Free Surface Flow
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Lecture 30

Welcome, students, to our last lecture of the sixth module about gradually varied flow. So here, we are going to finish a little bit of theory and solve some problems which would involve the calculation of normal depth because normal depth calculations are also required when we do the gradually varied flow profiles. So, that being said, we will proceed from where we left off in the last lecture. So, talking about the direct integration of the gradually varied flow differential equation. So, in terms of $\boldsymbol{K}$ and $\boldsymbol{Z}$, the differential equation for gradually varied flow.

For prismoidal channels, it can be written using this, which is $\frac{\partial y}{\partial x}$ is $S_0 \frac{(1-(K_0^2/K^2))}{(1-(Z_c^2/Z^2))}$. So, the important thing to note here is that this equation is a non-linear, first-order, ordinary differential equation. These are the three properties of this particular equation. And this can be integrated by analytical methods to get closed-form solutions only under certain very restrictive conditions. So, we cannot obtain a general solution for this, but analytically, we can get closed-form solutions for very special and restricted conditions.

So, there are several methods, the method due to Chow. So, there is a scientist, Chow, whose method is based on certain assumptions but is applicable with a fair degree of accuracy to a wide range of field conditions, is presented here. So, now what we are going to present is a method by Chow. So, this method is applicable for a wide range of field conditions. So, now looking at how he approached the problems, let it be required to find $y = f(x)$ in the depth range y_1 to y_2.

So, Chow had two assumptions that he made. And what are those two assumptions? The conveyance at any depth he assumed can be written as K^2 is equal to $C_2 y^N$. And therefore, at the depth y_0, where y_0 represents normal depth, K_0^2 can be written as $C_2 y_0^N$. This implies that in the depth range which includes y_1, y_2, and y_0, the coefficient C_2 and the second hyperbolic hydraulic exponent N, not the hyperbolic hydraulic exponent N, are constant.

So, what we are assuming is that the coefficient C_2 and N are constant. This is the first assumption. The second assumption is about the section factor Z. So, here the conveyance at any depth y, the section factor Z at any depth is also given by Z_2 is equal to $C_1 y^M$. And therefore, at critical depth or at normal depth, at any depth, say for example, why not normally is represented by normal depth. So, Z_c^2 is equal to $C_1 y_0^M$. If it is critical depth, then it should be Z_c^2 whole square is equal to $C_1 y_c^M$. Anyways, the idea is to just tell you that there are involved components like C_1 and M, the hydraulic exponent M. Now, this implies that in the depth range which includes y_1, y_2, and y_c, the coefficient C_1 and the first hydraulic exponent M are constant, the same as the one in the conveyance factor, the conveyance. Now, if we substitute the relationship by equations A1 through A4 in equation A, that means this one, if we substitute in this all these A1, A2, A3, and A4. We are going to obtain this equation which is $\frac{\partial y}{\partial x}$ is equal to $S_0 \frac{(1-(y_0/y)^N)}{(1-(y_c/y)^M)}$. So, this is N and this is M, this equation.

Now, if we put u equal to $\frac{y}{y_0}$, we can write dy equal to if we have this equation, then we can write $\frac{dy}{du}$ equal to y_0. y is equal to $y_0 u$. So, $\frac{dy}{du}$ is equal to or dy is equal to $y_0\, du$, same as this. And equation A5, this simplifies to this. So, $\frac{dy}{dx}$, this can be written as $\frac{\partial u}{\partial x}$ equal to $\frac{S_0}{y_0}$ using this condition here $\frac{(1-(1/u)^N)}{(1-(y_c/y)^M(1/u^M))}$ or and then we can bring I mean we can try to integrate dx and du on two different sides this one. So, this equation will turn to this particular equation. Now, if we integrate, we will get x equal to $\frac{y_0}{S_0}\Big[u - \int_0^u \frac{du}{(1-u^N)} + (y_c/y_0)^M + \int_0^u \frac{u^{N-M}}{1-u^N} + constant\Big]$. This is a normal integration. And if we try to look at this particular term here, we can simplify this integral, second integral as below.

Let us put $u^{N/J}$ where J is $\frac{N}{N-M+1}$. This is quite complicated maths to get dv equal to N/J. This is also differentiation of this. And this is the value that we are going to get. Now, it may be noted that ***F*** as a function of ***υ*** and ***J*** is the same function as ***F(u, N)***, with ***u*** and ***N*** replaced by ***υ*** and ***J***. So, these are the equations that come from mathematics.

So, we replace ***υ*** and ***J*** by ***u*** and ***N***. So, this equation A6 can now be written as ***x*** is equal to $\frac{y_0}{S_0}\left[u - F(u,N) + (y_c/y_0)^M \frac{J}{N} F(\upsilon, J)\right]$. And now, if we use this particular equation between two sections having coordinates (x_1, y_1) and (x_2, y_2), this will give us this particular long equation. Now, this function ***F(u, N)*** is known for varied flow conditions. I mean, the varied flow function. Extensive tables are readily available in the table.

So, this is a table that can be used for the calculation of the function ***F(u, N)***. Of course, this is quite a long and lengthy problem, a difficult one to understand. Do not bother in getting too much detail of this, but at least remember how this is solved. So, using this table, we can solve this particular equation that will give us the gradually varied profile for x_2 in the ***x*** direction. So, a method of obtaining the exact analytical solution of this one for integral and non-integral values of n is given by Gil.

So, a scientist called Gill, numerical integration can be performed easily on a computer to obtain tables of varied flow functions. And there are different scientists who have given a procedure for this. And you can look into the references that I have been showing in your previous slides to have more detail of how this numerical integration is done. However, in practical applications, since the exponents ***N*** and ***M*** are likely to depend on the depth of the flow, The average values of the exponents applicable to the range of values of depth involved must be selected.

Thus, the appropriate range of depth for ***N*** includes y_1, y_2, and y_0. ***N*** depends on y_1, y_2, and y_0. And for ***M***, it includes y_1, y_2, and y_0 again. In computing water surface profiles that approach the limits asymptotically, that means ***y*** going to y_0. The computations are usually terminated at ***y*** values which are within 1% of their limit values.

So, basically what I mean to say is that computer programs are generally used for the solution of this type of problem. Now, that being said, we will also have a slight, you know, a small look into the advanced numerical methods. So, we know the GVF equation normally dy/dx is equal to $S_o - S_f$ is equal to $1 - Fr^2$. This is a function of depth.

And this is exactly what is written here. Normally, it can be represented as dy/dx is a function of y in which $F(y)$ is $\frac{S_0 - S_f}{1 - \frac{Q^2 T}{g A^3}}$ and is a function of y only for given S_o, n, Q, and channel geometry. So, the important thing to understand is this equation here. It is nonlinear, and a class of methods which is particularly suitable for the numerical solution of the above equation is the Runge-Kutta method. So, one of the methods in which you solve this nonlinear equation is the RK method or Runge-Kutta method.

There are different types of Runge-Kutta methods, and all of them evaluate y at $x + \Delta x$ given y at x. So, if you know the value of y at x, then you can evaluate the value of y at x + a small distance Δx from that particular point. Now, if you use the notation y_i is equal to y of x_i, and x_i plus Δx is equal to $x_i +1$, hence y_i+1 is equal to y of x_i+1. The various numerical methods for the solution are given. I mean, I will just list them. One is the standard fourth-order Runge-Kutta method, where y_i+1 is given as this, right?

And what is K_1? K_1 is $\Delta x\, F(y_i)$, K_2 is $x_i\, F(y_i + \frac{K1}{2})$. So, it is an iterative procedure to find the value of y at the next grid, which is unknown, given the value of y at a known point. The beginning will be from the control point or the boundary condition. So, this is fourth-order and has more accuracy than the third-order equation. Kutta-Merson method.

So, all of them are evaluating y at the next grid when we know the value of the function or y at the grid that we know, right? For example, we start from here, a known quantity, and we can calculate this. For calculating this, we use this. Now, we know this. Using this, we calculate this. Using this, we calculate this. Using this, we calculate this. So, it is a So, it is a Marching scheme. So, see it has K_1, K_4, K_5, and see this is all quite complicated math. So, we will not get into much detail, but for your knowledge, I think this I am just covering what are the ways to solve numerically. So, using the above method, the channel, as I told, is divided into n parts of known length interval.

So, from starting 1, 2, 3, 4, this goes up to its equal length. It is divided right, N you say n parts are there if you start counting. And, this length interval is called Δx. Starting from the known depth, as I told before, the depth at other sections is systematically evaluated. For a known y_1 and Δx, the coefficients K_1, K_2 until K_N are determined by repeated calculations. And then, by substitution in the appropriate main equation, The value of y_{i+1} is found.

The SRK method involves the determination of $F(y)$ 4 times, while the KM method involves $F(y)$ to be evaluated 5 times for each depth determination. These two methods are direct methods, and no iteration is involved. The KM method possesses an important advantage in the direct estimate of its truncation error, which can be used to provide automatic interval and accuracy control in the computations. So, this is one way, but again, these types of methods are generally solved, you know, using a computer program. So, as of now, not in the scope of that particular assignment or exams.

Another method, which is called the trapezoidal method or TRAP method, is to see the whole point of estimation of y_{i+1}, which is the value of depth at the next step. In all these methods, we divide it into equal parts, so we divide it into Δx, Δx. If this is i, this is $i+1$, this is $i+2$, this is $i+3$, this is $i+4$. So, in this trapezoidal method, we use a trapezoidal approach. y_{i+1} is equal to y_i plus half of Δx $F(y_i)$ - $F(y_{i+1})$.

So, the calculation here, so basically it could also be F of y_{i+1}, or it could be. Or the other way around as well. The calculation starts with the assumption that $F(y_{i+1})$ is equal to $F(y_i)$ on the right-hand side of the equation here. So, this value can be assumed to be the same as $F(y_i)$. The value of y_i is evaluated from equation A13 and substituted into equation number 10, which is the main equation.

To get $F(y_{i+1})$, and this revised $F(y_{i+1})$ is then substituted into this particular equation, and this process is repeated. Thus, the ith iteration will be something like this. And this iteration proceeds until two successive values of, until what time does it go? Then two successive values of $F(y_{i+1})$ or y_{i+1} agree to a desirable tolerance. So, we do not find changes in the value of y_{i+1}.

So, this basically concludes our theoretical part, but the lecture does not stop here. We are going to solve one small problem from the uniform flow for the most efficient section of the trapezoidal section. So, after that, we will close this lecture and go to problem solving of gradually varied flow. Determine the most efficient section of a trapezoidal channel with with side slopes 1 vertical to 2 horizontal.

The channel carries a discharge of 11.25 and meter cube per second with a velocity of 0.75 meters per second. The question is, what should be the bed slope of the channel? Take Manning's $\boldsymbol{n}$ as equal to 0.025.

So, the solution The side slope of the channel section is 1:2. Discharge $\boldsymbol{Q}$ is equal to 11.25meter cube per second. The velocity of flow in the channel $\boldsymbol{V}$ is equal to 0.75 meters per second. Manning's $\boldsymbol{n}$ is given as 0.025.

And we have to find the bed slope for the most economical section. So, this is the given things that I have written down here. Now, for most, we have seen in our theory classes also, for the most economical trapezoidal channel section. We have $\frac{\boldsymbol{B+2zy}}{\mathbf{2}}$ is equal to $\boldsymbol{y}\sqrt{\mathbf{1}+\boldsymbol{z}^{\mathbf{2}}}$, that the top width by 2 is equal to the depth.

Or $\frac{\boldsymbol{B+2\times 2y}}{\mathbf{2}}$ is equal to $\boldsymbol{y}\sqrt{\mathbf{1}+\boldsymbol{z}^{\mathbf{2}}}$, which means $\boldsymbol{B + 4y}$ is equal to $\boldsymbol{2y}\sqrt{\mathbf{5}}$ or 4.472$\boldsymbol{y}$ or $\boldsymbol{B}$ is equal to 4.472$\boldsymbol{y}$ - 4$\boldsymbol{y}$ or $\boldsymbol{B}$ is equal to $\boldsymbol{B/y}$ is equal to 0.472, this is 1. As per continuity, we have $\boldsymbol{Q}$ is equal to $\boldsymbol{A} \times \boldsymbol{V}$, where $\boldsymbol{A}$ is equal to the wetted area. So, the area is going to be $\boldsymbol{Q/V}$, and that is 11.25/0.75, that is 15 meter square, and that will be equal to $\boldsymbol{B + zy} \times \boldsymbol{y}$ or (0.472y + 2$\boldsymbol{y}$) $\times\boldsymbol{y}$ is equal to 2.472$\boldsymbol{y}^{\mathbf{2}}$ implies $\boldsymbol{y}^{\mathbf{2}}$ is equal to 15/2.472, implies $\boldsymbol{y}$ is equal to 2.463 meters. Hence, $\boldsymbol{B}$ is equal to 0.472$\boldsymbol{y}$ or 0.472$\times$ 2.463 is equal to 1.163 meters. Now, the wetted perimeter $\boldsymbol{P}$ is equal to B+2 $\boldsymbol{y}\sqrt{\mathbf{1}+\boldsymbol{z}^{\mathbf{2}}}$ is equal to 0.472$\boldsymbol{y}$ +$\boldsymbol{2y}$ $\sqrt{\mathbf{5}}$ or 0.472$\boldsymbol{y}$+4.472$\boldsymbol{y}$ is equal to 4.944 $\boldsymbol{y}$ or weighted perimeter will be 4.944 $\times$ $\boldsymbol{2y}\sqrt{\mathbf{5}}$ is 2.463, this one is equal to 12.177 meters, this will be weighted perimeter. hydraulic radius $\boldsymbol{R}$ is equal to $\boldsymbol{A/P}$ is equal to 15/12.177 that is 1.232 meters. So, finally, we are going to apply Manning's formula. So, here it says $\boldsymbol{V}$ is equal to $\frac{\mathbf{1}}{\boldsymbol{n}}\boldsymbol{R}^{\mathbf{2/3}}\boldsymbol{S}^{\mathbf{1/2}}$ implies 0.75 that was the velocity is equal to 1/0.025 that is the Manning's parameter hydraulic radius is $\mathbf{1.232}^{\mathbf{2/3}}\boldsymbol{S}^{\mathbf{1/2}}$. or $\boldsymbol{S}^{\mathbf{1/2}}$ is equal to 0.75 $\times$ 0.025/($\mathbf{1.232}^{\mathbf{2/3}}$) implies $\boldsymbol{S}$ is equal to 0.0000266.

And this was what? So, you see quite a simple question with direct implication. I mean we need to just apply the direct formula. The most critical point here was the condition for the most economical trapezoidal channel section which we have already derived in our lectures. So, with this I think we will close this particular lecture and we are going to meet again at the beginning of the next module where we are going to mainly concentrate on the problem solving of gradually varied flow.

Thank you so much.