**Finite Element Method and Computational Structural Dynamics**
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**Lecture - 37**
**The Patch Test**

Hello friends. So, we have seen some variational crimes particularly inclusion of incompatible modes and the use of reduced or selective integration, to solve some of the problems that arise because of inconsistency of the fields of approximation or the interpolation of primary variables, and how those problems are tackled by using these tricks of the trade.

Now, whether these tricks of the trade variational crimes they work in practice or whether their use in the finite element such modified finite elements can be used for finite element analysis, and whether the convergence in the limited limiting case of idealized conditions whether the such modified elements be able to reproduce those simple conditions, which are essential for convergence of the finite element.

That is examined, that is tested by what we call as Patch Test, and we will see how those; what is a patch test, and how those patch tests are conducted. So, the unsurprisingly the patch test was developed by the initially by the same person who developed the isoparametric formulation ah Bruce Irons, and because these mapping and then distortion and use of variational crimes for solution. They are all interrelated concepts.

So, patch test coming from the same stable is no surprise and, but of course, the utility of patch test has moved beyond just the testing of elements modified elements and it has found many applications in the finite element arena. So, many of the enhancements, either in the form of additional polynomial terms in the inter to the interpolated field so, in case in the form of incompatible modes or in the numerical evaluation of integrals, in the by reduced order integration rule or diffuse of different orders of integration for evaluating different terms of the integrant.

So, these are some of the violations that happen into the basic tenets for of convergence of finite element solution. There are certain requirement, that is that needs to be adhere to ensure monotonic convergence of finite element solutions and these tricks they violate those requirements.

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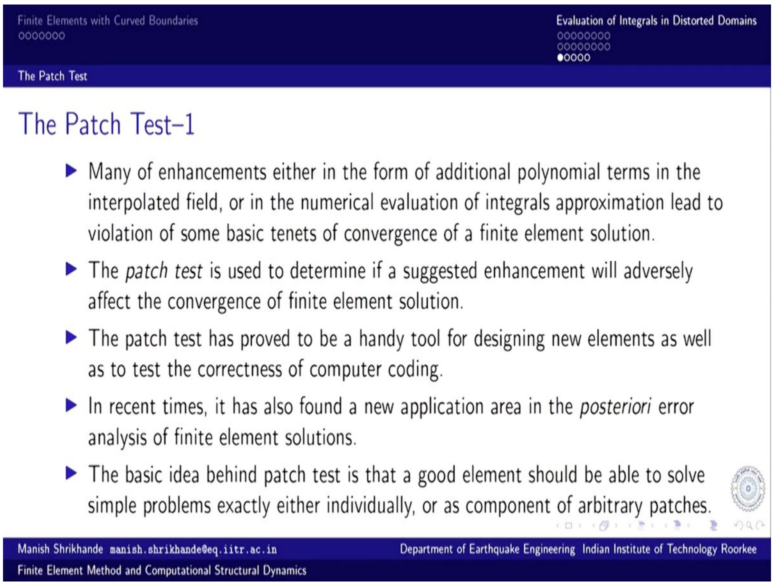


So, the patch test is used to determine if a suggested enhancement will adversely affect the convergence of finite element solution. And as a corollary, it can be used to suggest whether a particular enhancement that is to that is being considered, whether that will work in practice or not. It has proved to be a handy tool for designing new elements, as well as, to test the correctness of computer coding.

So, not only we can use the new finite elements. For example, as we discussed earlier, the degenerate elements by imposing certain conditions and which lead to obviously, very in case of deriving 3-nodded triangle from 4-node rectangle or deriving plate elements or shell elements from a continuum element.

All of these are some degenerative techniques are used and certain kinematic constraints are imposed on the displacement field. And it is quite useful patch test is quite useful to test whether these new elements that have been designed, that have been developed whether they will work in practice or not.

And, also using a new element, using an existing I mean very well-known element with known convergence characteristic, it can be patch test can be used to just test whether the element whether the finite element engine has been programmed correctly. If there has been an error in coding that will be reflected in the results of this patch test, and it will be easily caught. So, it is also a very good bug catching mechanism in the finite element engine.

And in recent times, it has also found a new application area in the posteriori error analysis of finite element solution. Now, posteriori error analysis is referred to post analysis. We run a finite element analysis, then ascertain the error in the solution, based on the analysis results.

And if the error is more, now this is very important to note because we do not, we will theoretically, we know error only when the true solution is known. But what we are trying to do here is, we are trying to ascertain error in the solution by using the approximate solution that has been just computed.

So, it is a very interesting way of ascertaining bounds on error in different zones, in different regions of the domain, and that gives us information whether refinement further refinement is needed if these bounds on error are acceptable or the mesh needs to be refined. Either the size has to be reduced or the polynomial degree of interpolation has to be increased to bring down the error.

So, this posterior error analysis is very important for adaptive finite element analysis. And patch test the patch of elements that we use here is a very very powerful tool in this posterior error analysis. So, the basic idea behind patch test is that a good element should be able to solve simple problems exactly, complex problems we get approximate solution that is fine, but an approximate technique has to first be able to reproduce very simple solutions, idealized situations which can be possibly be solved by manual calculation.

So, if a finite element model cannot solve a problem which can be solved analytically by just manual calculations, then obviously, there is something wrong. So, this that is the basic idea that we use very very simple, very very idealized conditions and for which we know the solution.

And then the patch of elements once it goes through the complete cycle of finite element analysis, the final result should be what we expect or what we know is the true solution. And if that is not achieved then obviously, there is a reason to look for sources of errors, why this is not happening? And the problems could be either the element is bad, or the programming is bad. So, what is involved in a patch test? So, patch test as the name suggest consists of a patch of elements.

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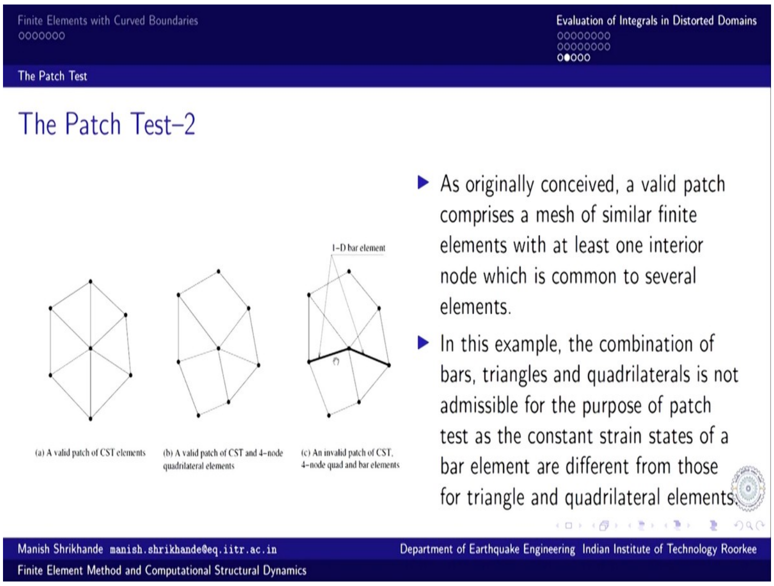


So, patch of elements as you can see here, these are I take here three different patches of elements. One is just an assembly of 3-node triangle, constant strain triangle, patch of 3-node triangles. Second one is a patch of triangle and quadrilaterals. So, 4-node quad and 3-node triangles and third one is a combination of 3-node quad 3-node triangles and 4-node quadrilaterals and 2-node 1-D bar element trace element.

And, in this I have call these two first two cases are the valid finite element patches, and this third one is an invalid finite element patch for conducting the patch test. The reason being the strain, the mechanism of deformation, the mechanism what we call as constant strain state is different in the elements that are used here. The notion of constant strain is identically similar for 3-node triangle and 4-node quadrilateral.

They are both from continuum elements, but constant strain, the concept of constant strain is different in one dimensional bar element or 2-nodded bar element. So, this combination of these different elements is an invalid patch so to say. So, we do not make this kind of patches, these kind of assemblies while conducting a patch test. So, as originally conceived a valid patch comprises a mesh of similar finite elements. So, that similar finite elements is the key word, with at least one interior node.

So, if you see here, there is one interior node here and one interior node in this second patch. So, at least one interior node should be there, which is common to several elements, need not be common to all. In this case, interior node is common to all the elements of the patch, but that is not necessary, but it should be the interior nodes, it should be common to several elements of the patch, that is the requirement.

So, in this example, the combination of bars, triangles and rectangle quadrilaterals is not admissible for the purpose of a patch test, as the constant strain states of bar element is different from those of the triangle and quadrilateral elements, right. So, by because of this, they become dissimilar elements and hence their use in the single patch is not admissible for constructing a patch test.

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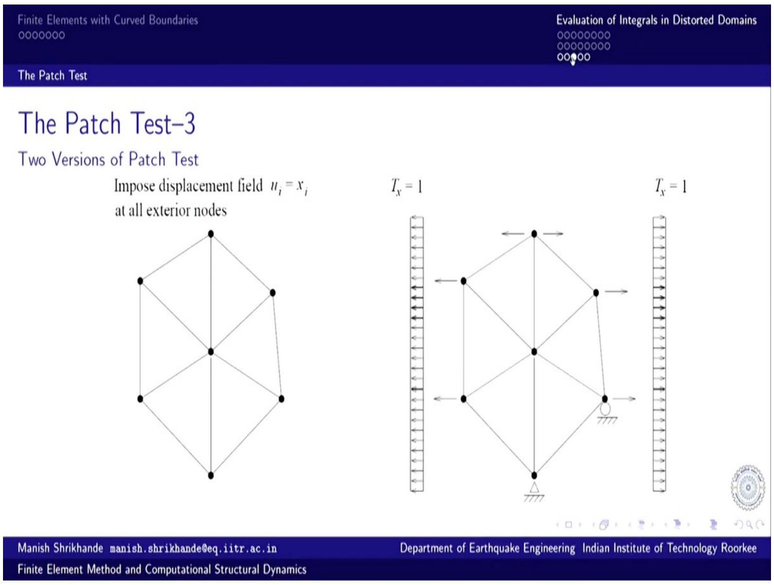


So, what are the variations? How the patch test is actually conducted? So, there are two variations, two types of patch tests. One is called the displacement patch test, other one is called the force patch test. So, in the displacement patch test, we impose the displacement field some known displacement field along the exterior nodes.

And, then these are considered as the prescribed boundary condition, geometry essential boundary conditions of the problem. And then the solution the system equilibrium equations are solved for the interior displacement field at the interior nodes. Now, because the displacement field is known, we know what displacement field has been applied. So, we can find out what is the we know what the displacement field should be at the interior nodes.

For example, we can impose the displacement proportional to their coordinates. So, the value of coordinates. So, the position of the point becomes same as the displacement. So, if that kind of displacement is being imposed and the system is linear so, the displacement at the interior nodes is also known and we can compute the displacements in the interior nodes and try to and compare whether it meets agrees with the imposed displacement field or not, if it does not then obviously, there is a problem at hand.

Second one, the force patch test is based on again at least one interior node, that valid patch is one thing that is of course there. In this before coming to patch force patch test we again discuss this displacement patch test, because the boundary conditions are imposed on all the exterior nodes. So, there is no problem here, it is always well constrained.

For the force patch test, in order to be able to solve, obviously, the system has to be restrained. So, we make use of minimal constraints that will allow the system of equations to be solved, right. So, in this case of 2-dimensional problem, we need to impose three constraints for to be able to restrain the rigid body modes.

So, these are the minimum constraints that are required to eliminate rigid body modes and then these constant tractions are applied. So, this is a constant stress. So, we apply this constant stress condition. So, constant $\sigma_x$ value so, the entire element is stretched, subjected to the same stress, constant state of stress.

And we evaluate for this constant state of stress, the equivalent nodal forces along these boundaries surface tractions and then solve the system of equations for these loads. And then check at the interior of the element, whether the computed stress state agrees with the imposed state of stress.

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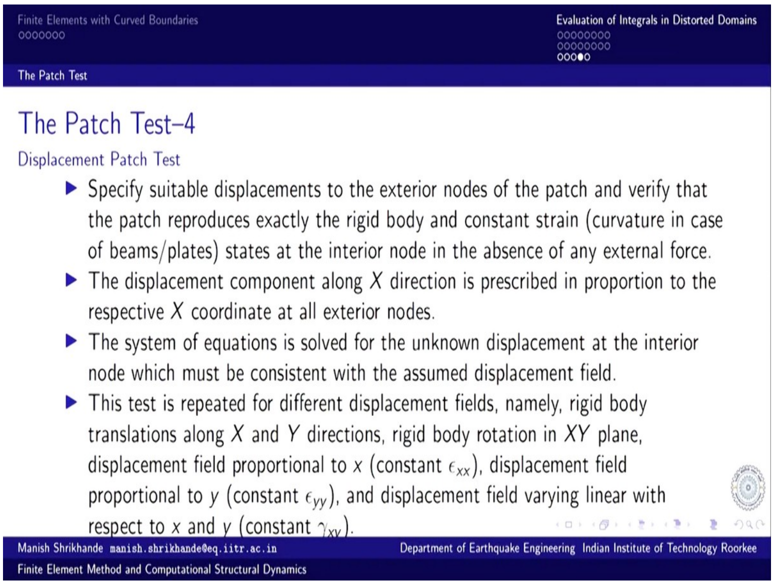


So, the displacement patch test as we discussed, specify suitable displacements to the exterior nodes of the patch and verify that the patch reproduces exactly that rigid body and constant strain states at the interior nodes in the absence of any external force. So, we can impose if we impose all the displacement uniformly, then the interior node should also meet the same displacement condition or if a constant strain state is imposed so, the displacement field is proportional to the coordinate, then the interior node should also have the proportional to coordinate displacement that should come as a solution of the finite element equations in the absence.

And there are no external forces applied it is only the imposed displacement field. So, this is along X and same thing is done for all other directions. So, constant displacement, I mean rigid body displacement in Y direction, linear displacement in Y direction which will represent constant strain along Y direction, epsilon yy and then again constant shear strain and the rotation rigid body rotation for the different displacements. So, all these have to be tested and verified.

And, if the patch passes this, then the elements can be said to have passed the displacement patch test. For all three I mean all three all the modes of deformations have to be explored and it has to pass the patch has to pass in individual modes of deformation.

So, rigid body mode, rigid boundary translation in X, rigid body translations in Y, rigid body rotation constant strain along X, constant strain along Y and constant shear strain all these have to be satisfied and for 3-dimensional obviously, the 6 components of strain and 6 rigid body motion. The force patch test apply we apply external forces on the exterior nodes which are consistent with the state of constant traction.

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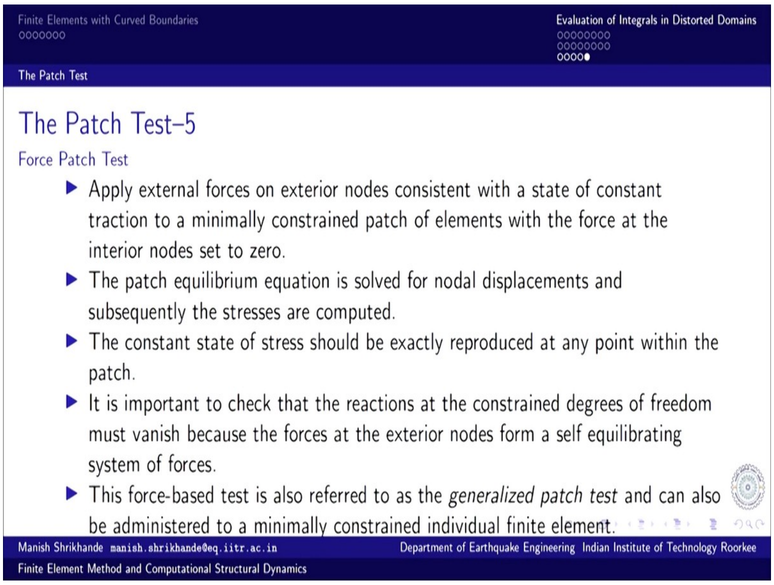


So, constant force, I mean uniform pull or push to a minimally constrained patch of elements. Those are required for the solution of finite element equations and the force at the interior nodes are zero. There are no external forces at the interior nodes, only the boundary tractions.

So, patch equilibrium equation is solved for nodal displacements and subsequently the stresses are calculated. So, the equilibrium the assembled global system of equilibrium are I mean the essential boundary conditions are imposed and then these equations are solved and then displacements and subsequently stresses in the elements are computed.

The constant state of stress should be exactly reproduced at any point within the patch. So, wherever I evaluate, wherever we evaluate, the stress it should always be representing the constant because the applied stress condition is that of a constant stress and this constant stress condition should be exactly reproduced wherever we check.

And, it is important to check that, the reactions at the constrained degrees of freedom must vanish. That is important. The reactions at the constrained degrees of freedom must vanish because the forces at the exterior nodes form a set of equilibrating system of forces. Because we are applying force on both the directions, is not it. So, if you look at it the patch test so, these tractions, they are self-equilibrating.

So, there is pull from on this side and there is pull on this side. So, these two sets are equilibrating to with respect to each other. So, whenever we convert these two equivalent nodal forces, they will all be cancelling each other so, equilibrium will always be maintained. So, the forces that we generate from these at these constrained reactions, they should all vanish because of the nature of the applied forces is such that they are self-equilibrating.

So, the constant state of stress should be exactly reproduced at any point, and it is important to check the values. And, this type of force-based test is also referred to as the generalized patch test and it can also be administered to a minimally constrained individual element so, a single. So, now, that is the idea of generalized patch test.

When we are using the force-based element, force-based test, then equilibrium for the from the equilibrium point of view a single element is also good enough. So, we can impose we can conduct this test on a single element with minimal constraints that will allow solution of equilibrium equations.

And then the single element based test should be conducted, and all these the stress constant stress condition the at any point in the element is checked and verified. So, individual element can also be tested. So, although this single patch test on a single element is a necessary condition, but it is not a sufficient condition. So, it is very easy test. While developing the element, we can individually we can just impose the minimal constraints and subject it to this constant state of stress and evaluate the results.

So, if it works fine, then it is subjected to a more broader patch of elements and more thorough investigation, but if it fails if the element fails the single element test then obviously, it fails and there is no need to proceed any further and that is just discarded, that formulation is discarded.

So, the patch test as I said we also use patch test I mean patch of elements for posterior analysis. What is done is we what it is called super convergent patch recovery of the stresses. The finite element, if we use standard finite element formulation that is based on the interpolation of the displacement. So, the displacement have a certain degree of completeness.

For example, 4-node quadrilateral has displacement field which is complete up to the first degree. So, the convergence of the displacement is of the order of error is of the order of 2 is not it. So, the error polynomial degree in error is of the order of 2. And if we go by the stresses and strains so, the strains are constant; the linear field is the derivative is constant. So, the error is of the order of first degree.

So, and that is a general rule. So, strain stresses and strains they converge slower, more slowly than the displacements. And displacement fields are more accurate than the stresses and strains that are computed. What is done is, actually at some point because the solution is in the minimum weighted residual so, it in the error is minimized in the least square sense.

So, there is I mean when we are looking at the stresses and strains at some point, we may be over estimating stress strains and at some points we may be under estimating strains. So, at some point, there may be some locations where the strains are actually very close to the true value of the strains.

And, it so happens that there are specific locations in the elements which are called Barlow stress points, where if they are evaluated, the stresses and strains they are the very close to the true values of stresses and strains corresponding to the loading condition.

So, once we evaluate those stresses and strains at those Barlow points, and then we construct an interpolation of those sampled values over the patch one patch of elements, and interpolate the stress field, that gives us a stress field which is of higher degree of accuracy then what is predicted by the finite element analysis.

And, this higher degree of accuracy that interpolated field from the sampled at Barlow stress points, and the difference from these stresses computed from the finite element run. That gives us an idea of the error in the finite element solution. So, this is what we call as posterior error analysis.

And again, we go patch by patch, we construct take a patch of elements from the mesh and construct this interpolated model of stresses at evaluated at the Barlow points, and then compare it find the difference between these higher I mean more accurate values of stresses from the stresses computed in the finite element run. And that gives us an idea where which zone is in need of refinement of the mesh, either size has to be reduced or the polynomial interpolation has to be increased.

So, that is what we call as adaptive finite element analysis and size refinement is referred to as h adaptive refinement, and polynomial increase is referred to as p adaptive refinement. So, that brings us to the end of this discussion of mapped elements and in general continuum based finite elements.

We will discuss briefly the finite elements for plate and shells, because that extends these concept these our discussion of variational crimes. So, plate elements for the analysis of plates and shells, they also present some of the very difficult locking problems that need to be resolved. And how do we deal with that, we will discuss in our next lecture.

Thank you.