

Free Surface Flow
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Lecture 8

Welcome, students, to this second lecture of our Module 2, which is on critical flow. So, we are going to start with specific force, exactly at the same point where we left off in the last lecture. Specific force. What is specific force? So, understand this.

We will see this particular diagram; we have seen that in our introduction part as well, you know, in Module 1. However, we will go again and try to, you know, I mean, this is quick, but I will again, you know, F_1 and F_2 are hydrostatic pressure forces, okay? F_3 is shear, and what is F_4 ? The component of the body force, and what are M_2 and M_1 ? The rate of change of momentum, all right. So, you see, if we have a short and horizontal and prismatic channel, F_3 and F_4 , horizontal means there will be no weight component. And for, you know, a short channel, that means we can neglect the frictional resistance, which means F_3 and F_4 will become 0. So, this equation will turn out to be $F_1 - F_2$ is equal to $M_2 - M_1$, simple.

And F_1 is $\rho g A_1 \bar{x}_1 - \rho g A_2 \bar{x}_2$ is equal to $m d \rho g$ by g , ρg is the unit weight of water or liquid. And this \bar{x} is the location of the centroid from the free surface. So, ρg if written as w , we can write $w A_1 \bar{x}_1 - w A_2 \bar{x}_2$ is equal to $\frac{w}{g} Q (V_2 - V_1)$. This is a standard momentum, you know, momentum formula. And w , as I said, is the unit weight of water.

So, I will just take away this one, even take away this one. So, $A_1 \bar{x}_1$, this w gets canceled from all sides like this one, this one, and this one. So, $A_1 \bar{x}_1 - A_2 \bar{x}_2$ is $\frac{Q}{g} \left(\frac{Q}{A_2} - \frac{Q}{A_1} \right)$, or

we can write, you know, $\frac{Q^2}{g A_1} + A_1 \bar{x}_1$ is equal to $\frac{Q^2}{g A_2} + A_2 \bar{x}_2$, and this is specific force.

Specific force is $\frac{Q^2}{gA} + A\bar{x}$; this is the definition of. Similar to specific energy, we have something called specific force.

Sorry, let me just bring this up together. So, you see this is a specific energy curve, right? This you have seen. So, now we have this specific force curve. So, for a given discharge Q , let us say we have a given discharge Q , this curve also has two limbs, CA and CB or CA and CB.

It is easier for you to understand like this. So, now this limb CA is approaching our x-axis like this asymptotically, asymptotically. Limb CB, this limb CB is rising upward and extends up to infinity, very important. So, for a given value of specific force, if you have one specific force here F_1 equal to F_2 , the curve has two depths at y_1 and y_2 . So, similar to specific energy, for one specific energy, we have two alternating depths, and similarly, for one specific force, two

alternating depths at y_1 and y_2 called sequent depths. Let us see, we are looking at this point C. Let me remove these two points. At point C, these two depths again become one. One depth called the critical depth, and here the specific force is minimum, and this also is called the critical flow. Condition, it is important.

Now, we will see to understand this, I was just a very small problem to solve. Now, the question is, show that the condition of minimum specific force at a given discharge represents the critical flow condition. How to solve that? So, if you look at the solution, we know that the specific force F is given by, in general, written here $\frac{Q^2}{gA} + A\bar{x}$, same as written in this particular slide. Now, you know when we try to find out maxima, minima, the best is to do finally using the way of differentiation.

So, we differentiate this particular equation with respect to y and we equate it to 0 for the minimum condition. So, what are we going to do? We are going to do d/dy of F , and in place of F , we write this. So, we do $\frac{d}{dy} \frac{Q^2}{gA} + \frac{d}{dy} (A\bar{x})$ is equal to 0.

And then, since d/dy is the thing, it becomes Q^2 , so only A is a function of y , right? So, what we do is $-\frac{Q^2}{gA^2} \frac{dA}{dy} + \frac{d}{dy}(A\bar{x})$, all right. Consider only the second term of the left-hand side. So, let us consider this term $\frac{d}{dy}(A\bar{x})$. Now, if we utilize the definition of centroid, what is the centroid?

\bar{x} is equal to we write $A\bar{x}$ can be written as $\int_A y dA$. And this is integrated over our area,

or we can write the integral of $\frac{d}{dy} \int_A y dA$ or simply A . Thus, equation 1 becomes

$-\frac{Q^2}{gA^2} \frac{dA}{dy} + A$. You see this particular term came from here, right? Or $-(Q^2)T$, you see

$\frac{dA}{dy}$ can be written as T , so. I mean, of course, I told that $\frac{dA}{dy} = T$ or $\frac{Q^2 T}{gA^3}$. If you realize,

this is the same condition or critical flow condition.

So, we were able to prove it. So, the question was we had to show That the condition of minimum specific force at a given discharge represents the critical flow condition. Here, we have been able to show exactly the same. Now, going into more detail into critical flow conditions.

We know that specific energy is minimum for a given discharge at critical flow condition. We know discharge is maximum for a given specific energy. We know specific force is minimum for a given discharge. This all we have seen and derived. Discharge is maximum for a given specific force.

And all the above conditions can be proved analytically from the basic equations. Now, the next step is we want to calculate how to find or how to calculate the critical depth that is y_c . So, for a rectangular channel, this is a rectangular channel, this is top width, this is width. So, critical flow condition is given by $\frac{Q^2 T}{gA^3} = 1$. So, in a rectangular channel area

is nothing but $B \cdot y_c$ and T is equal to B and capital Q/B is a small q discharge per unit width.

We can write this one as instead of T we write B and in place of A we write By_c . that

means, so $\frac{1}{g} \frac{Q^2}{B^2} \frac{1}{y_c^3} = 1$ and y_c^3 is equal to $\frac{q^2}{g}$ or the critical depth condition comes out

to be $y_c = \left(\frac{q^2}{g}\right)^{1/3}$. So, it depends only on the channel geometry and also the discharge

value. So, we also need to find the corresponding specific energy at critical depth.

So, you know energy is a critic means critical is just a subscript So, E_c is $y_c + \frac{V_c^2}{2g}$ or

$y_c + \frac{Q^2}{2gA_c^2}$ or $y_c +$ We just doing instead of so, we are what we are substituting is A_c is

equal to By_c and then we are trying to separate out. So, it becomes $y_c + \frac{Q^2}{B^2} \frac{1}{2gy_c^2}$ instead

of Q/B we write small q . So, we are able to write y_c here this becomes q square this g comes here.

So, $\frac{q^2}{g} \frac{1}{2y_c^2}$ or y_c plus, and you see, for critical flow condition, this $\frac{q^2}{g}$ was how

much? You see, $\frac{q^2}{g}$ was y_c^3 . So, this y_c^3 divided by $2y_c^2$, it becomes $y_c + 0.5 y_c$, which

gives $3/2 y_c$. So, critical energy is 1.5 of y_c . This is an explicit or direct solution of critical depth in a rectangular channel.

This is very simple because our channel geometry is very, very simple. So, let us solve a small problem with regards to this particular one. On the slide, of course, supercritical flow occurs at a Froude number of 2, at a depth y equal to 0.63 meters, in a rectangular channel. Find the critical depth. It is saying that supercritical flow, so we have to first take q from the question itself. It says supercritical flow, and it is confirmed because the Froude number is 2.

And it has given the depth also. Right? So, the depth is given as 0.63. Right? So, what do you think?

What will be the critical depth? Will it be more, or will it be less? See, supercritical. Right? Supercritical corresponds to less depth.

Therefore, critical, I mean just your way of thinking. So, critical depth will be greater than supercritical depth. So, critical depth will always be greater, should become 0.63 meters. So, when you find your answers, later we will see and then we try to see if we have got the correct answer.

This is just a logical way of thinking. So, let us look at the solution now. So, you see Froude number is equal to $\frac{V}{\sqrt{gD}}$ and D is equal to y for triangular section, right? D is hydraulic depth that is the y . So, Froude number was 2 here, right? So, Froude number was 2.

V we have to find out actually. g we know and D we know. So, supercritical velocity is 4.972. Now, next step would be probably to find what? small q small q is capital Q/B , AV/B , A is $B \times y$ so basically small q is nothing but $V \times y$ we have found out y also for supercritical state we have found out because small q remains same.

right. So, it comes to be 3.13 meters squared per second. Now, very simple we have got

q , we have g . So, y_c is $\left(\frac{q^2}{g}\right)^{1/3}$, $\left(\frac{3.13^2}{9.81}\right)^{1/3}$, it comes to be approximately 1 meter. So,

our critical depth comes to around 1 meter. Now the same thing we will try to see for a triangular section.

So for a triangular channel having a side slope of, let us say, m horizontal and one vertical. This is the figure. So, like a rectangular channel, now we have changed our cross-section to triangular. So, what will be the area? The area will be my^2 , and the top width T will be $2my$.

It is very simple to calculate using your geometry. So, we are going to substitute the critical flow condition, that is, $\frac{Q^2}{g}$ is equal to $\frac{A_c^3}{T_c}$ or m^3 . So, we substitute the value of A

here and T here. So, we get $\frac{m^3 y_c^5}{2m y_c}$, all in terms of $m^2 y_c$. Or we can simply write for a

triangular section, y_c can be written as $\left(\frac{2Q^2}{gm^2}\right)^{1/5}$ where m is the slope, m horizontal and

1 vertical.

Now, finding the specific energy at this critical depth is also simple. E_c , this is the

general formula, right? $y_c + \frac{V_c^2}{2g}$, move it in terms of Q . So, $y_c + \frac{m^2 y_c^5}{m^2 y_c^4}$, and you will

see m^2 and m^2 will get canceled, and this will remain y_c . In the next slide, we are going to see that. So, we are going to get this. So, I think there is some problem here; there should be this one by 4 actually.

But anyways, so now we see that E_c is equal to $1.25 y_c$. So, it is noted that the above equation is independent of the side slope m of the channel. Since A/T is equal to $y/2$, the Froude number for a triangular channel is defined as the Froude number for a triangular

V , not $\frac{V}{\sqrt{gy}}$, its characteristic length, right? So, $\frac{V}{\sqrt{g \frac{y}{2}}}$. So, $\frac{\sqrt{2}V}{\sqrt{gy}}$.

So, similarly, trying to see for a circular channel. So, we know the diameter D of the circular channel is given, and 2θ is the angle in the radius subtended by the water surface. This is the water surface. A is the area of the flow section. So, a very, very simple area of the sector plus, yeah.

So, we are simply trying to put areas. They see the equations become a little complex, that is true, but since the geometry is there, you can always try to derive it in this way. You cannot learn these equations for a circle and these things, but you can always try to derive it in this way. Find it out, you know, in the exam or anything is there. So, however,

this result is not that bad. A is equal to $\frac{D^2}{8}(2\theta - \sin 2\theta)$, and top width T is equal to $D \sin \theta$, that we know this is top width T . And 2θ , of course, then we can write $2\cos^{-1}(1 - 2y/D)$, that is, it is a function of both y and D . If we substitute this in this equation, so this is the main equation for critical flow.

And what are the things that we substitute A_c for circular and T_c also. For circular

conditions. So, $\frac{Q^2}{g}$ will come out to be $\frac{\left[\frac{D^2}{8}(2\theta_c - \sin 2\theta_c)\right]^3}{D \sin \theta_c}$. Since explicit solutions

for y_c cannot be obtained from this above equation, we can obtain a non-dimensional representation. So, we can have $\frac{Q^2}{g}$ equal to $0.0441942\theta_c$. See, basically all in terms of $\sin \theta_c$. D is there, but it gets cancelled. Basically, I do not suggest you to go for this one.

This is the main equation that you should be looking after. And there are certain tables that are going to help you out when trying to solve these problems. For example, these are elements of a circular channel, you know, for y/D , you have this radiance, you have so with radiance, you have A/D^2 , they are ready to use. In exams, we will not be expecting this much. I mean, the idea here is to say it is like a table, a table for different cross-sections.

There will be a table for circular channels, a table for triangular, trapezoidal, and so many for different values of y/D and θ . Here we have different values of different elements. You see, this is the table that is there. This is the table, just the continuation, and since A/T is a function of y/D , the Froude number for a given Q at any depth y will be very simple. Now, other than this, these two empirical equations have been proposed for quick and accurate estimation of critical depth in circular channels.

So, you can use this one in terms of the Froude number. So, the Froude number is, you know, y_c/D , this is by Swamee and this is by Straub. These are the two equations that can be used for quick estimation. If any questions are given, you will be provided these

formulas unless it is some basic type of derivation that you are required to do using your hand. All right.

So, I think I will end this lecture here, and in the next class, I will start with the trapezoidal channel. Thank you so much.