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Course Title

Finite element method for structural dynamic
And stability analyses

Lecture – 22

Axisymmetric models. Plate bending elements.

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Finite element method for structural dynamic and stability analyses

Module-7,8

Analysis of 2 and 3 dimensional continua
Plate bending and shell elements

Lecture-22 Axisymmetric models, Plate bending elements

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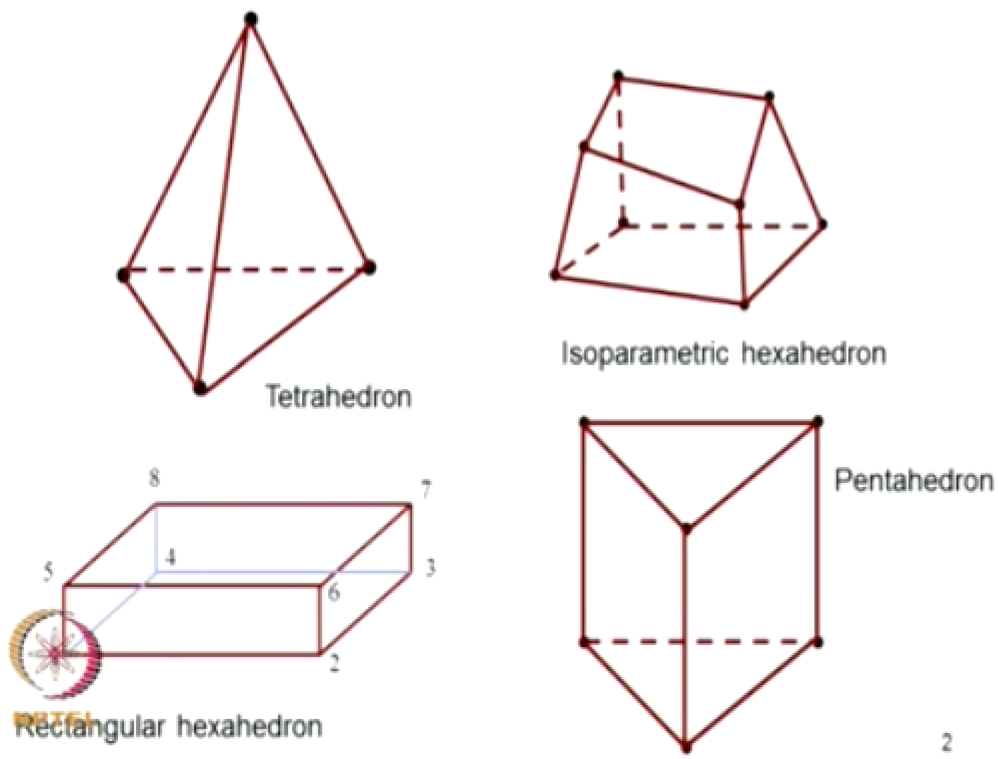
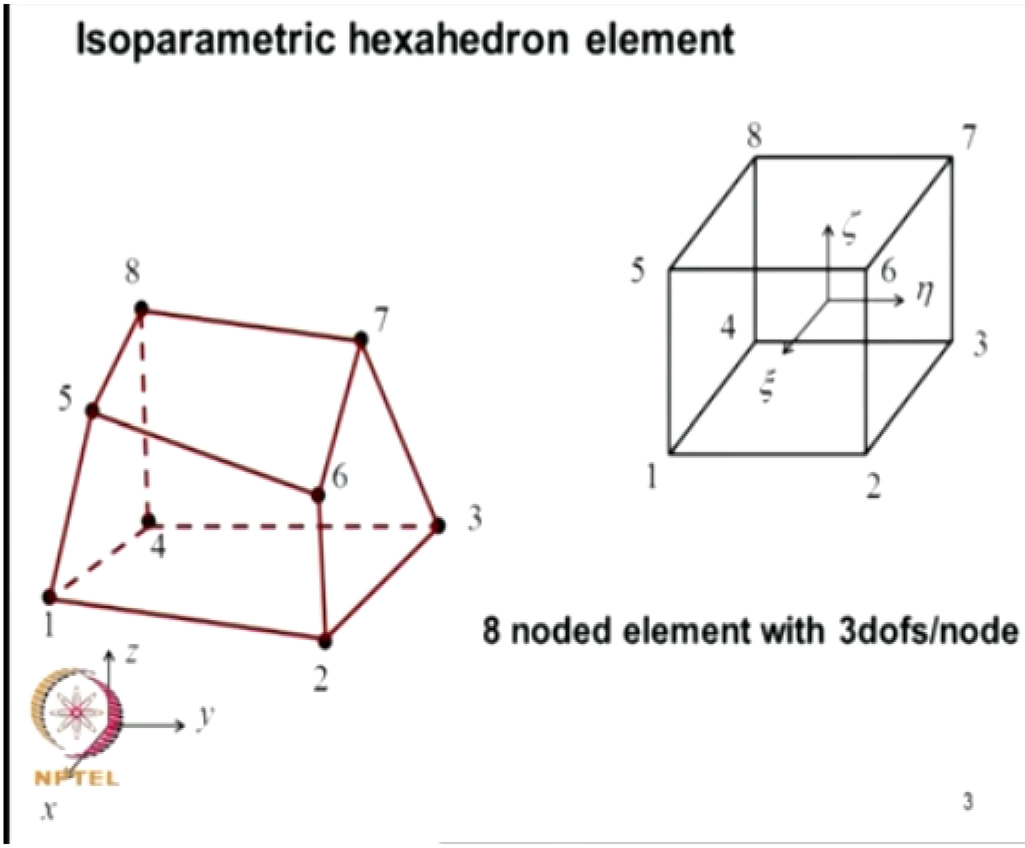
We have been discussing analysis of 2-dimensional continuum, and 3 dimensional continuum, so we will continue with that discussion, and we expect to close that discussion in this lecture. And we'll start discussing about problems of plate bending and shell elements, and this is the topics for today's lecture.

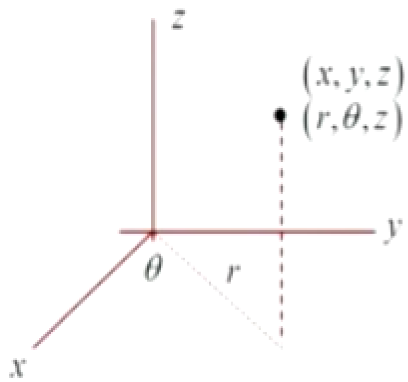


So in the previous class we considered 3 dimensional elements and derived the structural matrices, rectangular hexahedron, isoparametric hexahedron and tetrahedron elements, and we worked out a simple problem also.



Now we will continue with that so this is a 8 noded element with 3 degrees of freedom per node, so 24 degrees of freedom this is isoparametric hexahedron element for 3-dimensional solids.

Equations of elasticity in cylindrical polar coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Independent cordinates: (r, θ, z, t)

Stress components: $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{z\theta}, \sigma_{rz}$

Strain components: $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{r\theta}, \varepsilon_{z\theta}, \varepsilon_{rz}$

Displacements: $u_r, u_\theta, u_z \equiv u, v, w$

Body forces: F_r, F_θ, F_z

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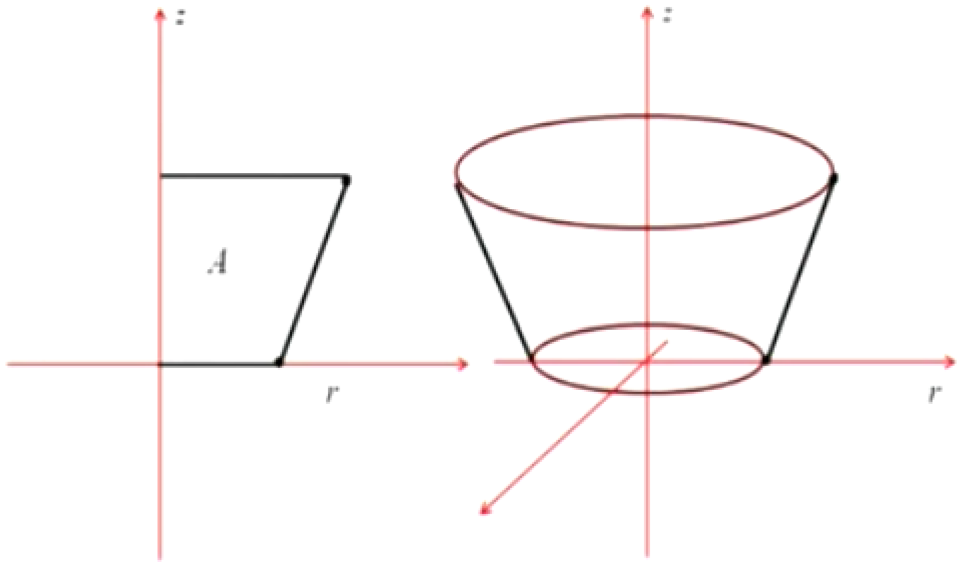
Now in today's class what we will do is we want to consider problems of, again a kind of 2 dimensional approximation to solids of revolution, solids which we obtained by revolving around one of the axis, so that kind of objects geometry is well treated using cylindrical polar coordinates, so we'll quickly review the equations of elasticity in cylindrical polar coordinates, so the coordinate system here is this is a Cartesian coordinate system X, Y, Z and in cylindrical polar coordinates we have R theta and Z, so this is the position vector R I mean coordinate R, this is angle theta, and this is height is Z.
Now the relationship between Cartesians and cylindrical polar coordinates is shown here, X is R cos theta, Y is R sin theta, and Z is same as Z. So in cylindrical polar coordinates the independent coordinates are R theta, Z and T, and stress components we denote as sigma RR, sigma theta theta, sigma ZZ, sigma R theta, sigma Z theta, and sigma RZ. Strain component correspondingly have the similar notation epsilon RR, epsilon theta theta, epsilon ZZ and so on and so forth. The displacement components are denoted by UR, U theta, UZ and we will agree to call them as simply a U, V, W. And the body forces upper case FR, F theta, and FZ.



A = Area of the generator plane

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So how this you know this axisymmetric solids are produced, suppose you have a generator plane with area A, and suppose if it revolves around axis Z it produces an object like this, so the focus of our discussion is on studying this type of objects. So in this case we need to model only this generator plane and we can complete the analysis for this object.

Equillibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)+F_r=0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{2}{r}\sigma_{r\theta}+F_\theta=0$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{1}{r}\sigma_{rz}+F_z=0$$

Strain displacement relations

$$\varepsilon_{rr}=\frac{\partial u}{\partial r}$$

$$\varepsilon_{\theta\theta}=\frac{u}{r}+\frac{\partial v}{\partial \theta}$$

$$\varepsilon_{zz}=\frac{\partial w}{\partial z}$$

$$2\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}$$

$$2\varepsilon_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}$$

$$2\varepsilon_{z\theta}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}$$

Stress-strain relations

$$e=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}$$

$$\sigma_{rr}=\lambda e+2v\varepsilon_{rr}$$

$$\sigma_{\theta\theta}=\lambda e+2v\varepsilon_{\theta\theta}$$

$$\sigma_{zz}=\lambda e+2v\varepsilon_{zz}$$

$$\sigma_{\theta z}=2G\varepsilon_{\theta z};\sigma_{zr}=2G\varepsilon_{zr};\sigma_{r\theta}=2G\varepsilon_{r\theta}$$

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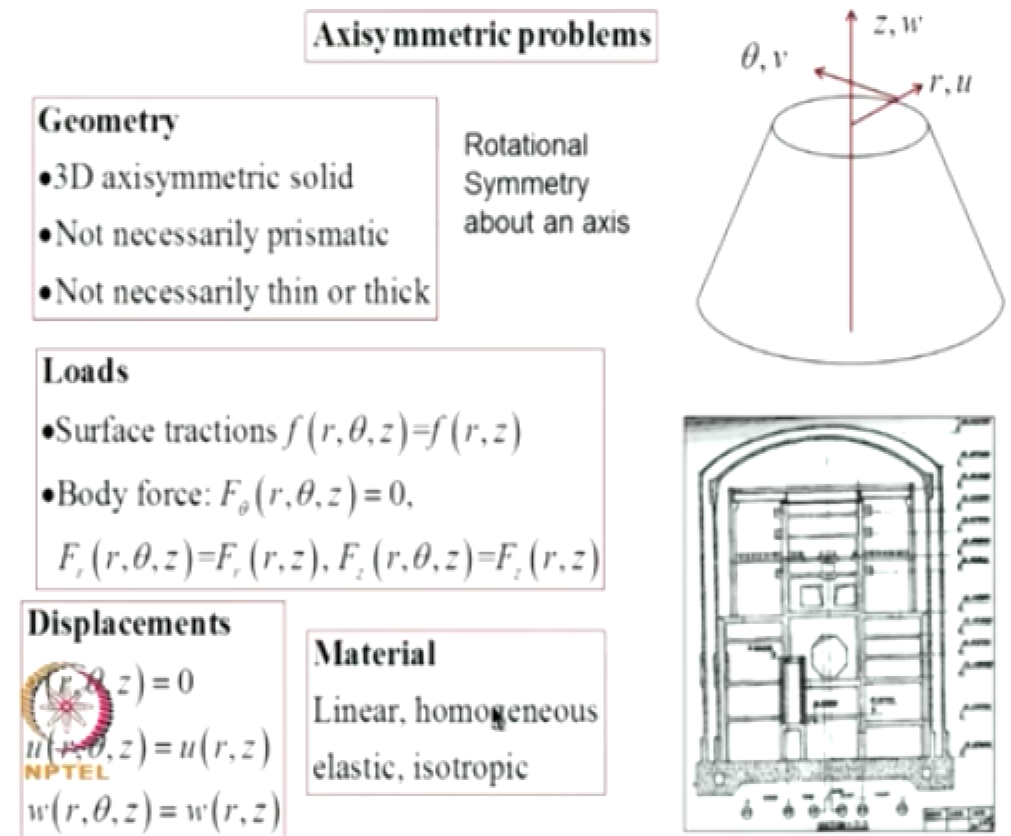
Now with that in mind let's review the equations of elasticity, we have seen these equations in Cartesian coordinates, so the equilibrium equations upon making using this transformation we can show that it gets transformed to 3 equilibrium equations are here, and the strain displacement relations again gets modified and they are shown here, and the stress-strain relation also gets modified and we are assuming the body is isotropic, a linearly elastic and isotropic, so the kind of objects that we are studying here have a rotational symmetry about an



axis, so Z axis is the axis of symmetry and there is a rotational symmetry about that, this type of structures as I was mentioning briefly in the previous lecture are encountered widely in engineering, this is a cross section of a nuclear reactor structure you can see this outer shell it's a cylindrical, it’s a cross section in plant it is circular so you can see that these shells have rotational symmetry about the vertical axis.

Now what exactly is a consequence of this symmetry on equations of elasticity? So the objects that we are considering are characterized by geometry with these features, they are 3 dimensional axisymmetric solids, they are not necessarily prismatic, not necessarily thin or thick, so these 2 conditions if you recall were necessary for implementing plane stress and plane strain models, we are relaxing that. The loads, the surface tractions are independent of theta, they’re only functions of R and Z. And similarly body forces there is no body force in the theta direction, and the other body forces in R and Z directions are independent of theta, under these assumptions the displacements in the theta direction will be 0, and the other two displacements U and W, U in R direction, W in Z direction will be independent of theta, so this is the postulate on displacements. The material itself we are going to assume linear, homogeneous, elastic and isotropic, so as a consequence of this what happens? Now we have V

$$\left.\begin{aligned} v(r,\theta,z) &= 0 \\ u(r,\theta,z) &= u(r,z) \\ w(r,\theta,z) &= w(r,z) \end{aligned}\right\} \Rightarrow \begin{aligned} &\varepsilon_{rr} = \frac{\partial u}{\partial r} \neq 0 \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{\partial \theta} = \frac{u}{r} \neq 0 \\ &\varepsilon_{zz} = \frac{\partial w}{\partial z} \neq 0 \quad 2\varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \neq 0 \\ &2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0; \quad 2\varepsilon_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \end{aligned}$$

$$\sigma_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\nu\varepsilon_{rr} \neq 0$$

$$\sigma_{\theta\theta} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\nu\varepsilon_{\theta\theta} \neq 0$$

$$\sigma_{zz} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 2\nu\varepsilon_{zz} \neq 0$$

$$\sigma_{\theta z} = 2G\varepsilon_{\theta z} = 0; \sigma_{zr} = 2G\varepsilon_{zr} \neq 0; \sigma_{r\theta} = 2G\varepsilon_{r\theta} = 0$$

Unknowns:
2 displ+4 strains+4 stresses
Equations
2 eqbm+4 strain-disp
+4 stress-strain

$$\left.\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + F_r = 0 \\ &\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + F_\theta = 0 \\ &\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + F_z = 0 \end{aligned}\right\} \quad \begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + F_r = 0 \\ &\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + F_z = 0 \end{aligned}$$

= 0, U is independent of theta, W is independent of theta, so the implications of this on strains if you look at epsilon RR is dou U/dou R it will be nonzero, epsilon theta theta is U/R + dou U/dou theta, now dou U/dou theta is 0, because V is 0 and this will be simply U/R and which is not 0. Epsilon ZZ of course derivative of W with respect to Z is nonzero, epsilon RZ again it involves derivative of U with respect to Z, and W with respect to R that will be nonzero, but the other shear strains epsilon R theta dou U/dou theta is 0 because U is independent of theta, dou V/dou R is 0 because V is 0, similarly this is 0 therefore this shear strain becomes 0. Similarly we can show that epsilon Z theta is also 0, so there are four nonzero strain components.

Now we can now look at the constitutive laws, we have sigma RR given by this, it is again nonzero, sigma theta theta is nonzero, sigma ZZ is nonzero, sigma theta Z is 2G epsilon theta Z, but epsilon theta Z is 0 therefore this is zero, sigma ZR is 2G into epsilon ZR, epsilon ZR is not 0, therefore this stress will be nonzero, but the other shear stress sigma R theta will be 0, because epsilon R theta is 0. So now we have four stress components, 4 strain components and 2 displacement components so there are 8 unknowns, and we need now 10 equations, so these are strain displacement relations 4 in number, this is stress strain relations 4 in number, now the other two equations are obtained from the equilibrium equations.

Now this are the equilibrium equations whatever I showed here are replicated here except that the terms which are 0 are now indicated in red, so consequently what happens, one of the equilibrium equation is satisfied identically and we are left with 2 equilibrium equation, so the summary is we have 2 unknown displacements, 4 strain components, 4 stress components which are nonzero and unknown, therefore there are 10 unknowns, we have 2 equilibrium equations, 4 strain displacement equations, and 4 stress strain relation, so these can be solved in the classical theory of elasticity there are ways of introducing a stress function and developing solutions based on this set of 10 equations.

$$\varepsilon = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ 2\varepsilon_{rz} \\ \varepsilon_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \\ \dfrac{u}{r} \end{Bmatrix}; u_e = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix}; \sigma = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{\theta\theta} \end{Bmatrix}; \sigma = D\varepsilon$$

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{(1-\nu)} & 0 & \dfrac{\nu}{(1-\nu)} \\ \dfrac{\nu}{(1-\nu)} & 1 & 0 & \dfrac{\nu}{(1-\nu)} \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ \dfrac{\nu}{(1-\nu)} & \dfrac{\nu}{(1-\nu)} & 0 & 1 \end{bmatrix}$$

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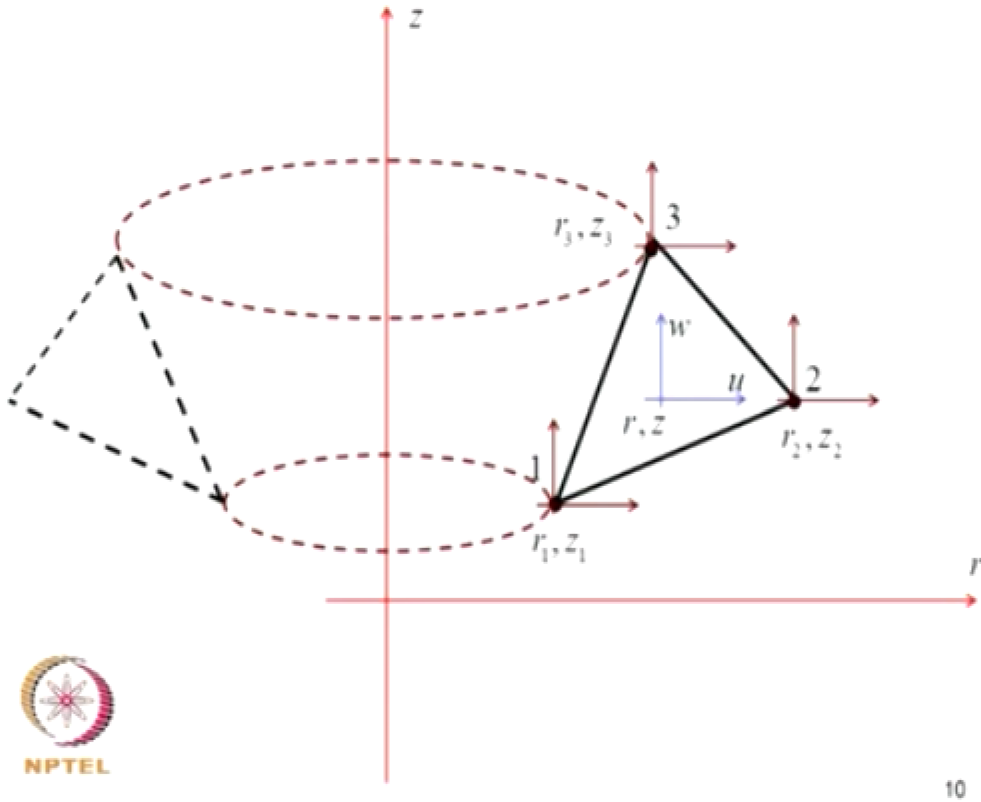
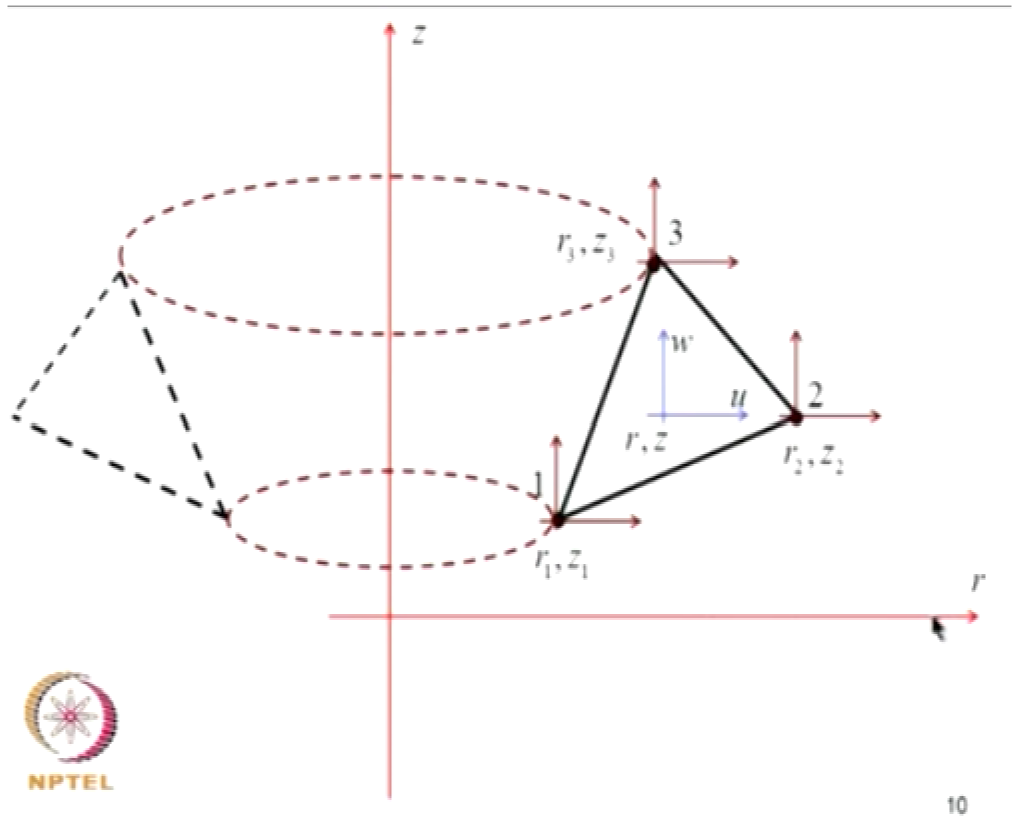
Now our objective is to develop the finite element model for this, so with that in view we'll just see now the strain will assemble them in a 4 x 1 vector as shown here, epsilon RR, epsilon ZZ, and epsilon theta theta, and they are related to displacement through these relations after introducing all the simplifications resulting from assumptions of axis symmetry of geometry loads and boundary conditions, so the element field variables will be URZ, WRZ, I am not showing T explicitly, it can be introduced subsequently, T is a time, so right now I am not doing that, stress will be these are the 4 stress components, so we have 4 strain components related to displacement, 2 displacement components which are not known, 4 stress components which are related to strain through this and using this relation they in turn get related to displacements. Now D is the matrix of, that relates a stress and strain, this is for an isotropic linearly elastic material, this is the D matrix, E is the Young's modulus, Nu is Poisson's ratio.



Now we want to now develop a finite element model for this, now we will start with a triangular element, so it has 3 nodes, and at each node there are 2 degrees of freedom shown here, and these 1, 2, 3 are the nodes and the nodal coordinates are R1 Z1, R2 Z2, and R3 Z3, this is Z axis, this is our R axis.



Now so the nodal degrees of freedom are U1 W1, U2 W2, U3 W3, so consequently we interpolate the field variables within the element in terms of these nodal coordinates, nodal degrees of freedom using this interpolation function, where encountered this triangular element

$$\begin{Bmatrix} u(r,z,t) \\ w(r,z,t) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u(x,y,t) \\ w(x,y,t) \end{Bmatrix} = [N]\{u\}_e \qquad \underbrace{\begin{Bmatrix} u \\ v \end{Bmatrix}}_{2\times 1} = \underbrace{[N]}_{2\times 6}\underbrace{\{u\}_e}_{6\times 1}$$

$$T = \frac{1}{2}\{\dot{u}(t)\}_e^t M_e \{\dot{u}(t)\}_e ; M_e = h\int_{A_0} \rho [N(r,z)]^t [N(r,z)] dA_0$$

earlier in the context of plane stress problem, so the same shape functions will be relevant here also, so we write this as UW is N into UE, this vector is 2 x 1, N is 2 x 6 and UV is 6 x 1, so based on this I write now the expression for kinetic energy, where ME is the mass matrix given by the H rho and transpose N, DA naught.

$$\varepsilon = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ 2\varepsilon_{rz} \\ \varepsilon_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \frac{u}{r} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \underbrace{[B]}_{4\times 6}\underbrace{\{u\}_e}_{6\times 1}$$

$$V = \frac{1}{2}\{u\}_e^t \left[h\int_{A_0} B^t DB dA_0 \right] \{u\}_e = \frac{1}{2}\{u\}_e^t K_e \{u\}_e ; K_e = h\int_{A_0} B^t DB dA_0$$

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So now we want to express the strain in terms of nodal displacements for that we first write the strain displacement relations for polar coordinate it is written in this form, this is actually the strain displacement relations we want to write it in this form and for UW I introduce N into UE, and this matrix into the matrix of shape functions is our B matrix, so this is 4 x 6 and this is 6 x 1, so that this is 4 x 1. Now based on this the strain energy is given by 1/2 UE transpose into the stiffness matrix UE, and the stiffness matrix is given by H into integral over A naught, B transpose, DB DA naught, so this development to parallels what we have done earlier for plane

Remark

Consider the situation in which the body is axisymmetic, it is supported axisymmetrically but the applied loads are not axisymmetric.

We will use Fourier representation for the loads as

$$\begin{Bmatrix} f_r \\ f_z \end{Bmatrix} = \underbrace{\sum_{n=0}^{\infty} \begin{Bmatrix} f_{rn} \\ f_{zn} \end{Bmatrix} \cos n\theta}_{\text{symmetric about } \theta=0} + \underbrace{\sum_{n=0}^{\infty} \begin{Bmatrix} \bar{f}_{rn} \\ \bar{f}_{zn} \end{Bmatrix} \sin n\theta}_{\text{antisymmetric about } \theta=0}$$

$$f_\theta = \sum_{n=1}^{\infty} f_{\theta n} \sin n\theta - \sum_{n=1}^{\infty} \bar{f}_{\theta n} \sin n\theta$$

$$\begin{Bmatrix} f_{r0} \\ f_{z0} \end{Bmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{Bmatrix} f_r \\ f_z \end{Bmatrix} d\theta$$

$$\begin{Bmatrix} f_{rn} \\ f_{zn} \end{Bmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{Bmatrix} f_r \\ f_z \end{Bmatrix} \cos n\theta d\theta; \quad \begin{Bmatrix} \bar{f}_{rn} \\ \bar{f}_{zn} \end{Bmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \begin{Bmatrix} f_r \\ f_z \end{Bmatrix} \sin n\theta d\theta$$

$$f_{\theta n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\theta \cos n\theta d\theta; \quad \bar{f}_{\theta n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\theta \sin n\theta d\theta$$

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elements, plane stress elements, so we can go ahead and use the shape functions and evaluate, if you recall for the triangular element we knew the shape functions there were linear functions and we were able to evaluate the mass and stiffness matrix exactly, so I am not going to develop those elements, but I am just giving a few steps so that if necessary you can proceed and evaluate these matrices.

Now we can consider now a slight relaxation on the requirements that we spelt out for analyzing axisymmetric solids, suppose if we consider the situation in which the body is axisymmetric, it is supported axisymmetrically but the applied loads are not axisymmetric, so the problem would still be 3 dimensional, can we solve this problem using 2 dimensional models is the question. Now since the system is linear, material is linear and we are assuming strain displacement relations to be linear, principle of superposition holds good, so the proposition here is that we will expand the applied loads surface tractions in a Fourier series in theta, okay, so if we apply, if lower case fr and fz are the surface tractions I expand them in Fourier series as shown here, and cosine terms will produce behavior which is symmetric about theta = 0, and sin terms will produce behavior which is anti-symmetric about theta = 0.

Now, and F theta it's a body force in theta direction will be expanded like this, I am assuming now that there is a surface traction in theta direction also not body force this. Now using the orthogonality property of sin and cosine functions we can evaluate these Fourier coefficients, FRN, FZN and FRN bar, FZN bar, F theta N, and F theta N bar, so this is straightforward, this is a simple application of Fourier's logic, so for N = 0 we get this, and for N = 1, 2, etcetera the sin and cosine terms etcetera are given here.

Now the idea of this formulation is, we will develop one finite element model for each of these terms, and it will produce an assemble of models, and we will synthesize there a total response by summing or the responses for each of the you know models, each one corresponding to one component of this excitations, so how to do that we will just quickly see a few steps. Now since surface tractions are expanded in Fourier series, we will also expand the displacements in the Fourier series, so this is written as nu N sin N theta + nu N bar sin N theta, this is V and UV is expanded in this form.

Displacements

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} u_n \\ w_n \end{Bmatrix} \cos n\theta + \sum_{n=0}^{\infty} \begin{Bmatrix} \bar{u}_n \\ \bar{w}_n \end{Bmatrix} \sin n\theta; v = \sum_{n=1}^{\infty} v_n \sin n\theta - \sum_{n=1}^{\infty} \bar{v}_n \sin n\theta$$

Strain displacement relations

$$\left.\begin{aligned} \varepsilon_{rr} &= \frac{\partial u}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{u}{r} + \frac{\partial v}{\partial \theta} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ 2\varepsilon_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\ 2\varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ 2\varepsilon_{z\theta} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \end{aligned}\right\}$$

$$\varepsilon_1 = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ 2\varepsilon_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} + \frac{\partial v}{\partial \theta} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix}$$

$$\varepsilon_2 = \begin{Bmatrix} 2\varepsilon_{r\theta} \\ 2\varepsilon_{z\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\ \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \end{Bmatrix}$$

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Now the strain displacement relations, this are the strain displacement, now we are considering the 3 dimensional strain displacement relations, what we do is we split the strain vector into a component with 4 x 1 elements and a 2 x 1 element, if the body is axisymmetric this will be 0, this is anyway it will be present even for axisymmetric solids or the general solids as well.

Stress-Strain relations

$$\sigma_1 = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix} = D_1\varepsilon_1 = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & 0.5(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ 2\varepsilon_{rz} \end{Bmatrix}$$

$$\sigma_2 = \begin{Bmatrix} \sigma_{r\theta} \\ \sigma_{z\theta} \end{Bmatrix} = D_2\varepsilon_2 = \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 2\varepsilon_{r\theta} \\ 2\varepsilon_{z\theta} \end{Bmatrix}$$

Energies

$$T_e = \frac{1}{2}\int_{V_1} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dV_0 \;\&\; V = \frac{1}{2}\int_{V_1} \left(\varepsilon_1^t D_1 \varepsilon_1 + \varepsilon_2^t D_2 \varepsilon_2\right) dV_0$$

Virtual work $\delta W = \int_S \left(f_r \delta u + f_\theta \delta v + f_z \delta w\right) dS$

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Now stress-strain relations, we will write separately for these 4 components and for these 2 components, they are not coupled in a linear isotropic elastic material those things won't be coupled, so sigma 1 I write it as D1 epsilon 1, sigma 1 is these 4 components are RR, theta theta, ZZ, RZ, this is given by this in terms of the strains, epsilon RR, epsilon theta theta, epsilon ZZ, and epsilon RZ. Similarly sigma 2 is consists of the shearing strain, sigma R theta and sigma Z theta and it is related to epsilon R theta and epsilon Z theta through this transformation.

Now the expression for energies, this is the kinetic energy, this is the strain energy, the strain energy has two parts, this is a breakup that is done to facilitate computation, there is no physical arguments for this, then similarly the virtual work done can also be written in this form.

$$T_e = \frac{1}{2}\pi\int_A \rho\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right) r\,dA \;\&\; V = \frac{1}{2}\pi\int_{V_0}\left(\varepsilon_1^t D_1\varepsilon_1 + \varepsilon_2^t D_2 \varepsilon_2\right) r\,dA$$

$$\begin{Bmatrix}u\\w\end{Bmatrix} = \sum_{n=0}^{\infty}\begin{Bmatrix}u_n\\w_n\end{Bmatrix}\cos n\theta + \sum_{n=0}^{\infty} \begin{Bmatrix}\bar{u}_n\\ \bar{w}_n\end{Bmatrix}\sin n\theta;\; v=\sum_{n=1}^{\infty} v_n \sin n\theta - \sum_{n=1}^{\infty} \bar{v}_n \sin n\theta$$

$$\Rightarrow T_e = \sum_{n=1}^{\infty} T_{en} \;\&\; V_e = \sum_{n=1}^{\infty} V_{en}$$

Consider $(n \neq 0)$

$$T = \frac{1}{2}\pi\int_A \rho\left(\dot{u}_n^2+\dot{v}_n^2+\dot{w}_n^2\right) r\,dA$$

$$V = \frac{1}{2}\pi\int_{V_0}\left(\varepsilon_{1n}^t D_1\varepsilon_{1n} + \varepsilon_{2n}^t D_{2n}\varepsilon_{2n}\right) r\,dA$$

$$\delta W = \pi\int_S \left(f_{rn}\delta u_n + f_{\theta n}\delta v_n + f_{zn}\delta w_n\right) r_s\,dS$$

r_s = value of r on S

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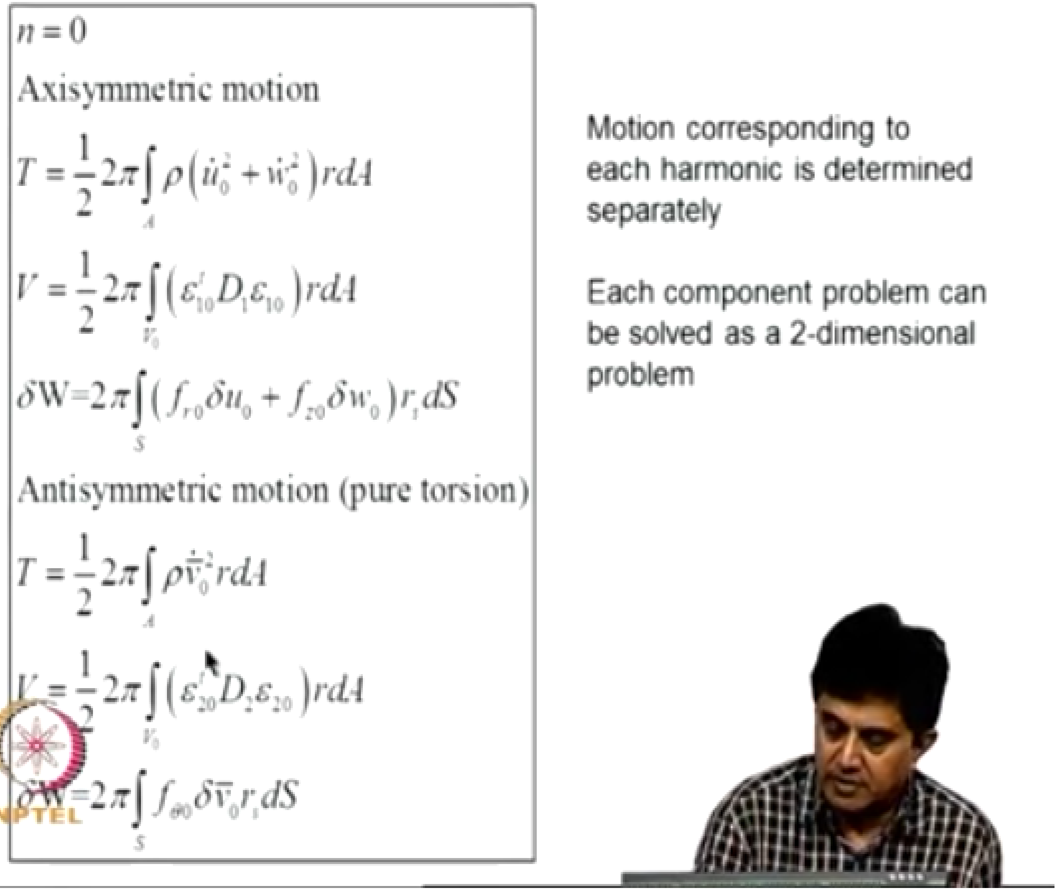
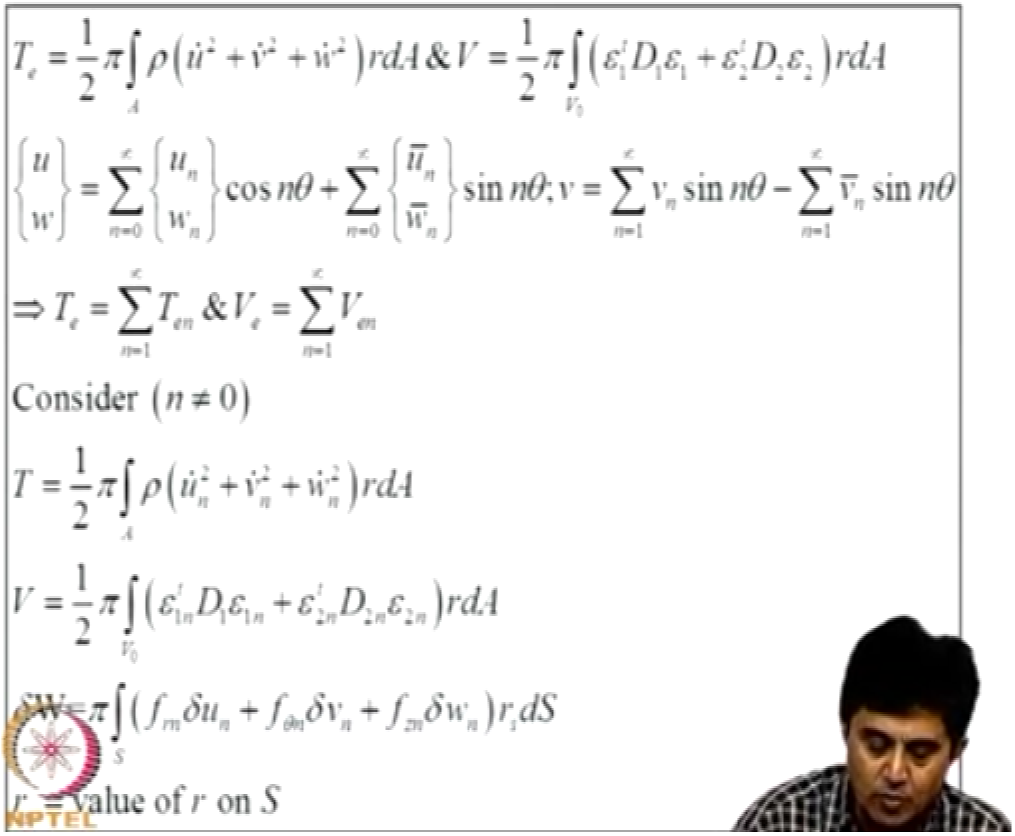
Now what is the implication of using Fourier decomposition on displacement components on the evaluation of energies? So now we can carry out the integration with respect to theta, this was over the volume element, now on theta if you can calculate the, you can carry out the integration and we get these expressions. Now TE the total element kinetic energy is taken to be a summation of kinetic energy for various Fourier components, given the orthogonality of these basis functions and its total basis functions and the fact that system is linear there won't be any coupling between the energy contributions from different Fourier terms, so this image, therefore I get if I sum all the kinetic energies over the Fourier coefficients I get the total kinetic energy, and similar statement is valid for this also.

Now we have to consider the nature of these terms for N = 0, N naught = 0 separately, for N naught = 0 this is the expression for kinetic energy, now if we now use, and similarly this is strain energy and this is the work done, so this is written for nth component in the Fourier expansion, so this is also written for the nth component in the Fourier expansion similarly the virtual work done also, this is a contribution from nth component in the Fourier series, here this integration we get RSDS, this is surface traction, the work done by the surface traction, this RS is actually value of R on the surface S.

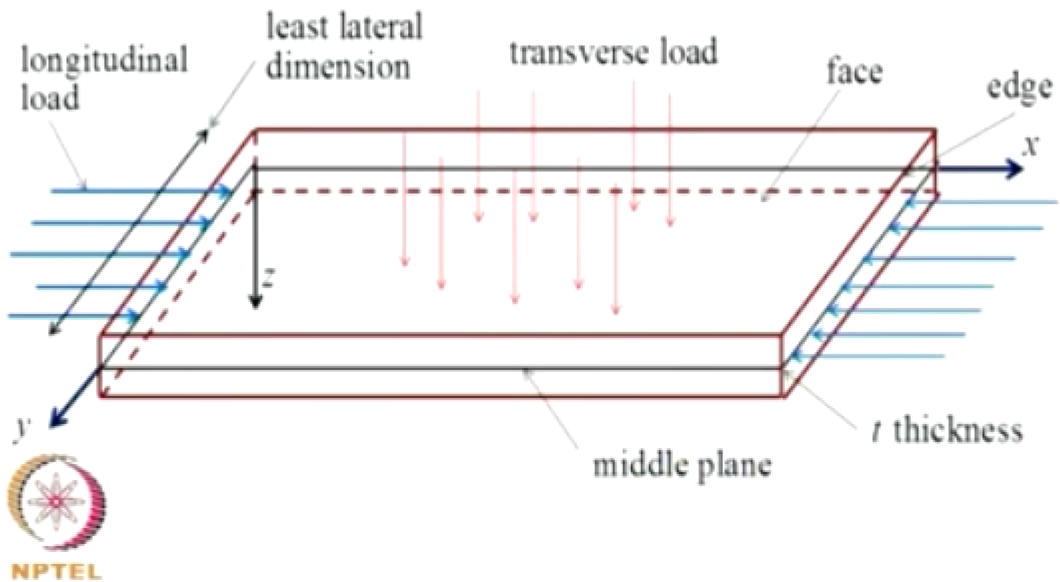


Now the axisymmetric motion, that is N = 0, sorry the N = 0 term needs to be handled separately, and it has axisymmetric and antisymmetric motion, this is pure torsion, and this consists of these terms, that is N = 0 is done separately. Now the idea of this decomposition is

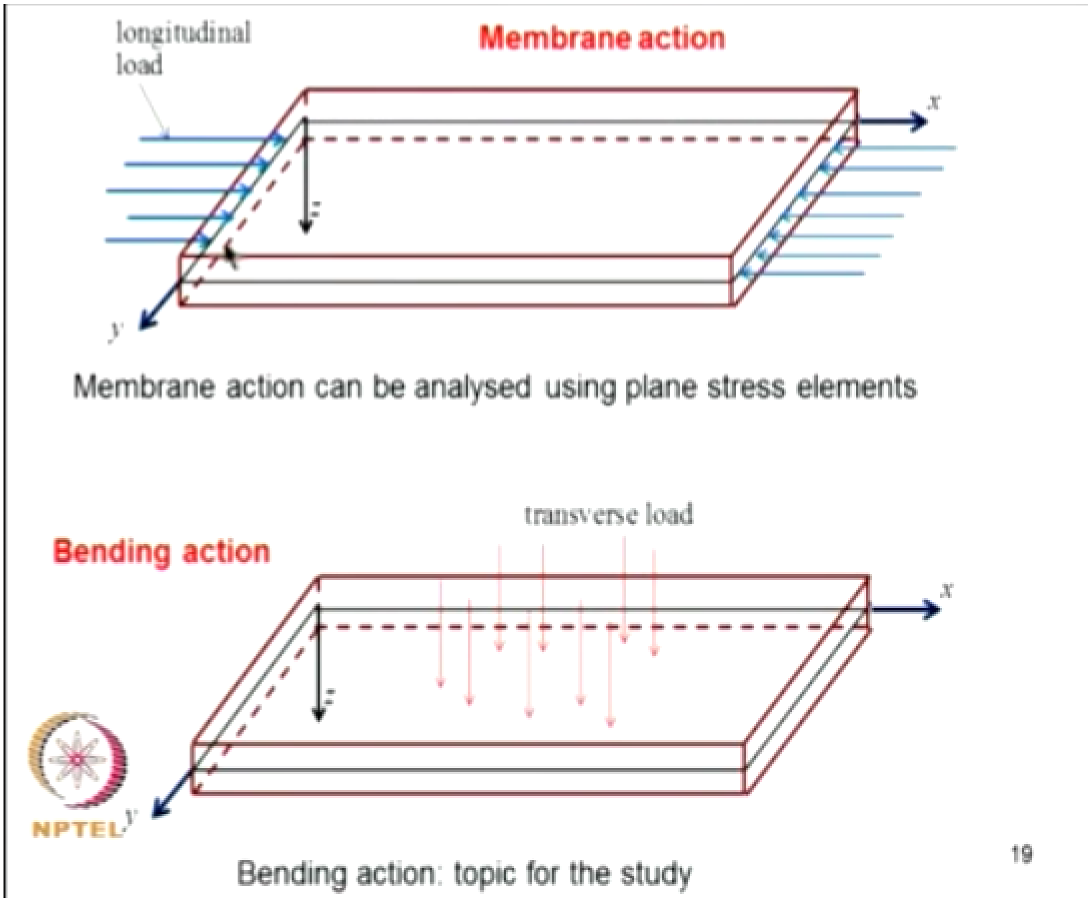


each one of these problems can be tackled as a 2 dimensional problem in you know analysis, so you can develop a 2 dimensional finite element model for each of these terms separately, so that can be you know systematically done I am not going to get into the details, I am just giving you the main idea on how to do that, so the motion corresponding to each harmonic is determined separately, and each component problem can be solved as a 2 dimensional problem, and we can develop finite element model for each of these component problems, because if you see now the structure of these energies they resemble that, resemble what we have handled earlier, so this is some simple ideas associated with analysis of axisymmetric solids.

Plate bending element



Now the next part of our discussion we will now move on to problems of a plate bending, so this figure explains the basic nomenclature that we use in the study of plates, so here is a rectangular plate, plate is a object which is bounded by two faces, T is the thickness and this plane is called a middle plane, and this is the least lateral dimension, and this is the thickness, this surface is called the edge, this top surface is called the face. Now in a problem like this the loads can act transverse to the middle plane or in line with the middle plane, the problem of loads acting in line with the middle plane can be tackled using plane stress models, whereas the question now is how to treat the analysis of this type of structures under transverse load, that is known as plate action. So this action where the structure deforms due to in plane loads is called



membrane action, so membrane action can be analyzed using plane stress elements. Now the bending action is due to transverse loads, this is what we are going to discuss in today's lecture.

Assumptions

1	The material of the plate is elastic, homogeneous, and isotropic	Not a fundamental assumption. Not a basic requirement. Alternative theories which relax these asumptions can be developed. Inelasticity and anisotropy may be desired for economic designs and optimal usage of material (e.g., inelasticity: earthquake engineering applications; anisotropy: composites, RCC)
2	The body is initially flat.	Valid for plate theory. In the theory of shells the body can be initially curved.
3	The thickness of the plate is small compared to its other dimensions. The smallest lateral dimension of the body is at least ten times larger than its thickness	$t << L$ = the least lateral dimension Most fundamental assumption from a physical view point $0.001 < \frac{t}{L} < 0.4$ $\frac{t}{L} \approx 0.001$: use membrane theory $\frac{t}{L} \approx 0.1$: thin plate theory $\frac{t}{L} \approx 0.4$: thick plate theory

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Now there are a few assumptions if we build a theory for behavior of these plates by making few assumptions, so I will run through these assumptions, so in this slide there is 2 columns here, the first column states the assumption, and the second column has few commentaries on these assumptions, so what we will do is first we will run through the assumptions and then discuss the consequences and comments on that, so the first is material of the plate is elastic, homogeneous and isotropic, the body is initially flat, that is the next assumption, the thickness of the plate is small, the thickness of the plate is small compared to its other dimensions, the smallest lateral dimension of the body is at least 10 times larger than the thickness.

4	The deflections are small as compared with the plate thickness. $\frac{t}{10}$ < max deflection< $\frac{t}{5}$: limits for validity of thin plate theory. The slopes of the deflected middle surface are small compared to unity. Terms with squares of slopes can be neglected.	The equations can be formulated in terms of initial undeformed geometry. Products of deformation parameters can be neglected. Validity can be tested by actual calculations in the course of solutions. If these conditions are violated, we can develop geometrically nonlinear theory by retaining all other assumptions. As magnitude of admissible displacements increases, the tendency for material nonlinear behavior needs to be taken into account.
5	There is no deformation of the middle surface during bending.	Defines the neutral plane. Not valid when in plane loads are present. Not valid for large deformations.
6	The deformations are such that straight lines initially normal to the middle plane remain straight and normal to the middle plane. The thickness of the plate remains unaltered.	The resulting errors are negligible for thin plates. Transverse shearing strains acting on planes normal to the middle plane are neglected. The assumptions can be relaxed. Normal strains are neglected.

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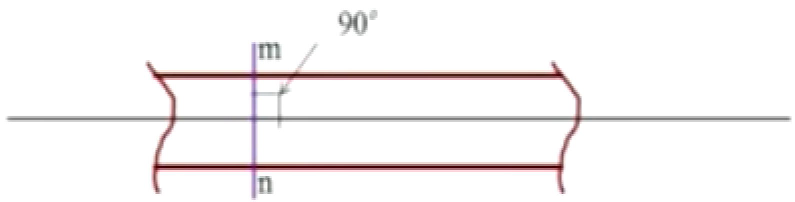
Next we make certain assumptions on magnitudes of deflections, the deflections are small as compared with the plate thickness, for example the maximum deflection is taken to lie between T/10 to T/5 this range can be viewed as limits for validity of thin plate theory. The slope of the deflected middle surface is small compared to unity, the terms the consequence of the terms with squares of the slopes can be neglected. Then there is no deformation of the middle surface during bending. Then the deformations are such that straight lines initially normal to the middle plane remains straight and normal to the middle plane, the thickness of the plate remains unaltered, okay.

7	The stresses normal to the middle surface are of negligible order of magnitude.	Valid for small t / L. Plate thickness does not change during deformation. Not valid in the vicinity of concentrated loads and as thickness increases.



Finally the stresses normal to the middle surface are negligible are of negligible order of magnitude, so these are the basic assumptions we make, we can quickly run through the implications of these assumptions, many of these assumptions are also made in analysis of beams, so some of the comments that I am going to make may be valid in that context also, so let's see. Now the material of the plate is elastic, homogeneous and isotropic, this is not a fundamental assumption, it is not a basic requirement to develop a plate theory, we can develop alternative theories which relax these assumptions, for example in the elasticity and anisotropy may be desired for economic designs and optimal usage of material, for example in problems of earthquake engineering we want to design structures to behave in elastically in a controlled manner. Similarly in composites and RCC etcetera we introduce anisotropy to you know

Assumptions

1	The material of the plate is elastic, homogeneous, and isotropic	Not a fundamental assumption. Not a basic requirement. Alternative theories which relax these asumptions can be developed. Inelasticity and anisotropy may be desired for economic designs and optimal usage of material (e.g., inelasticity: earthquake engineering applications; anisotropy: composites, RCC)
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enhance the performance of the structures, structural material, so this is a simplifying assumption if necessary these assumptions can be relaxed.

Now the body is initially flat, this is valid for plate theory, in the theory of shells the body can be initially curved, so for a plate theory this is required, then the thickness of the plate is small compared to its other dimensions, the smallest lateral dimension of the body is at least 10 times larger than its thickness. Now this is a fundamental assumption, this is what defines what a plate is, so T being much less than or equal to L, where T is the thickness being much less than the least lateral dimension is the most fundamental assumption from a physical point of it, if this condition is not satisfied we are not talking about plates, okay, so the typical range for T/L is this 0.001 to 0.4, if it is of the order of 0.001 we can use a membrane theory, like a string you know, 2 dimensional analog of a string, if it is in the range of about 0.1 thin plate theory can be used, if it is in the range of 0.4 then we need to use thick plate theory, so this is one of the important assumptions, the deflections are small as compared with the plate thickness and other description is given here. Now this enables us to write equations you know to form, the equations now can be formulated in terms of initial undeformed geometry, products of

4	The deflections are small as compared with the plate thickness. $\frac{t}{10}$ < max deflection< $\frac{t}{5}$: limits for validity of thin plate theory. The slopes of the deflected middle surface are small compared to unity. Terms with squares of slopes can be neglected.	The equations can be formulated in terms of initial undeformed geometry. Products of deformation parameters can be neglected. Validity can be tested by actual calculations in the course of solutions. If these conditions are violated, we can develop geometrically nonlinear theory by retaining all other assumptions. As magnitude of admissible displacements increases, the tendency for material nonlinear behavior needs to be taken into account.
5	There is no deformation of the middle surface during bending.	Defines the neutral plane. Not valid when in plane loads are present. Not valid for large deformations.
6	The deformations are such that straight lines initially normal to the middle plane remain straight and normal to the middle plane. The thickness of the plate remains unaltered.	The resulting errors are negligible for thin plates. Transverse shearing strains acting on planes normal to the middle plane are neglected. The assumptions can be relaxed. Normal strains are neglected.

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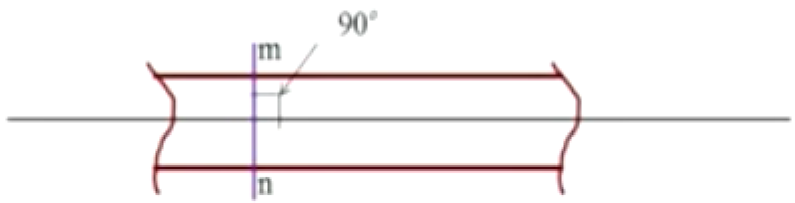
deformation parameters can be neglected, then validity actually if you make these assumptions and carry out a computation if you are interested you can test the validity of these assumptions by actually performing the calculations of a squares of displacement terms and products of slopes and things like that, in the course of solution we can really verify, for example the terms that we have ignored in the strain displacement relations we can compare with the terms that we have retained and see whether they are indeed small or not, so it is possible posteriorly kind of check if these assumptions are met or not. If these conditions are violated we can develop a geometrically nonlinear theory by retaining all other assumptions, all other assumptions means the earlier assumptions, okay, and some of the things that to follow.

As magnitude of the admissible displacement increases, their tendency for material non-linear behavior needs to be taken into account, see if you are including nonlinear strain displacement relations, and if the displacements are large the material has a tendency to enter into in elastic regimes as far as stress-strain relations are concerned, so that also has to be borne in mind. Next there is no deformation of the middle surface during bending, this actually helps us to define the neutral plane, and it is not valid when in plane loads are also present, it is not valid for large deformations. Then the deformations are such that straight lines initially normal to the middle plane remain straight and normal to the middle plane, the thickness of the plate remains unaltered. Now this is not going to be you know in a scientific sense satisfied, for example lines in which are initially normal to the middle plane will not remain normal to the middle plane after deformation, they may not remain straight and thickness need not remain unaltered, but the point is the resulting errors in the kind of situation that we are considering are negligible for thin plates, that is the range of applicability of the theory that we are developing, the transfer

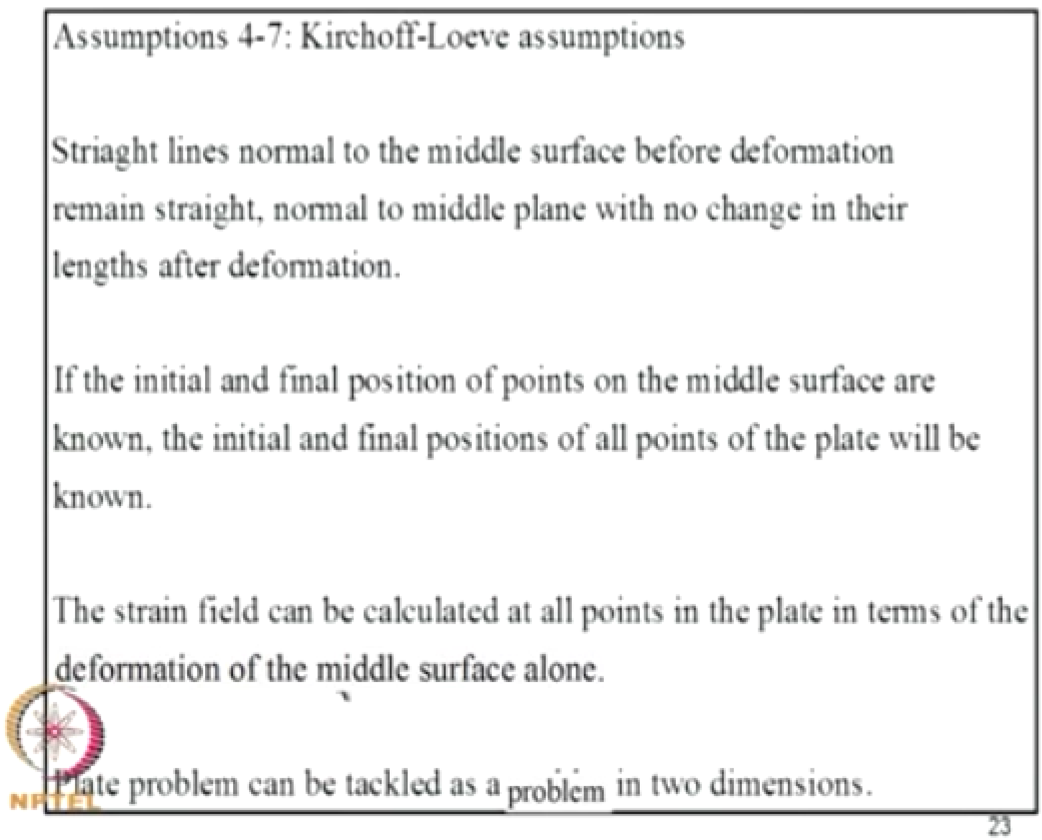
shearing strains acting on planes normal to the middle plane are neglected, the assumptions can be relaxed, these assumptions are also not fundamental you can relax these assumptions, and for example while discussing Euler-Bernoulli beam theory if you include a shear deformation, contribution of a shear deformation to transverse deflection we saw that we can develop an alternate theory known Timoshenko beam theory, so the assumptions that we are discussing here for in the context of plate are the Kirchhoff Love you know assumptions.
So if you relax some of these assumptions there are other theories like Mindlin plate theory which allows for thick plates, and the contribution of shear deformations to transverse deflections, so there are further higher order plate theories which are available in the existing literature. So the point is that this assumption can be relaxed and we can develop alternative theories, so the thickness of the plate remains unaltered then consequence is normal strains are neglected, the stresses normal to the middle surface are of negligible order of magnitude, this

7	The stresses normal to the middle surface are of negligible order of magnitude.	Valid for small t/L. Plate thickness does not change during deformation. Not valid in the vicinity of concentrated loads and as thickness increases.



again valid for small values of T/L, the plate thickness does not change during deformation, not valid in the vicinity of concentrated loads, and as the value of the thickness increases this assumption is not valid.
Now what is the meaning of this assumption, the deformations are such that straight lines initially normal to the middle plane remains straight and normal to the middle plane, so I have shown the undeformed cross-section of a plate, and MN is the line which is, actually it is a surface normal to the plane of this screen which, this is the middle plane, this is the middle plane and MN is at 90 degrees to the middle plane, this is before deformation, now after deformation so we can sketch this, so MN, the middle surface this is according to our assumption M bar, N bar, is where MN goes in the deformed geometry, and this angle remains as 90 degree, that's what we are saying. Now it is possible that it remains straight but it may be

at an angle, for example the MN may be like this, okay, so here there is a shear deformation, this angle alpha, okay, here there is no shear deformation but still this surface is remaining plane or this line is remaining as straight, but it can also deform in a more you know nonlinear way across the thickness, so then in which case we need higher order theories to be developed, so what we are doing in thin plate theory is this situation, in Mindlin plate theory we can use this situation, so when it comes to computation of kinetic energy we are considering a mass element and contribution to kinetic energy due to this deflection is taken into account, but a section like this, for example a section like this would rotate in the deformed configuration, and there can be inertia against this kind of rotations, so if thickness of the plate increases not only we need to allow for shear deformation, but also we need to take into account contribution to kinetic energy from rotary inertia terms, so some of these need to be included if you wish to develop you know refined theories for plate behavior, so the one that we are discussing is the theory that is valid for thin plates.


Assumptions 4-7: Kirchoff-Loeve assumptions

Striaght lines normal to the middle surface before deformation remain straight, normal to middle plane with no change in their lengths after deformation.

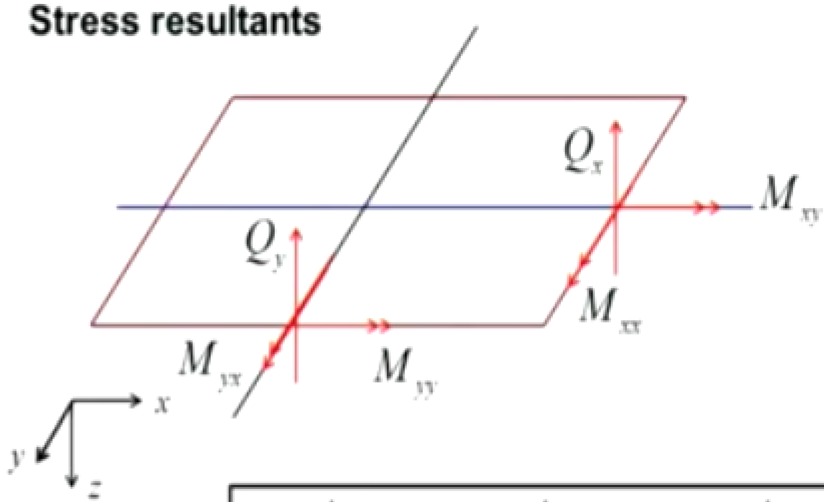
If the initial and final position of points on the middle surface are known, the initial and final positions of all points of the plate will be known.

The strain field can be calculated at all points in the plate in terms of the deformation of the middle surface alone.

Plate problem can be tackled as a problem in two dimensions.

So this assumption 4 to 7 listed in the table are known as Kirchoff-Love assumptions. Now the straight line normal to middle surface before deformation remains straight, normal to the middle plane with no change in their lengths after deformation that's what we have been emphasizing, if the initial and final position of the points on the middle surface are known, the initial and final positions of all points of the plate will be known, so there are consequence of the assumptions that we have made. The strain field can be calculated at all points in the plate in terms of the middle surface alone, okay, in terms of the deformation of, now the plate problem can be thus tackled as a problem in 2 dimensions, okay, so this is what will emerge now as we go through the analysis.

Stress resultants



$$M_{xx} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz;\ M_{yy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} z dz;\ M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} z dz$$

$$Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xz} dz;\ Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yz} dz$$

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Now in dealing with 3 dimensional elasticity problem we talked about stress and strain components, but in engineering theories like axially vibrating rod, beams, plates, etc., we talked about stress resultants, they are the integrals of stresses across the cross sections of the structure under consideration, so in a plate the stress resultants are shown here, there will be bending moments about X-axis twisting moment MXY, twisting moment MYX, bending moment MYY and shear force QY and QX, so these are the stress resultants in a plate problem, so how do you compute, how the normal stress and bending moments are related? So you integrate across the thickness of the plate as shown here, and you'll get what is a bending moment, that means sigma XX ZDZ, so you take the moments of the stress about the middle plane and you get this, similarly MYY on this face, we get this MXY is because of shearing stresses we get this, shear force again because of shearing stresses and they are integrals over the surface, so these are quantities per unit length.

Implications

Plane sections initially normal to the middle plane

- remain plane $\Rightarrow \varepsilon_{xz}(x,y,z) = \varepsilon_{xz}(x,y) \,\&\, \varepsilon_{yz}(x,y,z) = \varepsilon_{yz}(x,y)$
- and normal to the middle plane $\Rightarrow \varepsilon_{xz}(x,y) = 0, \varepsilon_{yz}(x,y) = 0$
- will have same length $\Rightarrow \varepsilon_{zz}(x,y,z) = 0$

$$\varepsilon_{zz}(x,y,z) = 0 \Rightarrow \frac{\partial w}{\partial z} = 0 \Rightarrow w(x,y,z) = w(x,y)$$

Consider $\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 0$

This may not be true. That is, $\varepsilon_{zz} = 0$ need not imply $\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$.

Abandon the constitutive law.

$$\ldots y) = 0 \Rightarrow \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \Rightarrow u(x,y,z) = -z\frac{\partial w}{\partial x}$$

$$\ldots y) = 0 \Rightarrow \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \Rightarrow v(x,y,z) = -z\frac{\partial w}{\partial y}$$

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Now let's again examine some of the implications in terms of our ability to formulate the problem, because we have made these assumptions what happens to the displacement stress and strain fields? Now plane sections initially normal to the middle plane remain plane the consequence of that is epsilon XZ, X, Y, Z is only function of X and Y, the dependence on Z is not there, and similarly epsilon YZ is function of X and Y only, it remains plane and normal to a middle plane that means the shearing strains are 0, if you make only this assumption the shearing strains are independent of Z, here you are going out and telling that it also remains normal to the middle plane therefore this is 0, and will have the same length would mean epsilon ZZ is 0, that means the thickness of the plate doesn't change, epsilon ZZ is therefore 0. Now epsilon ZZ = 0 means dou W/dou Z is 0, that means W(x,y,z) is independent of Z and it is W(x,y), so it is enough if we study this at the middle plane, that's what we will do.

Now you consider epsilon ZZ based on the constitutive law, epsilon ZZ is given by this, now we are assuming that epsilon ZZ is 0, but actually this may not be true, that is epsilon ZZ = 0 need not imply that sigma ZZ is equal to this, okay, so we abandon this constitutive law in our formulation, we won't really use this constitutive law. Now you look at now epsilon XZ is 0, the shearing strains are 0, that means dou W/dou X + dou U/dou Z = 0, consequently it means U(x,y,z) is – Z, dou W/dou X. Similarly epsilon YZ is 0, implies that V is - Z, dou W/dou Y, so I am basically able to express U and V in terms of W.

$$\left.\begin{aligned} \varepsilon_{xz}(x,y)=0 \Rightarrow \sigma_{xz} = 2G\varepsilon_{xz} = 0 \\ \varepsilon_{yz}(x,y)=0 \Rightarrow \sigma_{yz} = 2G\varepsilon_{yz} = 0 \end{aligned}\right\}$$

This cannot be true since we want that $\sigma_{xz} \neq 0$ & $\sigma_{yz} \neq 0$ so that the corresponding stress resultants Q_x & Q_y are $\neq 0$.

$\Rightarrow$ We need to abandon the following two constitutive laws:

$\sigma_{xz} = 2G\varepsilon_{xz}$ & $\sigma_{yz} = 2G\varepsilon_{yz}$

$\sigma_{zz} << \sigma_{xx}$ & $\sigma_{zz} << \sigma_{yy} \Rightarrow$ Neglect σ_{zz} in the constitutive laws.

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \approx \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \approx \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$2\varepsilon_{xy} = \frac{\sigma_{xy}}{G}$$

The remaining three constitutive laws are already abandoned.

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Now epsilon XZ = 0 means sigma XZ must be 0, similarly epsilon YZ = 0 means sigma YZ must be equal to 0, but this cannot be true since we want these stresses to be not equal to 0, so that the corresponding stress resultants are not 0, if we take this constitutive laws to be valid the implication is the shearing stresses will be 0, and therefore the stress resultants are also 0. Now this is not true, so what we do is we will abandon these 2 constitutive laws in our formulation. Now next we are assuming that sigma ZZ, sigma XX is much larger than sigma ZZ, and sigma YY is also much larger than sigma ZZ, so we can neglect sigma ZZ in the formulation of constitutive laws, so we will write epsilon XX actually it is given by this, now I will knock off this term and retain only this, similarly epsilon YY when I write I will use this, the shearing strain is given by this, okay, the remaining constitutive laws are already abandoned, there are only 3 constitutive laws which will be using with this simplification. Now therefore the constitutive law for thin plate theory is having these components sigma XX, sigma YY, sigma XY given by this.

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\ 2\varepsilon_{xy} &= \frac{\sigma_{xy}}{G} \end{aligned}\right\} \Rightarrow \begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \\ \sigma_{xy} &= 2G\varepsilon_{xy} \end{aligned}$$

These constitutive laws are identical to those applicable to plane stress element.

Plate is not in a state of plane stress.

For instance, $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ need to be nonzero and functions of $x, y, \& z$.

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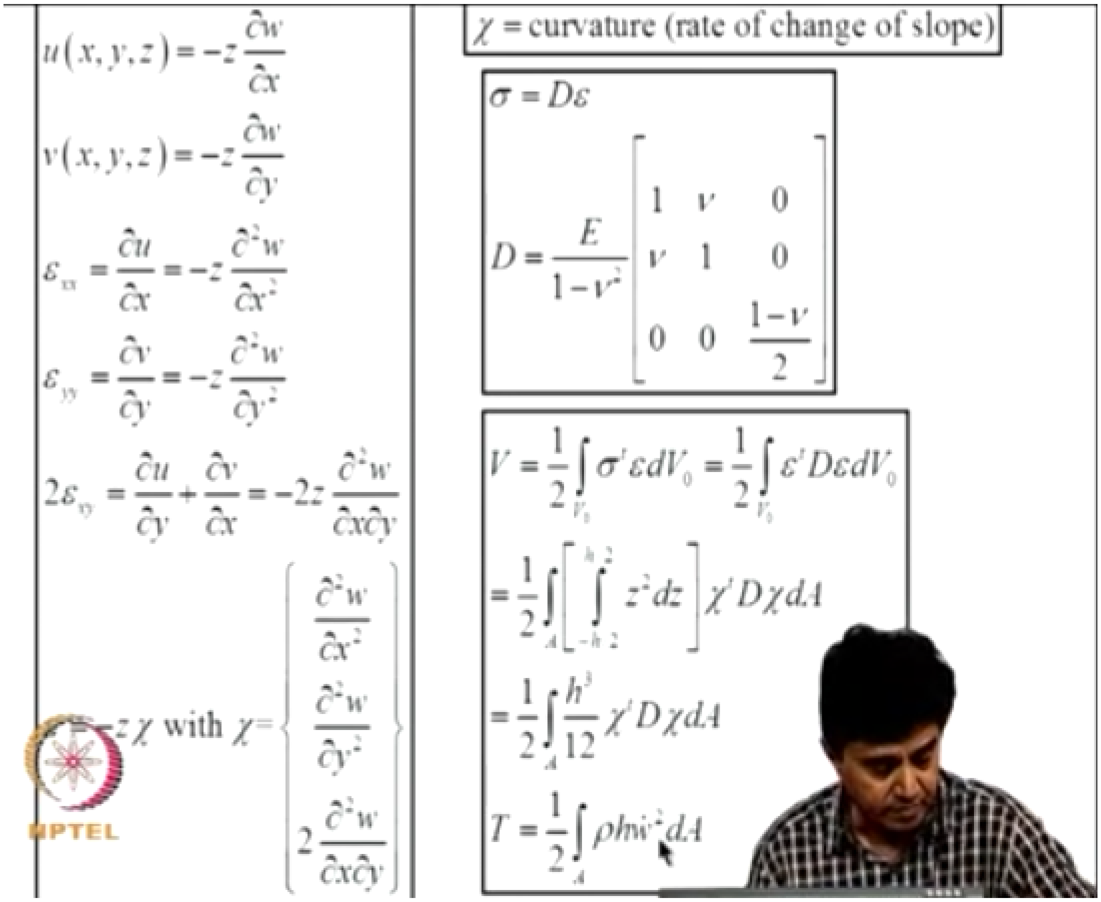
Now if you go back and check with the constitutive law that we use for plane stress elements, it turns out that the constitutive law that is emerging here is identical to what was used in plane stress theory, but you should be careful, you shouldn’t interpret bending of a plate as a problem in plane stress, there are many variations points for example sigma XX, sigma YY, XY, XZ, sigma YZ need to be nonzero in a plate bending problem, and functions of XY and Z, this is not true in a plane stress problem. The only point of commonality is that the relationship between stresses and strains that is these strains and the 3 corresponding stresses is identical to what was seen in a plane stress model.



Now we now need to compute the strain energies and kinetic energies, so that is where we are now moving towards, so we have now U is given by - Z dou W/dou X, V is given by - Z dou W/dou Y, epsilon XX now I can express in terms of W, it is - Z dou square W/dou X square, epsilon YY - Z dou square W/dou Y square, shearing strain is - 2Z dou square W/dou X dou Y, so we write now the strain in terms of this curvature, curvatures are rate of change of slopes dou square W/ dou X square, dou square W/dou Y square, and this dou square W/dou X dou Y, and we define epsilon as - Z into khi, where khi is the vector of curvatures as defined here.

Now sigma is D into epsilon, and D is this matrix, so we are now ready to write the expression for strain energy and kinetic energy, the strain energy is given by sigma transpose epsilon DV naught, and in terms of using this constitutive law it is epsilon D epsilon DV naught. Now I will use this now for this, and write for epsilon – Z khi and I will be able to write integrate over the depth, and that leads to this term H cube/12 and the remaining terms are khi transpose D khi DA, this is the strain energy. Kinetic energy is 1/2 rho H W dot square DA, as I already mentioned we are including only this contribution to kinetic energy from only W.

Thin rectangular element with 4 nodes, 3 dofs/node (Dofs=12)

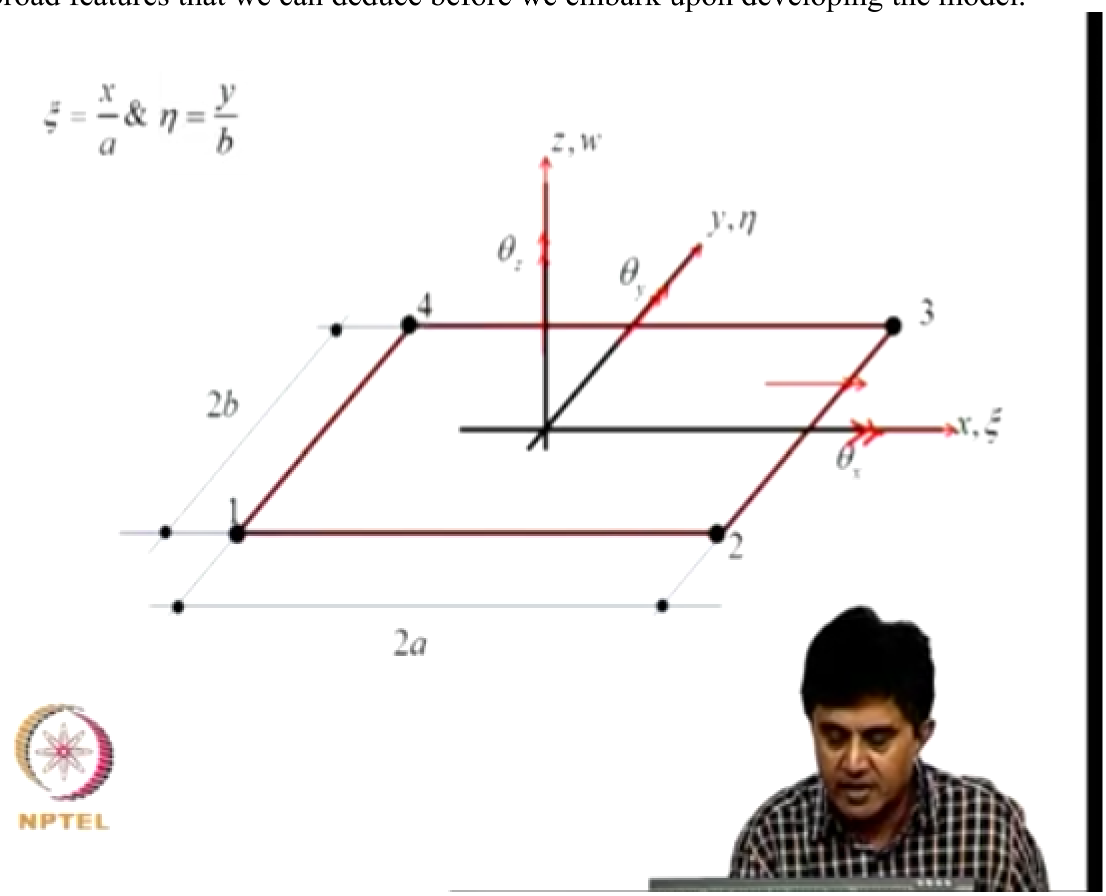
Field variable: $w(x,y,t)$

Order of the highest derivative present in the Lagrangian: 2

Dofs: $w(x,y,t), \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$

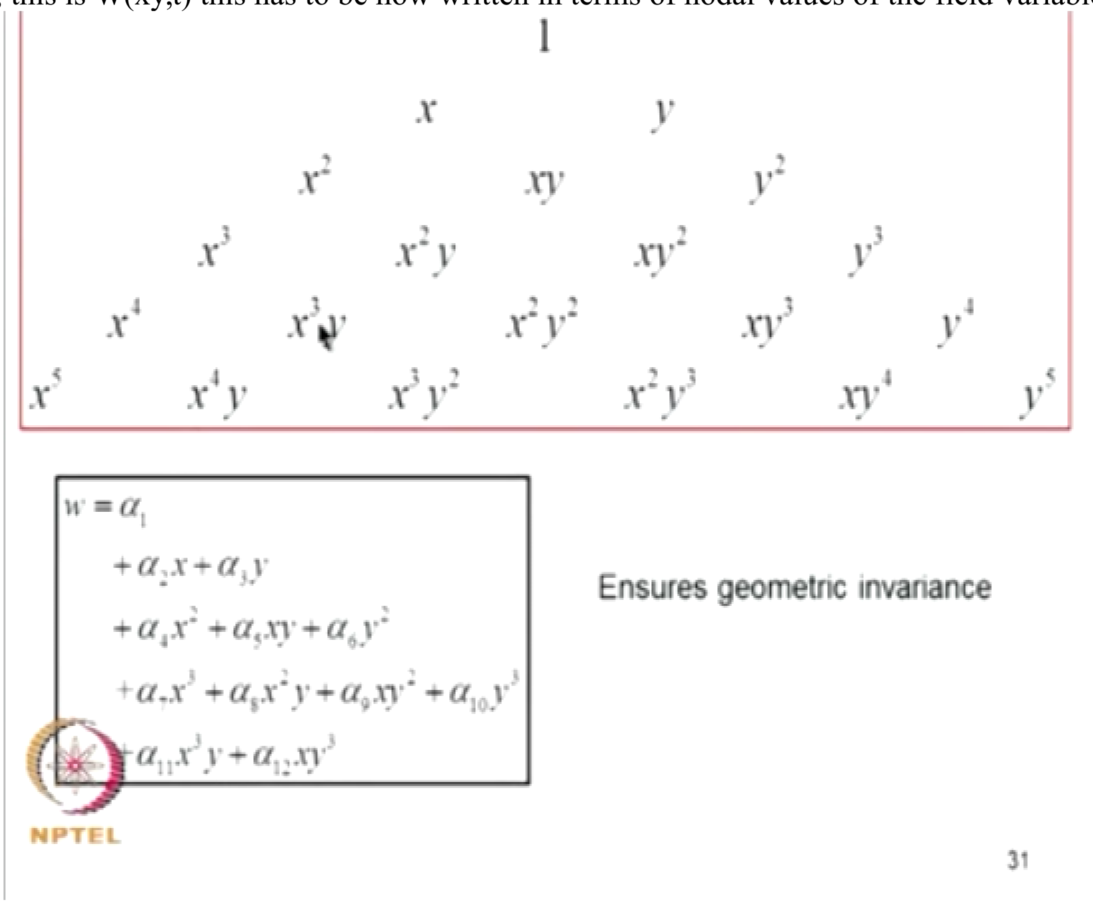
Number of generalized coordinates: 12

Now we can now consider the problem of formulating a finite element for finalizing plate structures, so we'll start with considering a thin rectangular element with 4 nodes, with 3 degrees of freedom per node and there are 12 degrees of freedom, the field variable is W(x,y,t), the order of the highest derivative present in the Lagrangian is 2, see the khi transpose D khi is there, khi itself has dou square W/dou X square the highest order of derivative of the field variable is 2. So the degrees of freedom should be, it should include derivatives up to 2 – 1, so there are now 2 special variables therefore at any node the degree of freedom will be W dou W/dou X and dou W/dou Y, since there are 4 nodes with each node having 3 degrees of freedom, the number of generalized coordinates in our representation must be 12, so these are the broad features that we can deduce before we embark upon developing the model.



So geometry of the model this is rectangular plate, these are the axis X, Y, Z and I introduce X/A as XI, and Y/B as eta, and those coordinates are also shown here, this is eta and XI, and this is Z coordinate, along Z we need not make any transformation this W is shown as a corresponding displacement here.

Now theta X is the twist about X axis, theta Y is the twist about Y axis, and theta Z is a twist about the Z axis, 1, 2, 3, 4 are the nodes, and we will now write the expressions for energies in such an element. Now the displacement field within this element, what is the displacement field, this is W(xy,t) this has to be now written in terms of nodal values of the field variables



which should be, the degrees of freedom should be W, dou W/dou X, dou W/dou Y, so there are 3 nodal coordinates here degrees of freedom here, so the W need to be represented in terms of these 12 nodal you know displacement values, so how do we represent that? So we will start with, this is a Pascal's triangle, we need 12 terms from this, so I will start with 1, XY, X square, XY, Y square, then X cube, X square Y, XY square, Y cube, when I reach this stage I have exhausted 10 terms, I have to pick 2 more terms now, so 2 more terms in this what we do is we select this term and this term, okay, so if I select this then I will not be able to, I have to select one more term then I will not be able to honor the requirements of geometric invariance that means if I were to change the nomenclature of X and Y axis, the representation of W will change because suppose if I retain this and retain any other element it is not symmetric in X and Y, so this is why we take these 2, so this is the representation that we can think of.

Now the steps for formulating the stiffness and mass matrices are quite similar to what we have done conceptually, but the questions you should now ask is, is this element a conforming element, okay, so by that I mean are the quantities W, dou W/dou X, and dou W/dou Y are continuous across the plate element boundaries, so it turns out that this element will not be a conforming element, so we will just see some of the details now. So I will now, I will have to

$$w = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2 + \alpha_7\xi^3 + \alpha_8\xi^2\eta + \alpha_9\xi\eta^2 + \alpha_{10}\eta^3 + \alpha_{11}\xi^3\eta + \alpha_{12}\xi\eta^3$$

$$w = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \end{bmatrix}\{\alpha\} = \left[P(\xi,\mu)\right]\{\alpha\}$$

$$\frac{\partial w}{\partial \xi} = \begin{bmatrix} 0 & 1 & 0 & 2\xi & \eta & 0 & 3\xi^2 & 2\xi\eta & \eta^2 & 0 & 3\xi^2\eta & \eta^3 \end{bmatrix}\{\alpha\}$$

$$\frac{\partial w}{\partial \eta} = \begin{bmatrix} 0 & 0 & 1 & 0 & \xi & 2\eta & 0 & \xi^2 & 2\xi\eta & 3\eta^2 & \xi^3 & 3\xi\eta^2 \end{bmatrix}\{\alpha\}$$

Evaluating $w, \frac{\partial w}{\partial \xi}, \& \frac{\partial w}{\partial n}$ at $\xi = \pm 1, \eta = \pm 1$, one gets

$$w_e = \left[A_e\right]\{\alpha\}$$

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select now this 12 generalized coordinates, so I will write, now I will write this W in terms of XI and eta and this is expression, there's a slight abuse of notation, these alpha 1 to alpha 12 are not necessarily this, this is in XY coordinates this is in XI eta coordinate, but it is alright because I am not going to use this representation further, this is to explain the concept I have used this, now this can be put in a matrix form as shown here, alpha is a vector of alpha 1, alpha 2, to alpha 12, and if I multiply this I get this function, this I call it as P into alpha.

Now I need to represent W, dou W/dou XI and dou W/dou eta, so I can differentiate this to get dou W/dou XI, I get this you can quickly see that 1 is 0, sai is 1, eta is 0, this is 2 sai, etcetera, etcetera, similarly dou W/dou eta is this, that means I am differentiating this with respect to eta, first 2 terms are 0, next is 1, 0, XI, 2 eta and so on and so forth. Now there are now 12 unknowns, but we know the value of W dou W/dou sai, and dou W/dou eta at the 4 nodes, that is XI = + - 1, and eta = +- 1, so there are 12 equations and 12 unknowns, so I can write this expression that means I can relate the nodal the vector of nodal displacements to these generalized coordinates alpha through a matrix which enforces these conditions, so if we do

$$\{\bar{w}_e\} = [A_e]\{\alpha\}$$

$$\{\bar{w}_e\} = \begin{bmatrix} w_1 & b\theta_{x1} & a\theta_{y1} & w_2 & b\theta_{x2} & a\theta_{y2} & w_3 & b\theta_{x3} & a\theta_{y3} & w_4 & b\theta_{x4} & a\theta_{y4} \end{bmatrix}^T$$

$$[A_e] = \begin{bmatrix}
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
0 & 0 & 1 & 0 & -1 & -2 & 0 & 1 & 2 & 3 & -1 & -3 \\
0 & -1 & 0 & 2 & 1 & 0 & -3 & -2 & -1 & 0 & 3 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & -2 & 3 & 1 & 3 \\
0 & -1 & 0 & -2 & 1 & 0 & -3 & 2 & -1 & 0 & 3 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 1 & 3 \\
0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 & -3 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
0 & 0 & 1 & 0 & -1 & 2 & 0 & 1 & -2 & 3 & -1 & -3 \\
0 & -1 & 0 & 2 & -1 & 0 & -3 & 2 & -1 & 0 & -3 & -1
\end{bmatrix}$$

that I will now call the nodal degrees of freedom in terms of, since I am introducing now XI and eta coordinates it will be like this, okay.

Now W1 B into theta X1 theta Y1 here is node 1, this is at node 2, node 3, node 4, this is AE matrix you can show that it turns out to be this, I mean it's a matter of simple coding you can verify this, so now alpha itself can be computed subsequently as AE inverse into W, so now W

$$\{\bar{w}_e\} = [A_e]\{\alpha\}$$

$$\Rightarrow \{\alpha\} = [A_e]^{-1}\{\bar{w}_e\}$$

We have

$$w = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \end{bmatrix}\{\alpha\}$$

$$= \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \end{bmatrix}[A_e]^{-1}\{\bar{w}_e\}$$

with

$$\{\bar{w}_e\} = \begin{bmatrix} w_1 & b\theta_{x1} & a\theta_{y1} & w_2 & b\theta_{x2} & a\theta_{y2} & w_3 & b\theta_{x3} & a\theta_{y3} & w_4 & b\theta_{x4} & a\theta_{y4} \end{bmatrix}^T$$

$$\Rightarrow w = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix}\{\bar{w}_e\}$$

$$N_j^T = \begin{bmatrix} \frac{1}{8}(1+\xi\xi_j)(1+\eta\eta_j)(2+\xi\xi_j+\eta\eta_j-\xi^2-\eta^2) \\ \frac{b}{8}(1+\xi\xi_j)(\eta+\eta_j)(\eta^2-1) \\ \frac{a}{8}(\xi+\xi_j)(\xi^2-1)(1+\eta\eta_j) \end{bmatrix}$$

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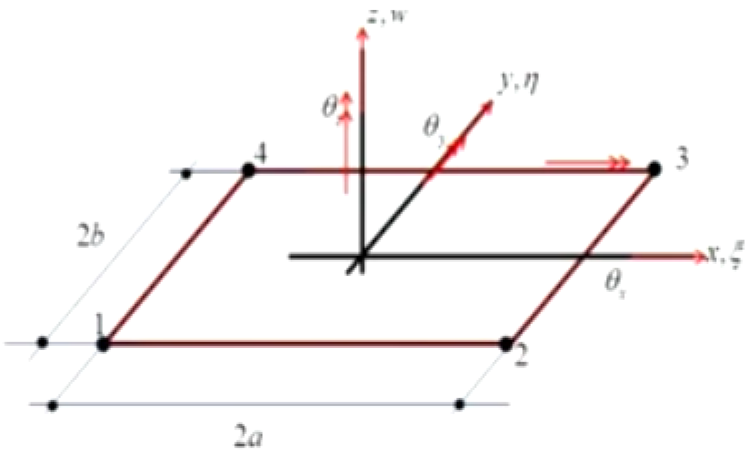
is given by this P into alpha, now this alpha is given by AE inverse into WE, so with this I will be able to now write WE which is this vector in terms of this product, this into AE inverse and that I have given this name as N1, N2, N3, N4 each one here is a 1 row and 3 element matrix, so the transpose of that is shown here, so this can be you know verified this requires some effort, this can be verified that we will get this, the symbolic you know softwares can be used to you know code this up and verify.



Consider the edge 2-3, that is, $\xi=1$

$$N_1^t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; N_2^t = \begin{bmatrix} \frac{1}{4}(1-\eta)(2-\eta-\eta^2) \\ \frac{b}{4}(\eta-1)(\eta^2-1) \\ 0 \end{bmatrix}; N_3^t = \begin{bmatrix} \frac{1}{4}(1+\eta)(2+\eta-\eta^2) \\ \frac{b}{4}(1+\eta)(\eta^2-1) \\ 0 \end{bmatrix}; N_4^t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now I raise this question of whether the element is conforming or not, so how do you verify that, so now let us consider the edge 2, 3, that is XI = 1, okay. Now along this edge now I will compute W theta X and theta Y, and see if I were to add one more element here what happens to these quantities across the edge 2, 3, so if you recall if these quantities theta X, W, and theta Y depend only on nodal coordinates at 2 and 3, then adding one more element will retain the continuity of these functions across this edge, but if the value of any of these variables also depend on values of these coordinates, then if you add one more element here there won't be a continuity here, because when I take a point on this edge and view it as a member of this element it is influenced by 1 and 4, similarly the point on this edge considered as a member of the neighboring element which I am going to add will depend on other two nodes which are outside this, so consequently there will be lack of a continuity of the field variable across the edge. So now the question is, is that condition satisfied or not, this is something that we need to verify, so what we do is we will consider now the edge XI = 1, and these functions NJ transpose J = 1, 2, 3, 4, J is the nodal coordinates, this XJ and eta J that you are finding are the coordinates for the jth node.

$$\{\bar{w}_e\} = [A_e]\{\alpha\}$$

$$\Rightarrow \{\alpha\} = [A_e]^{-1}\{\bar{w}_e\}$$

We have

$$w = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \end{bmatrix}\{\alpha\}$$

$$= \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \end{bmatrix}[A_e]^{-1}\{\bar{w}_e\}$$

with

$$\{\bar{w}_e\} = \begin{bmatrix} w_1 & b\theta_{x1} & a\theta_{y1} & w_2 & b\theta_{x2} & a\theta_{y2} & w_3 & b\theta_{x3} & a\theta_{y3} & w_4 & b\theta_{x4} & a\theta_{y4} \end{bmatrix}^T$$

$$\Rightarrow w = \begin{bmatrix} N_1(\xi,\eta) & N_2(\xi,\eta) & N_3(\xi,\eta) & N_4(\xi,\eta) \end{bmatrix}\{\bar{w}_e\}$$

$$N_j^T = \begin{bmatrix} \frac{1}{8}(1+\xi\xi_j)(1+\eta\eta_j)(2+\xi\xi_j+\eta\eta_j-\xi^2-\eta^2) \\ \frac{b}{8}(1+\xi\xi_j)(\eta+\eta_j)(\eta^2-1) \\ \frac{a}{8}(\xi+\xi_j)(\xi^2-1)(1+\eta\eta_j) \end{bmatrix}$$

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So now by putting J = 1, 2, 3, and 4 I can get this 4 functions, and we're using XI = 1, this is on this edge, so it is a simpler version of these quantities. Now using that for W along this edge I can write the expression W2 into this, beta theta X2 into this, W3 and beta theta X3 I get this

Along 2-3

$$w(1,\eta) = w_2\left\{\frac{1}{4}(1-\eta)(2-\eta-\eta^2)\right\} + b\theta_{x2}\left\{\frac{b}{4}(\eta-1)(\eta^2-1)\right\}$$

$$+w_3\left\{\frac{1}{4}(1+\eta)(2+\eta-\eta^2)\right\} + b\theta_{x3}\left\{\frac{b}{4}(1+\eta)(\eta^2-1)\right\}$$

$\Rightarrow$ w depends on nodal values $w_2, w_3, \theta_{x2}, \theta_{x3}$.

It can be shown that θ_x also depends only upon nodal values $w_2, w_3, \theta_{x2}, \theta_{x3}$.

$\Rightarrow$

w & θ_x will be continuous across edge 2-3 if another element were to be attached along this edge.

How about θ_y ?

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function, so now based on this you can see that W depends on nodal values of W2, W3, and theta X2, and theta X3, that means it is basically depending on W representations for W along XI = 1, depends on the values of degrees of freedom at 2 and 3, so consequently we can conclude that W depends on nodal values of this, and similarly we can also show theta X also depends only on these nodal coordinates, so it emerges that W and theta X will be continuous across the edge 2, 3, if another element were to be attached along this edge, so there is no problem, as far as W and theta X are concerned, but how about theta y? Now theta Y is given

How about θ_y ?

$$\theta_y = -\frac{1}{a}\frac{\partial w}{\partial \xi} = -\frac{1}{a}\left[\frac{\partial N_1}{\partial \xi} \quad \frac{\partial N_2}{\partial \xi} \quad \frac{\partial N_3}{\partial \xi} \quad \frac{\partial N_4}{\partial \xi}\right]\{w_e\}$$

$$N_j^t = \begin{bmatrix} \frac{1}{8}(1+\xi\xi_j)(1+\eta\eta_j)(2+\xi\xi_j+\eta\eta_j-\xi^2-\eta^2) \\ \frac{b}{8}(1+\xi\xi_j)(\eta+\eta_j)(\eta^2-1) \\ \frac{a}{8}(\xi+\xi_j)(\xi^2-1)(1+\eta\eta_j) \end{bmatrix}$$

Along ξ=1

$$\frac{\partial N_j^t}{\partial \xi} = \begin{bmatrix} \frac{1}{8}\xi_j\eta(1+\eta\eta_j)(\eta_j-\eta) \\ \frac{b}{8}\xi_j(\eta+\eta_j)(\eta^2-1) \\ -\frac{a}{8}(2+2\xi_j)(1+\eta\eta_j) \end{bmatrix}$$

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by – 1/8 dou W/dou XI and we have to evaluate this, so this requires some calculations, so if I now use this representation for XI = 1, and differentiate with respect to XI, I get these functions and based on that along these edge I need these four quantities I do this, and if you now

Along $\xi=1$,

$$\frac{\partial N_1^t}{\partial \xi}=\begin{bmatrix}\frac{1}{8}\eta\left(1-\eta^2\right)\\ -\frac{b}{8}(-1+\eta)\left(\eta^2-1\right)\\ 0\end{bmatrix};\frac{\partial N_2^t}{\partial \xi}=\begin{bmatrix}-\frac{1}{8}\eta\left(1-\eta^2\right)\\ \frac{b}{8}(-1+\eta)\left(\eta^2-1\right)\\ \frac{a}{2}(1-\eta)\end{bmatrix}$$

$$\frac{\partial N_3^t}{\partial \xi}=\begin{bmatrix}\frac{1}{8}\eta\left(1-\eta^2\right)\\ \frac{b}{8}(1+\eta)\left(\eta^2-1\right)\\ \frac{a}{2}(1+\eta)\end{bmatrix};\frac{\partial N_4^t}{\partial \xi}=\begin{bmatrix}-\frac{1}{8}\eta\left(1-\eta^2\right)\\ \frac{b}{8}(1+\eta)\left(\eta^2-1\right)\\ 0\end{bmatrix}$$

θ_y along $\xi=1$ depends upon values of w and θ_x at nodes 1,2,3, and 4 as well as θ_y at nodes 2 and 3.

For θ_y to be continuous between elements it should depend upon nodal displacements at nodes 2 and 3.

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carefully examine these expressions and you look at theta Y along XI = 1, it turns out that its value depends upon W and theta X at nodes 1, 2, 3, and 4, in addition to of course values of theta Y at nodes 2 and 3 okay, that means it is influenced by W and theta X also.

So consequently it so happens that it violates the condition that for theta Y to be continuous between element it should depend upon nodal displacements at nodes 2 and 3, so that means

$\Rightarrow$ The element is not a conforming element.

$\left[\text{ACM element after (Adini, Clough, and Melosh)}\right]$.

$$T_e = \frac{1}{2}\{\dot{w}_e\}^t [M]_e \{\dot{w}_e\}$$

$$[M]_e = \int_A \rho h N^t N dA = \rho hab \int_{-1}^{1}\int_{-1}^{1} N^t(\xi,\eta) N(\xi,\eta) d\xi d\eta$$

$$[M]_e = \frac{\rho hab}{6300}\begin{bmatrix} m_{11} & m_{21}^t \\ m_{21} & m_{22} \end{bmatrix}$$

$$m_{11} = \begin{bmatrix} 3454 & & & & & \\ 922b & 320b^2 & & & \text{sym} & \\ -922a & -252ab & 320a^2 & & & \\ 1226 & 398b & -548a & 3454 & & \\ 398b & 160b^2 & -168ab & 922b & 320b^2 & \\ 548a & 168ab & -240a^2 & 922a & 252ab & 320a^2 \end{bmatrix}$$

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this element is not a conforming element, this element is named ACM element after the scientists who formulated this element, it is still being used therefore in spite of this apparent undesirable feature the element is still used therefore we can go ahead and formulate the matrices, so we will see the consequence of that in the next class. So at this juncture we'll close this lecture, in the next lecture what we will see is we will complete this formulation and derive the mass and stiffness matrices, and then let us examine how to develop a conforming element, see one idea would be to use products of beam trial functions in the 2 directions, we have used the cubic polynomials for analyzing beams, so if you assume that plate is made up of 2, you know beams which are orthogonal to each other then we can utilize the products of beam functions that the shape functions that we use for the beams, and we can develop an element and let's see how that happens in the next class. We will conclude this lecture at this stage.

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