# Finite Element Method and Computational Structural Dynamics

**Prof. Manish Shrikhande**
**Department of Earthquake Engineering**
**Indian Institute of Technology, Roorkee**

## Lecture - 40
## Finite Elements for Plates and Shells - III

Hello friends. So, let us start our discussion, I mean how we develop the finite element model, practical Finite Element Model for Plates. As I told earlier, plates and shells finite element modeling of a plates and shells has attracted huge amount of interest and several I mean massive effort has been devoted to development of suitable elements, I mean working elements.

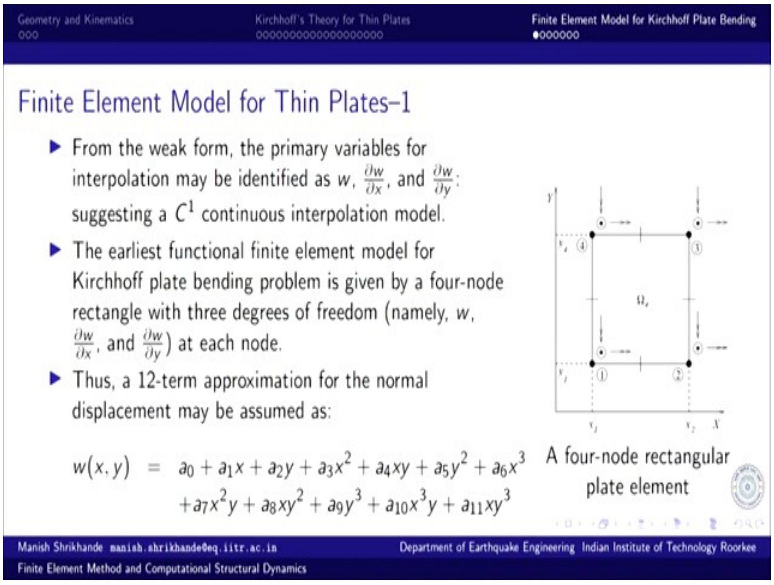
So, we will only scratch the surface of the massive amount of information that is available and the theoretical developments that have happened in this field. So, we discuss the formulation of one of the earliest successful finite elements for analysis of thin plates.

And it is surprising why this plate element works and when supposedly more theoretically, I mean as we will see during discussion, this particular finite element that we discuss, does not really agree with all the basic tenets of a good finite element with regard to compatibility and continuity.

And still, it does very well compared to other elements which are derived using more rigorous adherence to those basic tenets of finite element modeling. And that has led to again the emphasis on the importance of patch test. So, patch test can establish that which element can work and which element can have difficulty in achieving convergence.

So, even though this particular element has its own limitations and defects, it passes the patch test, and therefore, the convergence it has the capability to achieve convergence in finite element solution as the element size reduces, although the monotony convergence is of course compromised.

(Refer Slide Time: 03:02)




So, from the weak form, we know the primary variables for interpolation may be identified as transverse displacement, displacement w along z direction, and then the rate of change of displacement, so the deflection curve. So, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. So, these are the section rotations or deflection curves along x and y direction.

And this suggests, I mean because of this requirement of continuity of first order derivative with respect to x and with respect to y, it suggests a $C^1$ continuous interpolation model. So, $C^1$ first order derivative should be continuous. So, we have two both orthogonal directions and the earliest functional finite element model for Kirchhoff plate bending.

So, when I say Kirchhoff plate bending, it refers to thin plate where the Kirchhoff shear vanishing shear strength constraint is imposed. So, that is this earliest model is given by a four-node rectangle with 3 degrees of freedom. So, these primary variables defined at each node. So, three-nodes, three degrees of freedom at four-nodes, so that makes it for 12 total degrees of freedom, 12 degrees of freedom within an element.

So, if I look at the approximation, so of course, the derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, they will depend on what is the variation of w. So, I need to define the variation for w. So, w I can have the transfers or normal displacement w, defined using 12-term approximation.

12-term approximation, because I have 12 number of primary variables defined at the nodes. So I have 12 constraints that I can impose on the, that we can impose on the approximation.

So, the 12 part term approximation for normal displacement is assumed again picking up terms from Pascal triangle, symmetrically place terms and we pick up 12-terms starting from the lowest order term. So, constant term, then in two linear terms, then 3 quadratic terms, and then cubic terms and then two terms from the 4th order. So, x cube y and x y cube. So, they these complete.

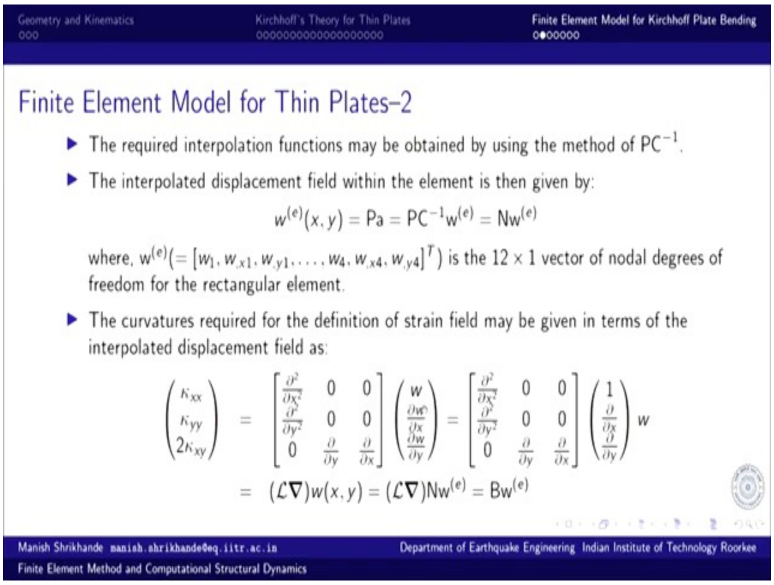
So, these are two spurious 4th degree terms, but it is complete up to 3rd degree polynomial. So, these are the 12-term approximation. And the element looks like this. So, four-nodes, 1, 2, 3, 4, and these are the degrees of freedom. So, normal displacement and rotation, section rotation, del w by del x and del w by del y.

And $\frac{\partial w}{\partial y} = -\theta_x$. So, $\frac{\partial w}{\partial x}$ will be equal to $-\theta_y$. So, these are the basic definitions, geometry or arrangement of the four-node rectangle element.

And the interpolation functions can now be derived by using the same method PC inverse. We can segregate these into polynomial terms and coefficient terms, and then impose the displacements and rotations at each degree of freedom at for each coordinate of these points.

And then, we have the nodal degrees of freedom, that will be equal to these coordinates the matrix of coordinates at which these are evaluated, and multiplied by these coefficients unknown coefficients a. And then, a can be calculated by inverting the matrix of coordinates. And substituting for a, that will lead to the basic approximation for the interpolation function. So, PC inverse, the P is the matrix of polynomial terms so that would be 1, x, y, x^2, xy, y^2, and so on up to xy cube.

(Refer Slide Time: 07:49)



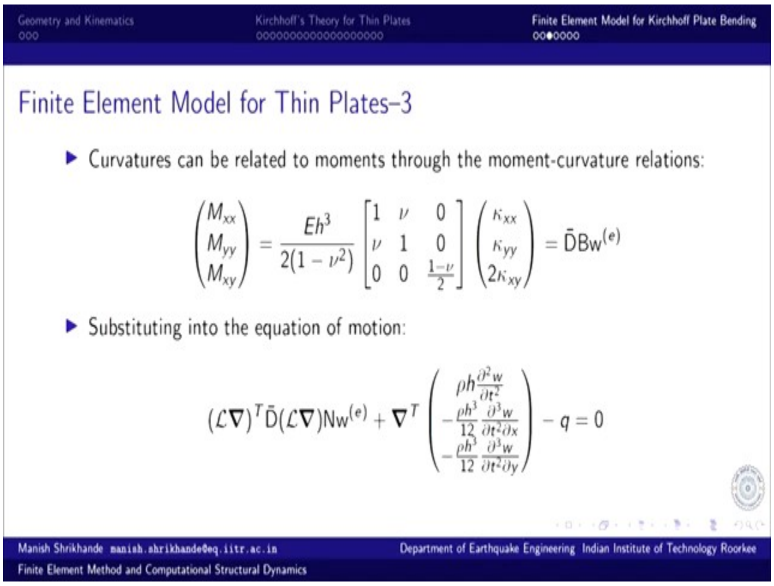

And $C^{-1}$ is the inverse of the coordinates, matrix of coordinates, and that would be evaluated at each of the four-nodes for different conditions for w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. So, interpolated displacement field within the element, it can be given by polynomial term multiplied by the coefficients a.

And coefficient a is the given by $C^{-1}$, coordinate matrix inverse multiplied by nodal variables of the primary variables, nodal values of primary variables. And this PC inverse is now referred to as the shape function matrix or interpolation function matrix multiplied by the nodal values of primary variables.

So, $w^{(e)}$ is the vector of nodal variable. So, w1, w I mean $\frac{\partial w}{\partial x}$ at node 1, $\frac{\partial w}{\partial y}$ at node 1 etc. So, this is the $w^{(e)}$ is 12 by 1 vector of nodal degrees of freedom for the rectangular element. So, the curvatures that are required from the definition of strain field can be again once we have the displacement variation, we can evaluate what is the curvature.

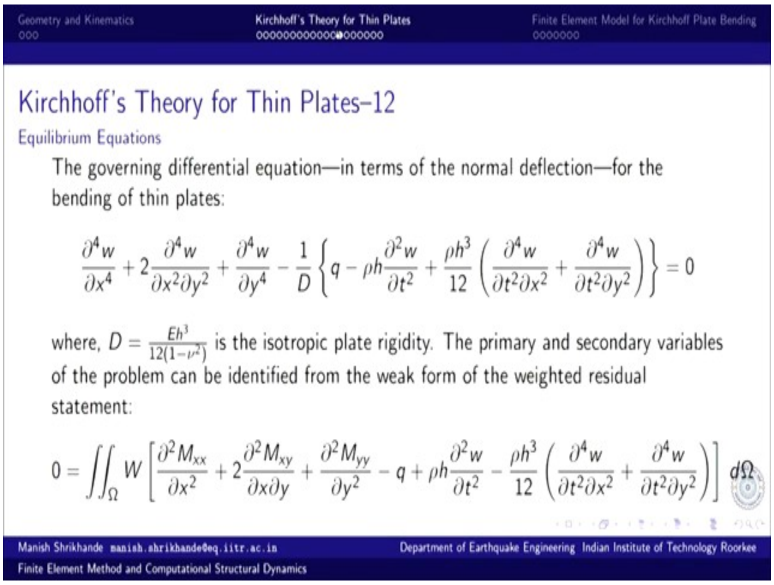
So, curvature is defined by the second order derivative of these primary variable, so, differential operator. And this second this primary variables can be obtained as a gradient vector of displacement. So, that is what gives us this essential derivation from curvature to displacement. And this B matrix, which is the this differential operator multiplied by this gradient vector and operating on the matrix of shape functions that gives us the B matrix. So, that is curvature displacement relationship.

(Refer Slide Time: 09:56)



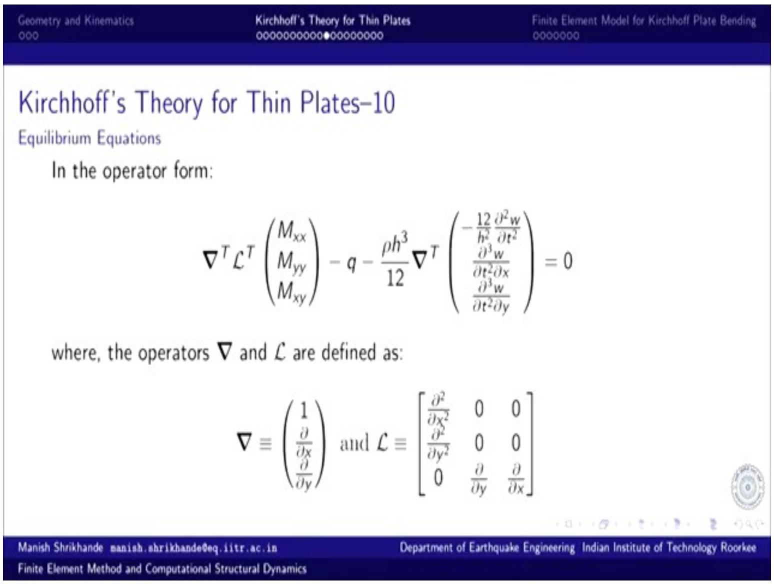

So, curvatures can be related to moments through the moment curvature relations. And moments, we already have the those expressions moment curvature relations and that can be represented as this constitutive matrix, we can call D bar as the constitutive matrix and B is the curvature displacement matrix and w is of course, the nodal variables.

(Refer Slide Time: 10:26)



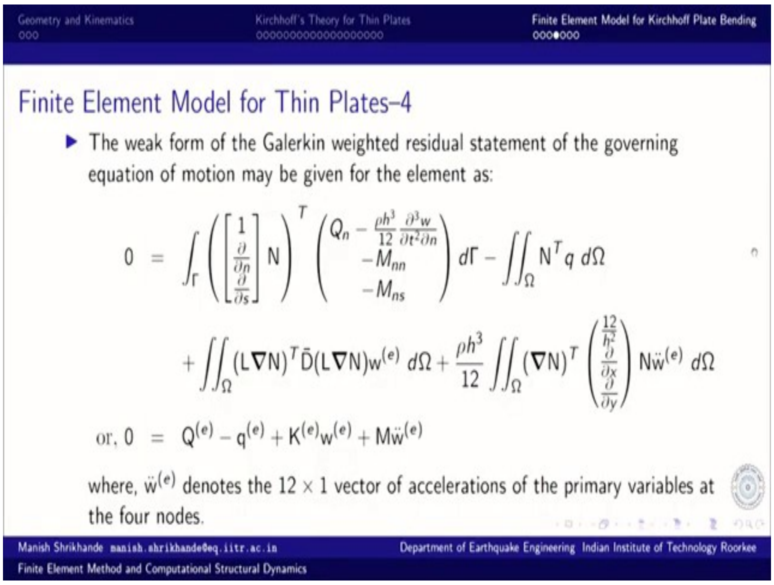
And these can be substituted in the governing equation of motion. The governing equation of motion is this is the governing differential equation in terms of displacements. So, we can and in terms of moments, we have the governing differential equation here.

(Refer Slide Time: 10:41)




And once we have the approximation for these Mx and My and Mxy, we can substitute from this our approximation for w and moment curvature relationship, and we can arrive at the governing differential equation. So, this that is the governing differential equation in operator form and now, again we go back to development of the weak form element level. For each element we developed the weak form, and this is what it leads to.

(Refer Slide Time: 11:22)

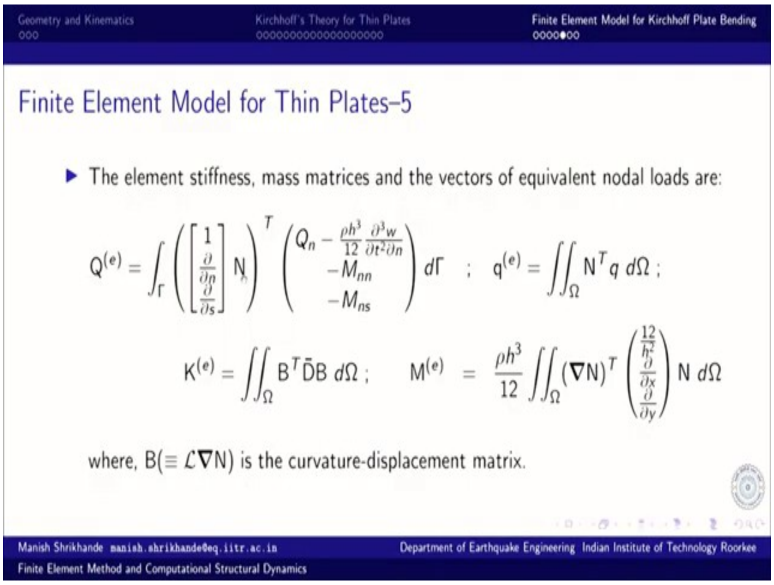



So, this is what the governing weak form will lead to, and boundary terms and the domain integral. And once we valuate these, so we end up with our familiar expressions of stiffness multiplied by generalized displacement, then mass multiplied by accelerations. So, mass or inertia terms, I mean moment of inertia, mass moment of inertia, then we have tractions and then the apply transverse loads.

So, that is the element level equilibrium equations w e, double dot w e represents the 12 by 1 accelerations of primary variables. And we will see how we interpolate, I mean how this the these nodal variables, nodal values of primary accelerations, they are transformed, and that is done by using this interpolation.

So, we assume acceleration field to be interpolated in the same way as the displacement because this N is the vector is the matrix or shape function for interpolation of displacement. So, we make this assumption that accelerations vary with respect to x y in the same way as the displacement w. So, w means and the transfers displacement w and $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, the node primary variables.

(Refer Slide Time: 12:58)



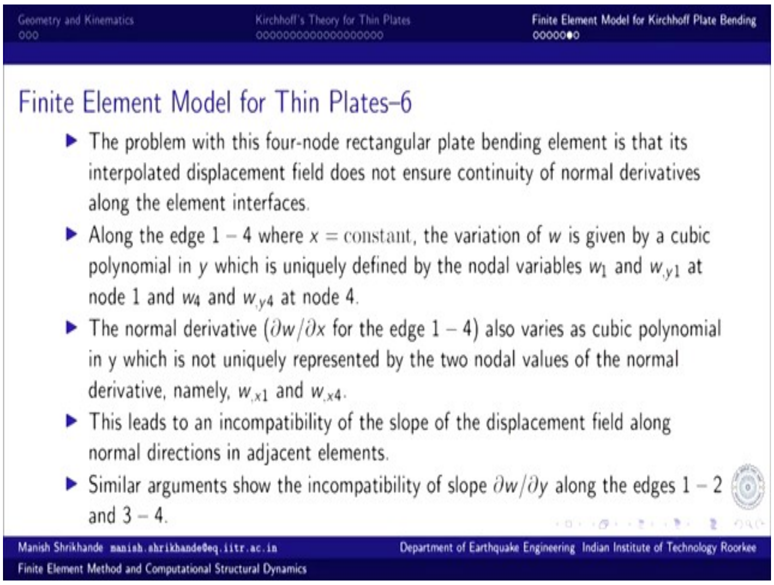

So, individually the terms are evaluated by using once we know the interpolation functions, they are known functions of x and y and these derivatives can be evaluated, and integrals can be computed boundary integrals and domain integrals can be computed once we know the element geometry. And that leads us to the surface traction terms, the equivalent nodal loads, the equivalent element stiffness matrix and mass matrix, mass or inertia matrix of the problem.

And that is equivalent that gives us with this element equilibrium equation. And we do this, I mean once we have this element level equilibrium equation, they are, these elements are put in place in the global system of equations for all the representing all degrees of freedom, and these local degrees of freedom are mapped on to global degrees of freedom, and representative these terms take their place in respective positions.

And they then next element the similar equations are derived and similar contributions from other elements are accumulated, until we cover the entire element, the discretization covering of covering the entire domain of the plate problem. And, at the end of that, we impose the essential boundary conditions of the problem. And finally, solve the system of equations. So, that in a sense is the problem of plate bending problem that is how we deal with it.

Now, as I said this is the earliest successful finite element model. And it was this element has been very successful even though it has some obvious defects. And we will see what those defects are and why that is not of a serious concern for us.

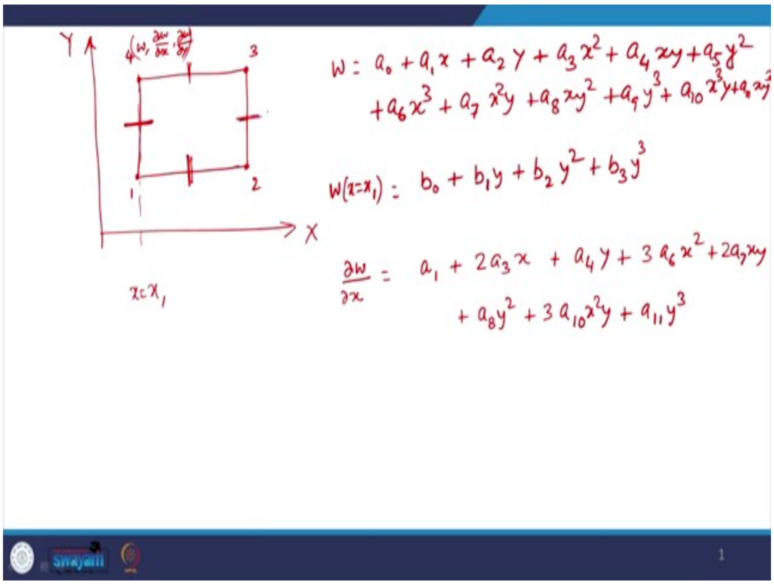
(Refer Slide Time: 15:23)




So, the problem with this four-node rectangular plate bending element is that the element interpolated displacement field within the element does not ensure continuity of normal derivatives along the element interfaces. So, normal derivatives are a problem. So, you see it is a two-dimensional problem. So, we can have derivatives with respect to x derivatives with respect to y, if we have x y system, right. So, we can have derivative with respect to x, we can have derivative with respect to y.

So, on this particular edge, let us say this edge aligns with the y axis then derivative with respect to x is a normal derivative for this. Similarly, if this edge aligns with x axis then derivative with respect to y is the normal derivative for this edge. And we will see that this is a problem here in this element. Why it is a problem? So, let us look at the element that we had. So, I will instead of going back to that slide, I will draw the element here.

(Refer Slide Time: 16:34)




So, these are the four-nodes. So, along this element edge 1-4, it is defined by constant value of x = x1. So, this edge is defined by x = x1. So, if I substitute the derivation the displacement is of course, w = 0 + a1 x + a2 y + a3 x^2 + a4 xy + a5 y^2 + a6 x^3, a7 x^2y + a8 xy^2, a9 y^3, a10 x^3 y + a11 xy^3.

Now, if I substitute x = constant here. So, I will end up with, this will also become a constant, I will end up with a term w at x = x1 will be equal to let us say 0 + b1 y + b2 y^2 + b3 y^3. That is all. So, 4 term, cubic term, cubic variation in y.

So, along this we have cubic variation for y, cubic variation in with respect to y, for the displacement. And this displacement is defined by w and $\frac{\partial w}{\partial y}$ . So, two nodes here, two values here w and w w, and so, these are the primary variables at the nodes. So, displacement w and displacement del w by rotation $\frac{\partial w}{\partial y}$ at node 4 and node 1, so they together there are 4 coefficient 4 terms and these 4 terms are sufficient to define these third degree variation.

But what happens to $\frac{\partial w}{\partial x}$ ? Let us work out what is $\frac{\partial w}{\partial x}$ . So, $\frac{\partial w}{\partial x}=a_1+2a_3x+a_4y+3a_6x^2+2a_7xy$ . So, if I evaluate $\frac{\partial w}{\partial x}$ at x = x1 constant, so again I will find that $\frac{\partial w}{\partial x}$ is also cubic in y, all x terms would be constants. So, we will be left with a cubic variation in y.

Now, this cubic variation in y I have only two terms here, $\frac{\partial w}{\partial x}$ at node 4 and $\frac{\partial w}{\partial x}$ at node 1. So, these two terms for this normal derivative, this is normal direction for this edge. So, del this is variations slope with respect to differential with the normal derivative.

So, this second, third degree variation with respect to y is not possible to represent exactly or uniformly by using only two values of $\frac{\partial w}{\partial x}$ at the two nodes. Similar arguments would hold for $\frac{\partial w}{\partial y}$ along the edge 1-2 and along the edge 3-4, and $\frac{\partial w}{\partial x}$ would be in not unique with the only two values are defined at the nodes 2 and 3, and that is what makes it problematic.

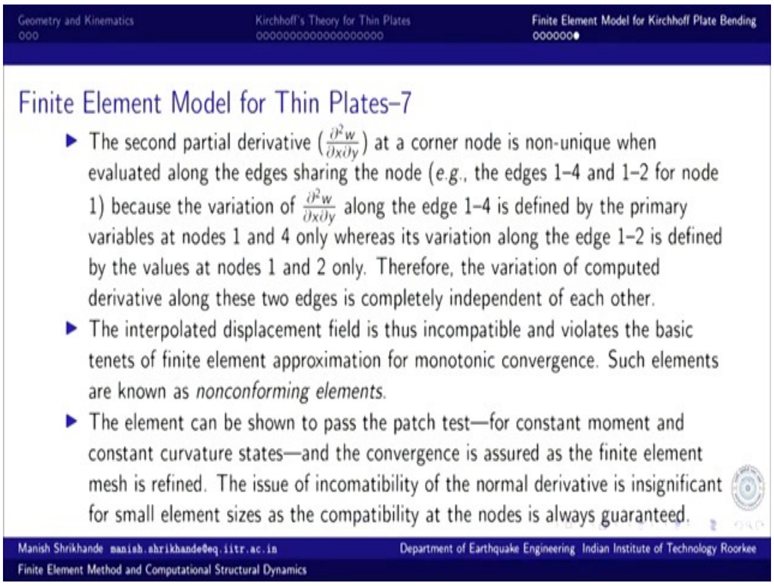
So, along edge 1-4, where x = a constant, the variation of w is given by a cubic polynomial in y which is uniquely defined by the nodal variables w1 and $\frac{\partial w}{\partial y}$ at node 1 and w4 and $\frac{\partial w}{\partial y}$ at node 4. This is a normal derivative, $\frac{\partial w}{\partial x}$ for the edge 1- 4 also varies as cubic polynomial in y. But for this, we have only two values that is $\frac{\partial w}{\partial x}$ at node 1 and $\frac{\partial w}{\partial x}$ at node 4. There is no other value which is available for mapping this.

So, this leads to incompatibility of the slope of the displacement field along normal directions in adjacent elements. So, different elements will have only I mean it is a non-unique variation. So, different elements can have different variation. So, it may be possible that if I have two elements two adjutant elements.

So, one element has a normal derivative of positive, another element will have a normal derivative of negative, and they would otherwise they will have continuous displacement w field would be continuous and $\frac{\partial w}{\partial y}$ would be continuous, but normal derivative would be discontinuous, there would be discontinuity in that.

So, the this leads to incompatibility of the slope of displacement field along normal direction in adjacent element. So, it is not completely incompatible that the edges are coming apart or anything like that. But still, it is not $C^1$ continuity in entire t, because this is of course, say first degree, first order derivative, and there is an incompatibility in the first order derivative. Similar arguments show the incompatibility of slope $\frac{\partial w}{\partial y}$ which is the normal derivative along edges 1-2 and 3-4.

(Refer Slide Time: 24:30)




There is another problem that is the second partial derivative of the at a corner node let us say node 1. So, $\frac{\partial^2 w}{\partial x \partial y}$ at corner node, let us say node 1, is non-unique if I evaluate it from edge 1-4 and if I evaluate it from edge 1-2, because the displacement field that govern this they are different, I mean the deformation, I mean for edge 1-2, the variation is a defined del w by del y is defined by the displacements at node 1 and 4.

And then I take what is the variation with respect to $\frac{\partial w}{\partial x}$ and evaluate that. Whereas, for element I mean edge 1-2, $\frac{\partial w}{\partial x}$ would be defined $\frac{\partial w}{\partial y}$ will be defined by the what is available at node 1 and 2, and then find out what happens at node 1, the second derivative.

So, because different variables control this evaluation in along different edges. So, the value is of course, going to be different. And this is again a problem of discontinuity of second partial derivative, and it may show up in the calculations and that may be required to be reconciled, I mean one has to be careful while interpreting these results.

So, the bottom line is the interpolated displacement field is thus incompatible and violates the basic tenets of finite element approximation for monotonic convergence. And such elements are known as nonconforming elements. So, those incompatible modes, addition of incompatible modes also makes the elements nonconforming. We have seen that before and yet its performance is often better than what is a rigorously developed conforming element formulation.

And in this case also, this nonconforming element apparently performs much better than many conforming elements that have been studied before this. So, this element can be shown to pass the patch test. It will pass the patch test for constant moment and constant curvature states.

So, moment and curvature they are analog to stress and strain, in case of plate bending problem. And because they can pass the patch test, so the convergence is assured as the finite element mesh is refined. And the incompatibility of the normal derivative is really insignificant for element size, as the compatibility of the nodes, I mean the primary variables are of course, compatible at the nodes. So, that is always guaranteed.

So, this seemingly incompatibility of the normal derivative is a small what would I say irritant so to say. But otherwise, the element performs very well and it provides very useful results for stress analysis and that can be used for advantage in many situations. So, this completes our discussion of thin plates or Kirchhoff plate elements.

So, if I have this thin plate formulation for Kirchhoff element, Kirchhoff plate bending, what constitutes the thick plate? So, thick plate, the we have this essential constraint, the Kirchhoff constraint of shear, vanishing shear strain, where we imposed that transverse shear strains are 0 in the shear strain in yz and zx planes are 0. So, that impose the condition that deflection or the slope of the deflected curve is related to the section rotations or the plane rotations.

And in case of thick plates, where the shear deformations are not 0, then there is no such a restriction and they can be treated independently. So, then, the primary variables then in that case I mean $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, they are essentially the section rotations, rotations about x axis and rotations about y axis.

So, instead of taking primary variables as $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, we can take the primary variables as w, $\theta_x$, and $\theta_y$. So, the transverse normal displacement w, and the section rotation, plane rotation about x axis, and plane rotation about y axis, and then develop the entire system a displacement field accordingly.

And then, because there is no vanishing shear strain constraint, so normal displacement is independent is treated independently of the section rotations, rotations about the axis theta x and theta y. So, they are all independently modeled and that actually leads to much simpler interpolation model because C 1 continuity is no longer required. It is only C 0 continuity for individual.

So, w only require is required to be continuous, theta x is required to be continuous independently, theta y is required to be continuous independently. So, that simplifies the interpolation model quite a lot. And we can very easy finite element models can be developed, and these are called first order shear deformation theory, and models developed on the basis of that are much simpler to in construction.

And, but these models again fail, I mean seemingly we have a general theory at hand if transverse displacement is the independent of the section rotation, so it is a more general theory. But this theory or the met elements derived on the basis of that they often fail in the case of when the when they are used to model the problems of thin plates, where we have this physical constraint of vanishing shear strain.

So, shear strains are actually very small. So, as to be negligible in case of thin plates. And that model typically without proper consideration for consistency for the field approximation, they fail. So, again consistency here comes that because these section rotations theta x and theta y are typically related to the first derivatives of displacements w.

So, interpolation for $\theta_x$ and $\theta_y$ has to be one order lower than that of the displacement w. So, if we can somehow achieve that, then these elements are very well they perform reason very well, in all cases for thick plate as well as thin plates, otherwise they will lock in case of shear I mean vanishing shear constraint, and that will again be by virtue of shear locking, excessive shear. I mean they represent shear strain when there is no shear strain in practice.

And again, we resort to similar familiar tricks of the trade, variational crimes, reduced integrals or selective integration for element formulation, integrate the shear terms with a reduced order of quadrature, then the flexural terms, and then the elements perform reasonably well. So, that is how thick plates are modeled and you can refer to standard text and even my text, my book for treatment of these first order shear deformation theory, and that plate element derived on the basis of that.

And shell elements are again developed, I mean a very rigorous development of shell element is rather involved and it is quite often done by using managing, as we said that shell element is a combination of membrane action and flexural action. So, it is often used, it is better many practical problems are solved by using what is called a facet element.

Facet element is a 3 noded plate element, and on top of it we superimpose a 3 node constant strain triangle. So, these two behavior, they sufficient, they provide sufficient capabilities to model the shell problems. And they are very commonly, very often used in practice from analysis of shells by using patchwork of facet element, triangular facet elements. And many of the finite element codes actually implement facet elements in their in the solution of shell problems.

So, with this we complete our discussion of finite element formulation. As we see that, the purpose of finite element formulation, purpose of finite element method is to transform partial differential equations into algebraic simultaneous equation, if the problem is the static. If it is a time variant problem, if the there are acceleration terms involved, the excitation forces are time varying, then that partial differential equation is transformed into an ordinary differential equation in time.

And that is we take up the time dimension separately from the space dimension because as we know, I mean as the dimension increases the problem size increases, and therefore, it is much more efficient, and much more convenient to separate the time variation from this space variation.

So, this partial differential equation in space time coordinate system is first treated by separation of variables, variation in space is separated from the variation in time. And then, finite element model is used for space variation part, and time dimension is taken separately as we will discuss in our next lectures.

Thank you.