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Course Title

Finite element method for structural dynamic And stability analyses

Lecture – 18

Substructuring schemes

By

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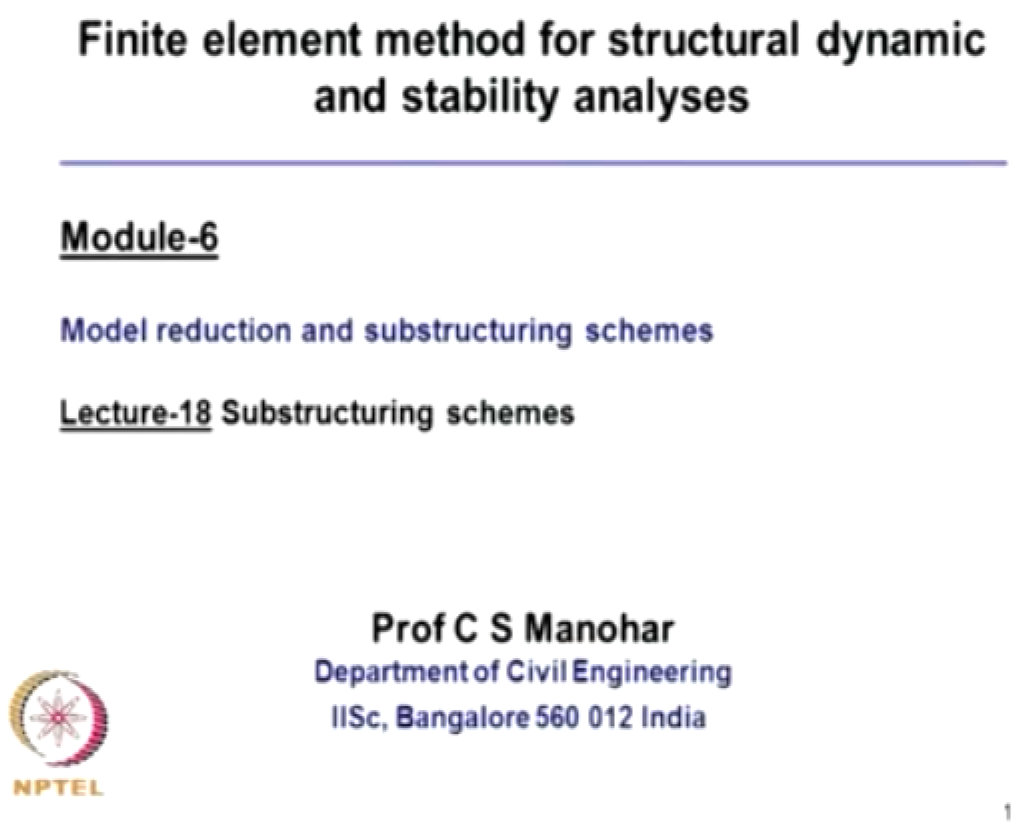

We have been discussing issues related to model reduction and substructuring, so we will continue with this topic in this lecture. So we can quickly recall what we have been discussing,

Recall : model reduction

$$M\ddot{X}+C\dot{X}+KX=F(t); X(0)=X_0 \,\&\, \dot{X}(0)=\dot{X}_0$$

$$X(t)=\begin{Bmatrix} X_m(t) \\ X_s(t) \end{Bmatrix}=\Psi X_m(t)$$

$$\Psi^t M\Psi\ddot{X}_m(t)+\Psi^t C\Psi\dot{X}_m(t)+\Psi^t K\Psi X_m(t)=\Psi^t F(t)$$

$$\Rightarrow M_r\ddot{X}_m+C_r\dot{X}_m+K_rX_m=F_r(t)$$

Static condensation $\Psi=\begin{bmatrix} I \\ -K_{ss}^{-1}K_{sm} \end{bmatrix}$

Dynamic condensation $\Psi=\begin{bmatrix} I \\ -\left[K_{ss}-\omega^2 M_{ss}\right]^{-1}\left[K_{sm}-\omega^2 M_{sm}\right] \end{bmatrix}$

SEREP $\Psi=\begin{bmatrix} \Phi_m \\ \Phi_s \end{bmatrix}\left[\Phi_m^t\Phi_m\right]^{-1}\Phi_m^t$

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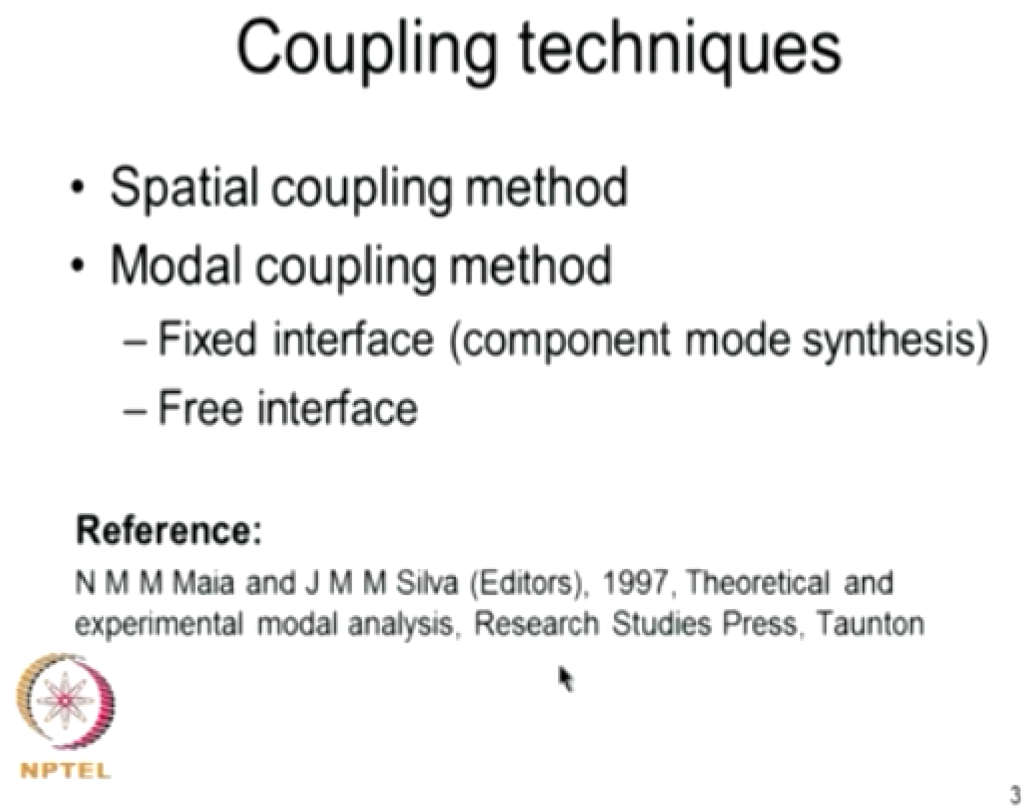
we are considering a large finite element model governed by equation of this kind, we are restricting our attention to linear time-invariant systems, the idea is that we want to reduce this model to a lower order model, R is for reduced here through a transformation X(t) is some sai into XM(t), that means we will partition this degree of freedom X(t) into a set of master degrees of freedom, and slave degrees of freedom and the state vector X(t) is related to the master degrees of freedom through this sai matrix. So once we find that we can substitute this into the governing equation, and we get this reduced, mass, damping and stiffness matrices and the reduce forces governing the master degrees of freedom.

There are different model reduction schemes they differ in their definition of this matrix sai. In static condensation technique the slave degrees of freedom are taken to be related to the master degrees of freedom through relations which are strictly valid only for static, under static conditions, so we derived the relation between X and XM using static equilibrium equations and this is a sai matrix. In dynamic condensation we again partition the degrees of freedom into master and slave, in addition we specify frequency at which the condensation is done, so the sai matrix is given as here. Now if one computes the natural frequencies and mode shapes of the reduced system the set of, you know, natural frequencies from this model need not agree with any of the natural frequencies of the larger model, that is certainly true in case of static condensation method. In dynamic condensation method if this omega is chosen such that it coincides with one of the natural frequencies of the larger system then one of the frequencies in the reduced model will match with that frequency.

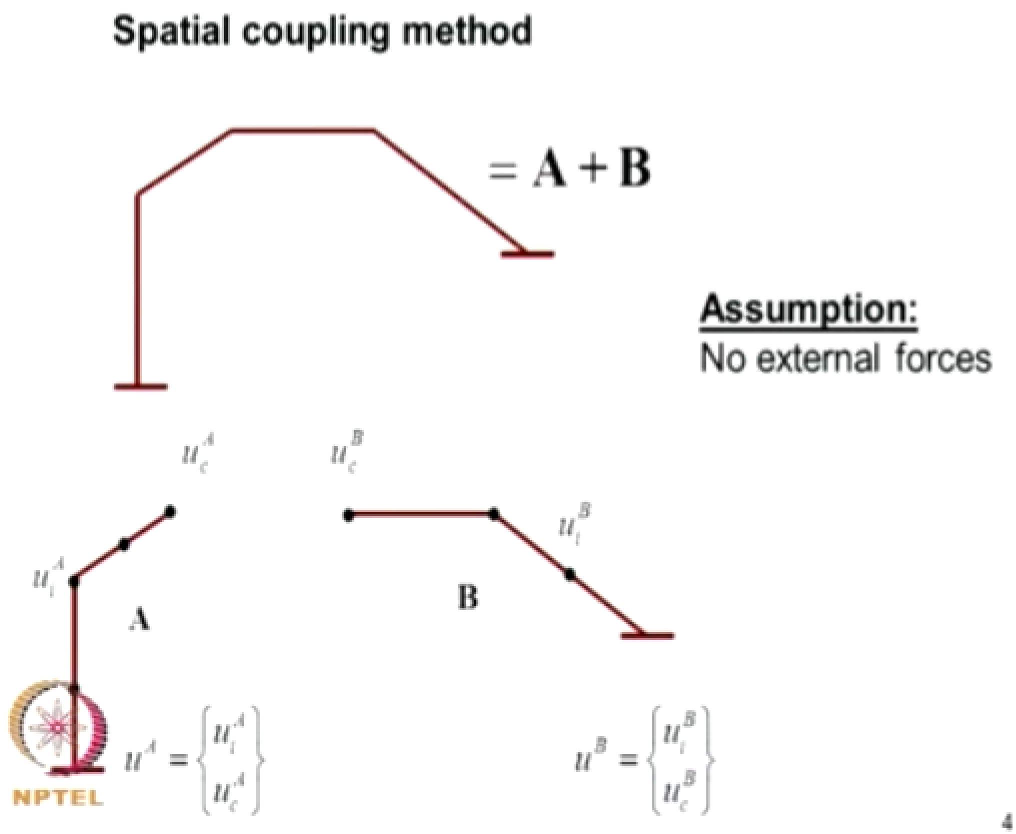
Now in a third method that is system equivalent reduction expansion process the sai matrix is constructed from the modal matrix of the larger system, so the modal matrix can be, I mean not all modes need to be evaluated first P modes, suppose if we include the modal matrix will be a rectangular matrix, now the transformation matrix there is synthesized from the partitioning of the available modal information and this is the sai matrix that is obtained here. Here again we need to partition the degrees of freedom to master and slave, in addition we need to also specify the modes of the larger model which needs to be replicated in the reduced model, SEREP has the feature that it exactly reproduces Eigen solutions of the larger model, a set of Eigen solutions of a larger model are retained in the smaller model, so these are the 3 techniques that we discussed.



Now we move on to the next topic in this that is known as coupling techniques or the substructuring techniques. The basic idea here is again we deal with large FE models and we want to devise some way of reducing the size of the problem while an computing the response, the idea here is that often a large finite element model may be needed to take into account geometric complexities of the structure, and a very detailed modeling may perhaps be not needed if one is interested in capturing the behavior of the system, a simpler model can capture the spatial variation of the system behavior, but a finer model is needed to capture all the geometric details, so in which case there is no point in working with a very large model, a smaller model would suffice.

The other situation where coupling techniques would be needed is when a built up structure is made up of different components and different engineers are involved in development of different components, so every engineer therefore if possible would develop a computational model as well as a experimental model for each of these substructures, so we need to synthesize the models developed by individuals to predict the behavior of the global structure, the testing on a global structure may or may not be possible, but finite element model indeed can be

constructed, so how to, what in such situation, so there are basically the coupling techniques can be classified into two categories spatial coupling method and modal coupling method.
In modal coupling method there are two further classification fixed interface and free interface, the main distinguishing feature of these two methods is that in modal coupling method the coupling matrices are derived in terms of Eigen solutions of the substructures, where as in spatial coupling method the coupling matrices are derived in terms of the substructure structural matrices, mass, stiffness, damping matrices, so there is no Eigen solutions that one need to compute if one is working with spatial coupling method. So we will see some of these details and a good reference for a discussion on this is a volume edited by Maia and Silva, the details are given here.



So we'll start with the discussion on spatial coupling method, will restrict our discussion by enlarge to free vibration analysis, suppose this is a built-up structure and I assume that this built-up structure is made up of 2 substructures A and B, so this built up structure is made up of A+ so called, I mean so this plus operation is combining these 2 systems. Now if we consider substructure A, the degrees of freedom in this model can further be classified as interior degrees of freedom and coupling degrees of freedom, coupling degrees of freedom occurs here where this node and this node are to be coupled to produce a built up structure, so the degrees of freedom at this node constitute UC of A, and other degrees of freedom in the interior are the interior degrees of freedom. Similarly for system B, I have UI of B and UC of B, clearly you can see that UC of A and UC of B need to be equal for compatibility relations to be you know obeyed.

Equation for subsystem A $[N_A \text{ dofs}]$

$$\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\begin{Bmatrix}\ddot{u}_i^A\\ \ddot{u}_c^A\end{Bmatrix} + \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A\end{bmatrix}\begin{Bmatrix}u_i^A\\ u_c^A\end{Bmatrix} = \begin{Bmatrix}0_i^A\\ f_c^A\end{Bmatrix}$$

Sizes: $u_i^A \sim (N_A - n_c)\times 1; u_c^A \sim n_c \times 1$

Equation for subsystem B $[N_B \text{ dofs}]$

$$\begin{bmatrix} M_{ii}^B & M_{ic}^B \\ M_{ci}^B & M_{cc}^B \end{bmatrix}\begin{Bmatrix}\ddot{u}_i^B\\ \ddot{u}_c^B\end{Bmatrix} + \begin{bmatrix} K_{ii}^B & K_{ic}^B \\ K_{ci}^B & K_{cc}^B\end{bmatrix}\begin{Bmatrix}u_i^B\\ u_c^B\end{Bmatrix} = \begin{Bmatrix}0_i^B\\ f_c^B\end{Bmatrix}$$

Sizes: $u_i^B \sim (N_B - n_c)\times 1; u_c^B \sim n_c \times 1$

Remark

The internal dofs may include all the unknown dof-s or could be the master dof-s obtained after a model reduction.

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Now let's consider the equation for subsystem A, so this subsystem A, the other degrees of freedom as I already said are partitioned into interior degrees of freedom and coupling degrees of freedom, so this partitioning of degrees of freedom induces a partition on structural matrices and this is depicted here KIIA, I stands for interior, C stands for coupling, it's not the I-th element of mass matrix that is not what is meant here, so the equation of motion is obtained in this form, we assume as I already said there are no external forces but nevertheless I should write FCA because these are the coupling forces that we need to consider later on.
Now if NC is the size of UC that is coupling degrees of freedom, then UIA will have NA – NC degrees of freedom, the size of this vector will be that. Similar equation for subsystem B with NB degrees of freedom can also be written which is as shown here. Now the interior degrees of freedom include all the unknown degrees of freedom or could be the master DOF's obtained after a model reduction, so that also could be that means while arriving at this one may use a condensation technique inbuilt into that if needed to be.

Equation for the coupled system

- Compatibility of displacements at the interface: $u_c^A = u_c^B$
- Equillibrium of forces at the interface: $f_c^A + f_c^B = 0$

$\Rightarrow$

$$\begin{bmatrix} M_{ii}^A & M_{ic}^A & 0 \\ M_{ci}^A & M_{cc}^A + M_{cc}^B & M_{ci}^B \\ 0 & M_{ci}^B & M_{ii}^B \end{bmatrix} \begin{Bmatrix} \ddot{u}_i^A \\ \ddot{u}_c \\ \ddot{u}_i^B \end{Bmatrix} + \begin{bmatrix} K_{ii}^A & K_{ic}^A & 0 \\ K_{ci}^A & K_{cc}^A + K_{cc}^B & K_{ci}^B \\ 0 & K_{ci}^B & K_{ii}^B \end{bmatrix} \begin{Bmatrix} u_i^A \\ u_c \\ u_i^B \end{Bmatrix} = 0$$

- The built up structure can now be analyzed using these assembled matrices.
- Size of assembled structural matrices: $(N_A + N_B - n_c) \times (N_A + N_B - n_c)$
- Suited when subsystems are studied using FEA
- Can be extended to >2 subsystems in a straightforward manner

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Now how do we get the equation for the coupled system? So the coupled system equation is obtained by considering the compatibility of displacements at the interface which demanded UCA must be equal to UCB, and similarly equilibrium of forces at the interface requires that some this FCA + FCB must be equal to 0, so if we use these conditions I get this method, so this is the built-up structural matrix, I mean for the built-up structure, this is a mass matrix, and the stiffness matrix. Now the built-up structure can now be analyzed using these assembled matrices, the size of assembled structural matrices will be NA + NB – NC, that is cross this, I mean this is the size of a vector displacement vector, now this method is well suited if subsystems are studied using a finite element analysis if you have access to all these matrices you can easily construct this matrices for the built-up structure.

Now I have illustrated this with respect to 2 subsystems, but it can be extended to a built-up structure having more than two subsystems then the extension is fairly straightforward.

Fixed interface modal coupling method (component mode synthesis)

Equation for subsystem A $[N_A \text{ dofs}]$

$$\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix} \begin{Bmatrix} \ddot{u}_i^A \\ \ddot{u}_c^A \end{Bmatrix} + \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A \end{bmatrix} \begin{Bmatrix} u_i^A \\ u_c^A \end{Bmatrix} = \begin{Bmatrix} 0_i^A \\ f_c^A \end{Bmatrix}$$

Sizes: $u_i^A \sim (N_A - n_c) \times 1; u_c^A \sim n_c \times 1$

Denote $n_i^A = N_A - n_c$

Assume fixed interfaces $\Rightarrow u_c^A = 0$

$\Rightarrow M_{ii}^A \ddot{u}_i^A + K_{ii}^A u_i^A = 0_i^A$

Determine the eigensolutions corresponding to this system.

Leads to $\omega_r^A, r = 1, 2, \cdots, n_i^A$ natural frequencies and $\phi_r^A, r = 1, 2, \cdots, n_i^A$ eigenvectors.

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Now I come to the other class of methods that is the so called modal coupling method, we will start with fixed interface method that is also known as component mode synthesis is one of the well-known methods, so we will again start with the equation for subsystem A, and we will partition the degrees of freedom as interior and coupling degrees of freedom and this equation is obtained.

Now I will denote this NA – NC as NIA, NI superscript A, now we begin by assuming that at the interfaces the degrees of freedom are made 0 that means the structure is fixed at the coupling point, okay. Then the governing equation therefore UC will be 0, so the governing equation will be UIA double dot + this KIIA UIA = 0, so this is 0 and this is 0 and this force is 0. Now we can do Eigen solution, Eigenvalue analysis associated with this equation and determine the system natural frequencies and mode shapes, okay. Now how do we, at the interface structure of course is not fixed so we need to correct for that, so in any case the using

$\Rightarrow u_i^A(t) = \left[\Phi_{ik}^A\right]\left\{p_k^A(t)\right\}$

where the subscript denotes the fact that we are retaining first k modes in the expansion.

Now release the dofs at the interface. $\Rightarrow u_c^A(t) \neq 0$.

Assume that $u_c^A(t)$ is related to $u_i^A(t)$ through relations that are strictly valid under static conditions; that is, use static condensation to eliminate $u_i^A(t)$ in terms of $u_c^A(t)$.

$$\Rightarrow \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A \end{bmatrix} \begin{Bmatrix} u_i^A \\ u_c^A \end{Bmatrix} = \begin{Bmatrix} 0_i^A \\ f_c^A \end{Bmatrix}$$

$$\Rightarrow u_i^A(t) = -\left[K_{ii}^A\right]^{-1} K_{ic}^A u_c^A(t)$$

Now consider the solution for the subsystem equation in terms of the expansion

$$\begin{Bmatrix} u_i^A \\ u_c^A \end{Bmatrix} = \begin{bmatrix} \Phi_{ik}^A & -\left[K_{ii}^A\right]^{-1} K_{ic}^A \\ 0 & I \end{bmatrix} \begin{Bmatrix} p_k^A(t) \\ u_c^A(t) \end{Bmatrix} = \Psi_k^A \begin{Bmatrix} p_k^A(t) \\ u_c^A(t) \end{Bmatrix}$$

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the modal matrix of the fixed interface system I obtained this representation in terms of these modal coordinates denoted as PKA(t), the subscript K here denotes that I am not retaining all the modes, I am retaining only K modes, so thereby I am already achieving a model reduction, okay.

Now we release the degrees of freedom at the interface, this analysis has been done with degrees of freedom fixed, now once I release that UC of A(t) is not equal to 0, now we make an assumption now that UC of A(t) is related to interior degrees of freedom through relations that are strictly valid under static condensation, that is we use static condensation to eliminate integral degrees of freedom in terms of coupling degrees of freedom, so by that I get this is the equation, I get integral degrees of freedom in terms of coupling degrees of freedom using this relation.

Now what I suggest, what we do is the solution is taken to be the sum of, this is a solution that we assume, the unknowns that we take are the modes corresponding to the fixed interface degrees of freedom, a fixed interface system and the coupling degrees of freedom obtained through a condensation, and this is a transformation matrix, is that okay? Now this is how we represent the assumed displacement, now we return to the governing equation and make this

Now returning to

$$\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\begin{Bmatrix}\ddot{u}_i^A \\ \ddot{u}_c^A\end{Bmatrix}+\begin{bmatrix}K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A\end{bmatrix}\begin{Bmatrix}u_i^A \\ u_c^A\end{Bmatrix}=\begin{Bmatrix}0_i^A \\ f_c^A\end{Bmatrix}$$

and using the transformation $\begin{Bmatrix}u_i^A \\ u_c^A\end{Bmatrix}=\Psi_k^A\begin{Bmatrix}p_k^A(t) \\ u_c^A(t)\end{Bmatrix}$

we get

$$\left[\Psi_k^A\right]^T\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\left[\Psi_k^A\right]\begin{Bmatrix}\ddot{p}_k^A(t) \\ \ddot{u}_c^A(t)\end{Bmatrix}+\left[\Psi_k^A\right]^T\begin{bmatrix}K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A\end{bmatrix}\left[\Psi_k^A\right]\begin{Bmatrix}p_k^A(t) \\ u_c^A(t)\end{Bmatrix}=\left[\Psi_k^A\right]^T\begin{Bmatrix}0_i^A \\ f_c^A\end{Bmatrix}$$

Consider

$$\left[\Psi_k^A\right]^T\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\left[\Psi_k^A\right]=\begin{bmatrix}\Phi_{ik}^A & -\left[K_{ii}^A\right]^{-1}K_{ic}^A \\ 0 & I\end{bmatrix}^T\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\begin{bmatrix}\Phi_{ik}^A & -\left[K_{ii}^A\right]^{-1}K_{ic}^A \\ 0 & I\end{bmatrix}$$

$$\begin{bmatrix}\Phi_{ik}^A & \alpha^A \\ 0 & I\end{bmatrix}^T\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\begin{bmatrix}\Phi_{ik}^A & \alpha^A \\ 0 & I\end{bmatrix}\left(\text{with } \alpha^A=-\left[K_{ii}^A\right]^{-1}K_{ic}^A\right)$$

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substitution, so once I substitute this I will pre multiply by sai KA transpose and I get this equation, and this leads to the definition of reduced, mass, and reduced structural matrices, so if we now consider this matrix transpose M into this and analyze this a bit we get terms involving phi IKA transpose into MIIA into phi IKA which I know is a diagonal matrix, because this is a modal matrix of fixed interface system whose mass matrix is MII of A, so making some of these simplifications we get the mass matrix for the substructure to be this.

$$\left[\Psi_k^A\right]^T \begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix} \left[\Psi_k^A\right] = \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}^T \begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix} \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \left(\Phi_{ik}^A\right)^T & 0 \\ \left(\alpha^A\right)^T & I \end{bmatrix} \begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix} \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \left(\Phi_{ik}^A\right)^T & 0 \\ \left(\alpha^A\right)^T & I \end{bmatrix} \begin{bmatrix} M_{ii}^A \Phi_{ik}^A & M_{ii}^A \alpha^A + M_{ic}^A \\ M_{ii}^A \Phi_{ik}^A & M_{ci}^A \alpha^A + M_{cc}^A \end{bmatrix}$$

$$= \begin{bmatrix} \left(\Phi_{ik}^A\right)^T M_{ii}^A \Phi_{ik}^A & \left(\Phi_{ik}^A\right)^T \left(M_{ii}^A \alpha^A + M_{ic}^A\right) \\ \left(\alpha^A\right)^T M_{ii}^A \Phi_{ik}^A & \left(\alpha^A\right)^T \left(M_{ii}^A \alpha^A + M_{ic}^A\right) + \left(M_{ci}^A \alpha^A + M_{cc}^A\right) \end{bmatrix}$$

$$= \begin{bmatrix} I & \left(\Phi_{ik}^A\right)^T \left(M_{ii}^A \alpha^A + M_{ic}^A\right) \\ \left(\alpha^A\right)^T M_{ii}^A \Phi_{ik}^A & \left(\alpha^A\right)^T \left(M_{ii}^A \alpha^A + M_{ic}^A\right) + \left(M_{ci}^A \alpha^A + M_{cc}^A\right) \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{M}_{kk}^A & \tilde{M}_{kc}^A \\ \tilde{M}_{ck}^A & \tilde{M}_{cc}^A \end{bmatrix}$$

Similarly this is the mass matrix, if we can work through this I get I mean if we multiply all this and work through this I get the elements of the mass matrix for subsystem A in this form. Now similarly the stiffness matrix we can work through, and here if we see here the Eigenvectors here this phi is the Eigenvector and it is orthogonal, and it is mass normalized, therefore phi transpose K phi is the square of the natural frequency, so at the matrix level we get phi IKA

$$[\Psi_k^A]^T \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & M_{cc}^A \end{bmatrix} [\Psi_k^A] = \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}^T \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A \end{bmatrix} \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} (\Phi_{ik}^A)^T & 0 \\ (\alpha^A)^T & I \end{bmatrix} \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A \end{bmatrix} \begin{bmatrix} \Phi_{ik}^A & \alpha^A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} (\Phi_{ik}^A)^T & 0 \\ (\alpha^A)^T & I \end{bmatrix} \begin{bmatrix} K_{ii}^A \Phi_{ik}^A & K_{ii}^A \alpha^A + K_{ic}^A \\ K_{ii}^A \Phi_{ik}^A & K_{ci}^A \alpha^A + K_{cc}^A \end{bmatrix}$$

$$= \begin{bmatrix} (\Phi_{ik}^A)^T K_{ii}^A \Phi_{ik}^A & (\Phi_{ik}^A)^T (K_{ii}^A \alpha^A + K_{ic}^A) \\ (\alpha^A)^T K_{ii}^A \Phi_{ik}^A & (\alpha^A)^T (K_{ii}^A \alpha^A + K_{ic}^A) + (K_{ci}^A \alpha^A + K_{cc}^A) \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda^A & (\Phi_{ik}^A)^T (K_{ii}^A \alpha^A + K_{ic}^A) \\ (\alpha^A)^T K_{ii}^A \Phi_{ik}^A & (\alpha^A)^T (K_{ii}^A \alpha^A + K_{ic}^A) + (K_{ci}^A \alpha^A + K_{cc}^A) \end{bmatrix} \text{ with } \Lambda^A = \mathrm{Diag}(\omega_A^2)$$

$$\begin{bmatrix} \tilde{K}_{kk}^A & \tilde{K}_{kc}^A \\ \tilde{K}_{ck}^A & \tilde{K}_{cc}^A \end{bmatrix}$$

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transpose KIIA into phi IKA to be a this lambda matrix, which is the diagonal matrix of natural frequencies of subsystem A in that fixed interface form, so these are the matrices that I get for substructure A, so this is the equation for substructure A at this stage in terms of, let me recall

$$\begin{bmatrix} \tilde{M}^{A}_{kk} & \tilde{M}^{A}_{kc} \\ M^{A}_{ck} & \tilde{M}^{A}_{cc} \end{bmatrix} \begin{Bmatrix} \ddot{p}^{A}_{k}(t) \\ \ddot{u}^{A}_{c}(t) \end{Bmatrix} + \begin{bmatrix} \tilde{K}^{A}_{kk} & \tilde{K}^{A}_{kc} \\ \tilde{K}^{A}_{ck} & \tilde{K}^{A}_{cc} \end{bmatrix} \begin{Bmatrix} p^{A}_{k}(t) \\ u^{A}_{c}(t) \end{Bmatrix} = \left[\Psi^{A}_{k}\right]^{T} \begin{Bmatrix} 0^{A}_{i} \\ f^{A}_{c} \end{Bmatrix}$$

Similarly, for subsystem B we get

$$\begin{bmatrix} \tilde{M}^{B}_{kk} & \tilde{M}^{B}_{kc} \\ M^{B}_{ck} & \tilde{M}^{A}_{cc} \end{bmatrix} \begin{Bmatrix} \ddot{p}^{B}_{k}(t) \\ \ddot{u}^{B}_{c}(t) \end{Bmatrix} + \begin{bmatrix} \tilde{K}^{B}_{kk} & \tilde{K}^{B}_{kc} \\ \tilde{K}^{B}_{ck} & \tilde{K}^{B}_{cc} \end{bmatrix} \begin{Bmatrix} p^{B}_{k}(t) \\ u^{B}_{c}(t) \end{Bmatrix} = \left[\Psi^{B}_{k}\right]^{T} \begin{Bmatrix} 0^{B}_{i} \\ f^{B}_{c} \end{Bmatrix}$$

$\Rightarrow$

Equation for the coupled system

- Compatibility of displacements at the interface: $u^{A}_{c} = u^{B}_{c}$
- Equillibrium of forces at the interface: $f^{A}_{c} + f^{B}_{c} = 0$

$\Rightarrow$

$$\begin{bmatrix} I^{A}_{kk} & 0 & \tilde{M}^{A}_{kc} \\ 0 & I^{B}_{kk} & \tilde{M}^{B}_{kc} \\ \tilde{M}^{A}_{ck} & \tilde{M}^{B}_{ck} & \tilde{M}^{A}_{cc} + \tilde{M}^{B}_{cc} \end{bmatrix} \begin{Bmatrix} \ddot{p}^{A}_{k}(t) \\ \ddot{p}^{B}_{k}(t) \\ \ddot{u}_{c}(t) \end{Bmatrix} + \begin{bmatrix} \omega^{2}_{A} & 0 & 0 \\ 0 & \omega^{2}_{B} & 0 \\ 0 & 0 & \tilde{K}^{A}_{cc} + \tilde{K}^{B}_{cc} \end{bmatrix} \begin{Bmatrix} p^{A}_{k}(t) \\ p^{B}_{k}(t) \\ u_{c}(t) \end{Bmatrix} = 0$$

This equation can further be analyzed to obtain natural frequencies and mode shapes of the combined system.

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once again the coupling degrees of freedom and PKA(t) are the modes emerging from fixed interface model for the substructure, so these together constitute the state vector.
Similarly for system B, subsystem B I can derive a similar equation, okay. Now I have to generate the equation for the coupled system these are equations for subsystem A and B derived independent of each other. Now I will impose this compatibility relations and this equilibrium equation, and I get the combined equation to be of this form where, for the combined system the state vector consists of normal modes of subsystem A, normal modes of subsystem B in the fixed interface format and the coupling degrees of freedom. The coupling degrees of freedom are equal by this compatibility requirement, so this is the state vector, and this is the mass matrix, and this is the stiffness matrix, stiffness matrix itself is in terms of the subsystem natural frequencies in the fixed interface form and the matrix at the, these matrices which we have derived. Now this is an equation that we need to analyze to produce the Eigen solutions for the built-up structure.

Remarks

- The method requires the knowledge of structural matrices of the substructures.
- Not suited for experimental studies since such matrices would not be easily available.
- Also creating fixed interfaces may not be feasible in an experimental substructure.

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Now we can make some observations, this method requires a knowledge of structural matrices of substructures, it is not suited for experimental studies since in experimental studies such matrices would not be easily available, what would be available will be a set of natural frequencies and mode shapes if you do an experiment. Now also creating fixed interfaces may not be feasible in an experimental substructure, okay so you can't weld a member and so on and so forth, so this method is suited for studies where substructures are studied using finite element method, but as I said one of the motivations for studying substructuring methods at least in the current scenario where computational power is fairly generously available, the reduction of the model in terms of degrees of freedom may or may not be that crucial but it becomes indeed crucial if some of the substructures are studied experimentally and they need to be combined with other substructures which may be modeled mathematically, okay and also this kind of reduction in computational efforts become relevant if you are doing uncertainty analysis where this, running of these programs forms a part of a Monte Carlo simulation run, where you need to repeatedly run this programs for, you know, nominally identical values of model parameters, so these methods remain relevant in such situations.

Free interface modal coupling method

Equation for subsystem A $\left[N_A \text{ dofs}\right]$

$$M^A\ddot{u}_A + K^A u^A = f^A$$

$$\begin{bmatrix} M_{ii}^A & M_{ic}^A \\ M_{ci}^A & M_{cc}^A \end{bmatrix}\begin{Bmatrix} \ddot{u}_i^A \\ \ddot{u}_c^A \end{Bmatrix} + \begin{bmatrix} K_{ii}^A & K_{ic}^A \\ K_{ci}^A & K_{cc}^A \end{bmatrix}\begin{Bmatrix} u_i^A \\ u_c^A \end{Bmatrix} = \begin{Bmatrix} 0_i^A \\ f_c^A \end{Bmatrix} \quad (*)$$

Sizes: $u_i^A \sim (N_A - n_c)\times 1; u_c^A \sim n_c \times 1$

$K^A\phi^A = \omega_A^2 M^A \Rightarrow N_A \times N_A$ matrices : $\Lambda^A = \text{diag}\left[\omega_{Ai}^2\right] \& \Phi^A$

such that $\left[\Phi^A\right]^T M^A\Phi^A = I \& \left[\Phi^A\right]^T K^A\Phi^A = \Lambda^A$.

Consider the k_A term expansion of u^A given by $u^A(t) = \Phi_{k_A}^A p_{k_A}(t)$

Write $\Phi_{k_A}^A = \begin{bmatrix} \Phi_{k_A i}^A \\ \Phi_{k_A c}^A \end{bmatrix}$

Substituting in $(*)$ we get

$$I\ddot{p}_{k_A}(t) + \Lambda_{k_A}^A p_{k_A}(t) = \left[\Phi_{k_A c}^A\right]^T f_c^A(t)$$

Now we'll move on to the next method known as free interface modal coupling method, here the objective is to produce a solution strategy where we bank on the Eigen solutions of substructures to construct the coupling matrices and the modal for the coupled system, so how does that work? So here again let us consider the equation for subsystem A, I write it in this form, and as before we will partition the degrees of freedom and to interior and coupling degrees of freedom and that induces partitioning of structural matrices and the forcing vector as shown here. Now the sizes are as shown here, we can perform now the Eigenvalue analysis corresponding to this system, I am not putting UCA to be 0 now, I am including it in the model. Now I will perform the Eigenvalue analysis and determine NA x NA modal matrix and diagonal matrix of Eigenvalues, okay, and I will make this modal matrix to be mass normalized so that phi transpose M phi is I, and phi transpose K phi becomes lambda for the subsystem A.

Now if I were to consider KA term expansion for the response, this is a full NA by NA matrix, but I may not include all the modes in a given calculation so out of capital N modes that are available I may retain K modes, so I will say that I am going to write UA(t) as phi KA PKA(t), where PKA(t) are the generalized coordinates. The subscript K here denotes the fact that I am retaining only K terms in the expansion, and KA means I'm talking about subsystem A.

Now I will now partition, U has been partitioned as interior and coupling degrees of freedom, and just as it has induced partitioning of stiffness and mass matrices it also induces a partitioning of modal matrix that is written here. Now we substitute this in, this equation and use the orthogonality relation I get the equation for subsystem A in this form.

Similarly, for sybsystem B we get

$$I\ddot{p}_{k_B}(t) + \Lambda^B_{k_B} p_{k_B}(t) = \left[\Phi^B_{k_Bc}\right]^T f^B_c(t)$$

The combined equation for the two systems can be written as

$$I\begin{Bmatrix} \ddot{p}_{k_A}(t) \\ \ddot{p}_{k_B}(t) \end{Bmatrix} + \begin{bmatrix} \Lambda^B_{k_B} & 0 \\ 0 & \Lambda^B_{k_B} \end{bmatrix}\begin{Bmatrix} p_{k_A}(t) \\ p_{k_B}(t) \end{Bmatrix} = \begin{bmatrix} \left[\Phi^A_{k_Ac}\right]^T & 0 \\ 0 & \left[\Phi^B_{k_Bc}\right]^T \end{bmatrix}\begin{Bmatrix} f^A_c(t) \\ f^B_c(t) \end{Bmatrix} \quad (**)$$

We need to now impose the conditions that $u^A_c(t) = u^B_c(t)$

$$\Rightarrow \Phi^A_{k_Ac} p_{k_A}(t) = \Phi^B_{k_Bc} p_{k_B}(t)$$

$$\Rightarrow \begin{bmatrix} \Phi^A_{k_Ac} & -\Phi^B_{k_Bc} \end{bmatrix}\begin{Bmatrix} p_{k_A}(t) \\ p_{k_A}(t) \end{Bmatrix} = 0 \Rightarrow Sp = 0 \text{ with } S = \begin{bmatrix} \Phi^A_{k_Ac} & -\Phi^B_{k_Bc} \end{bmatrix}$$

Partition S into a nonsingular square matrix S_d and remaining part S_i.

$$\begin{bmatrix} S_d & S_i \end{bmatrix}\begin{Bmatrix} p_d \\ p_i \end{Bmatrix} = 0 \Rightarrow p_d = -S_d^{-1} S_i p_i$$

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Similarly for subsystem B using the same logic I will get this equation, so just to recall I mean just to emphasize this capital Lambda is the natural frequencies of subsystem B, and the interface is kept free, and this is the corresponding modal matrix. Now the combined equation for the 2 systems is written in this form. Now we need to now impose the conditions that UCA(t) is UCB(t), so how do I get that? That is not implicit here, so this PKA and PKB contain information on coupling degrees of freedom, in some sense this equation need to be now constrained by this additional requirement, how do we do that? So UCA(t) is given by this in terms of degrees of freedom, generalized degrees of freedom PKA(t) and the modal matrix, so these should be equal, so this produces a constraint equation as shown here SP = 0, where S is the modal matrix corresponding to the coupling degrees of freedom for subsystem A and similar entity for subsystem B.

Now what I will do is, this combined vector P, I will now partition into PD and PI, this partitioning is done from a mathematical perspective, and I will select D degrees of freedom so that S, this matrix S gets partitioned into SD and SI, and SD remains as a matrix that can be inverted, it is a nonsingular matrix that can be inverted, consequently I can retain, I mean write PD as in terms of PI as – SD inverse SI PI, so this is a crucial step here. Now once I do this I

$$\Rightarrow \begin{Bmatrix} p_{k_A}(t) \\ p_{k_B}(t) \end{Bmatrix} = \begin{Bmatrix} p_d \\ p_i \end{Bmatrix} = \begin{bmatrix} -S_d^{-1}S_i \\ I \end{bmatrix} p_i = \Psi\{p_i\} \text{ with } \Psi = \begin{bmatrix} -S_d^{-1}S_i \\ I \end{bmatrix}$$

Noting that $f_c^A(t) = -f_c^B(t) = f_c(t)$,

$$\begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T & 0 \\ 0 & \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} \begin{Bmatrix} f_c^A(t) \\ f_c^B(t) \end{Bmatrix} = \begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T & 0 \\ 0 & \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} f_c(t) = \begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T \\ \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow \Psi\{\ddot{p}_i\} + \begin{bmatrix} \Lambda_{k_B}^B & 0 \\ 0 & \Lambda_{k_B}^B \end{bmatrix} \Psi\{p_i\} = \begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T \\ \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow \Psi^T\Psi\{\ddot{p}_i\} + \Psi^T \begin{bmatrix} \Lambda_{k_B}^B & 0 \\ 0 & \Lambda_{k_B}^B \end{bmatrix} \Psi\{p_i\} = \Psi^T \begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T \\ \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow M\ddot{q} + Kq = f_q$$

$$M = \Psi^T\Psi;\ K = \Psi^T \begin{bmatrix} \Lambda_{k_B}^B & 0 \\ 0 & \Lambda_{k_B}^B \end{bmatrix} \Psi;\ q = \{p_i\};\ f_q = \Psi^T \begin{bmatrix} \left[\Phi_{k_A c}^A\right]^T \\ \left[\Phi_{k_B c}^B\right]^T \end{bmatrix} f_c(t)$$

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can write the vector P which is PD PI in this form retaining only PI, and this matrix I denote as sai, that is sai is - SD inverse SI by I. Now noting that this equation is satisfied I get now the coupled equation in this form, this is purely in terms of PI which degrees of freedom which I am retaining, so I will pre multiply by sai transpose and carry out this operation, and I get the equation for the built up system M is sai transpose sai, K is sai transpose this into sai, Q is PI, FQ is this.

$$\Rightarrow \begin{Bmatrix} p_{k_A}(t) \\ p_{k_B}(t) \end{Bmatrix} = \begin{Bmatrix} p_d \\ p_i \end{Bmatrix} = \begin{bmatrix} -S_d^{-1}S_i \\ I \end{bmatrix} p_i = \Psi\{p_i\}$$

Noting that $f_c^A(t) = -f_c^B(t) = f_c(t)$,

$$\begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T & 0 \\ 0 & \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} \begin{Bmatrix} f_c^A(t) \\ f_c^B(t) \end{Bmatrix} = \begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T & 0 \\ 0 & \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} f_c(t) = \begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T \\ \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow \Psi\{\ddot{p}_i\} + \begin{bmatrix} \Lambda^B_{k_B} & 0 \\ 0 & \Lambda^B_{k_B} \end{bmatrix} \Psi\{p_i\} = \begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T \\ \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow \Psi^T\Psi\{\ddot{p}_i\} + \Psi^T \begin{bmatrix} \Lambda^B_{k_B} & 0 \\ 0 & \Lambda^B_{k_B} \end{bmatrix} \Psi\{p_i\} = \Psi^T \begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T \\ \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} f_c(t)$$

$$\Rightarrow M\ddot{q} + Kq = f_q$$

$$M = \Psi^T\Psi; K = \Psi^T \begin{bmatrix} \Lambda^B_{k_B} & 0 \\ 0 & \Lambda^B_{k_B} \end{bmatrix} \Psi; q = \{p_i\}; f_q = \Psi^T \begin{bmatrix} \left[\Phi^A_{k_A c}\right]^T \\ \left[\Phi^B_{k_B c}\right]^T \end{bmatrix} f_c(t)$$

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Now since no forces are taken to act at the interface I take the equation to be MQ double dot +

Since no forces are taken to act at the interface, $M\ddot{q} + Kq = 0$

Summary

$$M = \Psi^T\Psi; K = \Psi^T \begin{bmatrix} \Lambda^B_{k_B} & 0 \\ 0 & \Lambda^B_{k_B} \end{bmatrix} \Psi; \Psi = \begin{bmatrix} -S_d^{-1} S_i \\ I \end{bmatrix}$$

$$S = \begin{bmatrix} S_d & S_i \end{bmatrix} = \begin{bmatrix} \Phi^A_{k_A c} & -\Phi^B_{k_B c} \end{bmatrix}$$

Remarks

- The structural matrices for the built-up structure is constructed in terms of eigensolutions of the substructures.
- The eigensolutions for one or more of the substructures can be experimentally established.
- The knowledge of structural matrices for the substructures is not directly needed.

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KQ = 0, so to summarize I got the mass matrix as sai transpose sai, K as sai transpose this, and sai itself as this, and this S etcetera is as shown here. Now what needs to be noted here is that these matrices M and K are now derived purely in terms of the Eigen solutions of substructure A and B in the free interface form, you don't need the structural matrices for A and B while implementing this method, the Eigen solutions for one or more of the substructure can be experimentally established suppose A can be an finite element model, B can be an experimental model, or both the A and B can be experimental models, of course A and B can also be finite element models, so the knowledge of structural matrices for the substructures is not directly needed, if you have it so well and good, but it is not needed, so this is the method known as component mode synthesis.

Free interface modal coupling method with elastic coupling

Consider the situation when the two substructures are coupled elastically.

The equation for coupled system needs to be modified to take into account this feature.

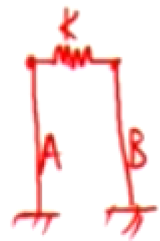
The compatibility condition $u_c^A(t) = u_c^B(t)$ is no longer applicable.

Consider the condition

$f_c^A(t) = -f_c^B(t)$

Let the matrix K_{CPL} represent the stiffness matrix such that

$$\begin{Bmatrix} f_c^A(t) \\ f_c^B(t) \end{Bmatrix} = K_{CPL} \begin{Bmatrix} u_c^A(t) \\ u_c^B(t) \end{Bmatrix} \text{ with } K_{CPL} = \begin{bmatrix} K_{CC} & -K_{CC} \\ -K_{CC} & K_{CC} \end{bmatrix}$$



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Now suppose we are considering a situation where 2 substructures are coupled elastically, that means there is one more coupling element, the equation for coupled system needs to be modified now to take into account this feature, so if you do free interface method the coupling element will not be easily represented. Now the compatibility condition in this case is no longer applicable, by this what I mean, suppose I have a system, this is one substructure, and this is another substructure, and this is a coupling element, this is A, and this is B, so I am talking about how to deal with this elastic coupling, this UCA(t) will not be equal to UB(t) because there is an elastic coupling between the two, okay, so it's not rigidly connected, so this displacement can be different from this.

Now on the other hand this equation remains valid, so what I do now is I will define a coupling matrix KCPL, let that represent the stiffness matrix such that at the coupling the displacements and the forces are related through this coupling matrix, with KCPL being this equal to this.

Now the equation for coupled system needs to be modified to take into account this coupling, Consider the situation when the two substructures are coupled elastically.

The equation for coupled system needs to be modified to take into account this feature.

The compatibility condition $u_c^A(t) = u_c^B(t)$ is no longer applicable.

Consider the condition

$f_c^A(t) = -f_c^B(t)$

Let the matrix K_{CPL} represent the stiffness matrix such that

$$\begin{Bmatrix} f_c^A(t) \\ f_c^B(t) \end{Bmatrix} = K_{CPL} \begin{Bmatrix} u_c^A(t) \\ u_c^B(t) \end{Bmatrix} \text{ with } K_{CPL} = \begin{bmatrix} K_{CC} & -K_{CC} \\ -K_{CC} & K_{CC} \end{bmatrix}$$

$$\left.\begin{matrix} u^A(t) = \Phi_{k_A}^A p_{k_A}(t) \\ u^B(t) = \Phi_{k_B}^B p_{k_B}(t) \end{matrix}\right\} \Rightarrow \begin{Bmatrix} u_c^A(t) \\ u_c^B(t) \end{Bmatrix} = \begin{bmatrix} \Phi_{k_A}^A & 0 \\ 0 & \Phi_{k_B}^B \end{bmatrix} \begin{Bmatrix} p_{k_A}(t) \\ p_{k_B}(t) \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} f_c^A(t) \\ f_c^B(t) \end{Bmatrix} = K_{CPL} \begin{bmatrix} \Phi_{k_A}^A & 0 \\ 0 & \Phi_{k_B}^B \end{bmatrix} \begin{Bmatrix} p_{k_A}(t) \\ p_{k_B}(t) \end{Bmatrix}$$

the compatibility condition is no longer applicable. Now you consider this and we get this, now UA(t) as before I will express in terms of this, and UB(t) in terms of this, so this is a representation I have. Now this forcing is related through to these degrees of freedom, through this relation, now consequently if you substitute these relations into the governing equation, I

Substituting these relations into

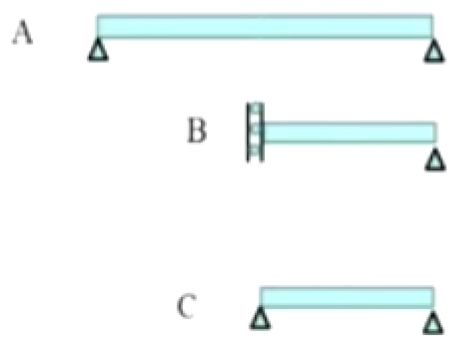
$$I\begin{Bmatrix}\ddot{p}_{k_A}(t)\\ \ddot{p}_{k_B}(t)\end{Bmatrix}+\begin{bmatrix}\Lambda^B_{k_B} & 0\\ 0 & \Lambda^B_{k_B}\end{bmatrix}\begin{Bmatrix}p_{k_A}(t)\\ p_{k_B}(t)\end{Bmatrix}=\begin{bmatrix}\left[\Phi^A_{k_A c}\right]^T & 0\\ 0 & \left[\Phi^B_{k_B c}\right]^T\end{bmatrix}\begin{Bmatrix}f_c^A(t)\\ f_c^B(t)\end{Bmatrix}$$

one gets the final equation of motion for the bulit-up system as

$$I\begin{Bmatrix}\ddot{p}_{k_A}(t)\\ \ddot{p}_{k_B}(t)\end{Bmatrix}+\left[\begin{bmatrix}\Lambda^B_{k_B} & 0\\ 0 & \Lambda^B_{k_B}\end{bmatrix}-\begin{bmatrix}\left[\Phi^A_{k_A c}\right]^T & 0\\ 0 & \left[\Phi^B_{k_B c}\right]^T\end{bmatrix}K_{CPL}\begin{bmatrix}\Phi^A_{k_A} & 0\\ 0 & \Phi^B_{k_B}\end{bmatrix}\right]\begin{Bmatrix}p_{k_A}(t)\\ p_{k_B}(t)\end{Bmatrix}=0$$

get equation of this form, and the final equation of motion, this is for system A and B together, and the final equation for the built-up system is obtained like this where the stiffness matrix for the coupled system is modified through these additional matrices, so this helps in dealing with situations where there is a elastic coupling between subsystems.

Use of symmetry in reducing model size



Instead of solving larger problem (A) solve two smaller problems (B and C) and synthesize the solution for the larger problem.

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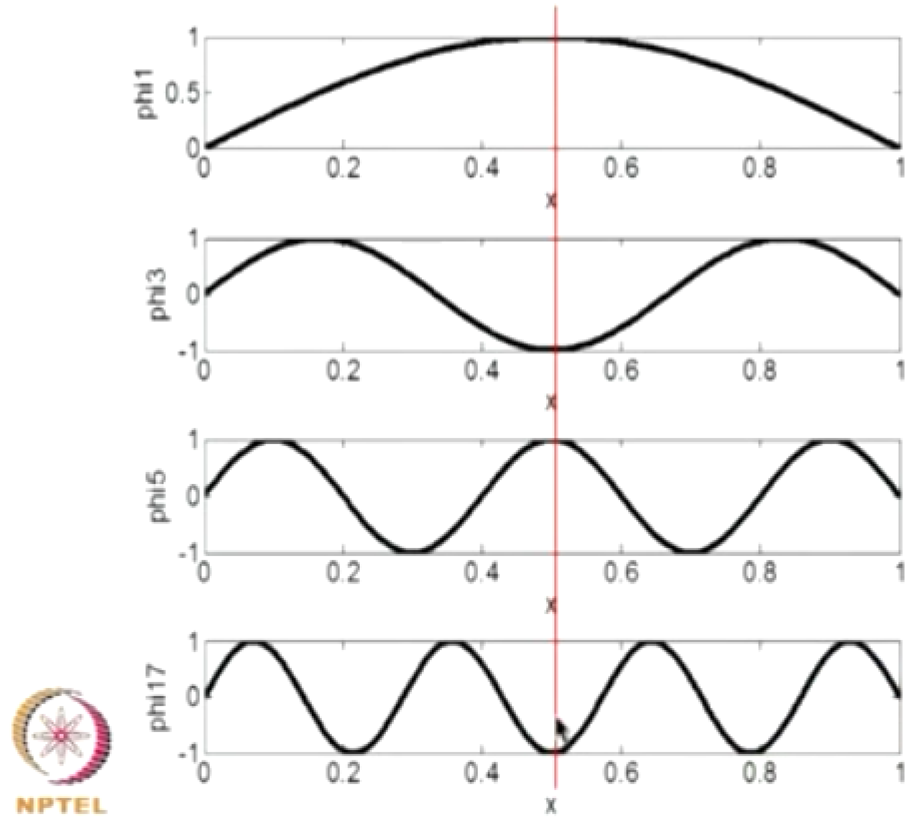
Now there are few other strategies where, I mean strategies become possible when in certain class of problems, for example if the structure has certain planes of symmetry, for example if you have a 1 span beam like this, can I construct the solution of this structure by analyzing a smaller problem, okay, so instead of solving the larger problem A, we will solve 2 problems B and C, and synthesize the solution of this problem, so what I do is I cut the structure here and I introduce this boundary condition, this is a roller and a hinge. So now if you look at mode



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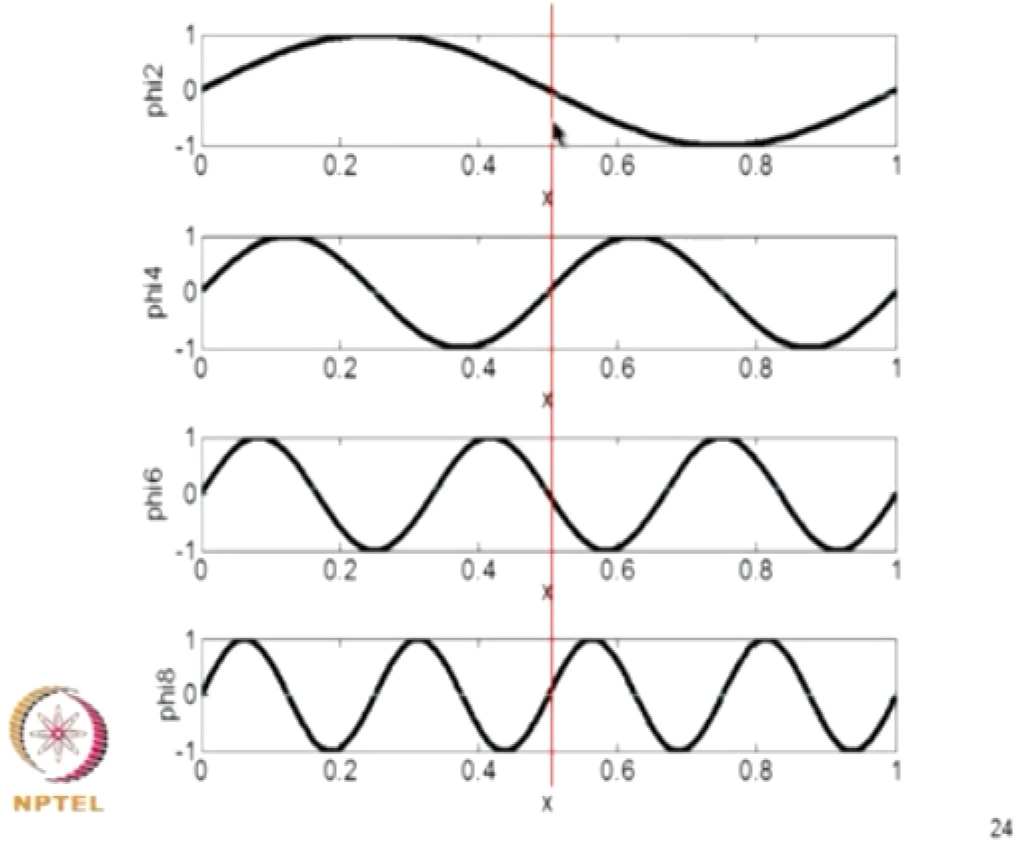
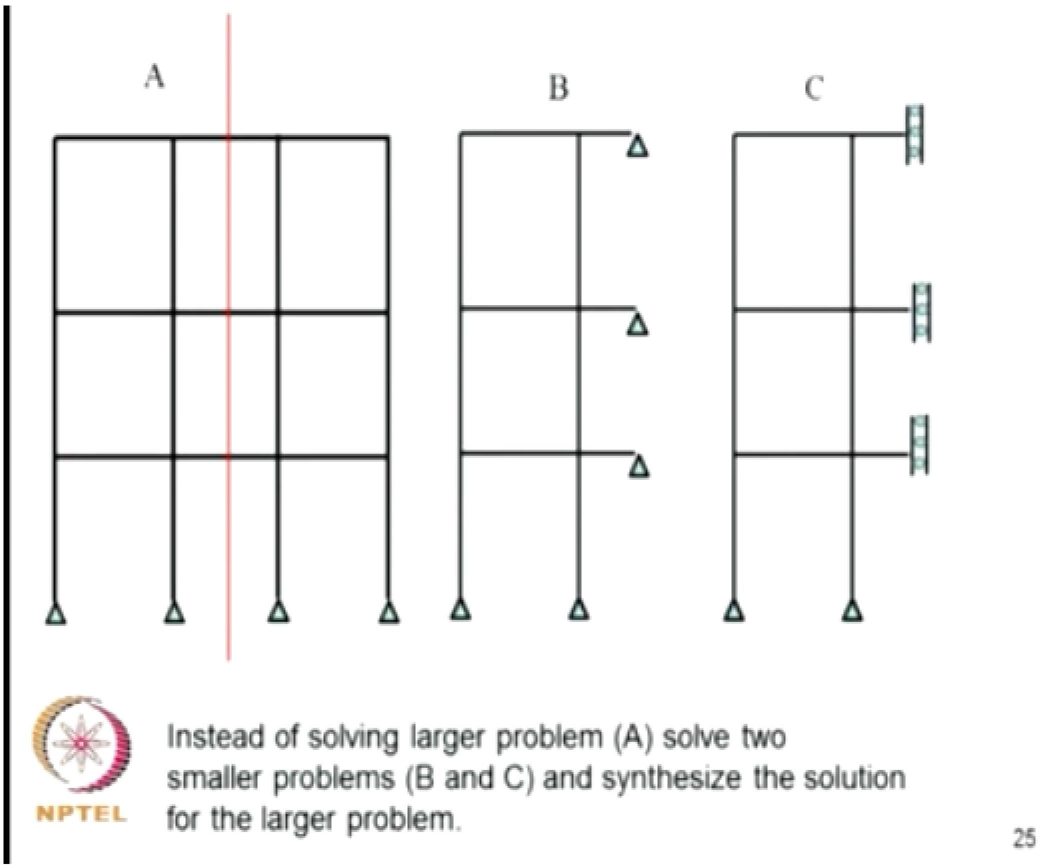
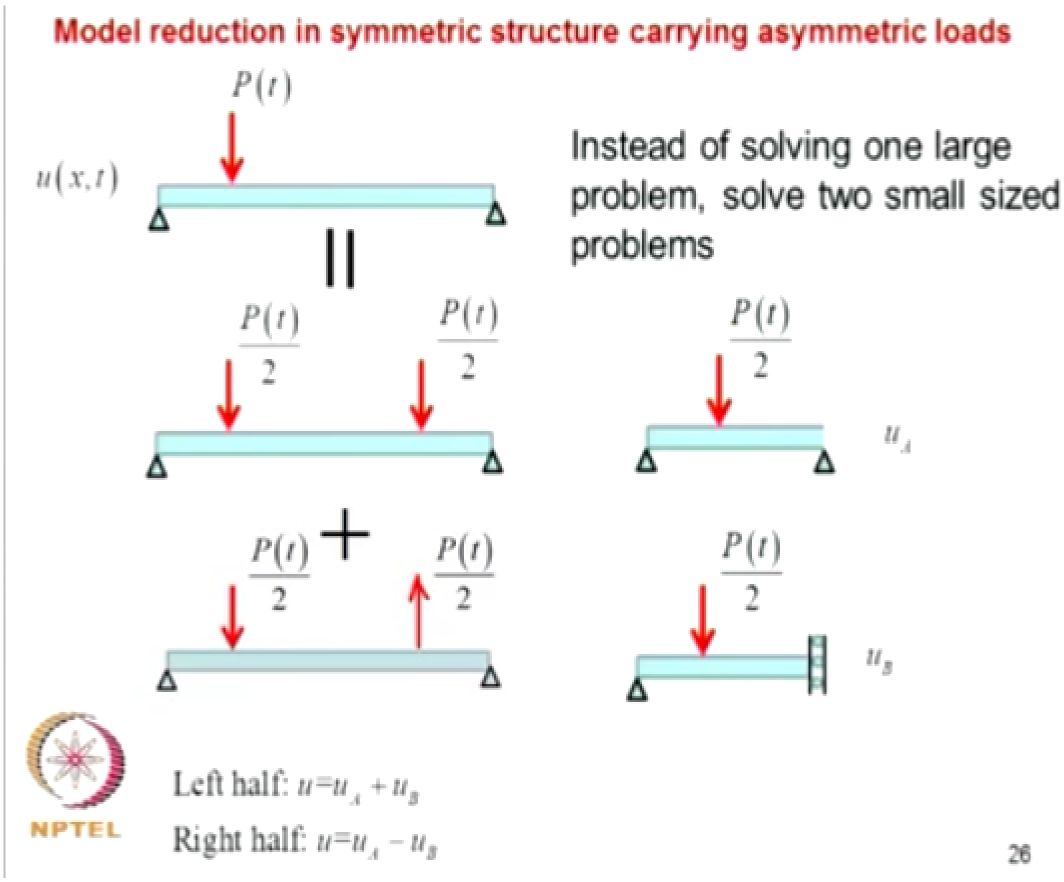
shapes of this simply supported beam all the modes 1, 3, 5 etcetera are symmetric about X = L/2, so these modes can be simulated by considering this situation. The displacement here you can see this is 0 is not 0, okay you followed I mean, right this is first mode, this is third mode, this is for fifth mode, and so on and so forth. So this can be simulated this, symmetric modes can be simulated like this, the even modes that is 2, 4, 6 etcetera are anti-symmetric, this is L/2



you can see here this mode shape is this and this is anti-symmetric, these anti-symmetric modes can be synthesized by considering this problem, so what I can do is I can analyze 2 small problems okay and find out mode shapes in for this and this, and by suitably using planes of symmetry I can construct the mode shape for this built up system, all the symmetric modes can be obtained from this, anti-symmetric modes can be obtained from this, and complete system modes can be thus constructed without solving a larger problem.



Now this is again illustrated here, suppose you have a multi-story building frame, this frame may require say 500 degrees of freedom to analyze the problem, so what I will do is, I will utilize the plane of symmetry which is this red line, and analyze 2 problems okay, so this will have 250 degrees of freedom, this is 250 degrees of freedom, so 2 such smaller problems will be solved and I will synthesize the solution for the built up system, okay that means instead of solving a larger problem A we solve 2 smaller problems B and C, and synthesize a solution for the larger problem, so this is another way of utilizing certain problem features to achieve model reduction, of course this is clearly possible only when such symmetries exist in the given structure.



Now suppose the symmetric structure is loaded asymmetrically, what happens, okay so that is not a problem, for example if you consider a simply supported beam loaded symmetrically by a force P(t), I will now consider 2 problems and this plus, so if you add these 2 you get solution for this problem. Now each of these problems can be handled by solving these 2 problems, okay, so this is you know advantage of using symmetric, so in principle can be used even when if structure is of course symmetric about X = L/2 in terms of boundary condition, geometry etcetera, but it is asymmetrically loaded, okay, so still we can utilize symmetry by following these arguments/

Element development

- Plane stress and plane strain models
- Axisymmetric solids
- 3D solids
- Plate bending elements
- Shell elements

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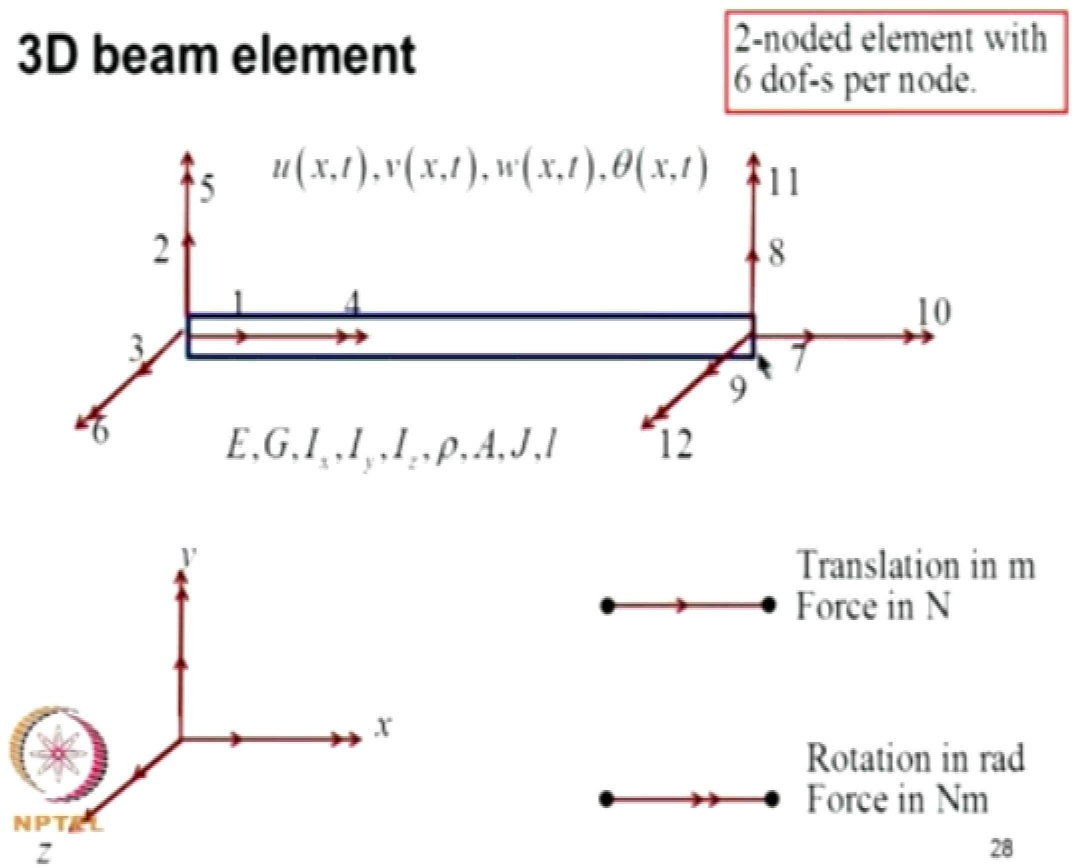
We will now return to the earlier, one of the themes that we have been pursuing that is development of structural matrices, so we considered a few lectures before, what we achieved



was we developed the mass and stiffness matrices for this 3D generalized, 3-dimensional beam with the 2 nodes and 6 degrees of freedom per mode, okay, so how did we do that? We considered this structure to you know the displacement fields that we assumed was axial deformation, deformation in plane, deformation out of plane, and the rotation about the longitudinal axis, so we accounted for all the energies, so this is energy due to axial

$$U=\underbrace{\frac{1}{2}\int_0^L AE\left(\frac{\partial u}{\partial x}\right)^2 dx}_{\text{Axial deformation}}+\underbrace{\frac{1}{2}\int_0^L GJ\left(\frac{\partial \theta}{\partial x}\right)^2 dx}_{\text{Twisting}}$$

$$+\underbrace{\frac{1}{2}\int_0^L EI_z\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx}_{\text{Bending@z}}+\underbrace{\frac{1}{2}\int_0^L EI_y\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx}_{\text{Bending@y}}$$

$$\left(\text{with } J=\int_A\left\{\left(-y+\frac{\partial \psi}{\partial x}\right)^2+\left(z+\frac{\partial \psi}{\partial z}\right)^2\right\}dA\right)$$

$$T=\underbrace{\frac{1}{2}\int_0^L m\dot{u}^2 dx}_{\text{Axial deformation}}\quad\underbrace{\frac{1}{2}\int_0^L I_m\dot{\theta}^2 dx}_{\text{Twisting}}+\underbrace{\frac{1}{2}\int_0^L m\dot{v}^2 dt}_{\text{Bending@z}}+\underbrace{\frac{1}{2}\int_0^L m\dot{w}^2 dt}_{\text{Bending@y}}$$

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$$\left(\text{with } I_m=\int_A \rho\left(y^2+z^2\right)dA\right)$$

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deformation, due to twisting, this is bending about Z, bending about Y, and similarly we computed the kinetic energy due to axial deformation, twisting, bending about Z, and bending about Y, and using this we constructed the Lagrangian and we interpolated these field variable in terms of their nodal values for U and theta we use linear interpolation functions, and for V and W use cubic interpolation functions, and we derived the 12/12 structural matrices.

Now we wish to now carry this exercise further, beyond 3D element now we can think of continuum problems, so in continuum problems we can think of plane, 2 dimensional problems first of all, so in 2 dimensional problems we have plane stress, plane strain, and axisymmetric problems, so then we can consider 3 dimensional solids and then plate bending elements, and some discussion on shell elements, so the general template for achieving this is basically to write the appropriate equations for strain energy and kinetic energy, identify the field variables interpolate, and you know take an element, identify the nodes, and degrees of freedom, and develop the, represent the field variables using polynomials in terms of the nodal degrees of freedom, substitute into the Lagrange's equation and derive the equation for the nodal degrees of freedom, and then question of coordinate transformation, assembling, imposition of boundary conditions, calculating external nodal forces, and thus deriving the final equilibrium equation will have to be done. Many of these steps we have already covered, the step beyond formulation of element level structural matrices has already been covered, okay, the problem of coordinate transformation, assembling of matrices, imposition of boundary conditions, computation of nodal forces and assembling all that has been earlier done, so to be able to extend that framework to more complicated problems we need to now expand our repertory of elements, so with that in mind in the next few lectures we will start discussing about some of these elements, and we'll begin by talking about 2-dimensional element, namely plane stress and plane strain elements.

Summary of equations of linear elasticity

Dependent and independent variables

$$\sigma(x_1,x_2,x_3,t)=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{bmatrix}(x_1,x_2,x_3,t)$$

$$\varepsilon(x_1,x_2,x_3)=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}(x_1,x_2,x_3,t)$$

$$u(x_1,x_2,x_3,t)=\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}(x_1,x_2,x_3,t)$$

Number of independent variables: 4

Number of dependent variables: 15

$$\sigma' = T^t \sigma T$$

$$\varepsilon' = T^t \varepsilon T$$

$$u' = T^t u$$

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Now in all this I'm assuming the structure behaves linearly, and material is isotropic and elastic, elastic, isotropic we are also going to assume homogeneity within an element, so the governing equations we will quickly recall from 3 dimensional linear elasticity, the state of stress at any point X1, X2, X3 is at given by this tensor, and the state of strain is given by this tensor, and the displacement fields U1, U2, U3 all these quantities are functions of X1, X2, X3, and time, so a number of independent variables is 4, that is X1, X2, X3 and T, and number of dependent variables is 15, there are 6 stress components, 6 strain components, and 3 displacements, so how do we tackle this problem. So upon coordinate transformation the stresses obey this rule of transformation, therefore second order tensors the displacement is a vector it follows this rule of transformation, where T is the coordinate transformation matrix.

Equations of elasticity for linear, isotropic, elastic continua

Equillibri um equations (3)

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + X_1 = \rho \ddot{u}_1$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + X_2 = \rho \ddot{u}_2$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + X_3 = \rho \ddot{u}_3$$

$$\sigma_{ij} = \sigma_{ji}; i, j = 1,2,3$$

Strain displaceme nt equations (6)

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}; \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}; \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right); \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right);$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\varepsilon_{ij} = \varepsilon_{ji}; i, j = 1,2,3$$

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Now we have 3 equilibrium equations which are given here, and this symmetry rho IJ = rho JI is assumed, that means we assume that there are no body moments acting on the system, so these are the 3 equilibrium equations. Similarly we have the 6 strain displacement relations as shown here, and as I said we are using basically linear models, so the strain displacement

Constitutive laws (6)

$$e = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\sigma_{11} = \lambda e + 2G\varepsilon_{11}$$

$$\sigma_{22} = \lambda e + 2G\varepsilon_{22}$$

$$\sigma_{33} = \lambda e + 2G\varepsilon_{33}$$

$$\sigma_{12} = 2G\varepsilon_{12}$$

$$\sigma_{13} = 2G\varepsilon_{13}$$

$$\sigma_{23} = 2G\varepsilon_{23}$$

$$E = \frac{G(3\lambda + 2G)}{\lambda + G}$$

$$\nu = \frac{\lambda}{2(G+\lambda)}$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\varepsilon_{11} = \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu I_1\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu I_1\right]$$

$$\varepsilon_{33} = \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu I_1\right]$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12}$$

$$\varepsilon_{13} = \frac{1}{2G}\sigma_{13} = \frac{1+\nu}{E}\sigma_{13}$$

$$\varepsilon_{23} = \frac{1}{2G}\sigma_{23} = \frac{1+\nu}{E}\sigma_{23}$$

Index notations

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right)$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

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relation would be linear and we assume material is hookean and also isotropic, therefore the relationship between stress and strain there will be 6 constitutive laws relating stress and strains, and they are given here, there are 2 elastic constants E and nu which is Young's modulus and person's ratio or lemma is constant and shear modulus, either we can express stress in terms of strains, or strain in terms of stresses. We can also use index notations, this I'm not going to do in this discussion, so for completeness I have mentioned here all these equations can be written in short form in this way. There is an alternative notation where the stress instead

Alternative notations

$$\sigma = \{\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{23}\}^t$$

$$\varepsilon = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad 2\varepsilon_{12} \quad 2\varepsilon_{13} \quad 2\varepsilon_{23}\}^t$$

$$D = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

$$C = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{\nu} \end{bmatrix}$$

$$\sigma = C\varepsilon$$

$$\tilde{C} = C^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\varepsilon = \tilde{C}\sigma$$

of being written as a matrix we write it as a vector 6 x 1 vector, this is the representation for stress, this is a representation for strain, and this is the matrix D that I will be needing in the formulation, and this 6 x 1 stress and strain matrices are related, for example stress is related to strain through this 6/6 matrix or strain is related to stress through this C tilde which is this, okay, so this is as I said material that obeys Hooke's law and which is isotropic.

$$D^t\sigma + X = \rho\ddot{u}$$

$$Du = \varepsilon$$

$$\sigma = C\varepsilon$$

$$\varepsilon = \tilde{C}\sigma$$

$$D^tC\varepsilon + X = \rho\ddot{u}$$

$$D^tCDu + X = \rho\ddot{u}$$

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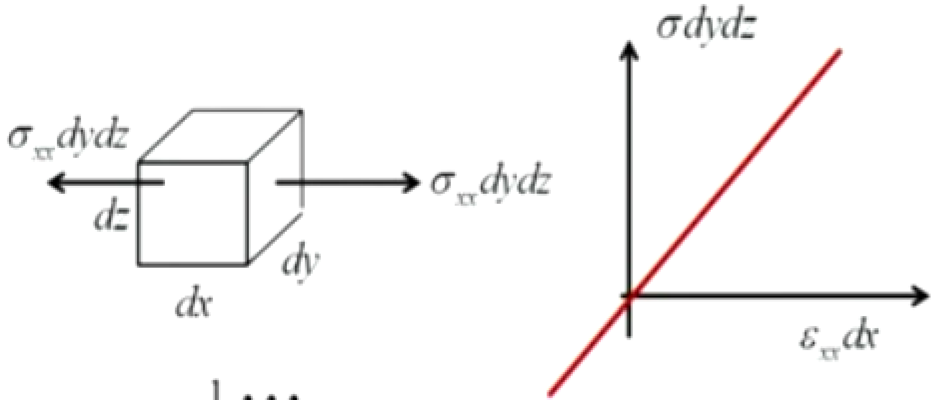
Now in terms of this matrix D the equilibrium equation can be written in this form, and this is the strain displacement relations, this is the constitutive law either these 2 represent constitutive law, and by eliminating using these relations I can write for sigma C into epsilon I get this equation, and for epsilon I can write in terms of DU so I will get this equation, and this is the governing equation for displacement in terms of operator D and matrix C, so this is the equation that we'll be solving.

Strain energy and kinetic energy



$$U = \frac{1}{2}\iiint \sigma_{xx}\varepsilon_{xx}\, dxdydz$$

$$\frac{1}{2}\int_V \sigma^t \varepsilon\, dxdydz \quad T = \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dxdydz$$

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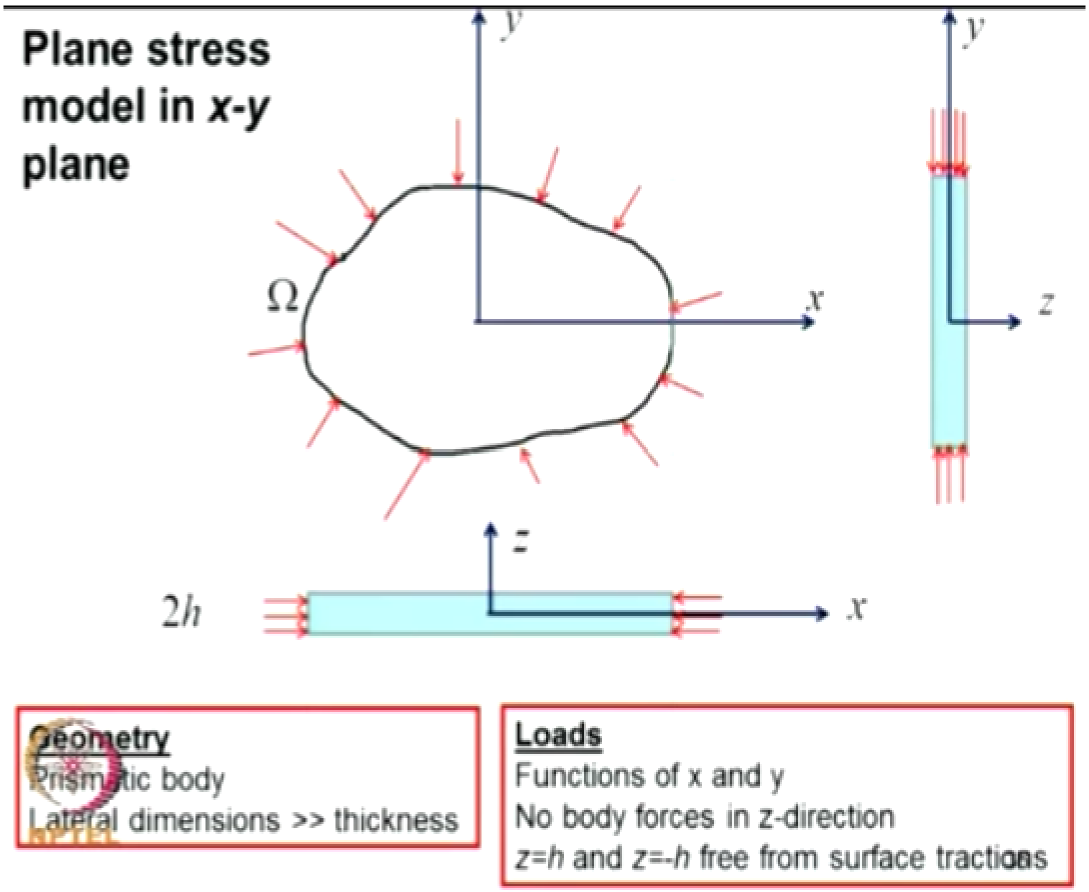
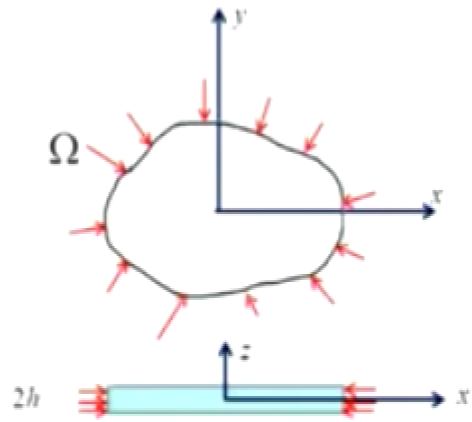
We also need to discuss about the energies in a 3 dimensional situation, what is the expression for energy? Now this can be explained by considering an infinitesimal element as shown here, and suppose if it is loaded you know if you consider this and only the axial stresses, the force on this phase is sigma XX into DY into DZ, and this force will deform the object, and therefore this force would have done some work on that deformation, and the question is what happens to that work done? It is stored as strain energy in the system, so this is given by this integral sigma XX into epsilon XX DX, this is the force and the elongation will be epsilon XX into DX, therefore this into this integrated or the entire volume will give us the total strain energy, so this is total strain energy. Kinetic energy is given in terms of displacement U, velocity U dot V dot and W dot as shown here.



Now we would like to simplify the problem of 3-dimensional elasticity, so as I said here we have 15 dependent variables, and we have 15 equations that is 3 equilibrium equations, 6 strain displacement relations, and 6 constitutive laws, so there are 15 equations, and 15 unknowns, now not all these 15 equations need to be solved for all situations, in some situations we can simplify. There are 2 aspects to this simplification, there are 4 independent variables X, Y, Z, and T, and there are 15 dependent variables, we can cut down on number of dependent variables and also we can cut down on the dependence of this variables on X, Y, Z, and T by you know suitable simplification, if there is no dynamics the T becomes irrelevant so it will be independent variables will be, 3 instead of 4, but can we use certain features of the problem and reduce these quantities further, so one such approach is the so-called plane stress model. So to consider a plane stress model in XY plane, we consider a continuum as shown here and the basic problem is, the red lines show the surface traction, there is a body force acting on this, and we want to analyze this continuum would be supported in some manner, and all that will be specified given the geometry of this continuum the way it is supported, and the surface tractions, and the body forces, and the constitutive law of the material which makes this continuum, what are the stresses, strains, and displacements, that is a problem we need to solve. Now the specification of geometry for plane stress model we restrict our attention to prismatic body, okay, so this is what is meant prismatic body along the Z axis the cross sectional area, the cross sectional property do not change, and lateral dimensions are much greater than thickness, so this thickness is smaller compared to the lateral dimensions, this is the restriction on geometry.

How about loads? The loads are only in the XY plane, and there is no body force in the Z direction, the loads can be surface tractions and body forces but they are restricted only to lie in the XY plane, so the surface Z equal to this thickness I have taken it as 2H, so Z = H, and Z = -H is free from surface tractions, so this top surface is free from surface tractions, so for these class of problems we can make a simplification to the 3-dimensional elasticity problem and that simplification is known as plane stress approximation. How does it work? Now let's consider



Boundary conditions

$\sigma_{zz}(x,y,\pm h)=0$

$\sigma_{zx}(x,y,\pm h)=0$

$\sigma_{zy}(x,y,\pm h)=0$

On Ω $\sigma_{xx},\sigma_{yy},\&\sigma_{xy}$ are independent of z

We interpolate these features into the interior

Boundary conditions

$$\left.\begin{aligned}\sigma_{zz}(x,y,z)&=0\\ \sigma_{zx}(x,y,z)&=0\\ \sigma_{zy}(x,y,z)&=0\end{aligned}\right\}\forall x,y,z\in\Omega$$

$$\left.\begin{aligned}\sigma_{xx}(x,y,z)&=\sigma_{xx}(x,y)\\ \sigma_{yy}(x,y,z)&=\sigma_{yy}(x,y)\\ \sigma_{xy}(x,y,z)&=\sigma_{xy}(x,y)\end{aligned}\right\}\forall x,y,z\in\Omega$$

the surface Z = +H and – H and as we said there are no surface tractions on that, therefore the stresses acting on those planes namely sigma ZZ, sigma ZX, and sigma ZY must be 0, at Z = plus minus H, for in for all X and Y. And on omega that means on this surface sigma XX, sigma YY and Sigma XY are independent of Z, because loading doesn't change with respect to Z, the loading is uniform across thickness, so that is independent of Z, in developing plane stress model what we do is we assume that these conditions prevail not just at the surface but in the interior also that means we interpolate these features into the interior by that what I mean strictly speaking sigma ZZ is 0 only on Z = plus minus H, but I assume that sigma ZZ is 0 throughout, and sigma ZX similarly is 0 only on these 2 outer surfaces, but I assume that it is 0 for all X, Y, Z. Similarly sigma XX and sigma YY and sigma XY these forces do not change with respect to Z on omega, but what I assume is they don't change anywhere in the interior as well, so this is the approximation.

Equilibrium equations

$$\left.\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+X=0\\
&\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+Y=0\\
&\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial x}+Z=0
\end{aligned}\right\}\Rightarrow
\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+X=0\\
&\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+Y=0
\end{aligned}$$

Constitutive law

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{xy}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix}$$

$\Rightarrow$

$$\varepsilon_{xx}=\frac{1}{E}\left(\sigma_{xx}-\nu\sigma_{yy}\right);\varepsilon_{yy}=\frac{1}{E}\left(-\nu\sigma_{xx}+\sigma_{yy}\right);\varepsilon_{zz}=-\frac{\nu}{E}\left(\sigma_{xx}+\sigma_{yy}\right)$$

$$\varepsilon_{xy}=\frac{1+\nu}{E}\sigma_{xy}=\frac{\sigma_{xy}}{G}\quad \varepsilon_{yz}=\varepsilon_{xz}=0$$

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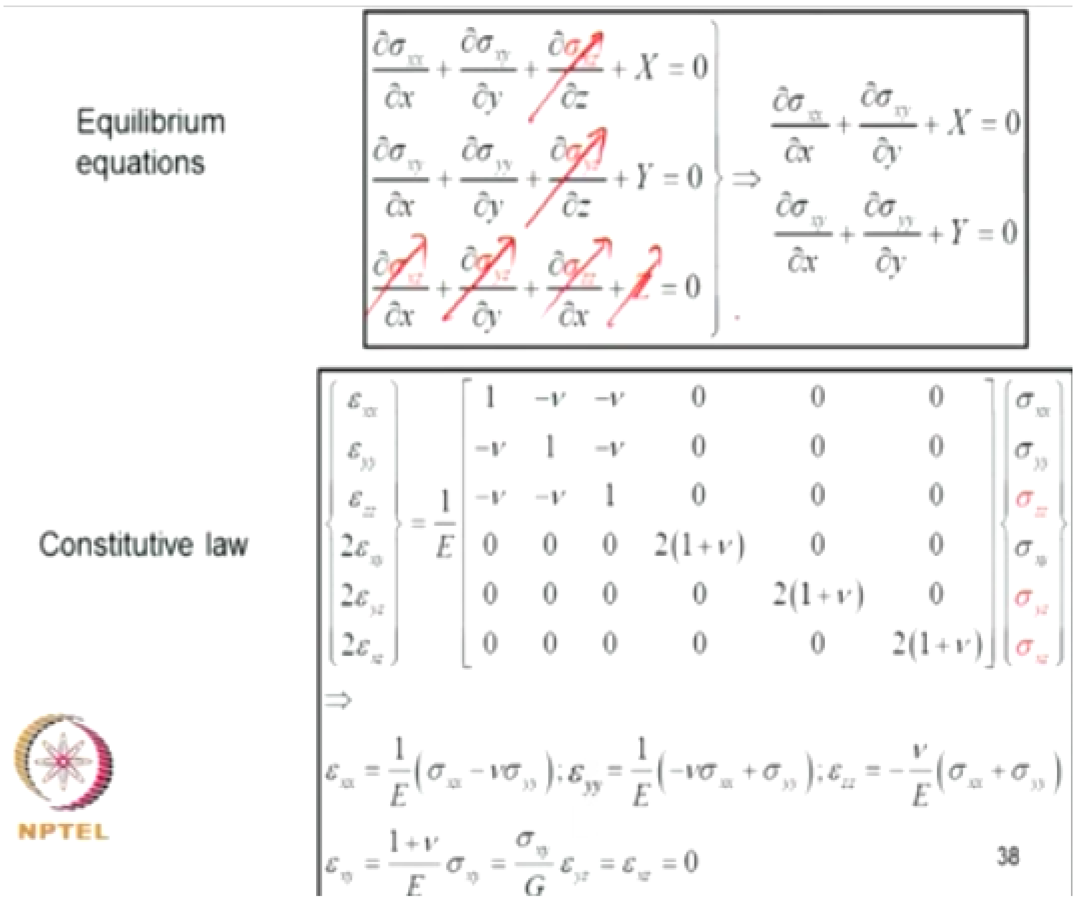
Now therefore at the end of this I have now certain quantities have become 0, and certain quantities have become independent of Z. Now let's see whether equilibrium equation is satisfied, so you go to the equilibrium equation right there 3 equilibrium equations you can see here the quantities that are written in the red are zeros as per our model, this is 0, so this drops off, this drops off, sigma XZ is 0 this drops off, YZ is 0 this drops off, ZZ is 0 this drops off,



and Z is 0 there is no body force that drops off so that is 0, so the remaining equations are the equilibrium equations which I need to consider further.

In the constitutive law again I have assumed stresses now I have to figure out what these strains are, so this is a relation between stress and strain but these quantities shown in red are 0, so once I expand I get epsilon XX and epsilon ZZ to be this, and epsilon XY is given this, but epsilon YZ and XZ become 0, so 2 of the strain components becomes 0, because certain stresses are 0. So at the end of it the stress matrix with all nonzero elements and dependencies is

Stress and strain tensors

$$\sigma(x,y,z)=\sigma(x,y)=\begin{bmatrix}\sigma_{xx}(x,y) & \sigma_{xy}(x,y) & 0\\ \sigma_{xy}(x,y) & \sigma_{yy}(x,y) & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\varepsilon(x,y,z)=\varepsilon(x,y)=\begin{bmatrix}\varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0\\ \varepsilon_{xy}(x,y) & \varepsilon_{yy}(x,y) & 0\\ 0 & 0 & \varepsilon_{zz}(x,y)\end{bmatrix}$$

Strain displacement equations

$$\varepsilon_{xx}=\frac{\partial u}{\partial x};\quad \varepsilon_{yy}=\frac{\partial v}{\partial y};\quad \varepsilon_{zz}=\frac{\partial w}{\partial z};\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right);$$

$$\underbrace{\varepsilon_{yz}=0=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right);\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0}_{\text{Ignored}}$$

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shown here, the original stress matrix is function of X, Y, Z it has 6 independent components, now this is 0 and these two are 0, so these are 0 as per our model, and these quantities are independent of Z, the nonzero strain components are shown here, so epsilon XZ is 0, YZ is 0, but epsilon ZZ will be 0. So now how many, come from stress to strain, now how do I get displacement? I have to use strain displacement relation, so the strain displacement relation epsilon XX is this, epsilon YY is this, epsilon ZZ is this, and epsilon XY is related to this. Now epsilon YZ I have got it to be 0, and epsilon XZ I have got it to be 0, therefore these quantities need to be equal to 0, but these will not be able to honor and we simply ignore these relations in further development, and that is where the approximations coming.

Unknowns

$\left.\begin{array}{l}\sigma_{xx}, \sigma_{yy}, \sigma_{xy} \\ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz} \\ u, v, w\end{array}\right\}$ 10 unknowns

Equations

2 equllibrium equations
4 stress-strain relations
4 strain displacement relations
} 10 equations

- Reduction in number of unknowns
- Reduction in number of independent spatial coordinates
- Not an exact model

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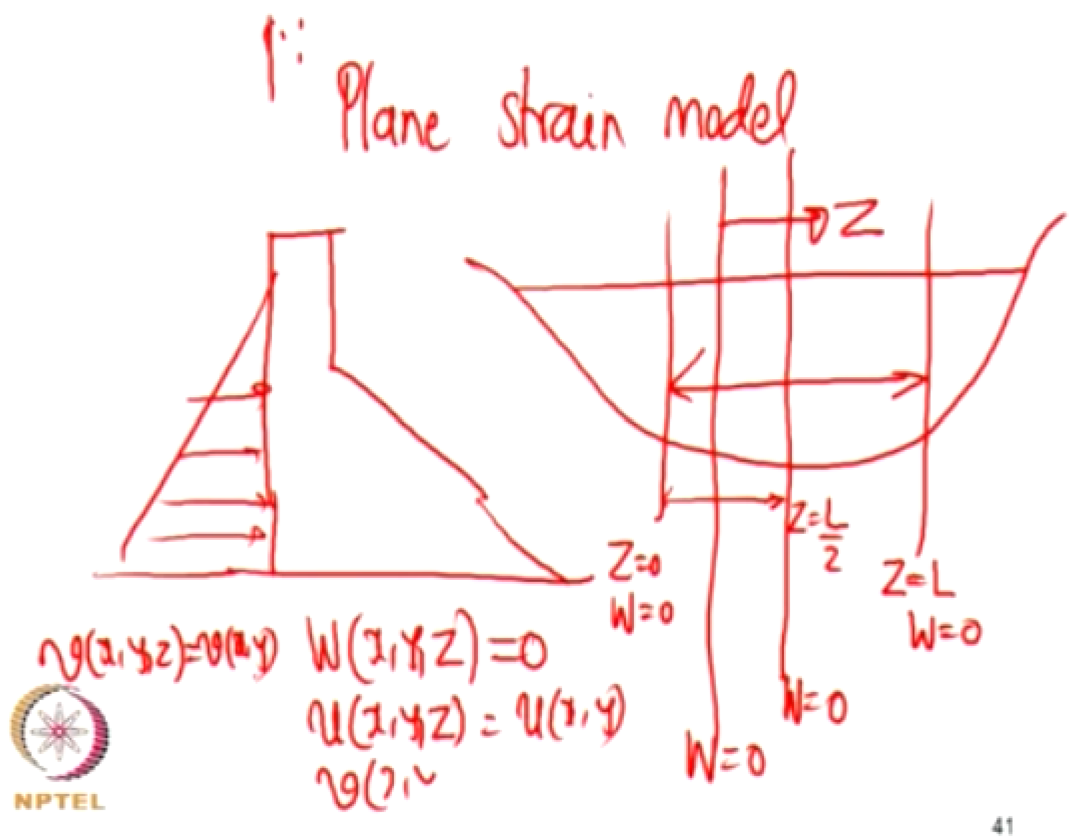
So now there are 10 unknowns that we are considering sigma XX, sigma YY, sigma XY and epsilon XX, epsilon YY, epsilon ZZ, and the U, V, W are the unknown displacement, so what are the equations? I have 2 equilibrium equations, 4 stress-strain relations, and 4 strain displacement relations there are 10 equations, so this is the simplification that we have achieved. What has happened? We have achieved a reduction in number of unknowns from 15 to 10, reduction in number of independent spatial coordinates so this X ,Y, Z becomes only XY and this because of the approximations in treatment of strain displacement relations is not an exact model, suppose if you substitute this into 3 dimensional equations of elasticity by that I mean suppose if you analyze the plane stress problem, and the solution that you obtain you substitute into the 3 dimensional equations of elasticity you will be able to satisfy equilibrium equations, constitutive laws, but you will have difficulty in satisfying certain compatibility relation, not all 6 compatibility equations will be satisfied, so that is one of the limitations of plane stress model.

Now in the next lecture what we will do is, we will consider another form of simplification that is known as plane strain mode, here the objective is again prismatic one of the example that we can give is that of a you know gravity dam subjected to hydrostatic force, so if this dam is in a valley like this we can assume that in this portion the dam cross section is prismatic, and the load doesn't change suppose this is the Z direction the load does not change in the Z direction. Now if at Z = 0, and Z = L if we assume that the dam is, the displacement W is 0 that means it is held fixed, then in plane strain model we deal with this type of situations where we again consider prismatic objects where the thickness, this is much larger than the lateral dimension it is the opposite of plane stress model, where in the plane stress model thickness was small in relation to the lateral dimension, whereas here it will be the opposite. Now the way we proceed is as follows now Z is 0 at W is 0 at Z = 0, and W is 0 at Z = L, now if you consider a mid plane the object is prismatic therefore it is symmetric about this mid plane, and the way the boundary conditions are applied on displacement that is also symmetric therefore if W is 0 here and W is 0 here, W must be 0 here, because load hasn't changed, geometry has not changed, boundary conditions have not changed, so this is 0.

Now next what I do is I consider a subsection which is half of this dam so I'll consider the quarter point. Now again the same logic at Z = 0 and Z = L/2, W is 0. Now the object is again symmetric about X = L/4, Z = L/4, boundary conditions are the same but loading is the same etcetera, etcetera, I get W = 0, so what happens is in this model by repeatedly using this argument we postulate that W(X,Y,Z) is 0. And next we assume that U(X,Y,Z) is function of U(X,Y) and similarly V(X,Y,Z) is V(XY) so this is a displacement field we will assume, W is 0 U is function of X and Y, V is a function of X and Y, we develop this model by considering



from this we'll go to now from displacement we go to strain, by using strain displacement relations and from this I use constitutive relations and come to stresses, so this model is known as plane strain model.

So in the next lecture we will consider the mathematical theory behind plane strain model and after having covered the two back ground to the theory of these two models, we will start developing finite element models for plane stress and plane strain continue, so we will take up that problem in the next class.

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