

Free Surface Flow
Dr. Mohammad Saud Afzal
Department of Civil Engineering
Indian Institute of Technology Kharagpur

Lecture 9

Hello students, welcome to this third lecture of our second module, which is critical flow, and we are going to start exactly from where we left off in our previous lecture. So, you remember we tried to calculate the critical depth for different types of channels: rectangular, triangular, and circular. Now, what are we going to do? We are going to see the same for the trapezoidal channel. So, for a trapezoidal channel, this is how a trapezoidal channel looks.

And you see, if this is the base B and the water depth is y and the slope here is 1 in m , the area of this trapezoid, the wetted area, will be $(B+my)y$, and the top width will be $(B+2my)$. And at the, you know, this is the condition for the critical flow, all right. So, we say at the critical flow, this condition should be satisfied. And therefore, $\frac{Q^2}{g}$ will be, we substitute the value of A_c and T_c in terms of, so A_c from here and T_c from here. Yeah.

And that comes to $\frac{(B+my_c)^3 y_c^3}{B+2my_c}$. Here. Here, an explicit expression for the critical depth y_c is not possible. This is very important to note. The non-dimensional representation of the above. So, this equation we will non-dimensionalize.

We generally non-dimensionalize. And how does it help? So, it helps in the solution of y_c with the aid of tables or graphs. Exactly, like we did for the circles. You remember from our last lecture.

So, just trying to, you know, do it in a, you know, a little bit more of, you know, non-dimensional quantities. So, we just rearrange our equations like this. And then, you know, where we substitute $\frac{my_c}{B}$ as zeta, and this equation gives $\frac{Q^2 m^3}{gB^6}$ in terms of zeta alone. Or we can also write it like this: $Q_m^{3/2}$. What we do is we take the square root of all this.

This gives the function ψ , and the above equation can be easily evaluated for various values of zeta and plotted as ψ versus zeta c. It may be noted that if alpha is equal to 1, ψ can be. So, this is for the uniform condition where alpha is equal to 1. But if we have alpha greater than 1, we can define ψ as this: $\alpha \left(\frac{Q^2 m^3}{g B^6} \right)^{1/2}$. The table which I am going to show below will give the values of ψ for different values of zeta. This table is very useful in the quick solution of problems related to the critical depth in trapezoidal channels.

So, this is how a typical value of, you know, the channels. This is ψ , you know. So, ψ , zeta, ψ , zeta. Okay. Zeta, ψ .

So, this is how it gives for different values of zeta. And since you see, zeta is $\frac{m y_c}{B}$ here. And this can easily be known because Q is known, m is known, g is known, and $B^{5/2}$ is known. So, ψ is known. What we need to calculate is eta, and then we can calculate y_c from here.

So, these are the tables that will be used for the solution of the problem. Now, we know since A/T is A , we substitute A and T as $B+2my$. The Froude number at any depth is given by $\frac{V}{\sqrt{g \frac{A}{T}}}$. And what we do is write V as Q/A , and that is a function of $\frac{my}{B}$ for a given

discharge Q . Now, I mean, these are the things that you know; the Froude number is also a function of $\frac{my}{B}$, and you can also calculate it like that. Now, we go to what the section factor Z is.

So, we know that the critical state of flow can be represented by the Froude number being equal to 1. Critical condition means the Froude number will always be equal to 1 or $\frac{V}{\sqrt{gD}}$ is equal to 1. This is the formula for the Froude number. Or, instead of V , we write Q/A , and \sqrt{gD} is there, equal to 1, or $\frac{Q}{A\sqrt{gD}}$ is equal to 1. This is a simple rearrangement.

And this A by, so $\frac{A}{D^{0.5}}$ is nothing but Z or the section factor. All right. Or you can also say

Z is equal to $\frac{Q}{g^{0.5}}$. So, this is the section factor, a very standard term in open channel

hydraulics. So, now we see the section factor Z is a function of the geometrical properties of the channel cross-section and y_c . You see the geometrical properties, right?

Okay. There is only one critical depth for a specified discharge in a given channel. If the section factor for the channel cross-section increases monotonically with y , this you can see directly from the equation. An explicit relation can be derived to determine the critical depth in a rectangular, triangular, or parabolic channel. However, for general applications, for example, trapezoidal, the critical depth may be determined by using design curves of Chow or by the trial and error method or by numerical solution of the nonlinear algebraic equation.

So, in some cases, it is possible to solve analytically or mathematically, but other times, you have to use the tables or the design curves by Chow or by trial and error; a numerical solution is definitely possible. Now, what are design curves? We talked about design curves just in the previous slide. So, design curves look something like this. You see, this is y_c ; this is the critical depth.

So, this is an example of a circular channel. This is for the trapezoidal channel. You see, circular, these are for trapezoidal, and this is rectangular. So, depending on this, this is

$\frac{Z_c}{D_o^{2.5}}$, that is raised to the power 2.5; this is D for circular, and this is for others,

rectangular or trapezoidal. All right.

So, depending upon $\frac{Z_c}{D_o^{2.5}}$, you can go ahead in the curve, you know, and then try to, you

know, just match it here and see what $\frac{y_c}{D_o}$ is and simply use the design curve for the

calculation of y_c . It is quite simple, and this has been done by Chow and many others later.

So, let us see, I mean, so I have already explained to you the terms, but let us see one by one. The value of discharge Q is known, and thus the section factor can be computed; Z_c

can be computed, right, because we know the value of discharge. Now, divide the value of the section factor by $B_o^{2.5}$.

For a trapezoidal channel of bottom width B_o or $D_o^{2.5}$ for a circular section of diameter D_o . These are steps; you know, these are actually steps to use in the design curve. So, if Q is known, you calculate Z_c , and then if you have Z_c divided by $B_o^{2.5}$, determine this value and this value. So, find the corresponding value as I told you, $\frac{y_c}{B_o}$ or $\frac{y_c}{D_o}$, depending upon the cross-sectional geometry like this or this or this.

So, this is quite simple to do, and then once you have obtained $\frac{y_c}{B_o}$, you know, or $\frac{y_c}{D_o}$, depending upon the channel cross-section, you can calculate y_c here. So, we see a problem, a sample problem. The question is to compute the critical depth in a trapezoidal channel for a flow of $30 \text{ m}^3/\text{s}$. The channel bottom width is 10 meters, side slopes are 2:1, and the bottom slope is negligible, right?

So, that means we have been given Q , we have been given B_o , and this is trapezoidal. So, the slope is also given. So, let us see, as I have written, Q is $30 \text{ m}^3/\text{s}$, B_o is 10 meters, and s is 2, I mean, as per our design curve. Now, we calculate Z_c as $\frac{Q}{g^{9.81}}$, Q was $\frac{30}{\sqrt{9.81}}$, this gives us 9.58. So, this value 9.58, we are going to look for s of 2:1.

So, first, we divide it by $B_o^{2.5}$. So, 9.58 divided by $10^{2.5}$ gives us 0.03. So, the X-axis is 0.03. Now, we will look into the design curve for s equal to 2. Here.

So, we have s equal to 2 somewhere here. We are going to look at it, and we will find that the value of $\frac{y_c}{B_o}$ corresponding to is 0.09. This we can look from the design curve. Again, just showing you how to So, for see, the value that came out was 0.03.

So, we look at 0.03 first. 0.01, 0.03 should be somewhere here. And then we take it up to s is equal to 2 somewhere here. And then we try to find out. And this value comes out to be

0.09, and hence y_c by B_o is 0.09, and y_c comes out to be 0.9 meters. This is the critical depth. This is one of the typical examples of finding the critical depth based on these design curves. So, the use of this design curve is quite an important thing. So, let me just maybe have a look here.

Let me erase this. So, now we are going to talk about something else, that is the first hydraulic exponent M . What is that? So, in many computations that involve a wider range of depth in a channel, such as in gradually varied flow computation, it is convenient to express the variation of Z with y in exponent form. Okay. The Z - y relationship can be written as a normal relation, which can be written as $Z^2 = C_1 y^M$. And this is found to be very advantageous.

In this equation, C_1 is a coefficient and M is an exponent called the first hydraulic exponent. It is found that generally M is a slowly varying function of the aspect ratio for most of the channel shapes. The variation of M and this $\frac{my}{B}$, that is zeta, for a trapezoidal channel is indicated in the figure that I am going to show you next. So, zeta is $\frac{my}{B}$, and this is the hydraulic exponent M . You see Z^2 is always in terms of y , right?

This is a general expression. But for rectangular, M is equal to 3, and for triangular, M is equal to 5. This is what we, I mean, just, this is what we are going to find. This is the variation of the first hydraulic exponent m in a trapezoidal channel, all right? So, the value of m for a given channel can be determined by preparing a plot of Z versus y on a log-log scale.

If m is constant between two points, that is Z_1, y_1 and Z_2, y_2 in this plot, the value of m is determined as m can be written as $\frac{2 \log(Z_2 / Z_1)}{\log(y_2 / y_1)}$. So, in this equation, the above equation, instead of Z , a non-dimensionalized Z value can also be used. For the trapezoidal channel equation, this equation can be used, and this represents a non-dimensionalized value of Z . If the suffix c is removed, and the slope of ψ versus $\frac{my}{B}$ on a log-log plot, such as in the

figure, I mean the figure before, can be used to obtain the value of m at any value of ψ . It may be noted that the m for a trapezoidal channel is a unique function of $\frac{my}{B}$ and will have a value in the range of 3 to 5. So, this is the most important value between 3 and 5; this value will exist.

An estimate of M can also be obtained by the relation; you can use this relation as well. These are all analytical or, you know, empirical equations, mainly empirical equations. So, this equation can also be used to determine the value of M for any cross section. Going further ahead, we are going to talk about channel transitions. So, the concepts of specific energy and critical depth are extremely useful in the analysis of problems connected with transitions. So, a transition is when a channel changes, you know, shape, maybe a contraction or any changes like that.

Like, for example, a channel with a hump is also a channel transition. So, this is a normal channel, right? If it encounters a hump like this, there is a transition, and then the water surface will, you know, fall down; the energy line will go down. So, let us say for subcritical flow, if we consider a horizontal frictionless rectangular channel of width B carrying Q at a depth y_1 . So, our depth is y_1 , okay?

And assume the flow to be subcritical. Our first assumption, as I said here, is that we are assuming a subcritical flow at section 1. Now we assume that at section 2 there is a smooth hump of height, ΔZ here. Since there is no energy loss between section 1 and section 2, the construction of the hump causes the specific energy at section 2 to decrease by ΔZ , right? Thus, the specific energy at sections 1 and 2 are given by E_1 , which can be written as

$$y_1 + \frac{V_1^2}{2g}, \text{ and } E_2, \text{ which is actually given by } E_1 - \Delta Z, \text{ all right?}$$

So this is something that you understand. So we write the specific energy at section 1 as this and the specific energy at section 2 as $E_1 - \Delta Z$. Since the flow is subcritical, the water surface will drop due to a decrease in specific energy. You see the specific energy is decreasing, right? And therefore, the water surface will drop. So, in this figure, the water surface, let us say, which was at point P.

At section 1 will, sorry here, so which was at section here, P will come down to R at section

2. And the depth y_2 will be given by E_2 as $y_2 + \frac{V_2^2}{2g} = y_2 + \frac{Q^2}{2gB^2 y_2^2}$. Now it is easy to see

from the figure that as the value of ΔZ is increased, the depth at section 2, which is y_2 , will decrease. The minimum depth is reached when the point R coincides with C. So, as you keep on increasing ΔZ , this P will start coming to R, and then the minimum that it can come to is this critical point. And that is at the critical depth point.

Now at this point, the hump height will be maximum. So, when so the energy cannot go below that, right? So, the maximum hump height will be ΔZ_m , and y_2 will be equal to y_c , which is the critical depth, and E_2 will be equal to E_c . Then the condition at ΔZ_m is given by the relation $E_1 - \Delta Z_m = E_2$, or because now E_2 has become critical, right? So, it becomes

$$E_c, \text{ and } E_c \text{ is nothing but } y_c + \frac{Q^2}{2gB^2 y_c^2}.$$

Now, the question that comes to our mind is: what happens when ΔZ is greater than ΔZ_m ? We can easily see that the flow is not possible with the given condition, that is, with the given specific energy. The upstream depth has to increase to cause an increase in the specific energy at section 1. If this is modified, the depth is represented by y_1' . So, for this particular thing, if we further increase the hump over ΔZ_m , the flow is not possible.

In that case, the depth here, you know, this depth here must increase. So, to allow the

increase in the hump. Now, $E_1' = y_1' + \frac{Q^2}{2gB^2 y_1'^2}$, with E_1' greater than E_1 and y_1' greater

than y_1 . Now, at section 2, the flow will continue at the minimum specific energy level, that is, at the critical condition. At this condition, again, y_2 is equal to y_c , that is, $E_1' - \Delta Z$ is equal to again E_c .

So, again, recollecting the various sequences, I mean, the sequences when ΔZ is between 0 and ΔZ_m , the upstream water level remains stationary. When we have a hump which is in between the maximum hump and 0, the upstream water level remains stationary at y_1 , while the depth of the flow at section 2 decreases with ΔZ reaching minimum. This is the first thing that we saw a couple of slides ago. Right, and that y_c at ΔZ_m . Now, if we further

increase the value of ΔZ , y_1 will change, so what will happen? The water level upstream will change to y_1' , while y_2 will continue to remain at y_c . So, all so, basically, what it means to say is, You can, so while keeping the upstream water level constant, you can increase the hump height from 0 to the maximum Z_m .

And if you further increase the hump size, so in the first condition, the water level was stationary upstream, but it was decreasing and decreasing downstream. But if you increase the hump size even further, then what happens is, At the downstream, the water depth will remain the same, that is, y_c , and the water at the upstream will keep on rising. Now, the question that comes to our mind is: what should be the minimum size of the hump for critical flow? Now, assuming a frictionless situation, we consider a smooth, frictionless, no shear stress.

A streamlined hump of height ΔZ is placed at a section in a rectangular channel carrying subcritical flow. We have been given much important information. The relationship between specific energies at the section upstream of the hump and at the section is given as very simple. $E_1 = E_2 + \Delta Z$, right? Now, let ΔZ_m be the height of the hump that would cause critical flow to occur over the hump without changing the upstream specific energy.

Any value of ΔZ greater than ΔZ_m would cause critical flow over the hump, but the upstream specific energy would change to a value greater than even. Thus, ΔZ_m could also be called the minimum height of a streamlined, frictionless hump that has to be provided to cause critical flow over the hump. Okay. So, an expression for the value of ΔZ_m is obtained as below. Since at $\Delta Z = \Delta Z_m$, E_2 is $\frac{3}{2} y_c$.

You remember E_2 was $\frac{3}{2} y_c$ for a rectangular channel. And

$$E_1 = E_c + \Delta Z_m \text{ or } \Delta Z_m = E_1 - \frac{3}{2} y_c . \text{ And then we divide both sides by } y_1 . \text{ We get } \frac{E_1}{y_1} - \frac{3}{2} \frac{y_c}{y_1}$$

. And we just substitute, you know, I mean, we equate this in terms of the Froude number at the depth, and we come up with an equation that is purely in Froude's number.

So, basically, this is the equation that comes for a frictionless case. Now, what happens if there is energy loss due to the hump? So, let us assume that the energy loss in the transition is H_L . So, the only difference is that we have introduced an additional term, H_L , here. Additional term.

Now, following the same procedure as in the frictionless case, E_2 will be E_c . We put $\frac{\Delta Z_m}{y_1}$

, and since this is there, we can write $\frac{\Delta Z_m}{y_1} + \frac{H_L}{y_1}$ in terms of the Froude number. This is

from using the using the frictionless case. Now, if we compare equation 1 with equation 2, it may be noted that the effect of energy loss in the transition due to shape and friction is equivalent to that of a hump placed in the downstream section.

So, the effect is that of a hump placed in the downstream section, alright, if we compare equation 1 and 2. See, in the case of frictionless, this was the hump. Now, this is also an additional hump. So, this we did for subcritical flow. Now, let us see what happens for supercritical flow.

So, if y_1 , if you do not recollect, just go and see for this particular case. This one, channel transition, we started for subcritical flow. Assuming the flow here was subcritical. So, what we are going to do now is we will see what happens if the flow this side is supercritical. Yeah, for supercritical flow.

So, if y_1 is in the supercritical flow regime, this figure here shows the variation of y_1 and y_2 in the supercritical flow over the hump. Now, what this figure exactly shows is that the depth of flow increases due to the reduction of specific energy. In the figure, this point, I mean the earlier figure point P', corresponds to y_1 , and point R' corresponds to y_2 . to the depth at section 2 up to the critical depth. y_2 increases to reach y_c .

Now, ΔZ is equal to ΔZ_m . For ΔZ greater than ΔZ_m , what will happen? The depth over the hump y_2 , equal to y_c , will remain constant. We are taking analogies from previous cases, and what will happen? The upstream depth variable will change.

It will decrease to have a higher specific energy. So, you remember in subcritical flow, the water depth was increasing to increase the specific energy. When the flow is supercritical, to have a hump greater than ΔZ_m for supercritical flow, the depth will have to decrease to have a higher specific energy E_1 . That is the main difference. So, the variation of depth y_1 and y_2 with ΔZ_m in the supercritical flow is shown in this figure.

All right. So, I think with this, I will end this particular lecture and continue with the transition involving a change in width in our next lecture. Thank you so much.