top end it is the boundary condition is given by this, one of the earlier lectures we've derived this, if I refer back and in the towards the end of this lecture also we are going to see something about this. Now

Orthogonality of buckling mode shapes

Consider n^{th} and k^{th} mode shapes, denoted respectively by $\phi_n(x)$ & $\phi_k(x)$. $\Rightarrow$

$$(EI\phi_n'')'' + P_n\phi_n'' = 0$$

$$(EI\phi_k'')'' + P_k\phi_k'' = 0$$

$$\Rightarrow \phi_n(x)(EI\phi_k'')'' + P_k\phi_n(x)\phi_k'' = 0$$

$$\Rightarrow \int_0^l \phi_n(x)(EI\phi_k'')'' dx + \int_0^l P_k\phi_n(x)\phi_k'' dx = 0$$

$$\Rightarrow \left\{(EI\phi_k'')'\phi_n(x) + P_k\phi_n(x)\phi_k'(x)\right\}_0^l - \int_0^l \phi_n'(x)(EI\phi_k'')' dx - \int_0^l P_k\phi_n'(x)\phi_k'(x) dx = 0$$

$$\Rightarrow \left\{(EI\phi_k'')'\phi_n(x) + P_k\phi_n(x)\phi_k'(x)\right\}_0^l - \left\{(EI\phi_k'')\phi_n'(x)\right\}_0^l + \int_0^l \phi_n''(x)(EI\phi_k'') dx - \int_0^l P_k\phi_n'(x)\phi_k'(x) dx = 0$$

the buckling mode shapes have like mode shapes in vibration, here also they have orthogonality properties that can be verified by considering a pair of Eigen solution, nth Eigen function, and k-th Eigen function, the Eigenvalue and Eigen function, the nth Eigen function will satisfy this equation k-th Eigen function will satisfy this, PN and PK are the nth and k-th Eigen values, so what we do, we multiply the first equation by the second equation by phi N(x) and integrate from 0 to L I get this equation, and these equations, these integrals I will integrate by parts and doing it twice we can show that this is something that we have done several times, so we can see that after we do this we get this equation reduces to this.

$$\left.\begin{array}{l}\left\{\left(EI\phi_k''\right)'\phi_n(x)+P_k\phi_n(x)\phi_k'(x)\right\}_0^l=0\\ \left\{\left(EI\phi_k''\right)\phi_n''(x)\right\}_0^l=0\end{array}\right\}\text{for classical boundary conditions}$$

$$\int_0^l \phi_n''(x)\left(EI\phi_k''\right)dx-\int_0^l P_k\phi_n'(x)\phi_k'(x)\,dx=0$$

Similarly, we get

$$\int_0^l \phi_n''(x)\left(EI\phi_k''\right)dx-\int_0^l P_n\phi_n'(x)\phi_k'(x)\,dx=0$$

$$\Rightarrow\left(P_k-P_n\right)\int_0^l \phi_n'(x)\phi_k'(x)\,dx=0$$

$$P_k\neq P_n\Rightarrow\int_0^l \phi_n'(x)\phi_k'(x)\,dx=0 \text{ for } n\neq k$$

$$\int_0^l EI\phi_n''(x)\phi_k''(x)\,dx=0 \text{ for } n\neq k$$

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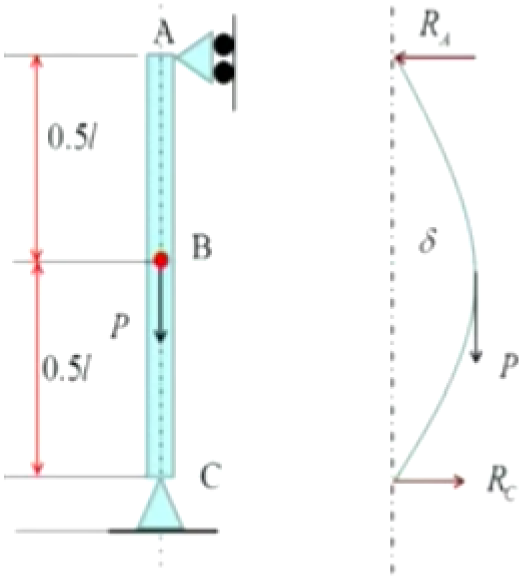
Now this terms inside this braces will be 0 for classical boundary conditions like clamped is simply supported free and on vertical rollers this will be 0, this we have seen earlier in context of vibration, so we need not have to get into the details again, so from this equation we have got now this equation, similarly by multiplying the first equation by phi K and integrating with respect to X and carrying out an integration by parts twice I get a similar equation where the subscripts N and K are reversed, so we get this if I subtract these 2 equations I get PK - PN into this equal to 0, and if we assume that PK is not equal to PN this integral 0 to L phi N prime X, phi K prime(x) DX = 0 for N not equal to K, and consequently using any one of these equations we can show that EI phi N double prime(x) and phi K double prime(x) DX is 0 for N not equal to K, so if you recall in vibration problems the mode shape that we obtained where orthogonal with respect to M and E are, here again we get similar orthogonality relations.

Example-1



$$P\delta \Rightarrow R_A = \frac{P\delta}{l} \Rightarrow R_C = -\frac{P\delta}{l}$$

Segment AB $\left(0 < x < \frac{l}{2}\right)$

$$EIy'' = -\frac{P\delta}{l}x$$

$$y = ax + b - \frac{P\delta x^3}{6EIl} = ax + b - \frac{\lambda^2 \delta x^3}{6l}$$

Segment BC $\left(\frac{l}{2} < x < l\right)$

$$EIy'' = -\frac{P\delta}{l}x + P(\delta - y)$$

$$EIy'' + Py = -\frac{P\delta}{l}x + P\delta$$

$$y'' + \lambda^2 y = \lambda^2 \delta\left(1 - \frac{x}{l}\right)$$

$$y(x) = c\sin\lambda x + d\cos\lambda x + \delta\left(1 - \frac{x}{l}\right)$$

Now what I will do now is I will consider a series of 4 or 5 problems, again aimed at gaining familiarity with how to deal with problems of stability analysis, many of these problems can serve as benchmarks when we develop finite element method to handle these problems, so what I am doing here will provide exact solutions within the scope of the beam theory that we are using, suppose I have now a propped hinge simply supported beam in which a load is applied at the mid-span okay, now again I assume that a neighboring equilibrium state is possible and I want to know at what value of P such equilibrium state is possible, so now if I consider segment A, B, I can write the equation EIY double prime is – P delta X/L the reactions are found here, you take moments about this here, RA into L is P into delta, so RA will be this, and RC will be by considering equilibrium in the horizontal direction I get this, so using that I will be able to write the equation for segment AB, and I have Y is AX square + B, this is the solution, and for segments BC I get a similar equation, but now when I am in segment BC when I write bending moment I should include effect of P also, so delta - Y will be the arm for taking moments, and that PY will come here, and consequently the complementary function here will be having sine and cosine terms, so we can analyze this problem, and we will be able to get the solution for X

$$y = ax^2 + b - \frac{P\delta x^3}{6EIl} = ax^2 + b - \frac{\lambda^2 \delta x^3}{6l} \text{ for } 0 < x < \frac{l}{2}$$

$$y(x) = c\sin\lambda x + d\cos\lambda x + \delta\left(1 - \frac{x}{l}\right) \text{ for } \frac{l}{2} < x < l$$

BCS

$y = 0$ at $x = 0$

$y = 0$ at $x = l$

$y = \delta$ at $x = \frac{l}{2}$

Continuity of $\frac{dy}{dx}$ at $x = \frac{l}{2}$

$$\tan\frac{\lambda l}{2} = \frac{3\lambda\left(\frac{l}{2}\right)}{\left(\frac{\lambda l}{2}\right)^2 - 9}$$

$$P_{cr} = \frac{18.7EI}{l^2}$$

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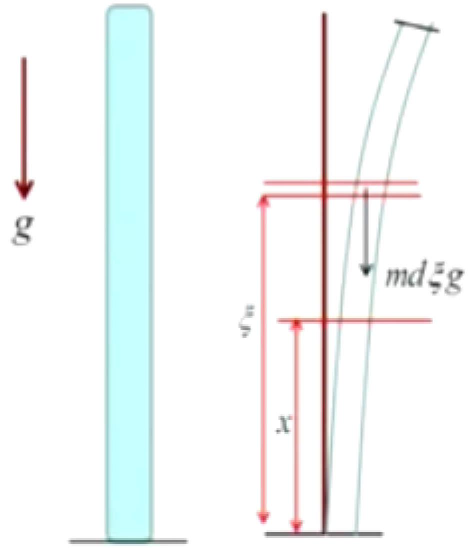
in the segment 1 and segment 2, and the boundary conditions now there are 4 constants of integration, the boundary conditions will be at X = 0 Y is 0, X = L Y is 0, and Y is delta at X = L, I am assuming the load is applied at this load is being applied at the mid-span so at Y = delta at X = L/2, and then there is a continuity of DY/DX at this X = L/2, so I have these 4 conditions and I can solve for that, and I will get the critical value to be given by this.

Example-2



$$EIy'' = \int_x^l mg\left[y(\xi) - y(x)\right]d\xi$$

$$EIy''' = \int_x^l -my'(x)\,g\,d\xi$$

$$\Rightarrow EIy''' + mg(l-x)y' = 0$$

$$z = \frac{2}{3}\sqrt{\frac{mg}{EI}(l-x)^3};\ u = \frac{dy}{dz}$$

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{1}{9z^2}\right)u = 0$$

$$(mgl)_{cr} = \frac{7.837EI}{l^2}$$

S P Timoshenko and J M Gere, 1963, Theory of elastic stability, McGraw-Hill, London, 100-103

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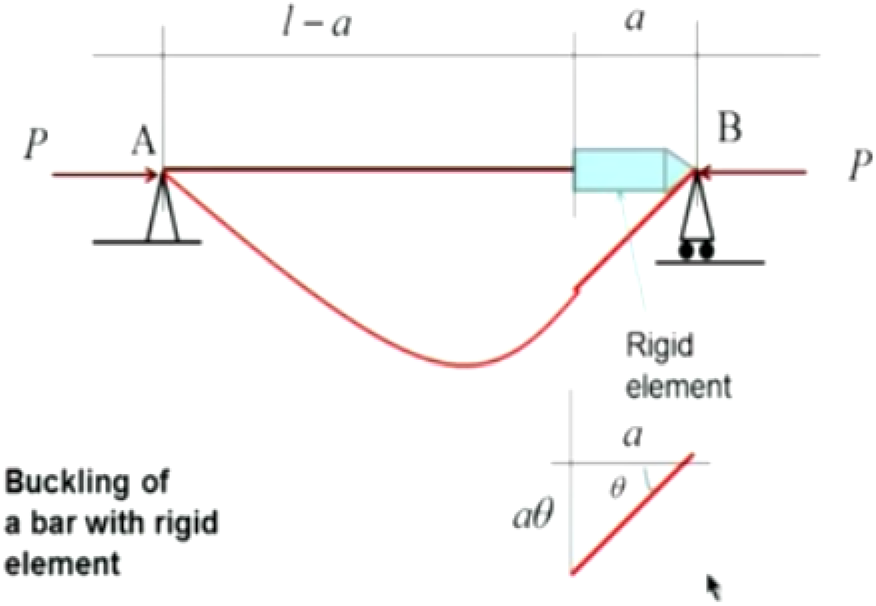
The next example is an example of a say cantilever beam which is loaded by its own weight, imagine there is a tree in a forest and how tall the tree can be before it buckles, is that type of question if we ask, this is a problem that is discussed in the book by Timoshenko and Gere, these are formulation from this resource, so what we do is we consider again the existence of a neighboring equilibrium position and write the X equation for bending moment, at a point X the bending moment is contributed by the weight into the corresponding arms, so for this infinitesimal, this thing is a D sai, so if my M is a mass per unit length, M into D sai is the mass and into gravity is the force, and bending moment at this section will be Y(sai) – Y(x) that is the arm, this is the MG into Y(sai) – Y(x) into D sai is the bending moment at this section due to this infinitesimal and if I integrate from X to L I will get the bending moment due to weight of the beam above this point, so that is what we have it here

Now we can simplify this when we differentiate this equation with respect to X and carry out a simplification we get this equation, and here you can see there is a X here so it has a variable coefficient, so one of the way to do it is to make these substitutions and we can show that this equation becomes this, and solution can be obtained using power series expansions and we can show that the critical length K is given by this, okay.

Example-3



Buckling of a bar with rigid element

Now another example this again to exercise our mind suppose we consider a beam and if we assume that at the end there is a rigid element, so the deflection of the beam will be such that this will be a straight line, this won't bend, it is not flexible, so it is again loaded by P, I want to know if this is the configuration of the structure what should be the P at which a neighboring equilibrium position becomes possible, so as I said this is a rigid element and this length is A, and this remaining length of flexible part is L – A, so how do we write this? Again the equation

$$EIy'' + Py = 0$$

$$y'' + \lambda^2 y = 0$$

$$y(x) = A\cos\lambda x + B\sin\lambda x$$

$$y(0) = 0 \Rightarrow A = 0$$

$$y(L-a) = a\theta \qquad (1)$$

$$y'(x) = B\lambda\cos\lambda x$$

Continuity of slope at $x = l - a$

$$\Rightarrow y'(l-x) = B\lambda\cos\lambda(l-a) = \theta \qquad (2)$$

$$\begin{bmatrix} \sin\lambda(l-a) & -a \\ \lambda\cos\lambda(l-a) & 1 \end{bmatrix} \begin{Bmatrix} B \\ \theta \end{Bmatrix} = 0$$

For nontrivial solution $\begin{vmatrix} \sin\lambda(l-a) & -a \\ \lambda\cos\lambda(l-a) & 1 \end{vmatrix} = 0$

$$\Rightarrow \tan\lambda(l-a) = a\lambda$$

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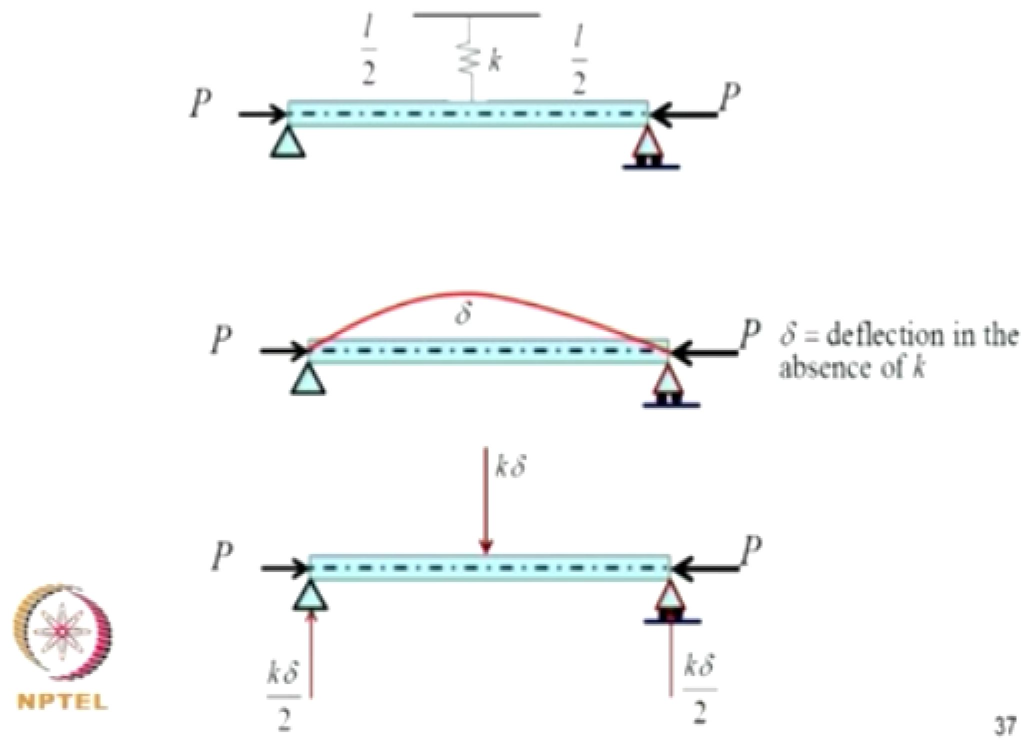
EIY double prime + PY = 0, and this is lambda square Y double prime + lambda square Y = 0, and this is a complementary function A cos lambda X + B sine lambda X, at X = 0 Y is 0, so A becomes 0.

Now the interesting thing is what happens here, so at X = L - A the deflection will be A theta, okay this deflection is A theta, and Y prime(x) is given by B lambda cos lambda X, so continuity of slope at X = L – A, I know the slope is theta because the rigid element I get this equation, so we can treat now B and theta as unknowns, and we have 2 equations and we can write this equation in this form, so for non-trivial solutions the determinant of this must vanish, and this is a characteristic equation, so once we solve this for different values of A, we will be able to find the critical value of P at which this structure buckles, or a neighboring equilibrium position becomes possible.

Example-4



Another problem, suppose there is a beam and I want to add a spring here as a in this example at the mid-span with a view to increase its axial load carrying capacity, so how far I can increase the critical load value by selecting this K? That is the question, so how do we tackle this problem? First I will ignore the presence of this spring and solve the problem I get this deflection delta say in the absence of that, and I will now select, I will apply a force K delta

$$EIy'' + Py = \frac{k\delta}{2}x$$

$$y'' + \lambda^2 y = \frac{k\delta}{2EI}x$$

$$y(x) = A\cos\lambda x + B\sin\lambda x + \frac{k\delta x}{2EI\lambda^2}$$

$$y(0) = 0 \Rightarrow A = 0$$

$$y'(x) = B\lambda\cos\lambda x + \frac{k\delta}{2EI\lambda^2}$$

$$y'\left(\frac{l}{2}\right) = 0 \Rightarrow B = -\frac{k\delta}{2EI\lambda^3\cos\frac{\lambda l}{2}}$$

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here so that this deflection becomes 0, so the governing equation will be EI Y double prime + PY = K delta/2 into X, and again dividing by P and using the notation lambda square we get this equation, this has a complementary function and a particular integral, at X = 0 Y is 0, therefore A is 0. Now Y prime(x) you can obtained by differentiating this, and at the mid span since structure is symmetric and spring is also place symmetrically this must be equal to 0, and this helps us to find the value of B, therefore this is Y(x).

$$y(x) = \frac{k\delta}{2EI\lambda^2}\left(x - \frac{\sin \lambda x}{\lambda \cos \frac{\lambda l}{2}}\right)$$

$$\Rightarrow y\left(\frac{l}{2}\right) = \delta = \frac{k\delta}{2EI\lambda^2}\left(\frac{l}{2} - \frac{\sin \frac{\lambda l}{2}}{\lambda \cos \frac{\lambda l}{2}}\right) \Rightarrow 1 = \frac{k}{2EI\lambda^3}\left(\frac{\lambda l}{2} - \tan \frac{\lambda l}{2}\right)$$

$$\Rightarrow 1 = \frac{k}{2EI\lambda^3}\frac{\lambda l}{2}\left(1 - \frac{\tan u}{u}\right) \quad \left(\text{with } u = \frac{\lambda l}{2}\right)$$

$$\Rightarrow 1 = \frac{kl}{4P}\left(1 - \frac{\tan u}{u}\right)$$

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Now at X = L/2 this is my delta, okay, now that would mean this if you take on the other side delta cannot be 0 therefore the term here must be equal to 0 and that leads to this characteristic equation, so we can solve this and find out the critical value of load U for a given value of K.

$$1 = \frac{kl}{4P}\left(1 - \frac{\tan u}{u}\right)$$

$$\frac{\tan u}{u} = 1 - \frac{4P}{kl}$$

$$k \to 0, u \to \frac{\pi}{2} \Rightarrow P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{9.869EI}{l^2}$$

$$k \to \infty, \tan u = u \Rightarrow u = 4.493 \Rightarrow P_{cr} = \frac{20.19EI}{l^2}$$

By inserting the spring $P = \frac{20.19EI}{l^2}$ is the highest value to which the load carrying capacity can be increased.

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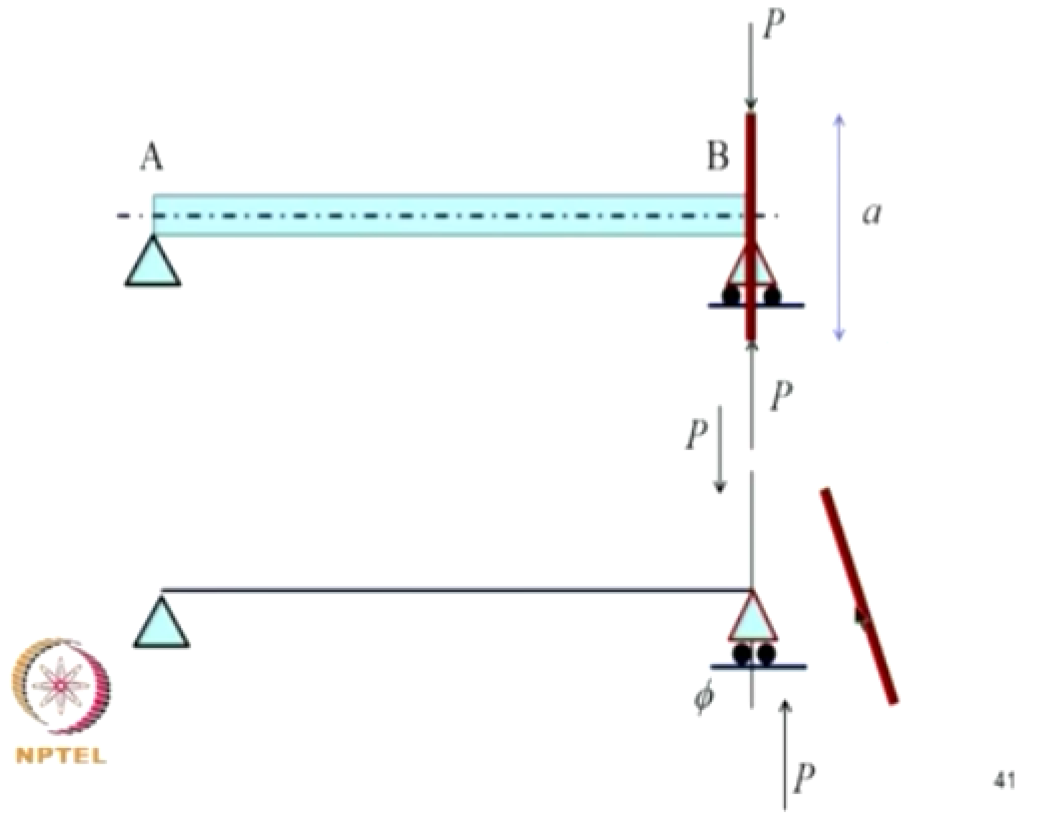
Now if you really solve this we can see that if you rewrite this equation, the characteristic equation I can write as Tan U/U = 1 – 4P/KL, if K goes to 0 then U goes to pi/2, and that is our Eulerian buckling load, this is this. Now as K tends to infinity then this is tan U = U, and U becomes 4.493 and for K tending to infinity the critical value will be given by this, that would mean by inserting the spring you cannot increase the carrying capacity beyond this, if you don’t have the spring your carrying capacity will be this, by inserting the spring no matter what you do you need not have to select a spring you know you cannot exceed this carrying capacity, okay, so this is the highest value to which the load carrying capacity can be increased.

Example-5



Another example, imagine there is a single span beam and in the end there is a rigid disc, and this disc is loaded by load P, okay, and this disc can rotate along with the beam, so in the deformed geometry the disc could have become like this, and the beam deflection will be something like this, that means if an adjoining equilibrium position is possible it may be like this.

Now we will find the reactions now see if this is angle is phi, and this is A, so this load will be

$$R_B l + P\left(l + \frac{a}{2}\sin\phi\right) - P\left(l - \frac{a}{2}\sin\phi\right) = 0$$

$$\Rightarrow R_B = -\frac{Pa\sin\phi}{l} \;\&\; R_A = \frac{Pa\sin\phi}{l}$$

$$EIy'' + \frac{Pa\sin\phi}{l}x = 0$$

$$y'' + \frac{\lambda^2 a\sin\phi}{l}x = 0$$

$$\Rightarrow y = Ax + B - \frac{\lambda^2 a\sin\phi}{6l}x^3$$

$$y(0) = 0 \Rightarrow B = 0$$

$$y(l) = 0 \Rightarrow A = \frac{\lambda^2 l^2 a\sin\phi}{6}$$

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acting at L - this distance, A sine phi, A/2 sine phi and this load will be acting at L + A/2 sine phi, and consequently the reaction, we can take moments and find out the reactions, these reactions are functions of the angle through which the disc has rotated which is the angle through which the beam has rotated at the right support, so this is a equation of bending moment, and the equilibrium equation is given by this once we find the reaction we can write the equation for the equilibrium and by dividing by EI etcetera I get this equation.
Now again the complementary function is this, and this is the particular integral and we impose a condition at X = 0, and X = L, Y must be equal to 0 and we get this value of A, and if we now

$$y(x) = \frac{\lambda^2 a \sin\phi}{6}\left(xl - \frac{x^3}{l} \right) \approx \frac{\lambda^2 a \phi x}{6l}\left(l^2 - x^2 \right)$$

$$\frac{dy}{dx} = \frac{\lambda^2 a \phi}{6l}\left(l^2 - 3x^2 \right)$$

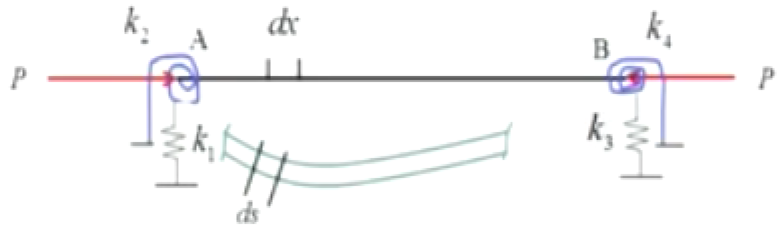
$$\Rightarrow -\phi = \frac{dy}{dx}(l) = -2l^2 \frac{\lambda^2 a \phi}{6l}$$

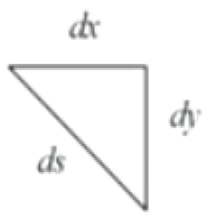
$$\phi\left(1 - \frac{\lambda^2 a l}{3} \right) = 0$$

$$\Rightarrow \lambda_{cr} = \frac{3}{al} \Rightarrow P_{cr} = \frac{3EI}{al}$$

approximate sine phi/phi I get Y(x) to be given by this. So now DY/DX, see phi is an unknown here still we have to find that, so I will find DY/DX and evaluate it at X = L, moment I do that and I know it is – phi, so I get in a characteristic equation, phi cannot be 0 therefore the terms inside the parentheses must be 0 and from this I get the load, critical value of the load parameter 3/AL, therefore P critical is given by this.

Energy formulation





$$ds^2 = dx^2 + dy^2$$

$$\Rightarrow ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} \approx dx\left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\bar{S} = \int_0^l \left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]dx = l + \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

$$\bar{S} - l = \Delta = \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

Now we will quickly review the equation of motion for axially loaded beam as shown here using Hamilton's principle, so we consider for sake of discussion, for purpose of discussion as a single span beam which is supported on 2 transfer springs K1 and K3, this K1, K2, K3, K4, K1 and K3 are translation springs, K2 and K4 are rotational springs, so suppose this beam has deflected like this, and an element DX would have become element DS, and using the geometry here, the geometry of this triangle we know that DS square is DX square + DY square, and we can reorganize this term to pull out, take the square root, take DX outside and I get this and by expanding this and retaining only the first term I get this equation, so this is the length of this, what was DX has become DS.

Now the total length of this curved beam profile is integration of DS, so it is 0 to L, DS DX integral, this is DS, integral DS which is nothing but integral of this quantity from 0 to L, so this I can write it as L + 1/2 integral 0 to L DY/DX whole square DX. Now S bar - L is the change in the length, that is delta, so that is given by this equation, why we need this now because we want to find out the work done by this load on the beam, right so we need to know how much the beam has moved, so this is very important unless P does work on the beam its effect will not be felt, so if you go back to the convention that we use in notations this is very important.

Kinetic energy: $T = \frac{1}{2}\int_0^l m\dot{y}^2(x,t)\,dx$

Strain energy due to bending $= \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)dx$

Strain energy stored in the springs:

$$\frac{1}{2}\left\{k_1 y^2(0) + k_2 y'^2(0) + k_3 y^2(l) + k_4 y'^2(l)\right\}$$

Work done by axial loads: $P\Delta = \frac{P}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx$

$$L = \frac{1}{2}\int_0^l m\dot{y}^2(x,t)\,dx - \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 y}{\partial x^2}\right)dx - \frac{P}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx$$
$$+\frac{1}{2}\left\{k_1 y^2(0) + k_2 y'^2(0) + k_3 y^2(l) + k_4 y'^2(l)\right\}$$

So the kinetic energy is 1/2 MY dot square this, strain energy has now contribution due to bending which is given by this and there is strain energy stored in the springs that is given here and there is work done by axial load which is P into delta which is given by this. So now therefore Lagrangian will be this, and this is the additional term due to the strain energy in the springs, so we can now use the variational argument this we have discussed in first or the second lecture of this course so that can be used, and if we do that we will get the governing

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+m\frac{\partial^2 y}{\partial t^2}+P\frac{\partial^2 y}{\partial x^2}=0$$

$$\left(EIy''\right)''+m\ddot{y}+Py''=0$$

Boundary conditions

$$EIy''(0,t)-k_2y'(0,t)=0$$

$$\left(EIy''\right)'(0,t)+Py'(0,t)+k_1y(0,t)=0$$

$$EIy''(l,t)+k_4y'(l,t)=0$$

$$\left(EIy''\right)'(l,t)+Py'(l,t)-k_3y(l,t)=0$$

Initial conditions

$$y(x,0)=y_0(x);\ \dot{y}(x,0)=\dot{y}_0(x)$$

- External excitations
- Time dependent boundary conditions
- Damping terms

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equation in this form, and if you use the convention of prime for a special derivative and dot for a time derivative we get this equation, and the set of 4 boundary conditions you can show that they are given by this, set of 4 equations so this needs to be augmented with the 2 initial conditions, initial displacement and velocity. To this of course we can add the effect of any external excitation or time-dependent boundary conditions as in case of earthquakes of course damping terms and so on and so forth, so this would be the governing equation would then be the generalization of static beam column for a dynamic situation, okay. So in our analysis subsequently we will consider some of this as we go along.

Now what we have discussed till now is although I discussed this the accounting equation of dynamical systems we have basically focused on elastic stability of statically loaded systems, so before we embark on finite element formulation what we will do is we will consider questions of stability of dynamical systems, so we will quickly review what are fixed points of a nonlinear vibrating system, what is meant by stability of those fixed points and that will have bearing on some of the discussions to follow. So in the next lecture we will consider nonlinear dynamical systems and investigate the fixed points and their stability, and in doing so we will introduce the notion of bifurcation and see how the ideas can be developed further, so with this will close this lecture.

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