Indian Institute of Science
Bangalore

NP-TEL
National Programme on
Technology Enhanced Learning

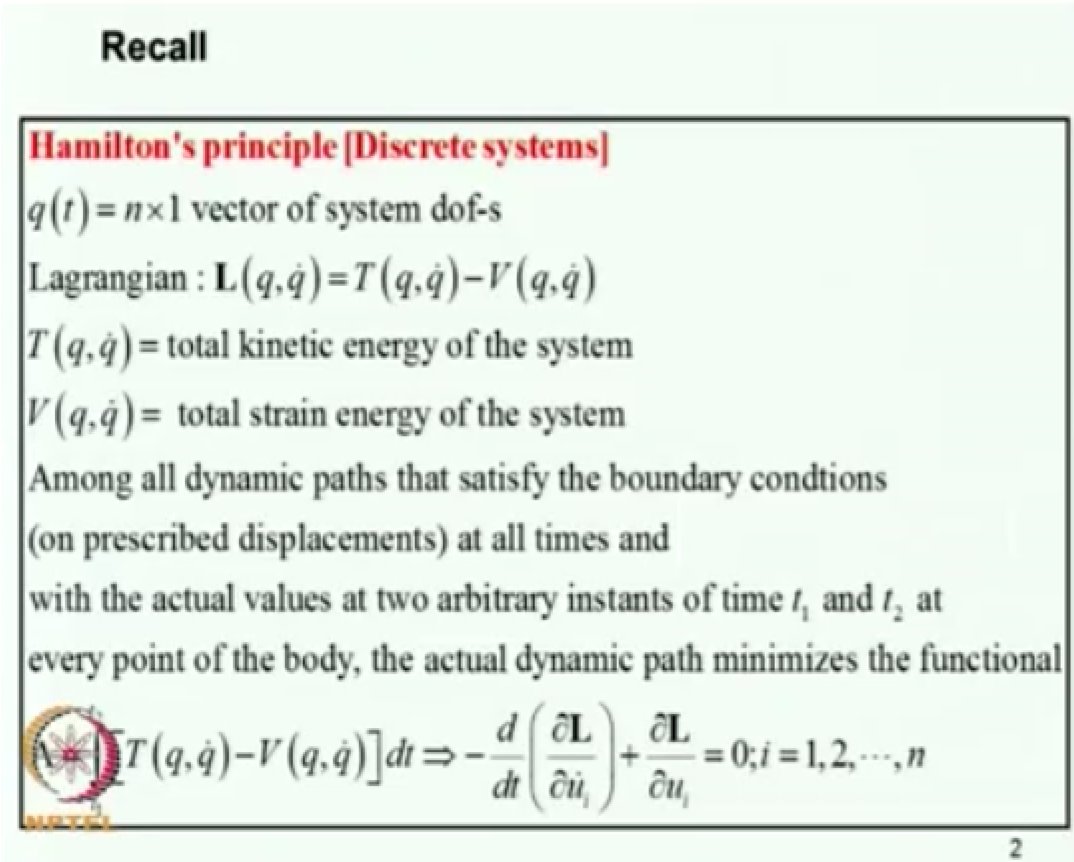
Course Title

Finite element method for structural dynamic And stability analyses

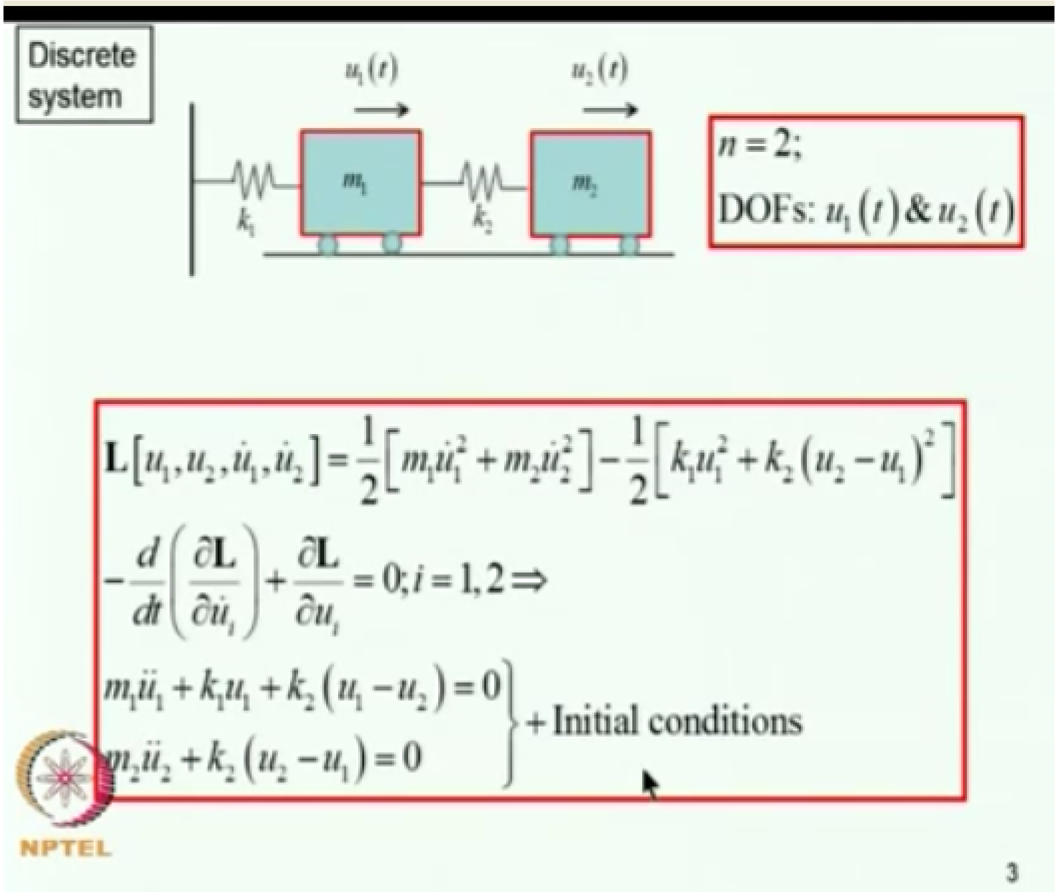
Lecture – 01
Equations of motion for continuous system and Rayleigh's quotient

By
Prof. GS Manohar
Professor
Department of Civil Engineering

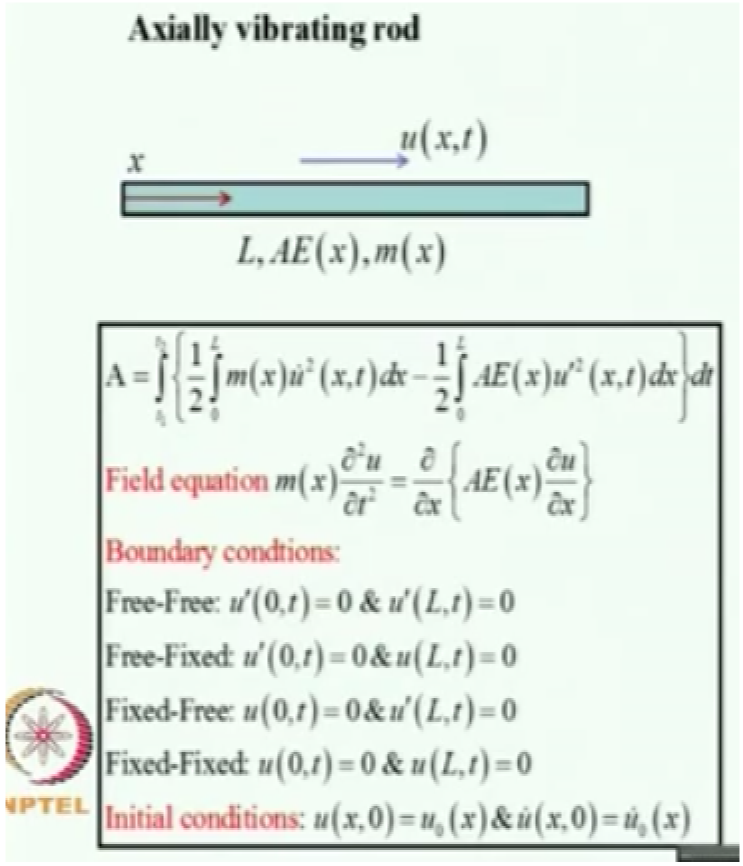
Indian Institute of Science,
Bangalore-560 012
India



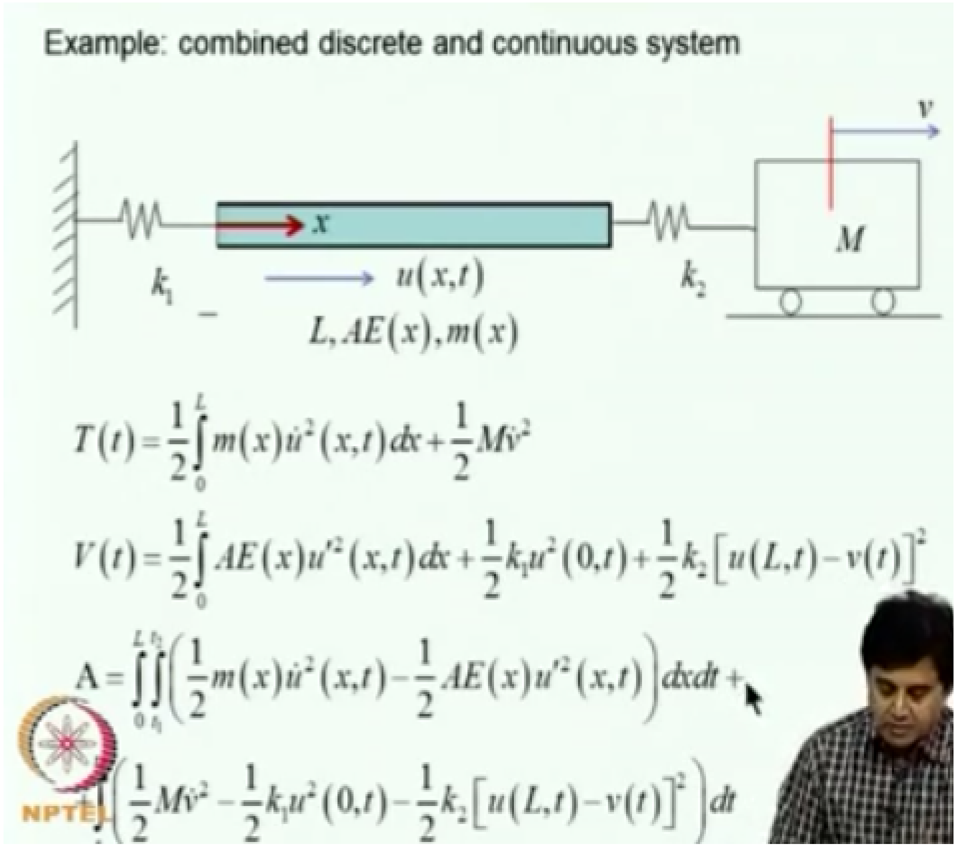
The previous lecture we looked at how to set up governing equations of motion for using Hamilton's principle and we defined a quantity known as Lagrangian, which is the difference of kinetic energy and potential energy, and we defined an integral known as action integral and according to Hamilton's principle motion takes place in such a way that this action integral is under minimum, so in a condition for stationarity of this action integral is obtained through this equation or obtained as this equation and this is known as the Lagrange equation.



Now we applied this principle to simple dynamical systems like a two degree of freedom, discrete multi-degree freedom system we set up the expression for the Lagrangian in terms of kinetic energy and the potential energy and we showed that, that approach leads to a pair of second order ordinary differential equations which are coupled to each other. Now this pair of equations could as well be obtained by drawing free body diagrams of M1 and M2 and using D' Alembert's principle, but this is a different approach where we deal with only scalar quantities like energies not the vector quantities that we will be using if we were to use D' Alembert's principle and draw free body diagrams and so on and so forth.



Now what we will do, we also covered the case of an actually vibrating rod, how the Hamilton's principle can be applied, so here the action integral is obtained as an integral or the domain of the rod that is integral 0 to L MU dot square DX and this is the strain energy and this is action integral, and by using the calculus of variations we obtain these are the field equations and I also pointed out that the variational approach also leads to the definition of the appropriate boundary conditions, and the four generic you know actually vibrating rod problems where the two ends are free, one end is fixed, and other end is free and both ends are fixed are obtained as, I mean the appropriate boundary conditions for these four cases are obtained as a part of this solution.



Now what we will do in today's class is we will extend this example, the approach to a few examples on actually vibrating rods and then we will apply the method to the problem of flexural vibration of beams, so the few examples that now I will consider helps you to analyze apart from setting up the field equations, we will also obtain the appropriate boundary conditions, the emphasis is on obtaining the boundary condition, so here we have a actually vibrating rod which is mounted on a spring here and at this end it carries a mass through a spring as shown here. Now the kinetic energy of this system consists of kinetic energy stored in this bar and the kinetic energy of this mass that is written here U(X,T) is the actual displacement of this bar and V is the displacement of this mass, so we get the kinetic energy as shown here. The potential energy is a contribution from strain energy stored here and the strain energy stored here, and energy stored here, and that is written in this form.

Now the action integral is obtained in terms of the energy stored in the various elements and that is as shown here.

$$\bar{u}(x,t)=u(x,t)+\varepsilon_1\eta_1(x,t)\ \&\ \bar{v}(t)=v(t)+\varepsilon_2\eta_2(t)$$

$$A(\varepsilon_1,\varepsilon_2)=\int_0^L\int_{t_1}^{t_2}\left(\frac{1}{2}m(x)\dot{\bar{u}}^2(x,t)-\frac{1}{2}AE(x)\bar{u}'^2(x,t)\right)dxdt$$

$$+\int_{t_1}^{t_2}\left(\frac{1}{2}M\dot{\bar{v}}^2-\frac{1}{2}k_1\bar{u}^2(0,t)-\frac{1}{2}k_2\left[\bar{u}(L,t)-\bar{v}(t)\right]^2\right)dt$$

$$\left.\frac{\partial}{\partial\varepsilon_1}A(\varepsilon_1,\varepsilon_2)\right|_{\varepsilon_1=0,\varepsilon_2=0}=0$$

$$\Rightarrow\int_0^L\int_{t_1}^{t_2}\left(m(x)\dot{u}(x,t)\dot{\eta}_1(x,t)-AE(x)u'(x,t)\eta_1'(x,t)\right)dxdt+$$

$$+\int_{t_1}^{t_2}\left(-k_1u(0,t)\eta_1(0,t)-k_2\left[u(L,t)-v(t)\right]\eta_1(L,t)\right)dt=0$$

$$\left.\frac{\partial}{\partial\varepsilon_2}A(\varepsilon_1,\varepsilon_2)\right|_{\varepsilon_1=0,\varepsilon_2=0}=0\Rightarrow\int_{t_1}^{t_2}M\dot{v}(t)\dot{\eta}_2(t)dt+\int_{t_1}^{t_2}k_2\left[u(L,t)-v(t)\right]\eta_2(t)dt=0$$

Now in this example I will run through the various steps again so we will assume that U(X,T) and V(T) are the unknown optimal solutions and U Bar (X,T) and V Bar (T) are the class of admissible solutions which are obtained by adding the variation, one variation ETA1 X,T other one ETA 2 (T) 2 U and V respectively. So now the action integral is now parameterize in terms of the two parameters epsilon 1 and epsilon 2 and we get this equation. Now that we know that the action integral would reach the stationary value, the necessary condition for that is the gradient with respect to epsilon 1 must vanish, and gradient with respect to epsilon 2 must vanish, and we know beforehand that this stationarity condition will be realized when epsilon 1 is 0, and epsilon 2 is 0, and similarly epsilon 1 and 2 are 0 because by definition U and V are the unknown optimal solutions. Now if I implement these two requirements we will get these two equations and here we have now terms involving time derivative of the variation and special derivative of the variation, the time derivative appears here as well as here in the second equation, so we will deal with these three integrals first and then come to the other part of the equation.

Consider

$$\int_{t_1}^{t_2} m(x)\dot{u}(x,t)\dot{\eta}_1(x,t)dt = \left[m(x)\dot{u}(x,t)\eta_1(x,t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m(x)\ddot{u}(x,t)\eta_1(x,t)dt$$

$$\int_0^L AE(x)u'(x,t)\eta_1'(x,t)dx = \left[AE(x)u'(x,t)\eta_1(x,t)\right]_0^L - \int_0^L \left\{AE(x)u'(x,t)\right\}'\eta_1(x,t)dx$$

$$\int_{t_1}^{t_2} M\dot{v}(t)\dot{\eta}_2(t)dt = \left[M\dot{v}(t)\eta_2(t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} M\ddot{v}(t)\eta_2(t)dt$$

We have, by definition of $\eta_1(x,t)=0$, $\eta_2(t)=0 @ t=t_1 \,\&\, t=t_2$

$$\Rightarrow \left[m(x)\dot{u}(x,t)\eta_1(x,t)\right]_{t_1}^{t_2}=0 \,\&\, \left[M\dot{v}(t)\eta_2(t)\right]_{t_1}^{t_2}=0$$

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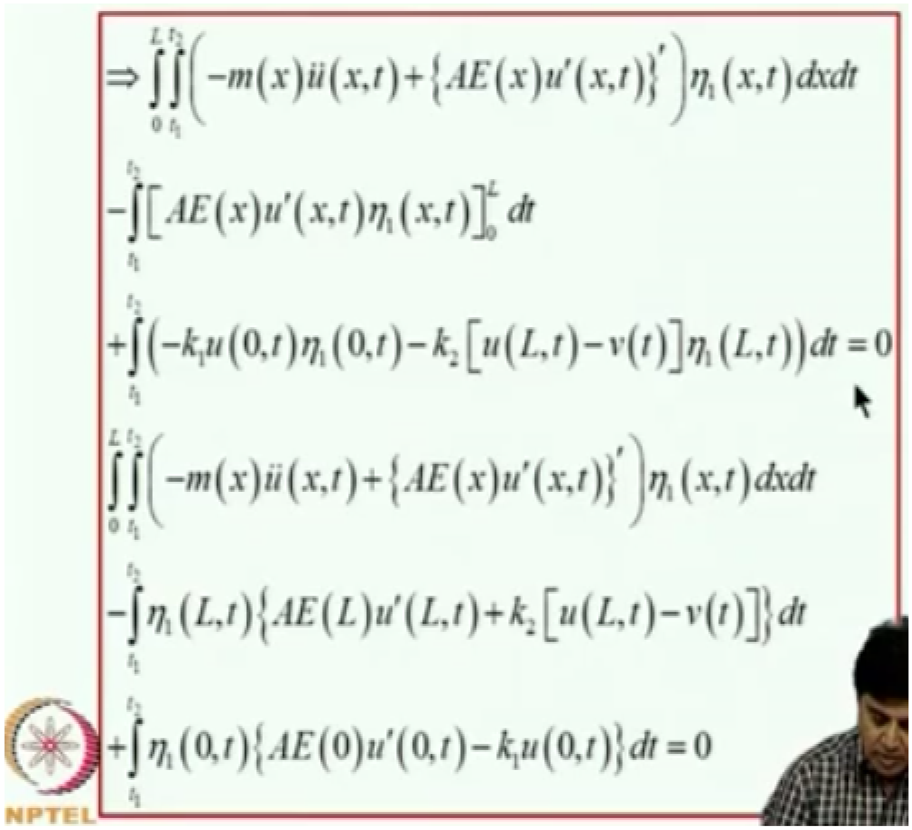
Suppose if you now consider T1 to T2, MU dot, ETA 1 dot, DT, I will integrate by parts and I will get the first term is this, and the second term is this, similarly on terms involving special derivative of the variation, we again integrate by parts, this is the first time I get, this is the second term. The second equation for the variable V dot (T) there is a ETA 2 dot (T) and integration by parts now leads to these two terms, now ETA 1 X, T and ETA 2 of T are arbitrary variations, and ETA 2 (T) is 0 at T1 and T2, so what happens is these two, two of these terms go to 0, see these terms are appearing here, and here, these two are 0, now we need to deal with the other terms.



Now reverting back to the original equation I get now the statement for stationarity of EI as this equation, and we have to now collect terms involving ETA 1 (X,T) which is done here, ETA 1 (0,T) and ETA 1 (L,T), and if we organize these terms we will get the field equation, since ETA 1 (X, T) is arbitrary, ETA 1 (L,T) is arbitrary, ETA 1 (0,T) is arbitrary, the only way this sum can be 0 for all ETA is that individual terms must be 0 and using that condition I get this as the field equation, and these two as the appropriate boundary conditions, so these boundary

$$\Rightarrow -m(x)\ddot{u}(x,t)+\left\{AE(x)u'(x,t)\right\}'=0$$

$$AE(L)u'(L,t)+k_2\left[u(L,t)-v(t)\right]=0$$

$$AE(0)u'(0,t)-k_1u(0,t)=0$$

Similarly

$$\int_{t_1}^{t_2} M\dot{v}(t)\dot{\eta}_2(t)\,dt+\int_{t_1}^{t_2} k_2\left[u(L,t)-v(t)\right]\eta_2(t)\,dt=0\Rightarrow$$

$$\int_{t_1}^{t_2}\left\{-M\ddot{v}(t)+k_2\left[u(L,t)-v(t)\right]\right\}\eta_2(t)\,dt=0\Rightarrow M\ddot{v}(t)+k_2\left[v(t)-u(L,t)\right]=0$$

$$M\ddot{v}+k_2\left[v-u(L,t)\right]=0$$

$$\left.\begin{aligned}&(AEu')'-m\ddot{u}=0\\&AEu'(0,t)-k_1u(0,t)=0\\&AEu'(L,t)+k_2\left[u(L,t)-v(t)\right]=0\end{aligned}\right\}+\text{ICs}$$

NPTEL

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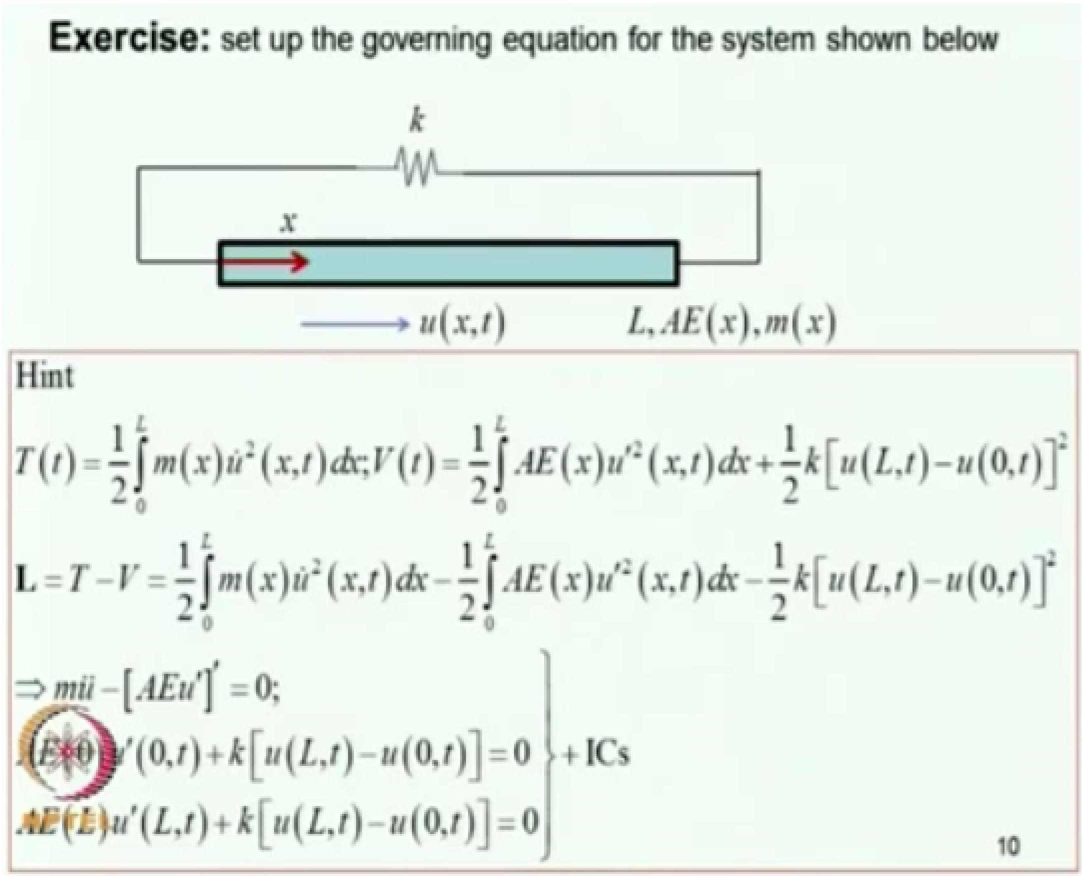
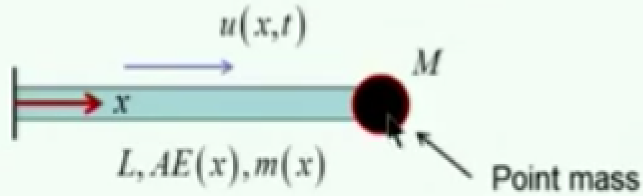
conditions for example at X = L, and now involve the force transferred through this spring, and X = 0 the force transferred through K1 as you can see here.

Now similarly for the point mass, if we do this second, if we use the second equation we get the governing equation to be this, this we have seen earlier for a simple single degree freedom system so the final equations these two are the field equations, these two are the boundary conditions, and we have to specify the appropriate initial conditions to solve the problem.



So the point that we can take from this exercise is that the variational method helps you to correctly identify the boundary conditions along with the derivation of the field equation, so I will run through similar, a few more examples I will not work out all the steps in detail, but I will mention, so this is one example where a spring K connects the two ends of the rod, so now strain energy will be given by strain energy stored in the bar and the energy stored in the spring as shown here, this is the strain energy stored in the bar, and this is the energy stored in the spring, this is a kinetic energy, this is only the energy stored in the bar, so this is the Lagrangian and if we apply now the variational argument we get the field equation to be this, this remains same as what we got earlier but the boundary conditions now at X = 0 will have this additional term which is the force transferred through this spring at this end, and similarly at X = L, I will get another force which is the force transferred at this end through the spring.

Exercise: set up the governing equation for the system shown below



Hint

$$T(t)=\frac{1}{2}\int_0^L m(x)\dot{u}^2(x,t)dx+\frac{1}{2}M\dot{u}^2(L,t)\;\&\;V(t)=\frac{1}{2}\int_0^L AE(x)u'^2(x,t)dx$$

$\Rightarrow$

$$\left.\begin{aligned} &m\ddot{u}-\left[AEu'\right]'=0\\ &u(0,t)=0\\ &AEu'(L,t)+M\ddot{u}(L,t)=0 \end{aligned}\right\}+\text{ICs}$$

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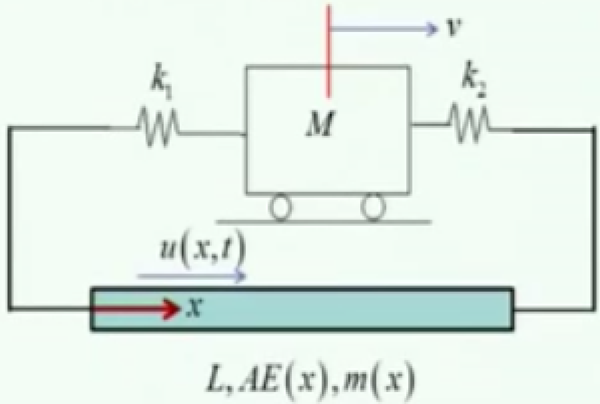
Now how about an inertial element being present at the boundary, suppose if I have now a point mass at the end, and we again now the kinetic energy will consist of the energy stored in the bar plus the kinetic energy of this mass which is given here, and the strain energy will be the energy stored in the bar so this is this expression, so if you apply the Hamilton's principle we will get this as a field equation, now at this end the bar is fixed, this is the boundary condition, at the other end now the axial thrust due to elastic action has now add, there is an additional term which is due to the inertia of this mass, so that will appear in the boundary condition so along with this of course we need to specify the appropriate initial conditions.

Exercise: set up the governing equation for the system shown below



Hint

$$T(t)=\frac{1}{2}\int_0^L m(x)\dot{u}^2(x,t)\,dx+\frac{1}{2}M\dot{v}^2$$

$$V(t)=\frac{1}{2}\int_0^L AE(x)u'^2(x,t)\,dx+\frac{1}{2}k_1\left[u(0,t)-v(t)\right]^2+\frac{1}{2}k_2\left[u(L,t)-v(t)\right]^2$$

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Now slightly more involved example, now there are two springs and another inertial element this is an exercise that you need to do, the hints are given here, this is the expression for the kinetic energy, this is expression for the potential energy, and I have worked out few steps if you work through this you will get the field equation, the equation of motion for the discrete

$$A=\int_{t_1}^{t_2}\int_0^L\left(\frac{1}{2}m(x)\dot{u}^2(x,t)-\frac{1}{2}AE(x)u'^2(x,t)\right)dxdt+\int_{t_1}^{t_2}\frac{1}{2}M\dot{v}^2dt$$

$$-\int_{t_1}^{t_2}\left\{\frac{1}{2}k_1\left[u(0,t)-v(t)\right]^2+\frac{1}{2}k_2\left[u(L,t)-v(t)\right]^2\right\}dt$$

$$\bar{u}(x,t)=u(x,t)+\varepsilon_1\eta_1(x,t)\;\&\;\bar{v}(t)=v(t)+\varepsilon_2\eta_2(t)$$

$$\Rightarrow A(\varepsilon_1,\varepsilon_2)=\int_{t_1}^{t_2}\int_0^L\left(\frac{1}{2}m(x)\dot{\bar{u}}^2(x,t)-\frac{1}{2}AE(x)\bar{u}'^2(x,t)\right)dxdt+\int_{t_1}^{t_2}\frac{1}{2}M\dot{\bar{v}}^2dt$$

$$-\int_{t_1}^{t_2}\left\{\frac{1}{2}k_1\left[\bar{u}(0,t)-\bar{v}(t)\right]^2+\frac{1}{2}k_2\left[\bar{u}(L,t)-\bar{v}(t)\right]^2\right\}dt$$

$$\frac{\partial}{\partial\varepsilon_1}A(\varepsilon_1,\varepsilon_2)\Big|_{\varepsilon_1=0,\varepsilon_2=0}=0\;\&\;\frac{\partial}{\partial\varepsilon_2}A(\varepsilon_1,\varepsilon_2)\Big|_{\varepsilon_1=0,\varepsilon_2=0}=0\Rightarrow$$

$$\left.\begin{aligned}&M\ddot{v}+k_1\left[v-u(0,t)\right]+k_2\left[v-u(L,t)\right]=0\\&m\ddot{u}-\left[AEu'\right]'=0\\&-AEu'(L,t)+k_2\left[v-u(L,t)\right]=0\;\&\;AEu'(0,t)+k_1\left[v-u(0,t)\right]=0\end{aligned}\right\}+\text{ICs}$$

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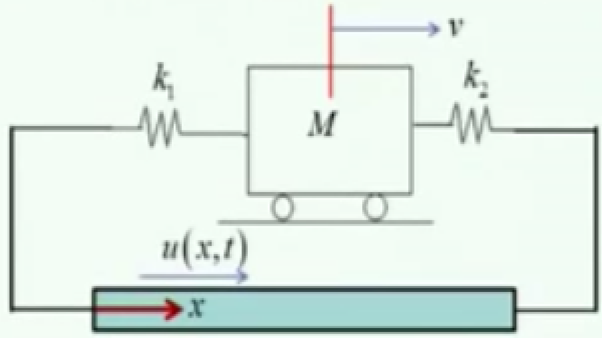
mass that is this is given by this and for the bar this is this, and the boundary conditions as you

Exercise: set up the governing equation for the system shown below



$L, AE(x), m(x)$

Hint

$$T(t)=\frac{1}{2}\int_0^L m(x)\dot{u}^2(x,t)dx+\frac{1}{2}M\dot{v}^2$$

$$V(t)=\frac{1}{2}\int_0^L AE(x)u'^2(x,t)dx+\frac{1}{2}k_1\left[u(0,t)-v(t)\right]^2+\frac{1}{2}k_2\left[u(L,t)-v(t)\right]^2$$

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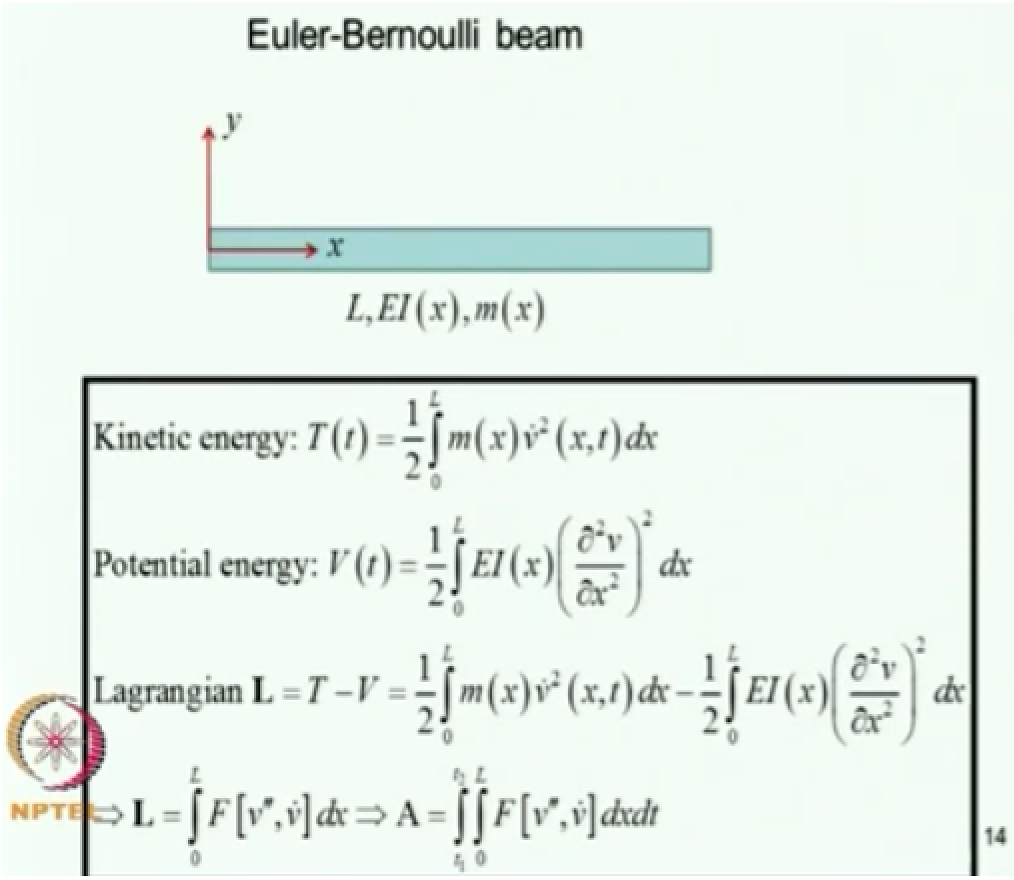
can expect there will be a term involving the force transferred through the K1 at this end and a term involving force transferred through K2 at this end, so that is what you see at L = T, there is a force through K2 and at L = 0, there is a force through K1.



Now this logic can be extended now to other structural elements like Euler-Bernoulli beam plates and plane stress elements and so on and so forth, so for purpose of illustration I will consider one example that is case of an Euler Bernoulli beam, so this beam has span L, flexural rigidity EI (X), and mass per unit length M (X), the kinetic energy is given by this expression M (X) into V dot square DX, I am considering the underlying beam theory is Euler-Bernoulli beam theory therefore there is no contribution to kinetic energy from rotary inertia, similarly when I write potential energy this is the strain energy due to bending and there is no contribution to this energy from shear deformation, so if you include rotary inertia and shear deformation we will get the corresponding formulation for a Timoshenko beam. But right now we will focus on Euler Bernoulli beam and this is the Lagrangian T – V, and this is the action integral, so we can see that the Lagrangian is now a function of V double prime that appears here, and V dot which appears here, so I can write action integral in terms of this function as F (v double prime v dot).

$$\bar{v}(x,t) = v(x,t) + \varepsilon\eta(x,t)$$

$$A(\varepsilon) = \int_{t_1}^{t_2}\int_0^L F\left[\bar{v}'', \dot{\bar{v}}\right] dx\,dt$$

$$\left.\frac{dA}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow \int_{t_1}^{t_2}\int_0^L \left\{\frac{\partial F}{\partial v''}\eta''(x,t) + \frac{\partial F}{\partial \dot{v}}\dot{\eta}(x,t)\right\} dx\,dt = 0$$

$$\int_0^L \frac{\partial F}{\partial v''}\eta''(x,t)\,dx = \left[\frac{\partial F}{\partial v''}\eta'(x,t)\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v''}\right)\eta'(x,t)\,dx$$

$$= \left[\frac{\partial F}{\partial v''}\eta'(x,t)\right]_0^L - \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v''}\right)\eta(x,t)\right]_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)\eta(x,t)\,dx$$

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \dot{v}}\dot{\eta}(x,t)\,dt = \left[\frac{\partial F}{\partial \dot{v}}\eta(x,t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)\eta(x,t)\,dt$$

$$\Rightarrow$$

$$\int_{t_1}^{t_2}\int_0^L \left\{\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)\right\}\eta(x,t)\,dx + dt\int_{t_1}^{t_2}\left(\left[\frac{\partial F}{\partial v''}\eta'(x,t)\right]_0^L - \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v''}\right)\eta(x,t)\right]_0^L\right)dt = 0$$

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Now as before we consider this to be the unknown optimal solution and this is the variation and this is the class of admissible function which I have now been parameterized in terms of Epsilon, so A as a function of epsilon is this, now we know that optimal value of A as a function of epsilon is reached to an epsilon equal to 0, so we apply this condition, so DA by D epsilon is Dou F by Dou epsilon which will be Dou F by Dou V double Prime into ETA double prime + Dou F by Dou V dot into ETA dot, so the bar we are not maintaining distinction between V Bar and V because this calculation is being done at epsilon equal to zero, and in which case V Bar and V are identical, so this is the equation. Now we again consider these two terms separately, so we have 0 to L, Dou F by Dou V double Prime ETA double prime X,T, so one integration by parts will lead to this equation, and on the second equation we will again perform another integration by parts so I will end up with these three terms. Similarly the integral, if you consider this term and consider integration from T1 to T2 we get again integration by parts, we get these two.

Now at T = T1 and T2 the variation is 0, so this term drops off, so we have to still interpret the other terms which will still remain till we complete their interpretation so we carry them further in our interrogation and we get, this is now the equation which is a condition for stationarity of A.

$$\int_{t_1}^{t_2}\int_0^L \left\{\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)\right\}\eta(x,t)\,dx+dt\int_{t_1}^{t_2}\left(\left[\frac{\partial F}{\partial v''}\eta'(x,t)\right]_0^L-\left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v''}\right)\eta(x,t)\right]_0^L\right)dt=0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)=0$$

$$\left[\frac{\partial F}{\partial v''}\eta'(x,t)\right]_0^L=0\ \&\ \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v''}\right)\eta(x,t)\right]_0^L=0$$

$$F=\frac{1}{2}\left(m\dot{v}^2-EIv''^2\right)$$

$$\frac{\partial F}{\partial v''}=-EIv''\Rightarrow\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)=-\left(EIv''\right)''$$

$$\frac{\partial F}{\partial \dot{v}}=m\dot{v}\Rightarrow\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)=m\ddot{v}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)=0\Rightarrow\left(EIv''\right)''+m\ddot{v}=0\ \text{[Field equation]}$$

$$\left[EIv''\eta'(x,t)\right]_0^L=0\ \&\ \left[\left(EIv''\right)'\eta(x,t)\right]_0^L=0\ \text{[Boundary conditions]}$$

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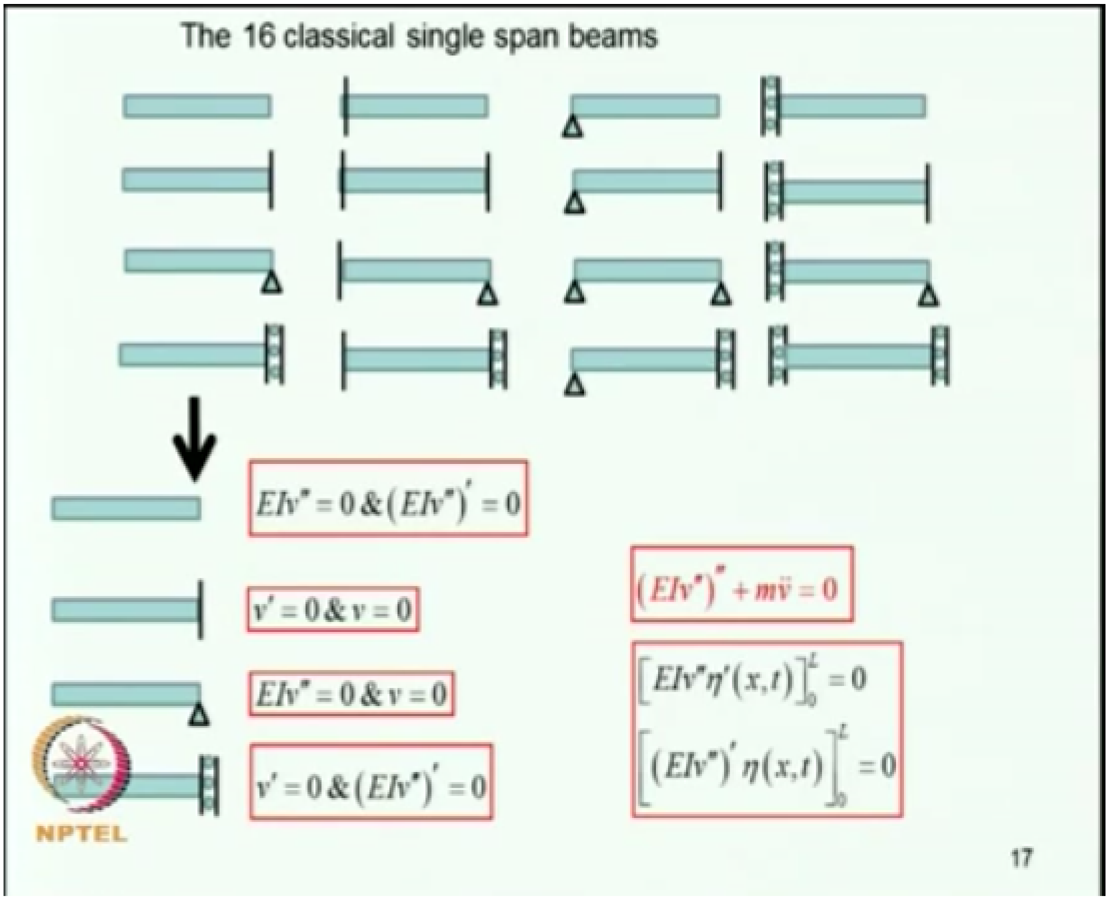
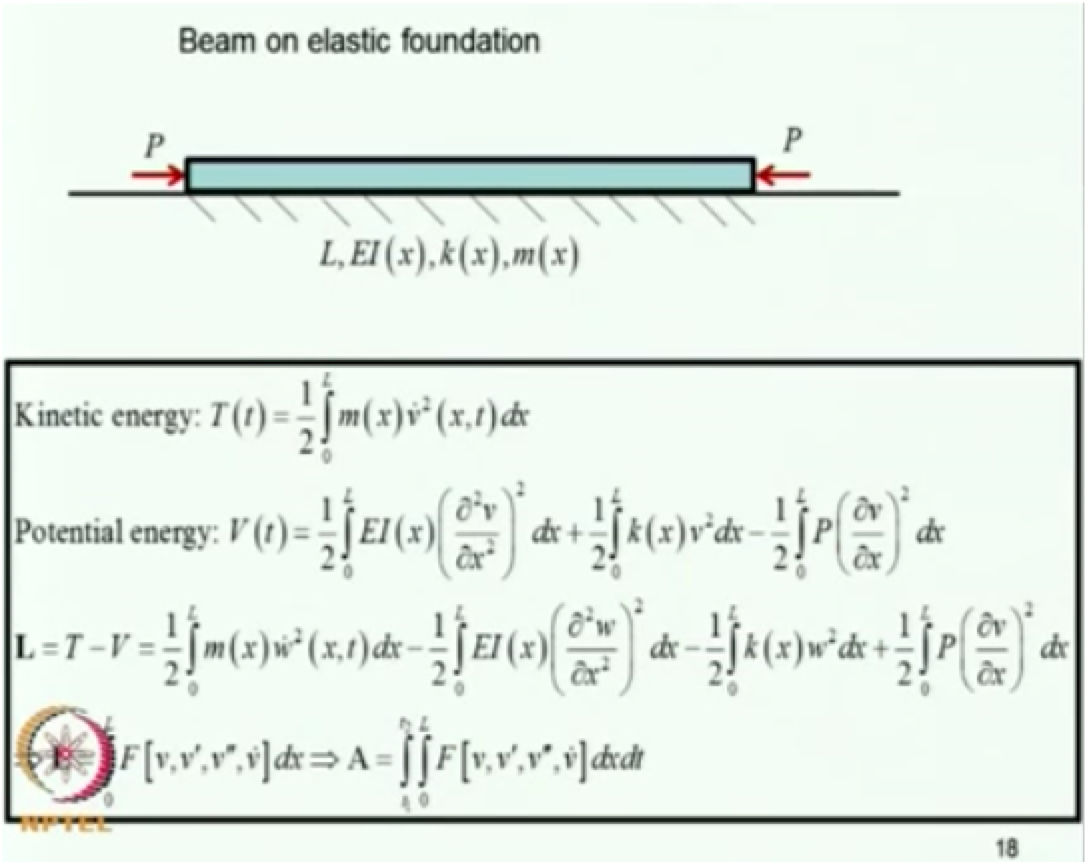
Now we'll again you know use the arguments that variation is arbitrary therefore the only way this term can be equal to 0 is when each of these terms are individually equal to 0, so since ETA (X,T) is arbitrary, the term inside this brace must be equal to 0 I get this, and similarly term inside the bracket must be 0, and term inside this bracket must be zero. So now this is still F is remaining, the notation for F is, F is this, so if you now insert all this calculation you know Dou F by Dou V double prime, Dou F by Dou V dot etcetera in this you will get the field equation which is the well-known Euler Bernoulli beam equation, this is a field equation, now the boundary conditions are to be derived from these conditions. Now at free end when displacements are not specified we will not be able to assign any value to the variation therefore the term that multiplies that must be equal to 0, so at a free end neither displacement can be specified nor slope can be specified, therefore for this term to be equal to 0 it is natural that this multiplier must be equal to 0.

So on the other hand at the, if X = 0 or L if displacements are specified, for example in a simply supported beam the translation is specified to be 0 in which case this will be 0 and similarly at the fixed end the slope will be 0, therefore this will be 0, the variation would be 0, so there are



now 16 combinations of one span beams that are possible and I've just displayed them here, this is both ends are free, this is one end is fixed, other end free and so on and so forth. So now we can examine suppose we are focusing on one of these boundaries and I look at this boundary and consider these expressions we can see here that the slope cannot be specified therefore EIV double prime must be equal to 0, we know that EIV double prime represents a bending moment, bending moment is 0, and similarly the shear force is 0 which comes from this equation because displacement cannot be specified so this is 0, at the clamped end we know that the displacement is 0, therefore the boundary conditions emerge as V prime = 0, V = 0 which again satisfies this equation because the variation must satisfy these conditions. Similarly it is simply supported end the bending moment is 0, and the displacement is 0, at this roller end the slope is 0 and shear force is 0, so therefore see that consistent with the variational formulation we can think of 16 possible single span beams and we can set up all the appropriate boundary conditions.



Now slightly more involved example supposed an Euler Bernoulli beam is resting on a Winkler's foundation and it also carries an axial load P, now what is the kinetic energy? The kinetic energy will be due to flexure of the beam and this is given by this, the potential energy now will be due to the flexure of the beam and due to the elastic foundation and contribution

$$A = \int_{t_1}^{t_2}\int_{0}^{L} F\left[v, v', v'', \dot{v}\right] dx\,dt$$

Euler-Lagrange's equation

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right) = 0$$

$$F = m\dot{v}^2 - EIv''^2 - kv^2 + Pv'^2$$

$$\Rightarrow$$

Field equation: $m\ddot{v} + \left[EIv''\right]'' + Pv'' + kv = 0$

Boundary conditions: $\left[EIv''\right]' + Pv' = 0 \;\&\; EIv'' = 0$ at $x = 0, L$

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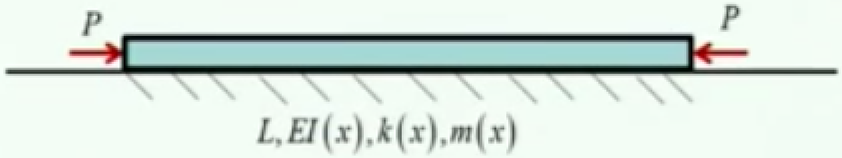
from axial loads, so if we write these expressions this is flexure, this is from Winkler Foundation, and this is due to the axial compression load, so we can form the Lagrangian and again construct the action integral.

Beam on elastic foundation



Kinetic energy: $T(t)=\frac{1}{2}\int_0^L m(x)\dot{v}^2(x,t)\,dx$

Potential energy: $V(t)=\frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx+\frac{1}{2}\int_0^L k(x)v^2dx-\frac{1}{2}\int_0^L P\left(\frac{\partial v}{\partial x}\right)^2 dx$

$$\mathbf{L}=T-V=\frac{1}{2}\int_0^L m(x)\dot{w}^2(x,t)\,dx-\frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx-\frac{1}{2}\int_0^L k(x)w^2dx+\frac{1}{2}\int_0^L P\left(\frac{\partial v}{\partial x}\right)^2 dx$$

$$\Rightarrow \mathbf{L}=\int_0^L F\left[v,v',v'',\dot{v}\right]dx\Rightarrow \mathbf{A}=\int_{t_1}^{t_2}\int_0^L F\left[v,v',v'',\dot{v}\right]dx\,dt$$

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Now the F function that is the Lagrangian now has, because of Winkler Foundation and the axial loads we have now in addition to V double Prime and V Prime we also have V and V Prime,

$$A = \int_{t_1}^{t_2}\int_0^L F\left[v, v', v'', \dot{v}\right] dx\,dt$$

Euler-Lagrange's equation

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right) = 0$$

$$F = m\dot{v}^2 - EIv''^2 - kv^2 + Pv'^2$$

$$\Rightarrow$$

Field equation: $m\ddot{v} + \left[EIv''\right]'' + Pv'' + kv = 0$

Boundary conditions: $\left[EIv''\right]' + Pv' = 0 \,\&\, EIv'' = 0$ at $x = 0, L$

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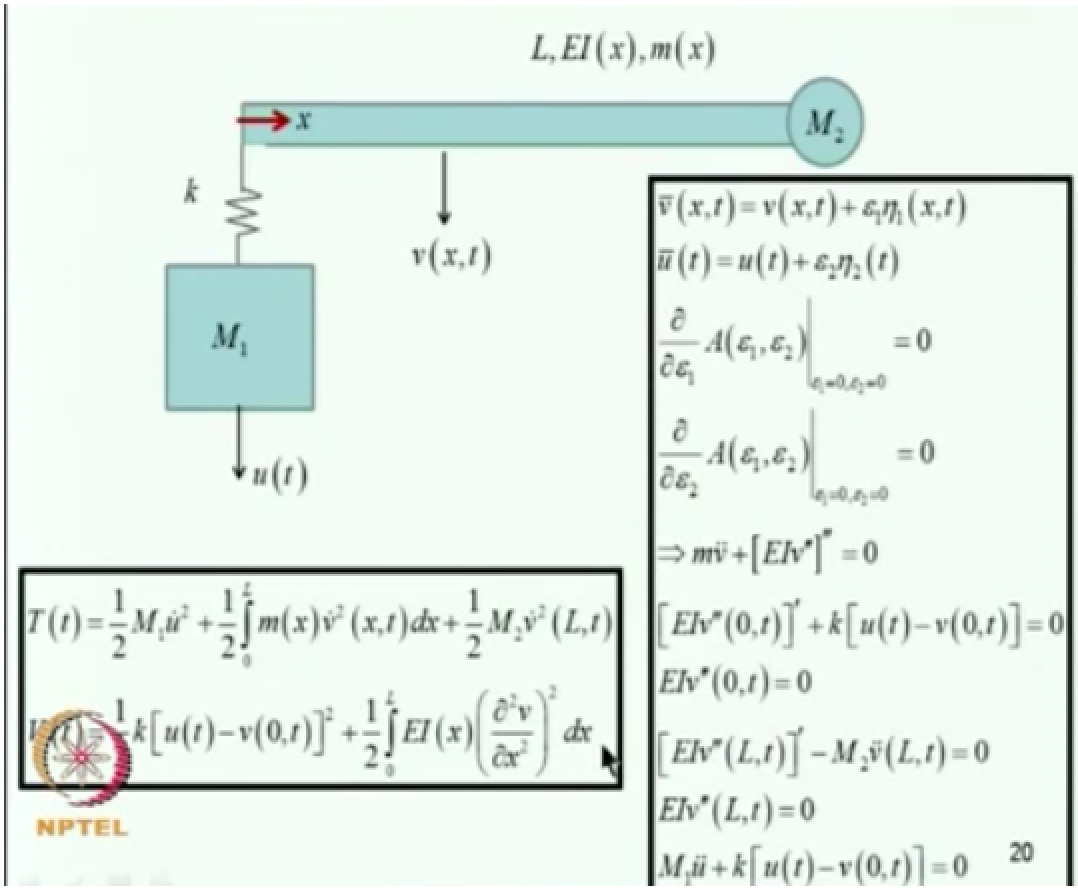
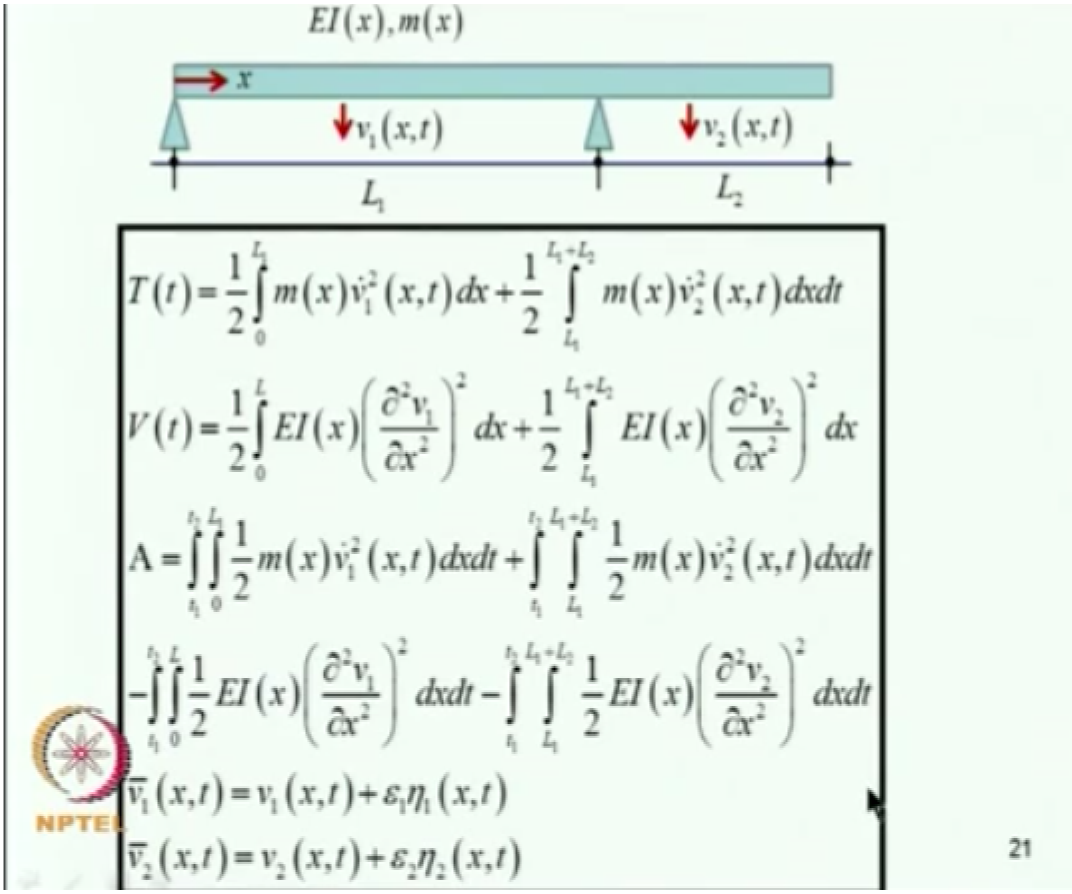
so we can do this, we get this as Euler Lagrange equation and the field equation turns out to be this, and the boundary conditions now we'll involve for example at the two ends the axial thrust will appear in the boundary conditions and the bending moment would be 0 at the free ends, okay, this is the shear force at the two ends, okay, so I leave the intermediate steps for you to work out.



So now a beam carrying some inertial elements and a spring this again helps you to you know appreciate different facets of this formulation so the kinetic energy now has contribution from this mass, contribution from this beam, and from this mass M2, M1 and M2 are point masses so this is the expression for the kinetic energy U (T) is the displacement of this mass and V (X,T) is the displacement of the beam. The potential energy now has the one that is stored in the beam and the one that is there in the spring and we get this, so now you apply the variational formulation you will get one equation for U, another equation for V, and the equation for V reads as shown here, and the equation for U is this, and the boundary conditions at X = 0, we can see that bending moment emerges to be 0, but the shear force will have contributions from at this end from this mass and at this end there will be a shear force contribution from this spring, so these are the appropriate boundary conditions.



Now how about a beam with a overhang, so there are now three places where we need to worry about the boundary conditions, so what I do is I call displacement field in this portion as V1 and in this portion as V2 and we will assume that EI and M are functions of X as X varies from 0 to L1 + L2, so the kinetic energy now has contribution from this segment of the beam 0 to L1 and L1 to L1 + L2, similarly the strain energy flexure in this part plus flexure in this part, the action integral is now therefore T - V is this expression, so now I will take V1 to be a class of parameterize admissible, set of functions is formed for both V1 and V2 as shown here, okay?

$$A(\varepsilon_1,\varepsilon_2)=\int_{t_1}^{t_2}\int_{0}^{L_1}\frac{1}{2}m(x)\dot{\bar{v}}_1^2(x,t)\,dx\,dt+\int_{t_1}^{t_2}\int_{L_1}^{L_1+L_2}\frac{1}{2}m(x)\dot{\bar{v}}_2^2(x,t)\,dx-$$

$$\int_{t_1}^{t_2}\int_{0}^{L_1}\frac{1}{2}EI(x)\left(\frac{\partial^2\bar{v}_1}{\partial x^2}\right)^2 dx-\int_{t_1}^{t_2}\int_{L_1}^{L_1+L_2}\frac{1}{2}EI(x)\left(\frac{\partial^2\bar{v}_2}{\partial x^2}\right)^2 dx$$

$$\frac{\partial}{\partial\varepsilon_1}A(\varepsilon_1,\varepsilon_2)\bigg|_{\varepsilon_1=0,\varepsilon_2=0}=0\ \&\ \frac{\partial}{\partial\varepsilon_2}A(\varepsilon_1,\varepsilon_2)\bigg|_{\varepsilon_1=0,\varepsilon_2=0}=0\Rightarrow$$

$$\int_{t_1}^{t_2}\int_{0}^{L_1}m(x)\dot{v}_1(x,t)\dot{\eta}_1(x,t)\,dx\,dt+\int_{t_1}^{t_2}\int_{L}^{L_1+L_2}m(x)\dot{v}_2(x,t)\dot{\eta}_2(x,t)\,dx\,dt-$$

$$\int_{t_1}^{t_2}\int_{0}^{L_1}EI(x)v_1''(x,t)\eta_1''(x,t)\,dx\,dt-\int_{t_1}^{t_2}\int_{L}^{L_1+L_2}EI(x)v_2''(x,t)\eta_2''(x,t)\,dx\,dt=0$$

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Now so this is the action integral in terms of epsilon 1 and epsilon 2, and these are the conditions for stationarity of this, so if we run through this calculation you will have to do this differentiation and the corresponding integration by parts and if you carry out these calculations

$$\Rightarrow -\int_{t_1}^{t_2}\int_{0}^{L_1}\left\{m(x)\ddot{v}_1(x,t)+\left[EI(x)v_1''(x,t)\right]''\right\}\eta_1(x,t)\,dx\,dt$$

$$-\int_{t_1}^{t_2}\int_{L_1}^{L_1+L_2}\left\{m(x)\ddot{v}_2(x,t)+\left[EI(x)v_2''(x,t)\right]''\right\}\eta_2(x,t)\,dx\,dt$$

$$-\int_{t_1}^{t_2}\left[EI(x)v_1''(x,t)\eta_1'(x,t)\right]_0^{L_1}dt-\int_{t_1}^{t_2}\left[EI(x)v_2''(x,t)\eta_2'(x,t)\right]_{L_1}^{L_1+L_2}dt$$

$$+\int_{t_1}^{t_2}\left[\left(EI(x)v_1''(x,t)\right)'\eta_1(x,t)\right]_0^{L_1}dt+\int_{t_1}^{t_2}\left[\left(EI(x)v_2''(x,t)\right)'\eta_2(x,t)\right]_{L_1}^{L_1+L_2}dt=0$$

The first two terms lead to

$$\Rightarrow m(x)\ddot{v}_1(x,t)+\left[EI(x)v_1''(x,t)\right]''=0;0<x<L_1\ \&$$

$$m(x)\ddot{v}_2(x,t)+\left[EI(x)v_2''(x,t)\right]''=0;L_1<x<L_1+L_2$$

we get the field equations for X varying from 0 to L1 and X varying from L1 to L1 + L2 separately and they are shown here. And now the boundary conditions you will have to consider what happens here, what happens here, and what happens here, so the required conditions of compatibility and equilibrium of forces etcetera is automatically satisfied in this formulation and we get I mean you can go through these steps, these are simple manipulations

$$-\int_{t_1}^{t_2}\left[EI(x)v_1''(x,t)\eta_1'(x,t)\right]_0^{L_1}dt-\int_{t_1}^{t_2}\left[EI(x)v_2''(x,t)\eta_2'(x,t)\right]_{L_1}^{L_1+L_2}dt$$

$$+\int_{t_1}^{t_2}\left[\left(EI(x)v_1''(x,t)\right)'\eta_1(x,t)\right]_0^{L_1}dt+\int_{t_1}^{t_2}\left[\left(EI(x)v_2''(x,t)\right)'\eta_2(x,t)\right]_{L_1}^{L_1+L_2}dt=0$$

$$\Rightarrow -EI(L_1)v_1''(L_1,t)\eta_1'(L_1,t)+EI(0)v_1''(0,t)\eta_1'(0,t)$$

$$-EI(L_1+L_2)v_2''(L_1+L_2,t)\eta_2'(L_1+L_2,t)+EI(L_1)v_2''(L_1,t)\eta_2'(L_1,t)$$

$$+\left(EI(L_1)v_1''(L_1,t)\right)'\eta_1(L_1,t)-\left(EI(0)v_1''(0,t)\right)'\eta_1(0,t)$$

$$+\left(EI(L_1+L_2)v_2''(L_1+L_2,t)\right)'\eta_2(L_1+L_2,t)-\left(EI(L_1)v_2''(L_1,t)\right)'\eta_2(L_1,t)=0$$

$$EI(0)v_1''(0,t)=0\ \&\ v_1(0,t)=0$$

$$EI(L_1+L_2)v_1''(L_1+L_2,t)=0\ \&\ \left(EI(L_1+L_2)v_2''(L_1+L_2,t)\right)'=0$$

$$-EI(L_1)v_1''(L_1,t)\eta_1'(L_1,t)+EI(L_1)v_2''(L_1,t)\eta_2'(L_1,t)+\left(EI(L_1)v_1''(L_1,t)\right)'\eta_1(L_1,t)$$

$$-\left(EI(L_1)v_2''(L_1,t)\right)'\eta_2(L_1,t)=0;\ -EI(L_1)v_1''(L_1,t)+EI(L_1)v_2''(L_1,t)=0$$

$$v_1'(L_1,t)=v_2'(L_1,t);\ v_1(L_1,t)=0;\ v_2(L_1,t)=0$$

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and we get the boundary conditions as shown here, so this is at X = 0, bending moment would

$$m(x)\ddot{v}_1(x,t)+\left[EI(x)v_1''(x,t)\right]''=0;0<x<L_1$$

$$m(x)\ddot{v}_2(x,t)+\left[EI(x)v_2''(x,t)\right]''=0;L_1<x<L_1+L_2$$

$$EI(0)v_1''(0,t)=0$$

$$v_1(0,t)=0$$

$$EI(L_1+L_2)v_1''(L_1+L_2,t)=0$$

$$\left(EI(L_1+L_2)v_2''(L_1+L_2,t)\right)'=0$$

$$-EI(L_1)v_1''(L_1,t)+EI(L_1)v_2''(L_1,t)=0$$

$$v_1'(L_1,t)=v_2'(L_1,t);$$

$$v_1(L_1,t)=0;$$

$$v_2(L_1,t)=0$$

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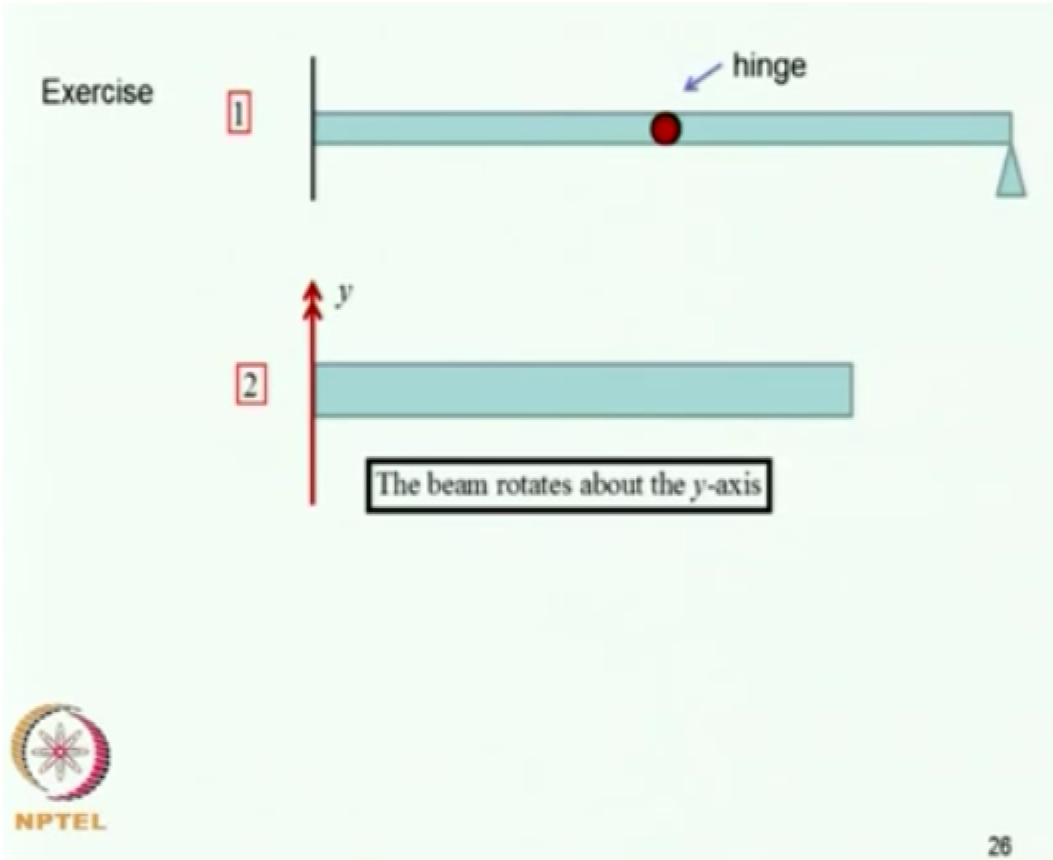
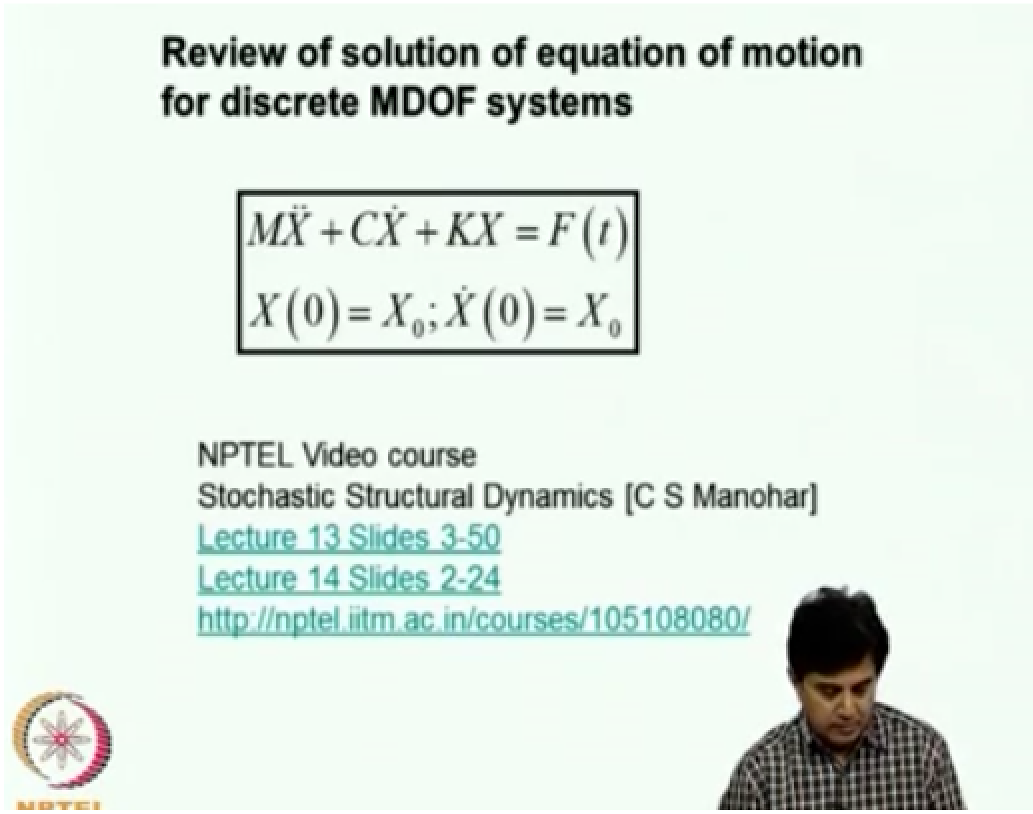
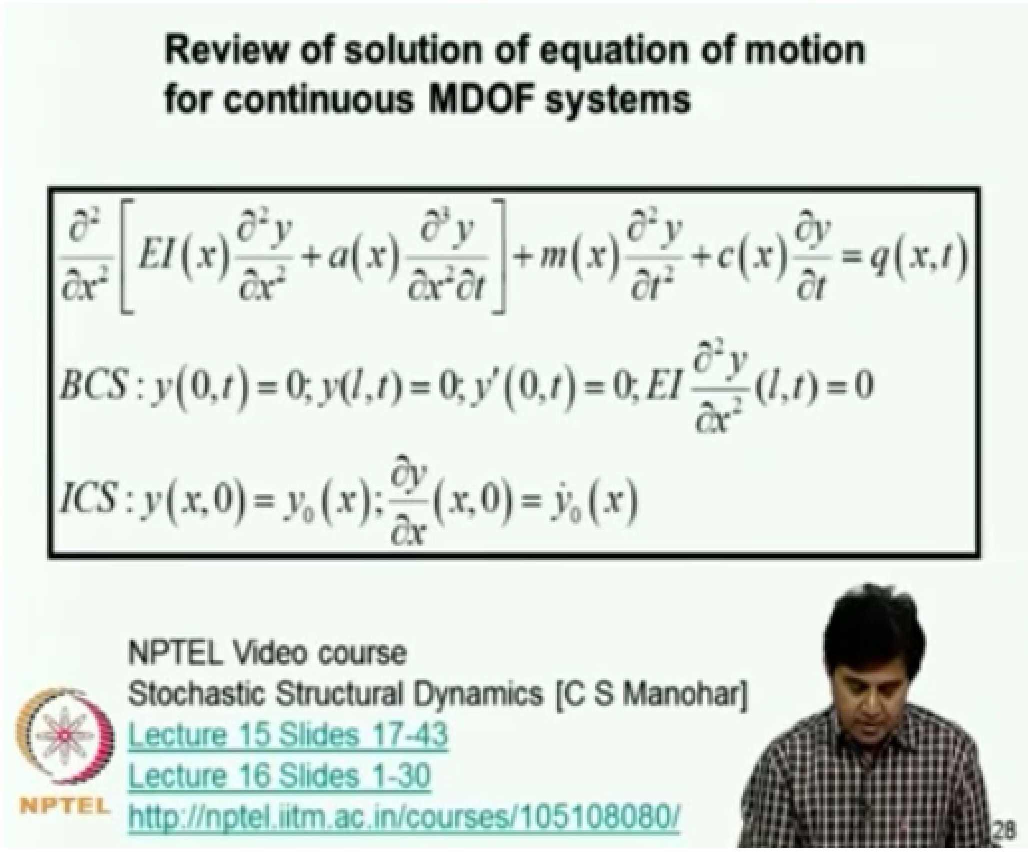
be 0 and displacement would be 0, here the bending moment would be 0, shear force would be 0, here what happens? So at X = 0 these are the boundary conditions at the X = L1 + L2, that is the right hand free end, this is the bending moment and shear force are 0, at the intermediate point this is the equilibrium of moments and this is compatibility and V1 and V2 must be equal to 0, right, so we have now 8 boundary conditions and 2 fourth order equations and along with that we should specify the required for initial condition, so this gives the formulation for a beam with or overhang.



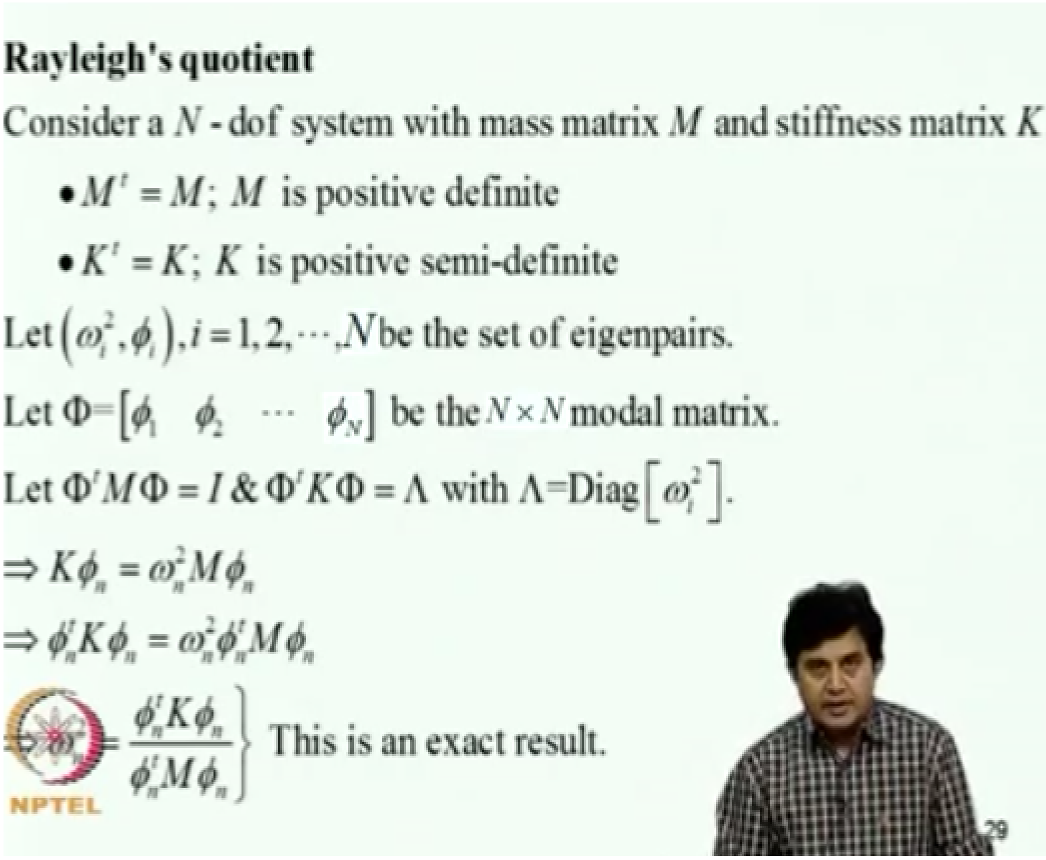
So I suggest two exercises you can consider a cantilever beam with a hinge in between you now formulate the equation of motion and obtain the appropriate boundary conditions, the other example is a beam that is spinning about this Y axis, okay, so you have to correctly write the expression for kinetic energy and the energy due to bending and derive the appropriate field equations and boundary conditions.



Now at this stage it is useful to quickly review the solution of equation of motion for discrete multi-degree freedom systems, so this is a generic form of equation of motion for linear systems, time invariant linear systems, MX double dot + CX dot + KX = F (T) X is displacement vector, M is mass matrix, C is damping matrix, K is the stiffness matrix, and these are the initial conditions. Now what I will, I don't intend to do this review as part of this course because this review is already available in another NPTEL course which I have taught and this is the website where this is available and if you go to lecture 13, slides 3 to 50, and lecture 14 slides 2 to 24 you will get this review, so I will assume that when you reach this point you will go to these lectures and familiarize yourself with the contents of this lecture before proceeding further as a part of this course.

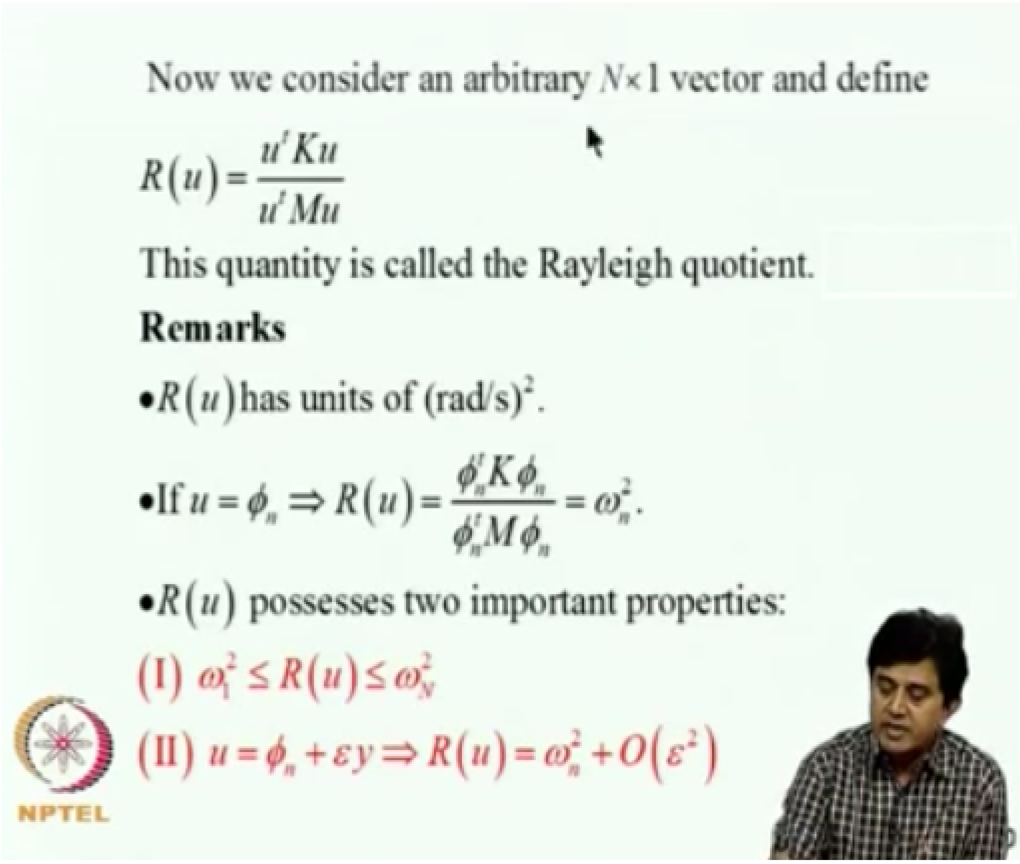


Similarly in the same course that I mentioned in lectures 15 and 16 there is a review of solution of equation of motion for continuous multi-degree freedom systems, specifically in the context of an Euler Bernoulli damped Euler Bernoulli beam and all the steps involved are again completely outlined here, they run over two lectures, two lectures, lecture 15 and 16 and these are the slide numbers, so again I presume that before you proceed further in this course you will visit these two lectures and familiarize yourself with them contents of those lectures.



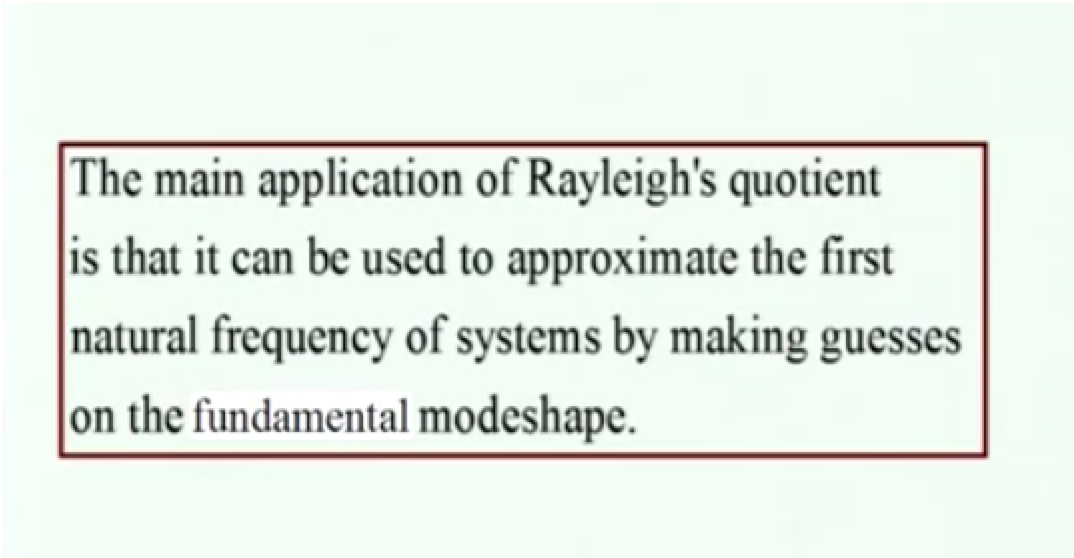
Now the next topic that I would like to discuss is known as Rayleigh's Quotient. Now the strategy that we are using right now is we want to first outline what are the approximate methods for analyzing vibrating systems, and we will outline a few methods and that will lead to, that will provide the motivation for us to understand how the finite element method is developed, so as a prelude to that will start with what is known as Rayleigh's quotient. Now if you consider N degree of freedom system with mass matrix M and stiffness matrix K, where M is symmetric and positive-definite, K is symmetric and positive semi-definite we will denote by Omega I and Phi I, the ayat Eigenpair so these are obtained by solving the eigenvalue problem associated with these two matrices, and phi is the modal matrix so it is the matrix of the eigenvectors. So this is N cross N matrix each column represents one eigenvector, and I assume that these eigenvectors have been normalized so that Phi transpose M Phi's an identity matrix that means these normal modes are mass normalized and phi transpose K Phi is a diagonal matrix where the diagonal entry is omega I square, okay, so these are the orthogonality relations and the normalization conditions used.

So if you take now the nth eigenvalue and nth eigenvector the equation that it satisfies is given by this. Now if you pre multiply by Phi N transpose and on both sides I get this equation and K is N by N matrix, Phi N is a N cross 1 matrix, so this will be a scalar because this is 1 Cross N, N cross N, N cross 1, therefore this is a scalar, so I can divide, write this expression for Omega N square as Phi N transpose K Phi N divided by Phi N transpose M Phi N, so this is an exact result, okay. Now instead of considering this nth eigenvector, that eigenvector I will now replace by an arbitrary N cross 1 vector, I form the ratio U transpose KU divided by U

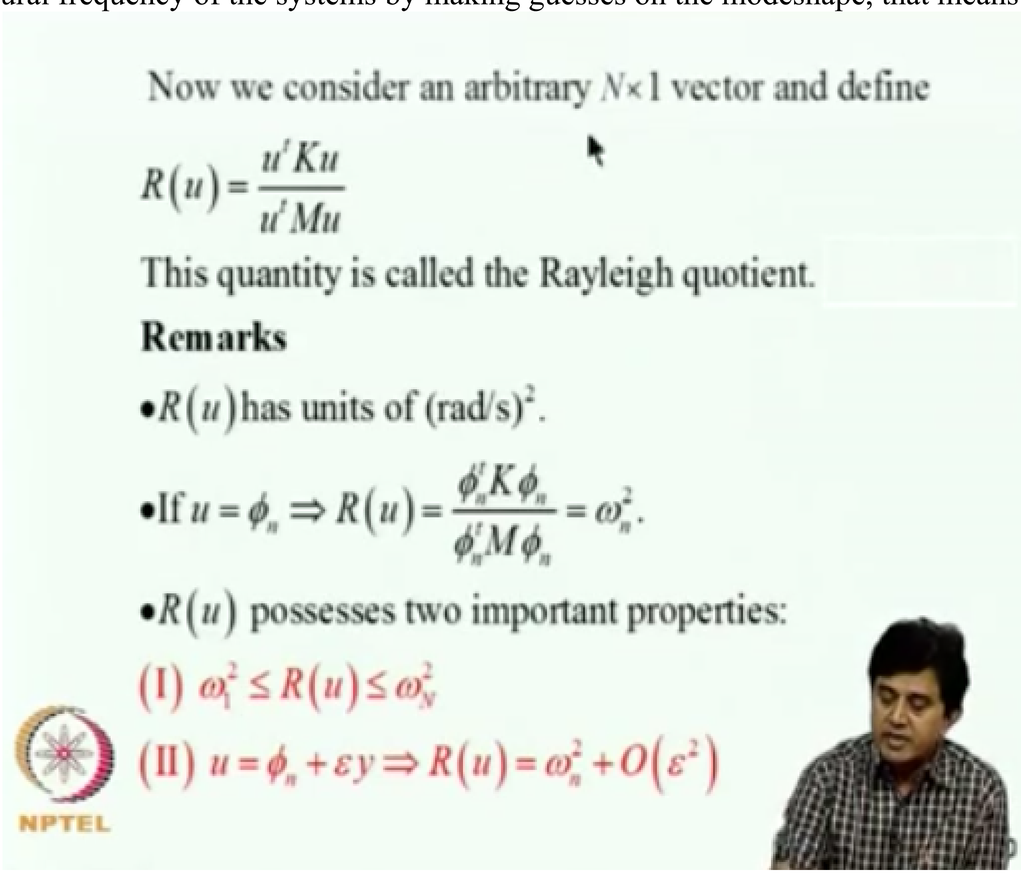


transpose MU, but definition I call this quantity as Rayleigh's quotient, so clearly we denote it as R(U), U is a vector and it has clearly the units of radian per second whole square.

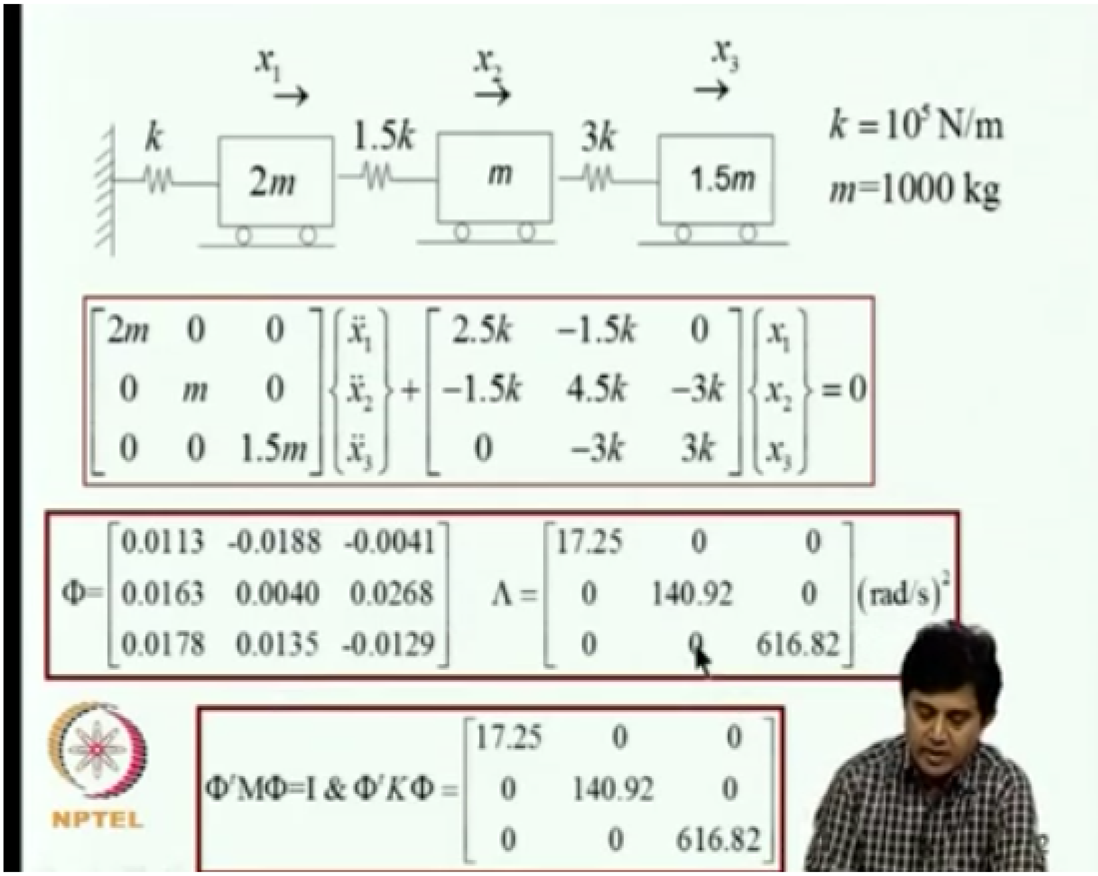
Now if you were to coincide with the nth eigenvector R(U) would be simply the nth natural frequency, okay, now why we are interested in this quantity? This quantity has two important properties namely this is bounded between the first natural frequency, first eigenvalue and the last eigenvalue, and also the second property says if U is in the neighborhood of Phi N, if U is Phi N + epsilon Y, where Y is a vector, which is perturbation, R(U) which is the eigenvalue will be in the neighborhood of omega N square, but this correction is order epsilon square, the error in the vector U, the error I mean the departure from the true eigenvector if it is order epsilon, the error in the eigenvalue will be of the order epsilon square, so this helps us to evaluate the eigenvalue is an approximate manner by making guesses on the eigenvector, so if you make an error of order epsilon in making the guess, you will end up with an order epsilon square in your estimate of that eigenvalue, so this is an advantage, so we'll see with a few examples and also I will now show why those two statements are true.



Actually the main application of Rayleigh's quotient is that it can be used to approximate the first natural frequency of the systems by making guesses on the modeshape, that means we are



going to use this part of the result, so if my objective is to estimate the first natural frequency I will start by making a guess on the first modeshape and substitute that in this expression and I will get an estimate for the first natural frequency, okay.



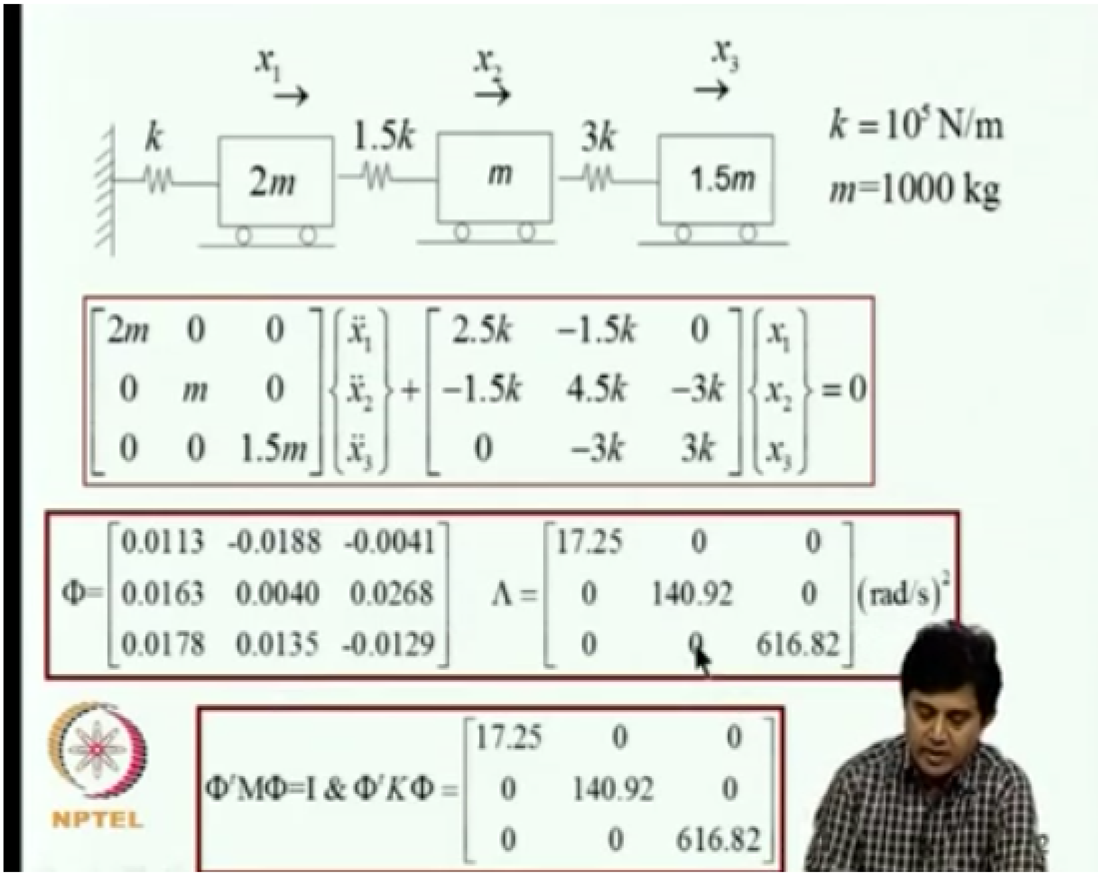
Now we will see that through an example, suppose if you consider now a three degree freedom system and these are the parameters you can easily write the equation of motion, so this is the equation of motion, this is a mass metric, this is the stiffness matrix, and if you carry out the eigenvalue analysis you will see that the modal vector is, modal matrix is given by this, this is the diagonal matrix of squares of natural frequencies and this modal matrix is mass normalized you can verify that, and the Phi transpose K Phi is nothing but this lambda, so this is an exact result.

u	$R(u)$
$(1 \quad 1 \quad 1)^t$	22.22
$(1 \quad 2 \quad 3)^t$	28.20
$(1.1 \quad 1.6 \quad 2)^t$	18.80
$(1 \quad -1 \quad -1)^t$	422.22

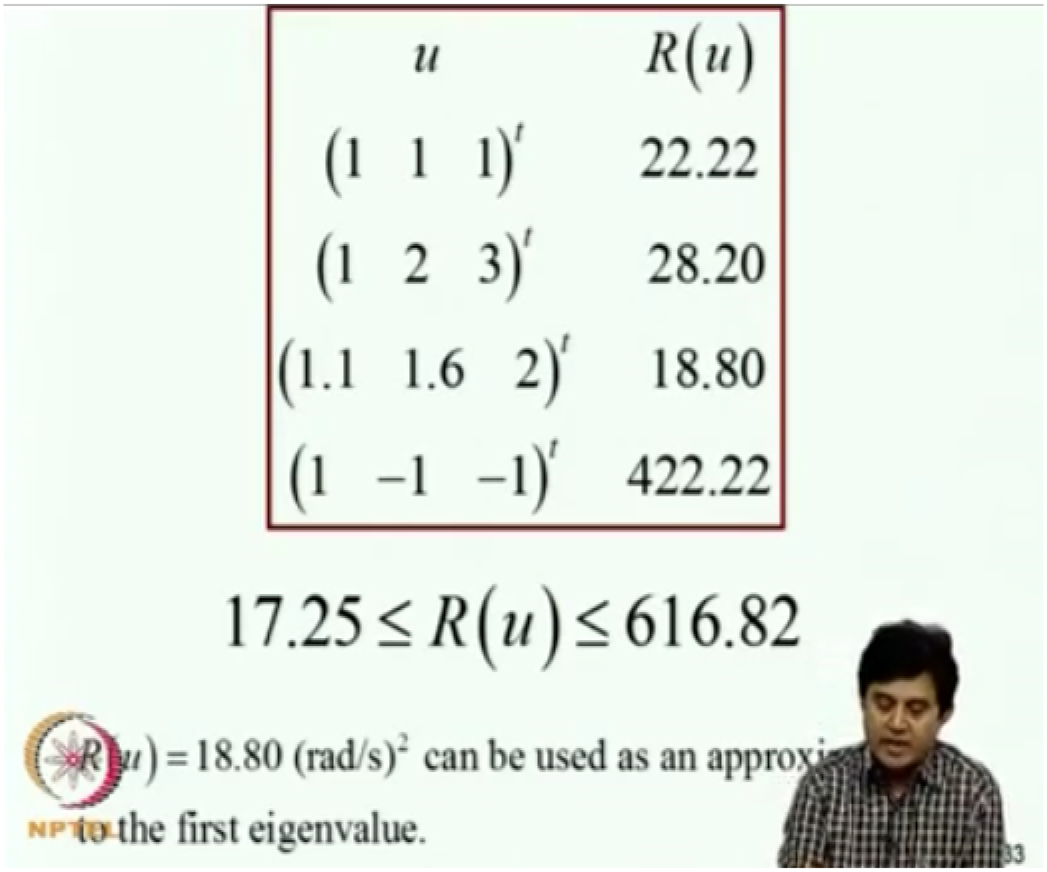
$$17.25 \le R(u) \le 616.82$$

$R(u) = 18.80\ (\text{rad/s})^2$ can be used as an approxi

to the first eigenvalue.

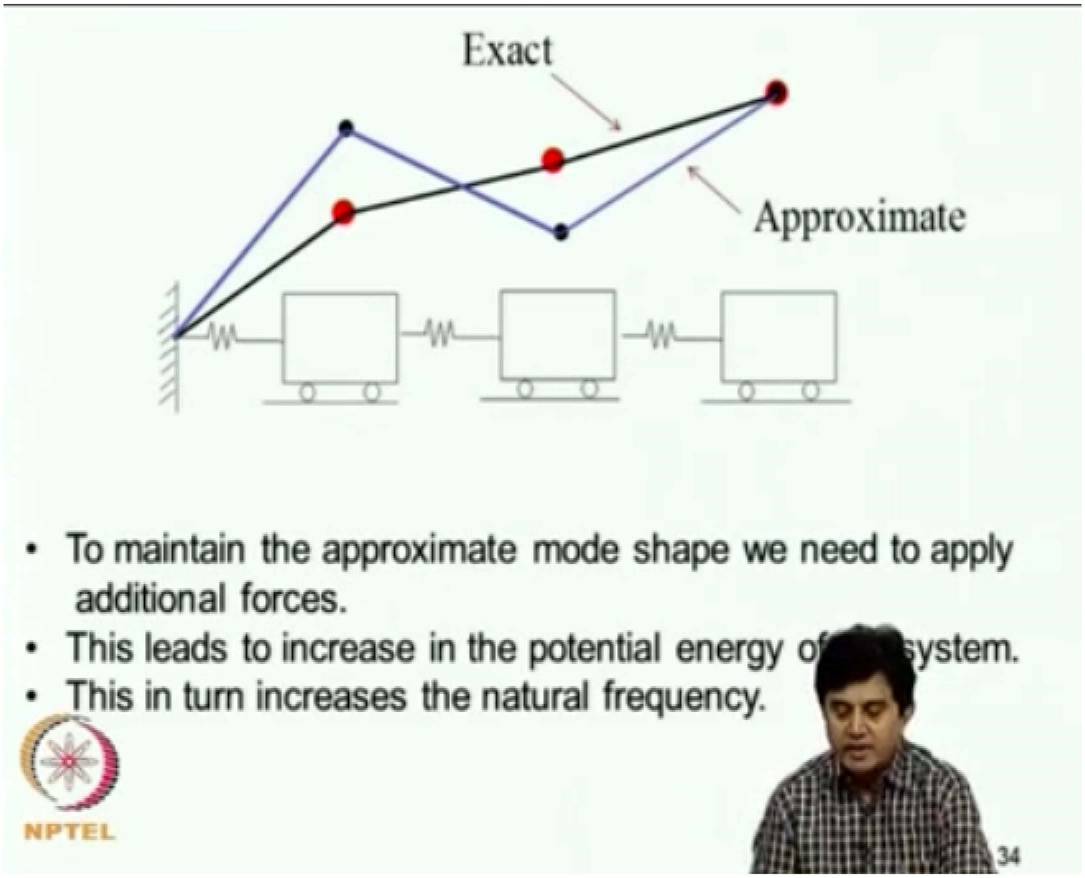


Now if I now make a guess U for example 1, 1, 1, what will be R(U) you can compute that, that means I am assuming actual mode shape if for example is 1.13, 1.63, 1.78, so what I will do is, I will take that as 1, 1, 1, so that's a guess on my modeshape. Then my approximation to R(U) turns out to be this number 22.22, so this is if you are interested in estimating the first natural frequency this will be there you are estimate of the square of the first natural frequence, if you take something like 1, 2, 3, right this will be the estimate. On the other hand if you take

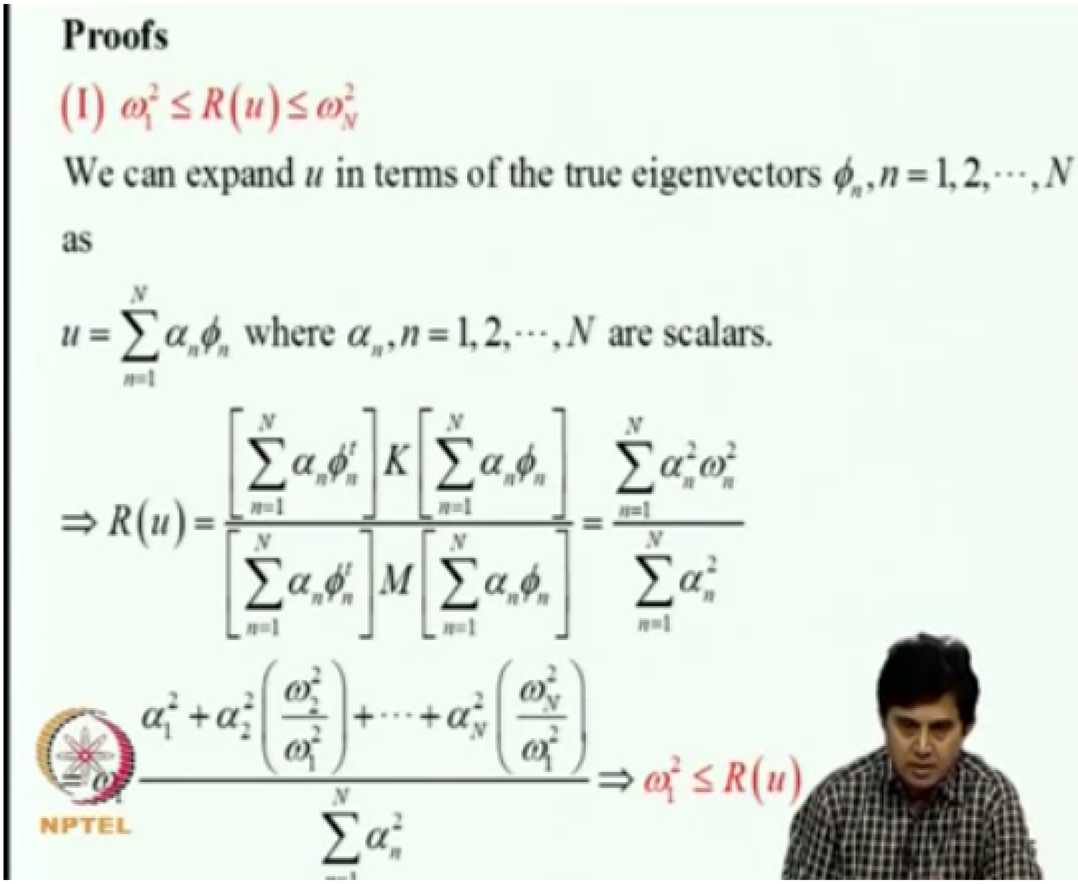


say 1.1, 1.62 you come closer to the true natural frequency. Now if you take 1 – 1 – 1 this will be a bad guess on the mode shape if you are familiar with how mode shapes behave, this will not be a good guess on the first mode shape therefore this is depicted in the Rayleigh’s quotient, but no matter what you are doing we see that Rayleigh’s quotient is bounded between the first natural frequency square and this last natural frequency square in all these examples.

Now among these various trials on your guesses on the first mode shape this one gives the lowest Rayleigh’s quotient, so if you actually in this case I know 17.25 the answer, but if you are applying this method under situation where you don't know the exact answer you should accept this as the best possible approximation to the square of the natural frequency because of this bounding property R(U) can at best be only equal to the first natural frequency square, it cannot drop below that so because of that bounding property the lowest among this is the best possible estimate for the first eigenvalue.

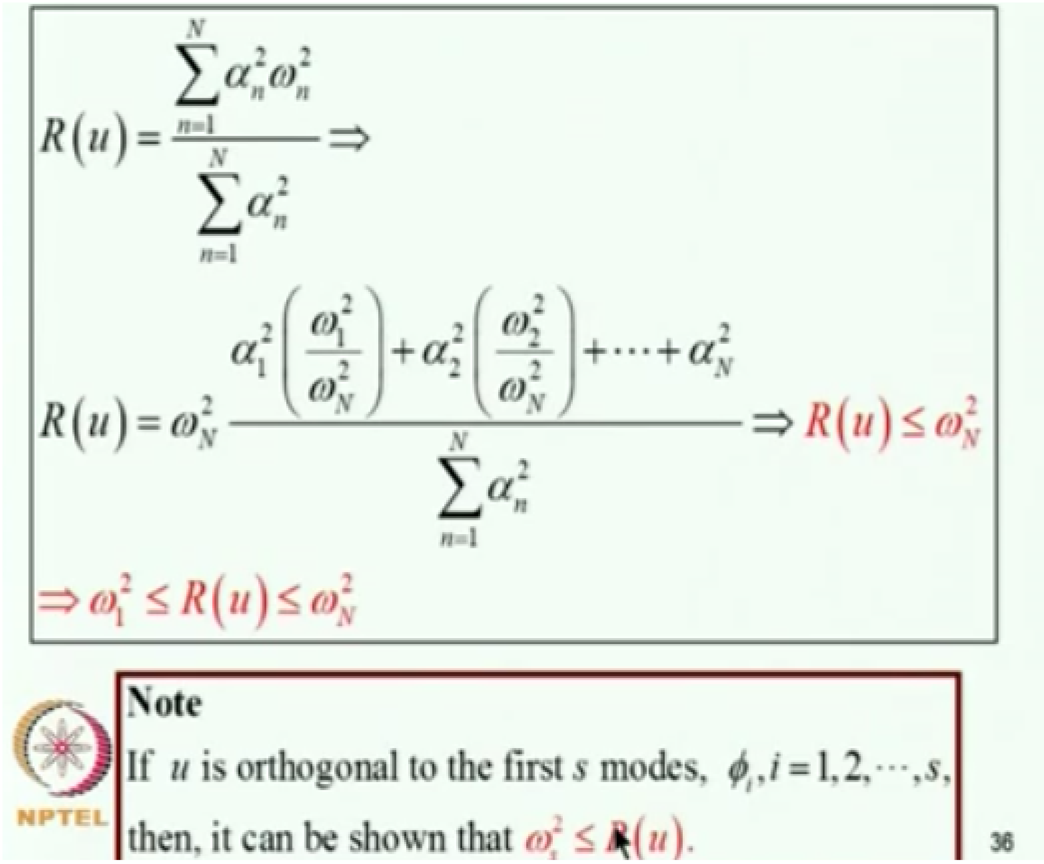


Now, why this bounding occurs? First I will give a physical explanation and then we will work through the mathematics of it, suppose for this system, this is a schematic, it is not a sketch drawn to the scale, suppose this is the true mode shape, now if I make this as a initial guess, to maintain this as a normal mode for the system I should supply additional forces which bring back this point here, brings back this point here, this point to this, you will be supplying additional forces to the system and that does additional work and that contributes the potential energy so therefore the natural frequencies go up, okay, so to maintain the approximate mode shape we need to apply additional forces and this leads to increase in the potential energy of the system which in turn increases the natural frequency, that explains why the Rayleigh's question provides the bound.

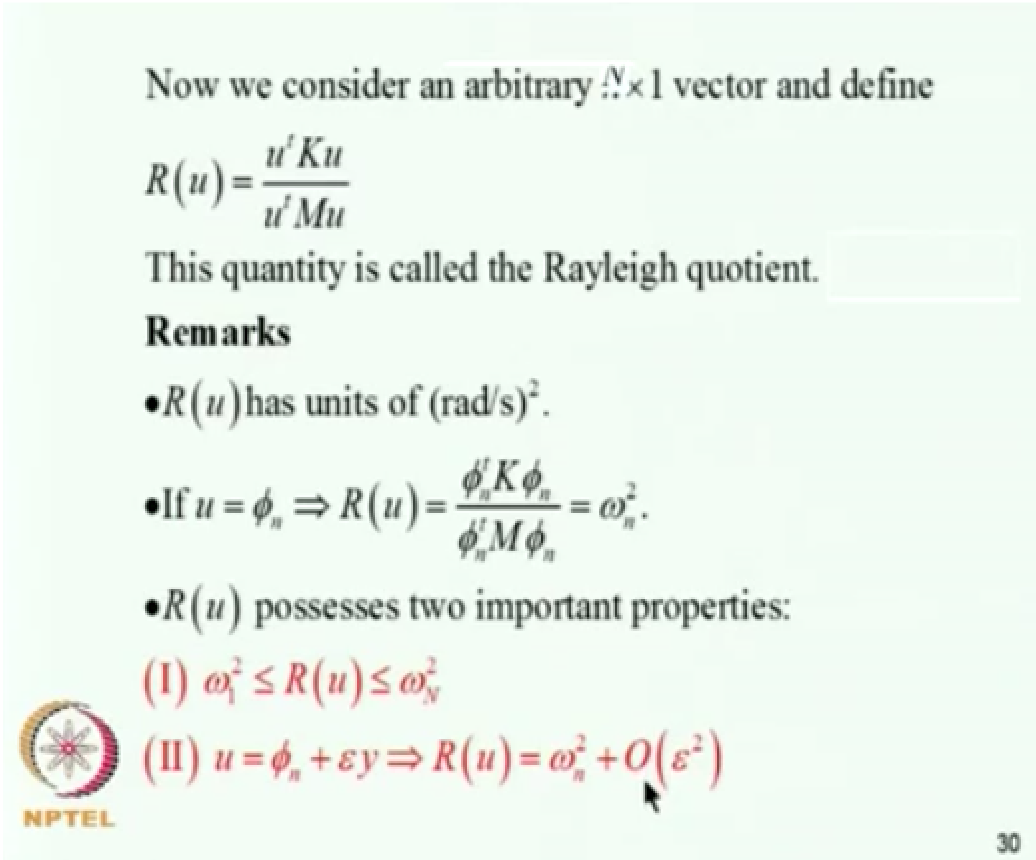


Now we can quickly argue out why these results are true, so I will start with the first result that R(U) is bounded between first natural frequency, the highest natural frequency, how do I show that? Now U is a vector, N cross 1 vector, therefore it can be expanded in terms of these true mode shapes, so I can write U as a linear combination of these mode shapes, where A alpha N takes value from 1 to capital N, now using this R(U) can be written in this form. Now Phi N's are true mode shapes and they are orthogonal with respect to K and M, okay and their mass normalized if you expand this out, this will become a double summation but all the cross terms will go to 0 and I will be left with this in the denominator. Similarly in the numerator the diagonal terms will have Omega N square multiplying alpha N square, so this will be the Rayleigh's quotient in terms of the unknown true natural frequencies and this unknown alpha N's.

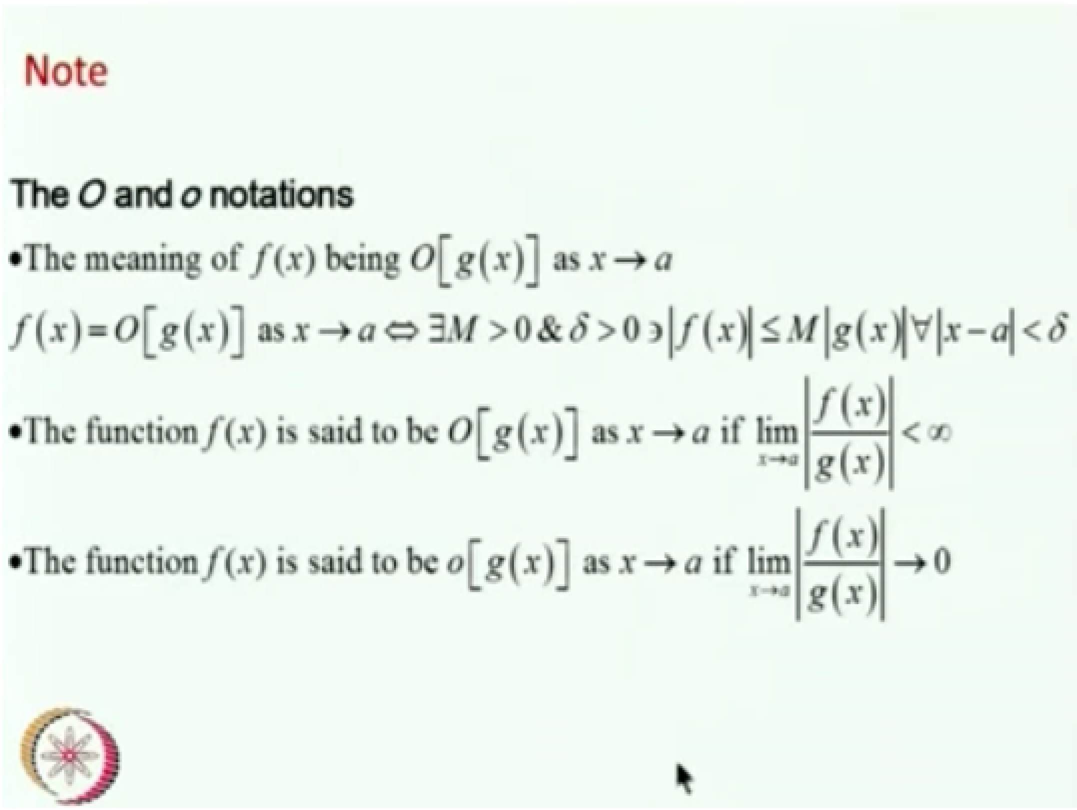
Now I will manipulate this, what I will do is I will expand this, and pull out Omega 1 square outside, so first term will be alpha 1 square as it is, Omega N square has gone outside and this is other terms are this and the denominator remains the same. Now there is an ordering of natural frequencies Omega 1 is less or equal to Omega 2 less than equal to Omega 3 and so on and so forth that is how the labels on eigenvalues appear, so consequently we can see that these terms inside the bracket here will be a greater than or equal to 1, so consequently we see that this inequality holds, okay.



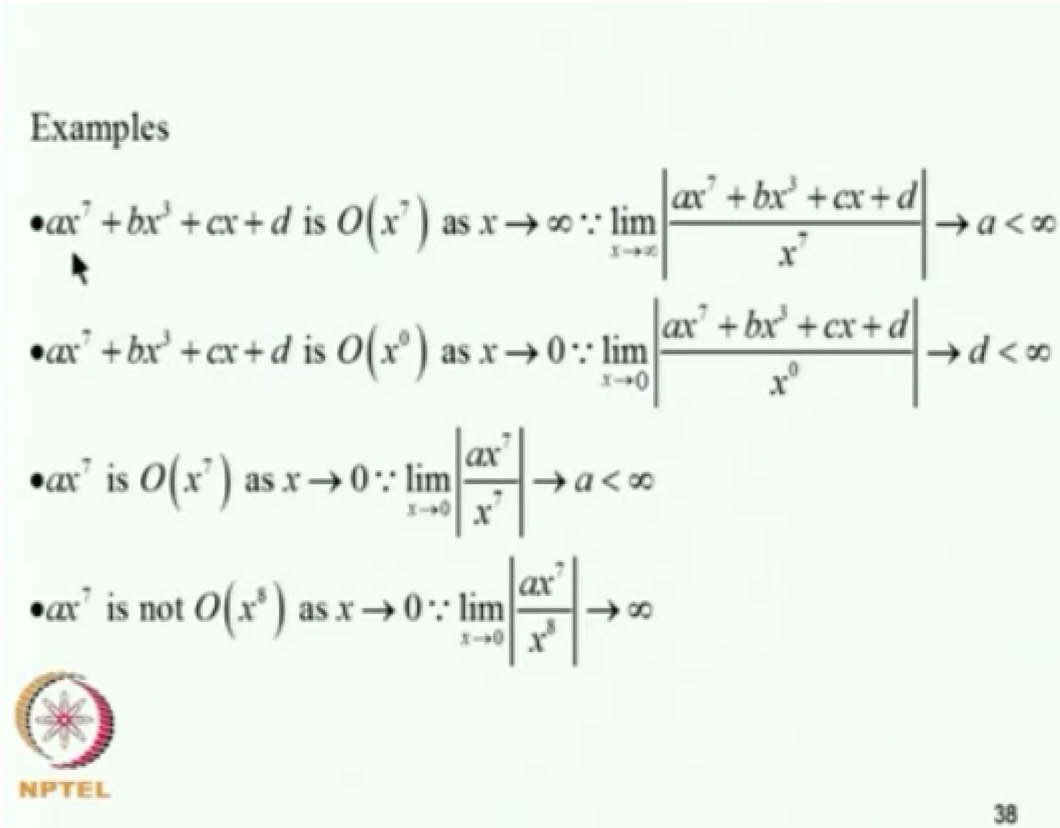
Now to get the other bound what I will do is now I will pull out the last frequency outside and divide by Omega N square all other terms, so this will be, this multiplier will be 1, now by the same logic now all these numbers inside this bracket will be less than or equal to 1, so consequently this bound emerges so we get this that result that Rayleigh’s quotient is bounded between first eigenvalue and the last eigenvalue. Now we can work through this if the U that we are selecting is orthogonal to the first S modes, it so happens that you know first three modes and you select the, I mean in a 10 degree freedom system for example if you select your trial function to be orthogonal to the first 3 modes then you are, you will end up getting an upper bound on the fourth natural frequency, so that is the third natural frequency, this result can be shown using similar arguments I leave this as an exercise for you to complete.



The second result that I stated has this notation order epsilon square, okay, now what I want to do is first I want to define what exactly is this notation means, so we will see that first then before going to the proof, so we use what are known as the capital O and a lowercase o



notations, now if we consider a function F(X) and if we say that it is order G(X) as X goes to A what it means is if X(A) is in the neighborhood, X is in the neighborhood of A by this distance Delta then this F(X) will be less than or equal to this M x G(X) where for every A there exists M greater than 0 and Delta greater than 0 so that this inequality is whole square, operationally to verify that what we need to do is we have to divide F(X) by G(X) and take the limit extending to A, if this is finite then we say that F(X) is order G(X), as X goes to A. The lowercase order G(X) means this actually goes to 0, it is not just finite but it actually goes to 0, okay.



Now some few examples, for example we have this X to the power of 7 + BX cubed + CX + D, now the question is, is it order X to the power of 7 as X goes to infinity, how do I verify? I will form this ratio and you can see that this first term will be A and all this will be 1 by X to the power of 4 etcetera, as X tends to infinity and this goes to A it is finite, therefore this is true. Now is this order X to the power of 0, as X goes to 0, again we see this as X goes to 0 all these terms will drop off there is a D here, which is finite therefore this is also true, so AX to the power of 7 is order X to the power of 7 by this logic again, whereas AX to the power of 7 is not order X to the power of 8, because this will be 1 by X and as X goes to 0 this becomes unbounded. A few more example sin X is order X, as X goes to 0, because sin X by X, as X

Examples

- $\sin(x)$ is $O(x)$ as $x \to 0 \because \lim_{x \to 0}\left|\frac{\sin(x)}{x}\right| \to 1 < \infty$
- $\sin(x^2)$ is $O(x^2)$ as $x \to 0 \because \lim_{x \to 0}\left|\frac{\sin(x^2)}{x^2}\right| \to 1 < \infty$
- $\cos(x)$ is $O(x^0)$ as $x \to 0 \because \lim_{x \to 0}\left|\frac{\cos(x)}{x^0}\right| \to 1 < \infty$
- $\sin(x)$ is $o(x^0) = o(1)$ as $x \to 0 \because \lim_{x \to 0}\left|\frac{\sin(x)}{x^0}\right| \to 0$
- $\cos(x)$ is $O\left(x^{-\frac{1}{2}}\right)$ as $x \to 0 \because \lim_{x \to 0}\left|\frac{\cos(x)}{x^{-\frac{1}{2}}}\right| = \lim_{x \to 0}\left|\sqrt{x}\cos(x)\right| \to 0 < \infty$

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goes to 0 is 1 which is finite, similarly sin X square is order X square because of this identity, so there are few more examples you can verify and this order notation will be using soon in this part of the lecture but later on when I come to numerical integration of equilibrium equations I will be using these notations again, so that's why I have introduced at this stage.

Proof

(II) $u = \phi_n + \varepsilon y \Rightarrow R(u) = \omega_n^2 + O(\varepsilon^2)$

$$R(\phi_n + \varepsilon y) = \frac{(\phi_n + \varepsilon y)^t K (\phi_n + \varepsilon y)}{(\phi_n + \varepsilon y)^t M (\phi_n + \varepsilon y)} = \frac{\phi_n^t K \phi_n + 2\varepsilon \phi_n^t K y + \varepsilon^2 y^t K y}{\phi_n^t M \phi_n + 2\varepsilon \phi_n^t M y + \varepsilon^2 y^t M y}$$

$$R(\phi_n + \varepsilon y) = \frac{\omega_n^2 + 2\varepsilon \phi_n^t K y + \varepsilon^2 y^t K y}{1 + 2\varepsilon \phi_n^t M y + \varepsilon^2 y^t M y}$$

We can write $y = \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i \phi_i$

$$\Rightarrow \phi_n^t K y = 0; \phi_n^t M y = 0; y^t K y = \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \omega_i^2; y^t M y = \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2$$

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Now what is the result that we are talking about? If U is in the neighborhood of the true and eigenvector and the departure is epsilon Y, that is order epsilon then the Rayleigh's quotient will be Omega N square + order epsilon square, so how do I show that? So you can again write R (phi N + epsilon Y) you write this and you expand this, and Phi N is a true mode shape therefore it will satisfy that, I know this is the square of the natural frequency and this is 1, so that's what I write and I am left with these two terms. Now this Y is the perturbation, now I will take Y to be in terms of all the modes except the nth mode that is how we can write this. So if you now substitute this we can show that Phi N transpose KY is 0, because we are excluding I not equal to N that is 0, and Phi N transpose MY is 0 and this is 0 and we get Y transpose Y as this which is, this is here and Y transpose KY, Y transpose MY is here, Y transpose KY is here, the term I not equal to N now in this summation I cannot take value of N.

Now if you go back and substitute you get this ratio and the denominator I can write in this form and if I do a binomial expansion of this and retain only the first few terms I get this expression and we can see that the Rayleigh's quotient is possesses the property that I stated, that means this term is order epsilon square, okay, so if you make a reasonably good guess on eigenvector you will get reasonably good, much better answer on eigenvalue that is the story,

$$\Rightarrow R(\phi_n + \varepsilon y) = \frac{\omega_n^2 + \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \omega_i^2}{1 + \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2} = \left(\omega_n^2 + \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \omega_i^2 \right) \left(1 + \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \right)^{-1}$$

$$= \left(\omega_n^2 + \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \omega_i^2 \right) \left(1 - \varepsilon^2 \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 + \varepsilon^4 \left(\sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \right)^2 + \cdots \right)$$

$$\Rightarrow R(\phi_n + \varepsilon y) = \omega_n^2 + \varepsilon^2 \left(\sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \omega_i^2 - \sum_{\substack{i=1 \\ i \neq n}}^{N} \alpha_i^2 \right) + \text{higher order terms}$$

$$\Rightarrow R(u) = \omega_n^2 + O(\varepsilon^2)$$

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$$u = \phi_n + \varepsilon y \Rightarrow R(u) = \omega_n^2 + O(\varepsilon^2) \Rightarrow$$

- Rayleigh's quotient has a stationary value in the neighbourhood of an eigenvector.
- It takes minimum value in the neighbourhood of the fundamental mode.

Rayleigh's principle

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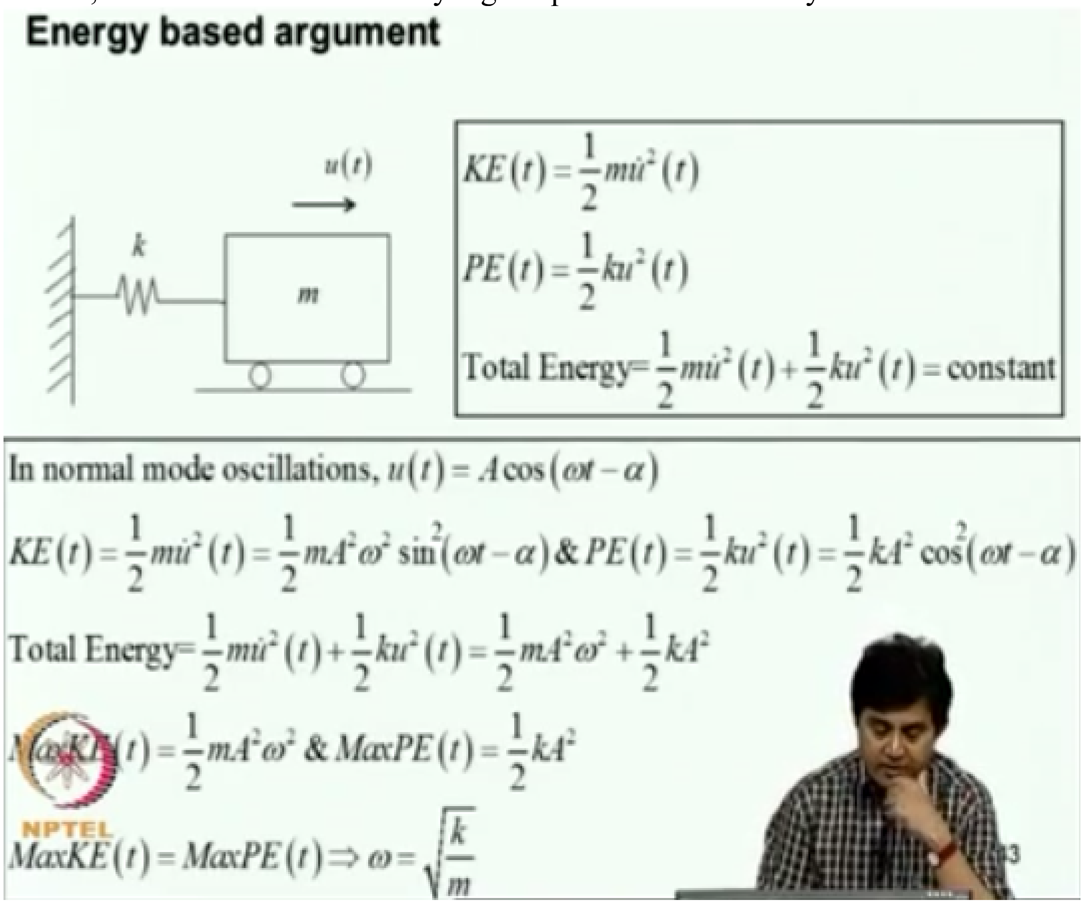
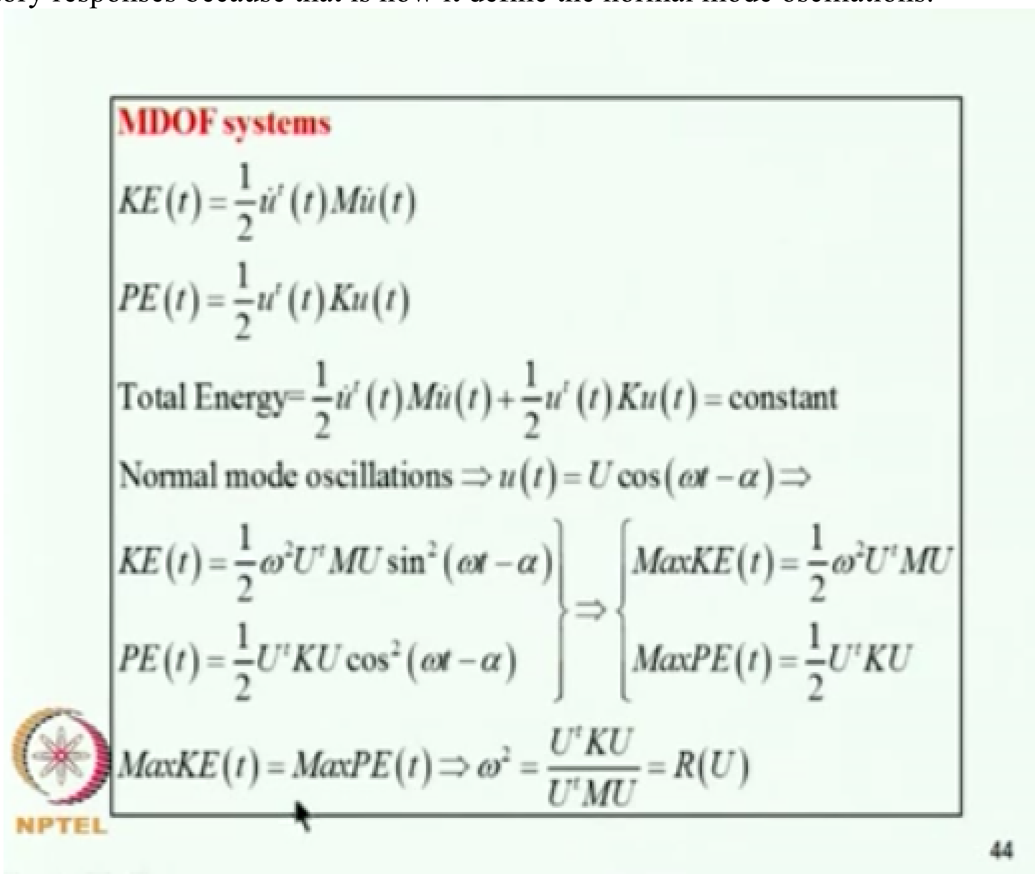
so this also means that this Rayleigh's quotient has a stationary value in the neighborhood of an eigenvector, it indeed takes the minimum value in the neighborhood of the fundamental mode this result, this is known as Rayleigh's principle, okay. Now this will be useful in some of our formulations, now the discussion on Rayleigh's quotient can also be you know based on a



slightly different argument and that is based on energies, so if I now consider for purpose of illustration and consider a simple system a mass spring system there is no energy dissipation here, therefore it is a conservative system, the total energy is conserved, so this is the expression for kinetic energy, this is the expression for potential energy, if I add these two energies I get the total energy which must be a constant because there is no energy dissipation in the system.

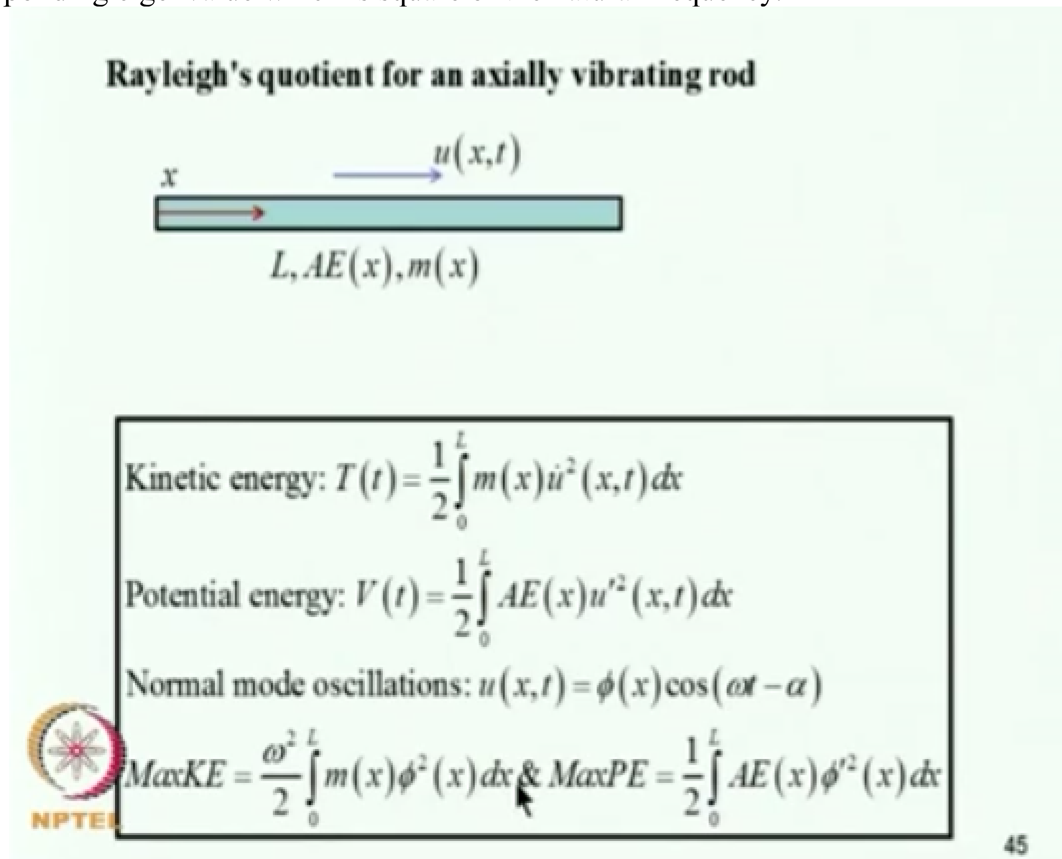
Now in normal mode oscillations we are interested in harmonic oscillations, so in a multi degree freedom system we say that a system is executing normal mode oscillations if all points on the structure are vibrating harmonically at the same frequency, so in this case it's a very structural case so if it is executing harmonic motions U(T) will be given by this. Now let us substitute this into the expression for kinetic energy I get kinetic energy to be MA square Omega square by 2 sin square Omega T and this is cos square, Omega T – alpha.

Now total energy is sum of this, if you add now this sin square and cos square will add up and I get this is a constant that I am talking here, now this would automatically means that when kinetic energy reaches its maximum value the potential energy must be 0, and similarly when potential energy reaches its maximum value the kinetic energy will be 0, so if we write that you can see here the expression for kinetic energy has sin square Omega T – alpha, the potential energy has cos square Omega T – alpha, the maximum value that function sin square Omega T - alpha can take is 1, similarly cos square Omega T - alpha the maximum value that this function can take is also 1, so I get the maximum kinetic energy to be this, maximum potential energy to be this, and since they are equal I get the expression for the natural frequency, so I have been able to write this expression for the natural frequency without writing the equation of motion, I am simply using the fact that I am looking for harmonic oscillations, harmonic oscillatory responses because that is how it define the normal mode oscillations.



Now how about the similar argument for a multi degree freedom system, so if U dot is the velocity vector, M is the mass matrix, we can write the kinetic energy as 1/2 U dot transpose MU dot, and potential energy as 1/2 U transpose KU, the total energy will be the sum of these two and again this must be a constant. In normal mode oscillations all points on the structure vibrate harmonically at the same frequency, this is the amplitude of those oscillations and this is a frequency and this phase difference could be either 0 or phi in an un-damped system that means it's a synchronous motion all points vibrate harmonically at the same frequency so that the ratio of oscillations at any two points is independent of time that also means all points reach their respective maximum value simultaneously, maximum and minimum values simultaneously that mean phase is either 0 or Phi that means points can be perfectly in phase or perfectly out of phase, there's only two phase difference is possible.

Now for this type of motions which are the normal mode oscillations I can write the kinetic energy this is given by this, potential energy is given by this, now the maximum value of kinetic energy is reached when sin square Omega T - alpha is 1, that gives me this and maximum potential energy is reached when cos square omega T – alpha is 1 that gives me this expression, so if I equate these two I get the expression for Omega square which is the U transpose KU, U transpose MU and now you will recognize that this is nothing but the definition of Rayleigh's quotient for a trial vector U, okay, clearly when U coincides with the true natural frequency of, true normal mode of the system this will be the corresponding natural frequency or the corresponding eigenvalue which is square of the natural frequency.



Now how about for a continuous system, now let me consider this actually vibrating bar U(X,T) is the axial displacement, L is the span, A is axial rigidity, M(X) is a mass per unit length, so we have derived these expressions for kinetic energy and potential energy which is strain energy, so again in normal mode oscillations all points on the structure vibrate harmonically at the same frequency and Phi(X) is now that eigen function which is not known, okay. So now I will substitute this into the expression for kinetic energy and again notice that the maximum value will be reached when cos square Omega T - alpha is 1, and sin square Omega T - alpha is 1, respectively for the potential energy and the kinetic energy I will get maximum kinetic energy to be this, and maximum potential energy to be this, so these two need to be equal, so

$$MaxKE = MaxPE \Rightarrow \omega^2 = \frac{\int_0^L AE(x)\phi'^2(x)dx}{\int_0^L m(x)\phi^2(x)dx} = R[\phi(x)]$$

Note

$\omega_1^2 \leq R[\phi(x)]$

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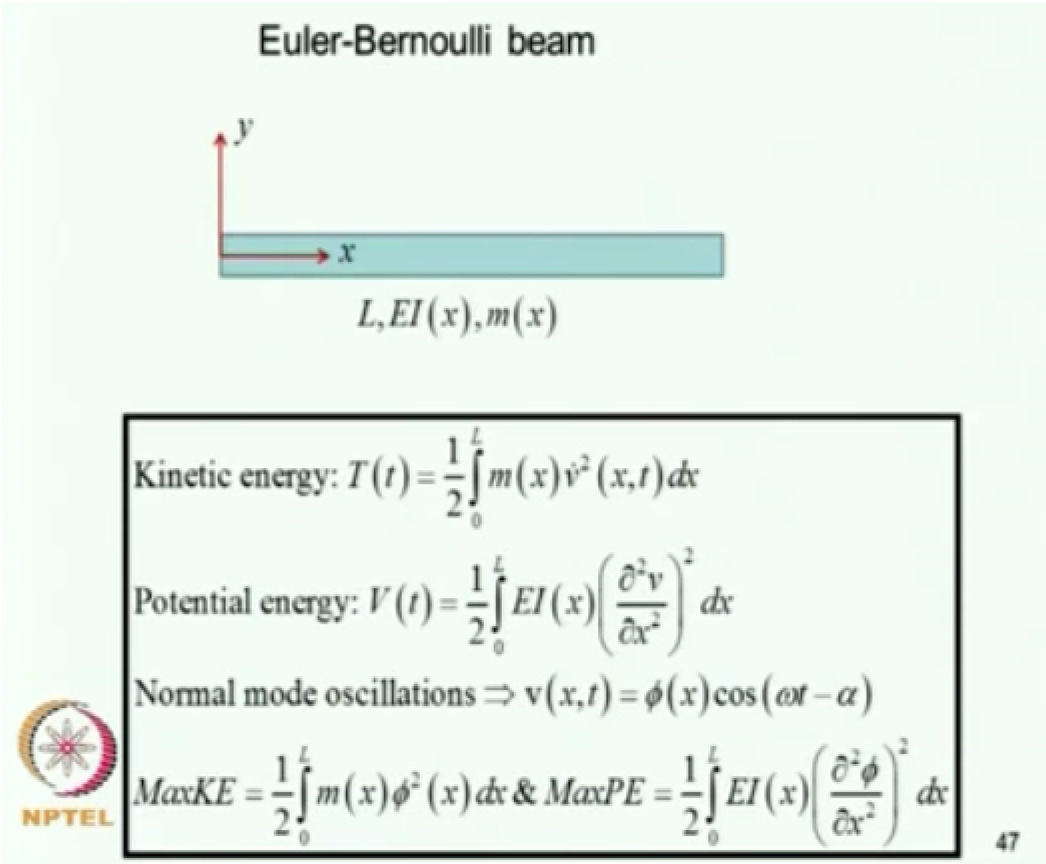
based on that I get now the expression for the frequency of oscillation and this is defined as, now the Rayleigh's quotient for actually vibrating bar.

Now unlike in discrete multi-degree freedom system this Rayleigh's quotient here only satisfies the bound at this end because the highest natural frequency of a distributed parameter system goes to infinity, there is no bound on this side, it virtually means that Rayleigh's quotient is less than infinity, but that is not worth stating, so we only get bound on one side, how about the



Euler Bernoulli beam? Again we consider this beam system L is a span, EI is the flexural rigidity, M(X) is a mass per unit length, and displacement in the Y direction is V(X,T) so this is the expression for kinetic energy, this is the expression for the potential energy. Now in normal mode oscillations again we take the response to be Phi (X) cos Omega T – alpha, so this is the expression for the maximum kinetic energy, this is the expression for maximum potential

$$MaxKE = MaxPE \Rightarrow \omega^2 = \frac{\int_0^L EI(x)\phi''^2(x)dx}{\int_0^L m(x)\phi^2(x)dx} = R[\phi(x)]$$

Note

$$\omega_1^2 \le R[\phi(x)]$$

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energy. These two must be equal, so I get the definition of Rayleigh's quotient for a beam in terms of flexural rigidity and mass per unit length as shown here, okay, we notice that the Rayleigh's quotient is greater than or equal to Omega 1 square.

So in the next class what I will do is I will apply this idea of Rayleigh's quotient for a few class of systems and show how it is useful in characterizing the natural frequency of I characterized obtaining approximation to the fundamental mode of vibrating systems.

Now the success of this quotient, Rayleigh's quotient as an approximation to the first eigenvalue depends on how well you can approximate the mode shape, so it is not always that you can make intelligent guesses, right, so you need to have a systematic procedure to lower the Rayleigh's quotient. Now how do we lower the Rayleigh's quotient, so to be able to do that we should introduce certain undetermined parameters in my definition of the trial function, this is the assumed mode shape, so if I assume Phi(X) as a series, if I do expand this unknown trial function in a set of a say orthogonal functions with unknown, undetermined coefficients then I can select those undetermined coefficients so that the Rayleigh's quotient is minimized, right? So that gives us a systematic way to lower the Rayleigh's quotient and therefore obtain better approximation to the first natural frequency.

Now in doing so we not only get a better approximation to the first natural frequency but we will also end up getting approximations to the higher natural frequency, this method that I am talking about is known as Rayleigh Ritz method, so in the next part of the lecture after illustrating the applications of Rayleigh's quotient we will develop the Rayleigh Ritz method and then again apply to a few methods, I will conclude this lecture at this point.

Programme Assistance

Guruprakash P
Dipali K Salokhe

Technical Supervision
B K A N Singh
Gururaj Kadloor

Indian Institute of Science
Bangalore