Free Surface Flow
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Lecture 26

Welcome, students, to the beginning of the new module, which is gradually varied flow. In this first lecture, we are going to slowly introduce you to the concept of gradually varied flow. In previous classes, especially in Module 1, we have talked a lot about the different types of flows. Flows can be steady or unsteady, which means it depends on whether the flow properties are changing with time or not. Uniform and non-uniform flow is when the flow parameters, flow velocity, or flow depth changes in space or not.

If it does not, that is uniform flow, which we covered in the previous two modules. One was mainly about the theoretical part, and the other module was about solved examples and the numericals that you tend to encounter in uniform flow. Before that, we covered critical flow. However, this gradually varied flow module today is part of non-uniform flow. Non-uniform flow is predominantly of two types but can be said to be three as well.

One is gradually varied flow, the other is spatially varied flow, which is a variation of gradually varied flow, or rapidly varied flow. The major one is rapidly varied flow. So, basically, there are two main types: gradually varied flow and rapidly varied flow. With that being said, I think now it is time to start the introduction part to give you a better idea of what gradually varied flow is. So, a steady non-uniform flow in a prismatic channel with gradual changes in its water surface elevation is called gradually varied flow.

This is just one of the properties of water surface elevation. So, if I mean the way that we define gradually varied flow is mainly through the water surface elevation or depth y, it could be other properties as well. So, gradually varied flow, or more famously called GVF. So, one of the practical examples of gradually varied flow is backwater produced by a dam or weir across a river. So, the surface slope is varying very, very slowly and very steadily with a very little slope of the water surface, and that is one of the examples in the backwater produced by a dam or weir.

In a river is one of these examples. So, you see, this is just a figure which shows that, you see, this is a structure. It could be anything. So, far, far away, this is far away Far away from this structure, we have uniform flow.

A little closer, we have GVF. Upstream of the structure, and this one rapidly varied flow just after the structure or downstream. So, this figure shows all three types of flow, and the major culprit here is the presence of this structure. And so, near the structure, the flow GVF and RVF are types of non-uniform flow. So, this entire thing is non-uniform flow.

Now, what are some of the assumptions of gradually varied flow? So, the pressure distribution at any section in GVF is assumed to be hydrostatic, and this was the same assumption as uniform flow. However, this assumption will not be taken when we have a region of high curvature from the gradually varied flow analysis. So, this assumption excludes the region of high curvature from the gradually varied flow analysis GVF.

The second assumption is that the resistance to flow, that is, the frictional resistance at any depth, is assumed to be given by the corresponding uniform flow equation. So, we are going to use the uniform flow equation for writing down the resistance to the flow, such as what are the resistance formulas or equations that we will be able to use. For example, Manning's formula and with the condition, what is the condition that the slope term The condition that the slope term to be used in the equation is the energy slope and not the bed slope. It is important that the slope that we are going to use in the equation is the energy slope and not the bed slope.

Thus, the energy slope can be written as S_f is equal to $\frac{n^2V^2}{R^{4/3}}$. So, basically, if you see Q was $\frac{1}{n}R^{2/3}{S_o}^{0.5}$ If you square on both sides, you are going to get, this was the bed slope.

In traditional Manning's equation, Q^2, sorry, it is not Q, sorry, the equation was V. So, V^2 is equal to 1/n^2 R^4/3 So Now, bring this side and this side, S_o will be $\frac{n^2V^2}{R^{4/3}}$. So, in traditional Manning's equation, this was S_o. However, if you want to apply this particular equation, this instead of S_o, we write S_f and what is S_f ?

Energy slope. So, energy slope is written using our Manning's equation. So, the use of Manning's is very important. Changes compared to the uniform flow. Now, that being said, we need to go to the differential equation of gradually varied flow.

Look at this figure first. So, this is the bed, this is the bed slope, this is the datum level. This is already driven, but I am just making sure that you understand this is the water surface elevation. This is the datum head. This is the depth, and this is the velocity head, and therefore, this is the energy line, and this is the slope of this line is S_f.

So, under these conditions, I will rub it down so that this figure looks much cleaner. Now, this H is Z, this is $Z + E$, or E is specific energy, and E is what? $y + \frac{V^2}{2g}$. So, this is the equation which is quite important. Now, since the water surface varies in the longitudinal direction, in this direction, the water surface is varying.

How is it varying? Changing slowly, but changing. The total energy and flow depth are also functions of x. Since the water surface varies in the x direction, basically, then energy is also E, and the water depth is also a function of x. Now, if we differentiate this equation, this one in x, we get $\frac{dH}{dx}$ is equal to $\frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\frac{V^2}{2g}$, very simple. And now, $\frac{dH}{dx}$ represents the energy slope.

So, basically, strictly speaking, $\frac{dH}{dx}$ is $-S_f$ because energy is falling in the direction of the flow. So, the negative sign, as I told, signifies that the total energy of the flow always decreases in the direction of motion. This is important to note. Now, $\frac{dz}{dx}$ is the water surface slope relative to the channel bottom. $\frac{dy}{dx}$ represents the bed slope. So, or mathematically speaking, again, dz. So, mathematically, yeah, so yeah.

$\frac{dz}{dx}$ mathematically represents the bed slope So, or mathematically represents the channel bottom. That is, $\frac{dz}{dx}$ is actually $-S_o$, and this $\frac{dy}{dx}$ is the water surface slope relative to the channel bottom. Just to clarify again, $\frac{dz}{dx}$ is $-S_o$, and this $\frac{dy}{dx}$ is the water surface slope relative to the channel bottom. And here also, the negative sign signifies that the bed elevation decreases along the downstream direction, same. Now, the third term on the right-hand side of equation 2, what is the third term? This is the third term.

So, the third term on the right-hand side of equation 2 can be written as $\frac{d}{dx}\frac{V^2}{2g}$. First, we differentiate V with respect to x or write Q is equal to VA, or V can be written as $\frac{Q}{A}$. So, this $\frac{V^2}{2g}$ becomes $\frac{Q^2}{2gA^2}$, and then we differentiate this. Then, so $\frac{Q^2}{2g}$ comes out, and then it reduces to $\frac{d}{dx}$ of $\frac{1}{A^2}$. And then, if you differentiate A^2, that will become $-\frac{1}{A^3}\frac{dA}{dx}$, just simple differentiation. But we already know that dA is equal to Tdy.

So, instead of dA, we can put Tdy, where T is the channel top width. Therefore, we can write $\frac{d}{dx}\frac{V^2}{2g}$ as $-Q^2$. So, $-Q^2\frac{Tdy}{gA^3dx}$, then this becomes $\frac{dy}{dx}$ or $-Q^2\frac{T}{gA^3}\frac{dy}{dx}$, this equation. Or now we know that Froude number squared is nothing but $Q^2\frac{T}{gA^3}$.

So, this term here is replaced by $F^2{}_r$. Hence, equation 2, this equation 2, this equation 2, $\frac{dH}{dx}$ can be rewritten as $-S_f$ instead of $\frac{dH}{dx}$ is equal to $-S_o + \frac{dy}{dx} - -Q^2\frac{T}{gA^3}\frac{dy}{dx}$. So, it is very simple. In the next step, we are going to take this side.

Then we can write $S_o - S_f$ is equal to $\frac{dy}{dx}$ will take common $1 - Q^2\frac{T}{gA^3}$. Let us see. Or this you can bring down in the next step, another next step, $\frac{dy}{dx}$ is $\frac{S_o - S_f}{1 - Q^2\frac{T}{gA^3}}$. And this is the dynamic equation of gradually varied flow.

Important equation, one of the most important equations of gradually varied flow. Now, we will see one small type of, you know, derivation, we have to show that for a gradually varied flow in a frictionless rectangular channel having a bed slope S_b, we can write x as $\frac{y}{S_b}\left(1+0.5\left(\left(\frac{y_c}{y}\right)\right)^3\right)+C$. This is again a sort of derivation, not exactly a problem. So, we know

The dynamic equation for this particular case, the previous case, what was the equation? This was the equation. So, bed slope, so basically we have $\frac{dy}{dx}$ is equal to $\frac{S_o-S_f}{1-Q^2\frac{T}{gA^3}}$. Here it is written this is S_b, correct. So, we write $\frac{dy}{dx}$ is equal to $\frac{S_b-S_f}{1-Q^2\frac{T}{gA^3}}$.

Now it is given frictionless channel. Frictionless channel means S_f is equal to 0. So, S_f is 0, $\frac{dy}{dx}$ becomes $\frac{S_b}{1-Q^2\frac{T}{gA^3}}$. Another information is given; it is given as rectangular channel. So, for a rectangular channel, we know

The top width is nothing but B, that is the width of the rectangular channel. So, for a rectangular channel width B is equal to top width T. So, we get instead of T we put B and instead of T we have put B and area we have put By. So, we get $\frac{dy}{dx}$ is equal to $\frac{S_b}{1-Q^2\frac{T}{gA^3}}$. Or this $1B$ gets cancelled, $\frac{S_b}{1-Q^2\frac{B}{gB^2y^3}}$. Simple, B B gets cancelled, $1B$.

Or $\frac{dy}{dx}$ is nothing but $\frac{S_b}{1-\frac{q^2}{gy^3}}$. So, this $\frac{Q^2}{B^2}$ is small q. So, it becomes $\frac{S_b}{1-\frac{q^2}{gy^3}}$, where q is the discharge per unit width. And we also know that for a rectangular channel, the critical depth is given by y_c is equal to $(\frac{q^2}{g})^{1/3}$, or ${y_c}^3$ is equal to $\frac{q^2}{g}$, or y_c cube is equal to the one that I have written here. All right, so what we are going to do in the next step, instead of

this $\frac{q^2}{g}$, we are going to put ${y_c}^3$. So, using 2 in 1, we get $\frac{dy}{dx}$ is equal to $\frac{S_b}{1-(\frac{y_c}{y})^3}$. Or $1-(\frac{y_c}{y})^3$. So, what we do is we take it this side and we take it this side. So, it becomes $(1-(\frac{y_c}{y})^3)dy$ is equal to $S_b dx$.

And the next step is to integrate from 0, or you just do indefinite integration. So, this is So, this 1 will become y on integration, this becomes half of $\frac{{y_c}^3}{y^2}+c1$ is equal to $S_b x$, this integration gives $S_b x + c2$, where $c1$ and $c2$ are the constants of integration. Now, this equation can be rearranged further, because what we need to prove here is, x is equal to.

So, we need to write just in terms of x. $y + 0.5y(\frac{y_c}{y})^3$ is equal to $S_b x + c2 - c1$, or we take $y(1 + 0.5(\frac{y_c}{y})^3$ is equal to $S_b x + c2 - c1$. This, this you take this side, and S_b you take whole down, and this is another constant. So, this is what we needed to, hence proved. Now, we will see another form of the differential equation of gradually varied flow. Let us say if k is the conveyance at any depth y, and k_o be the conveyance corresponding to a normal depth y_o.

So, we are talking about uniform flow as denoted by subscript 0. So, the conveyance at normal depth is given by k_o, and at any general conveyance is given by k. And what we know by definition, k is $\frac{Q}{{S_f}^{0.5}}$ and k_o is $\frac{Q}{{S_o}^{0.5}}$. So, why S_f because we assume in GVF, the second assumption that So will be replaced by the

energy line slope that is S_f, and this is the second one is uniform flow. So, if we divide this and this, or you can do the other ways as well, we divide this like this. So, and do square root, Q will get cancelled, and this is the remaining equation that will remain with us. $\frac{S_f}{S_o}$ is equal to $\frac{{K_o}^2}{K^2}$.

Similarly, if Z is equal to the section factor at depth y, this is what we did for the conveyance factor. Similarly, we have another thing called the section factor at depth y, and Z_c is the section factor at critical depth. Here, in the conveyance factor, we try to compare it with the normal depth. Here, for Z, we are going to compare it with the critical depth. So, we know the formula Z^2 is nothing but $\frac{A^3}{T}$, and Z_c is $\frac{A^3}{T_c}$ or $\frac{Q^2}{g}$ because this term can also be written as $\frac{Q^2}{g}$.

If we divide that, if we divide this by this, sorry, if we divide this one by this one, 1 divided by 2, we will get this equation: $Q^2 \frac{T}{gA^3}$ is equal to $\frac{{Z_c}^2}{Z^2}$. And what is this? So, if we use the above equations, the basic differential equation can now be written as $\frac{dy}{dx}$ is equal to S_o into 1. So, if you take S_o common, then $\frac{1-\frac{S_f}{S_o}}{1-Q^2\frac{T}{gA^3}}$. So, this is the regular GVF equation or You see $\frac{S_f}{S_o}$ from the conveyance factor $\frac{K_o}{K}$ and g^2 is $\frac{Z_c}{Z_o}$.

So, this equation is useful in developing direct integration techniques. If Q_n represents the normal discharge at depth y and Q_c denotes the critical discharge at the same depth y. So, I mean, we also write Q_n is equal to $K{S_o}^{0.5}$. And Q_c can be written as $Zg^{0.5}$. Using these definitions, the differential equation of GVF can be written as $\frac{dy}{dx}$ is equal to $S_o \frac{1-(\frac{Q}{Q_n})^2}{1-(\frac{Q}{Q_c})^2}$. And this is an alternative formulation of GVF, gradually varied flow.

I think this will be good enough for this lecture, and in the next class, we will start with the differential energy equation of gradually varied flow. Thank you so much. See you in the next lecture.