Rock Mechanics and Tunneling
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Lecture 30
Analysis of Stresses (Continued) and Stress-Strain Relationship

Hello everyone! I welcome all of you to the 4th lecture of Module 6. So, in Module 6, we are discussing about the rock and rock mass failure criteria. At present, we are discussing about the analysis of stresses and today we will discuss about the stress strain relationship.

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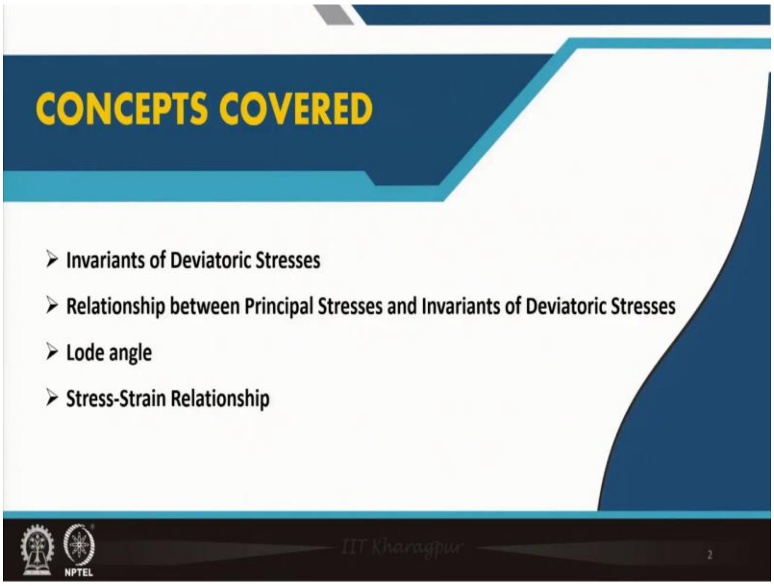

So, today we will discuss about the invariants of deviatoric stresses, relationship between principal stresses and invariants of deviatoric stresses, then Lode angle. Finally we will discuss the stress-strain relationship.

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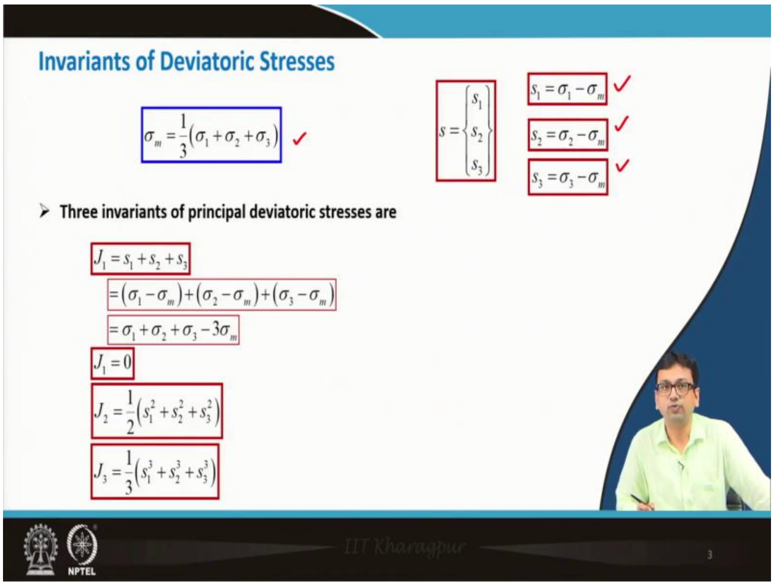

So, we already know that $\sigma_m = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$, where σ_1, σ_2, and σ_3 are the principal stresses, and the corresponding deviatoric stresses are S_1, S_2, and S_3, where $S_1 = \sigma_1 - \sigma_m$, $S_2 = \sigma_2 - \sigma_m$, and $S_3 = \sigma_3 - \sigma_m$. These things we have learnt in the previous lecture

Then, accordingly the three invariants of principal deviatoric stresses are $J_1 = S_1 + S_2 + S_3$ $= (\sigma_1 - \sigma_m) + (\sigma_2 - \sigma_m) + (\sigma_3 - \sigma_m) = (\sigma_1 + \sigma_2 + \sigma_3) - 3\sigma_m$. So, the final expression will be $J_1 = 0$. It is also discussed in the previous lecture. Then, $J_2 = \dfrac{1}{2}(S_1^2 + S_2^2 + S_3^2)$ and $J_3 = \dfrac{1}{2}(S_1^3 + S_2^3 + S_3^3)$.

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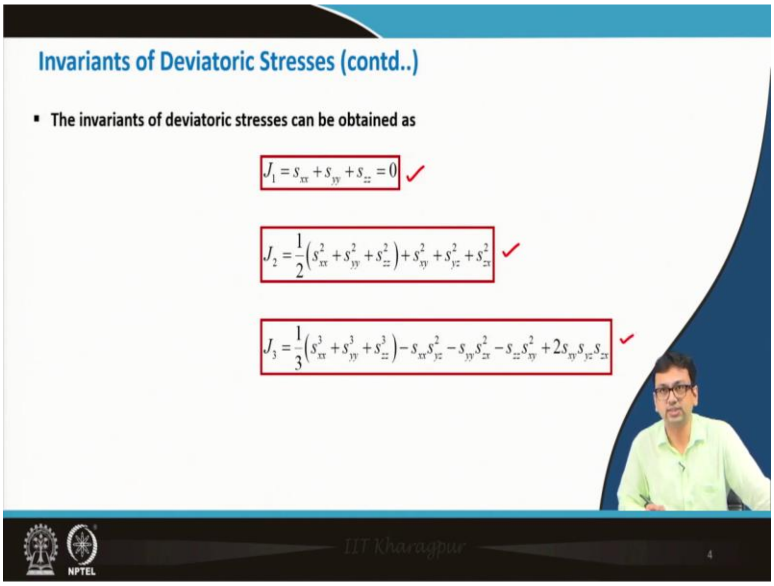

Now, for the generalized case, expressions for J_1, J_2, and J_3 are given as follows:

$$J_1 = S_{xx} + S_{yy} + S_{zz} = 0$$

$$J_2 = \frac{1}{2}(S_{xx}^2 + S_{yy}^2 + S_{zz}^2) + S_{xy}^2 + S_{yz}^2 + S_{zx}^2$$

$$J_3 = \frac{1}{3}(S_{xx}^3 + S_{yy}^3 + S_{zz}^3) - S_{xx}S_{yz}^2 - S_{yy}S_{zx}^2 - S_{zz}S_{xy}^2 + 2S_{xy}S_{yz}S_{zx}$$

So, we have discussed these also in detail in our previous lecture.

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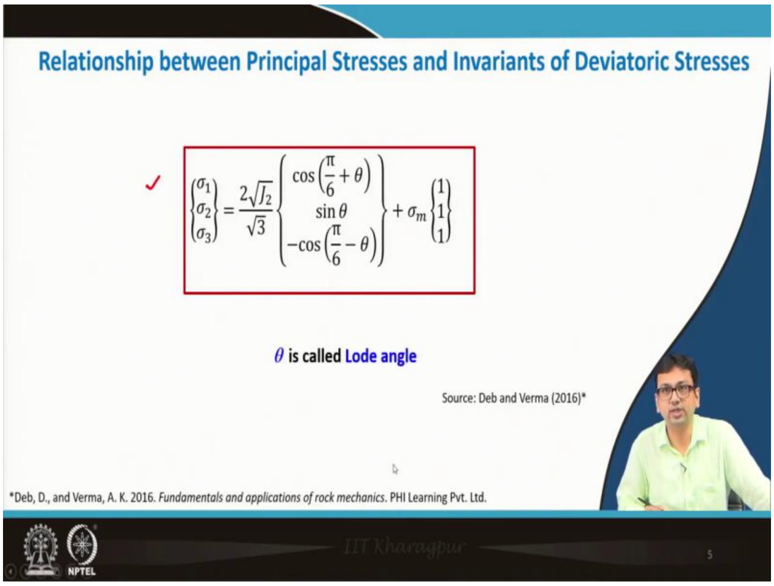

Now, let us see this the relationship between the principal stresses and invariants of deviatoric stresses which is given as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_2 \end{Bmatrix} = \frac{2\sqrt{J_2}}{\sqrt{3}} \begin{Bmatrix} \cos\left(\frac{\pi}{6}+\theta\right) \\ \sin\theta \\ -\cos\left(\frac{\pi}{6}-\theta\right) \end{Bmatrix} + \sigma_m \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

In the above equation, σ_1, σ_2, and σ_3 are the principal stresses, J_2 is the second invariant of the deviatoric stress and σ_m is the mean stress, and the new term, θ is the Lode angle.

So, it is a very important relationship between the principal stress and the invariant of the deviatoric stresses and we are introducing a new term, which is called the Lode angle (θ).

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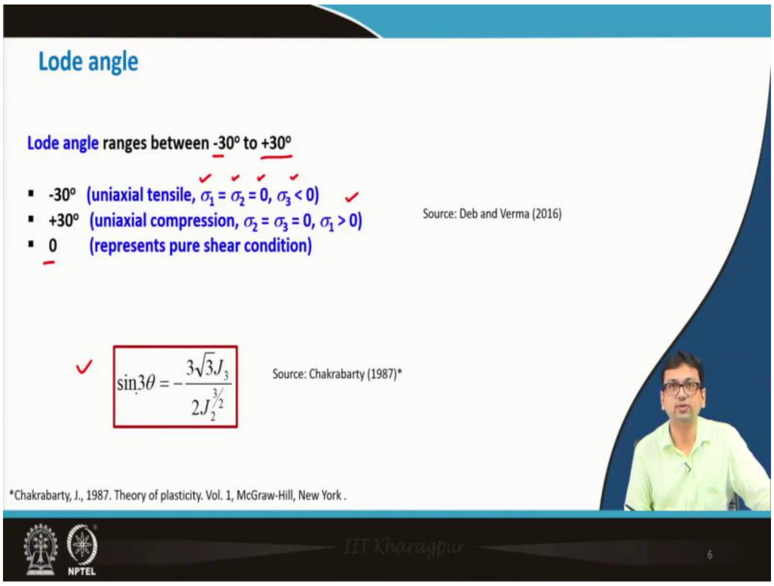

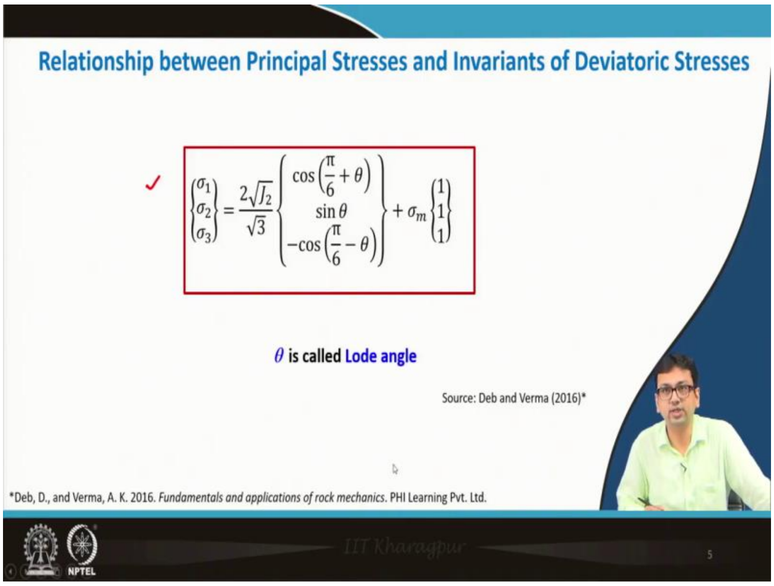

Now, let us discuss briefly about the Lode angle. So, the Lode angle ranges between -30º to +30º. Now, θ will be -30º, when the uniaxial tensile load is applied, i.e., $\sigma_1 = \sigma_2 = 0, \sigma_3 < 0$, and the θ will be +30º, when there is uniaxial compression, i.e., $\sigma_1 > 0, \sigma_2 = \sigma_3 = 0$. Finally, θ will be zero under pure shear condition.

As per the equation $\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_2\end{Bmatrix} = \frac{2\sqrt{J_2}}{\sqrt{3}}\begin{Bmatrix}\cos\left(\frac{\pi}{6}+\theta\right)\\ \sin\theta\\ -\cos\left(\frac{\pi}{6}-\theta\right)\end{Bmatrix} + \sigma_m\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}$, the principal stresses $\sigma_1, \sigma_2, \sigma_3$ are the functions of Lode angle (θ).

So, the expression for θ is as follows:

$\sin 3\theta = -\frac{3\sqrt{3}J_3}{2J_2^{3/2}}$, where J_2 and J_3 are the second and the third invariant of the deviatoric stress tensor, respectively. Hence, using the equation, we can find out the Lode angle (θ), and we have also seen that the range of Lode angle (θ) is in between -30º to +30º.

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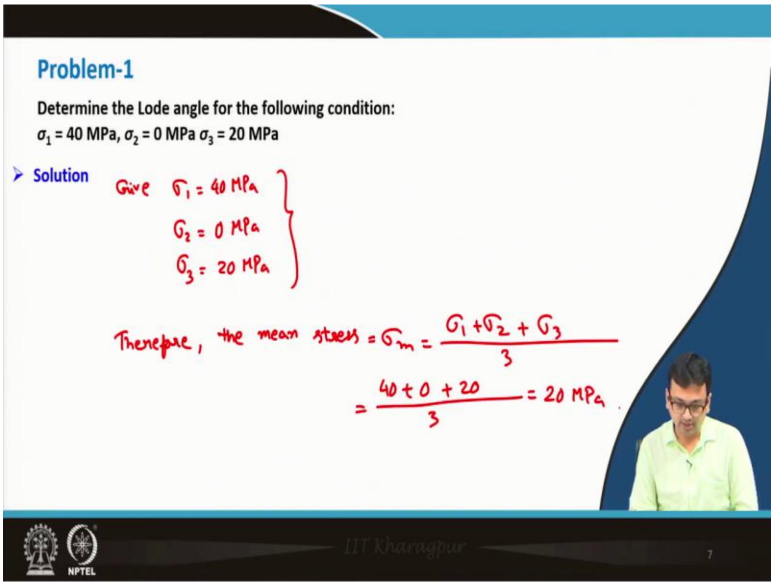

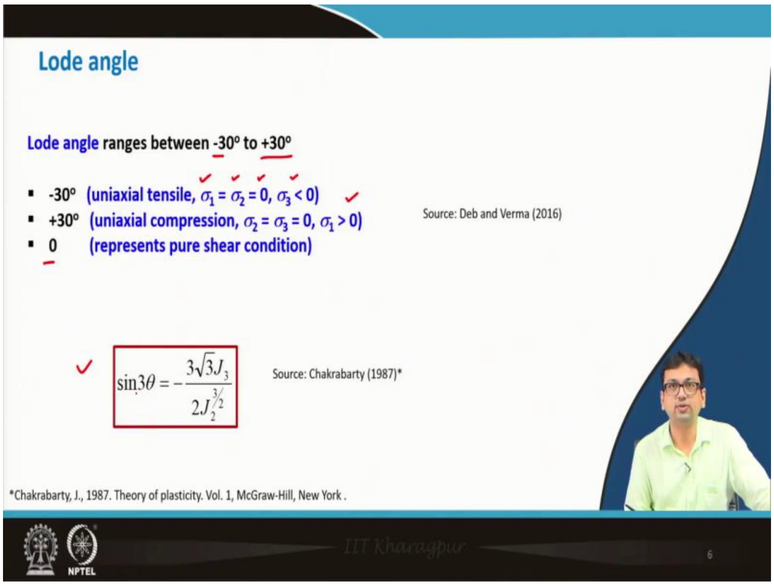

Now, let us solve a problem on the Lode angle (θ). The problem statement is to determine the Lode angle for the following condition:

$$\sigma_1 = 40 \text{ MPa}, \sigma_2 = 0 \text{ MPa}, \sigma_3 = 20 \text{ MPa}$$

So, to find out the value of Lode angle (θ), we need to know J_2 and J_3. We know that, $J_1 = 0$ and J1 is also not coming in this equation. Hence, we have to evaluate the J_2 and J_3.

Now, in order to find out J_2 and J_3, we need to know the S_1, S_2, and S_3. Again, in order to evaluate S_1, S_2, and S_3, we need to find out the mean stress (σ_m) as we know that $S_1 = \sigma_1 - \sigma_m$. Likewise, $S_2 = \sigma_2 - \sigma_m$ and $S_3 = \sigma_3 - \sigma_m$.

Now, the mean stress, $\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{40 + 0 + 20}{3} = 20\,\text{MPa}$.

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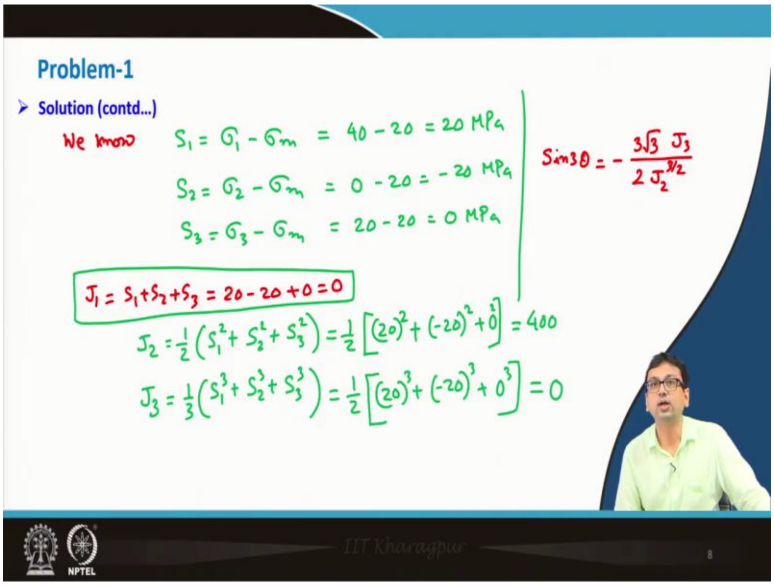

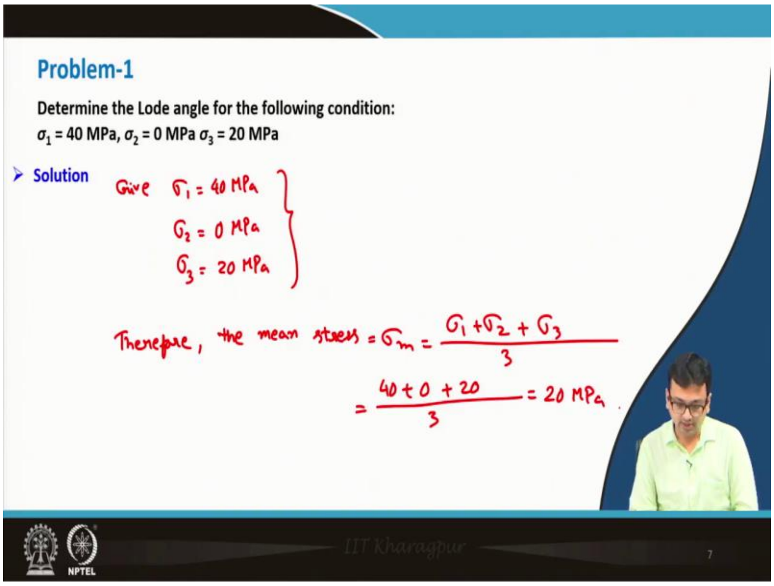

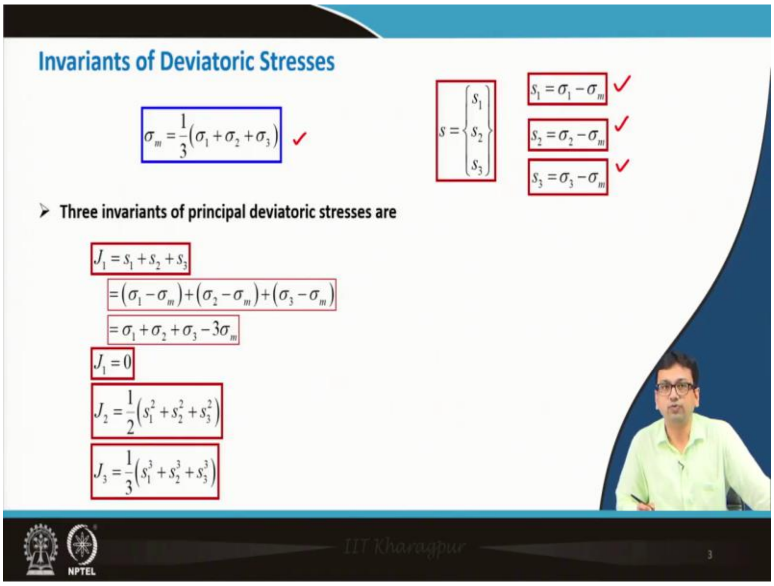

Now, we know that, $S_1 = \sigma_1 - \sigma_m = 40 - 20 = 20$ MPa

Similarly, $S_2 = \sigma_2 - \sigma_m = 0 - 20 = -20$ MPa and $S_3 = \sigma_3 - \sigma_m = 20 - 20 = 0$ MPa.

So, we know the following expression:

$\sin 3\theta = -\dfrac{3\sqrt{3}J_3}{2J_2^{3/2}}$. Here, we need to find out J_2 and J_3.

Anyway, we know that $J_1 = 0$, and we also know that the J_1 is not required to find out the Lode angle. However, let us check whether J_1 is coming out to be zero or not. So, $J_1 = S_1 + S_2 + S_3 = 20 + (-20) + 0$. So, $J_1 = 0$

So, the J_1 is calculated for the understanding only. It will not be used to evaluate the Lode angle (θ).

Since, we know that, $J_2 = \dfrac{1}{2}(S_1^2 + S_2^2 + S_3^2)$

So, $J_2 = \dfrac{1}{2}[(20)^2 + (-20)^2 + (0)^2] = 400$

The expression for J_3 is $J_3 = \dfrac{1}{2}(S_1^3 + S_2^3 + S_3^3)$

So, $J_3 = \dfrac{1}{3}[(20)^3 + (-20)^3 + (0)^3] = 0$

So, we have J_2 and J_3. Now, we can easily find out the $\sin 3\theta$ and from there, we can find out the θ.

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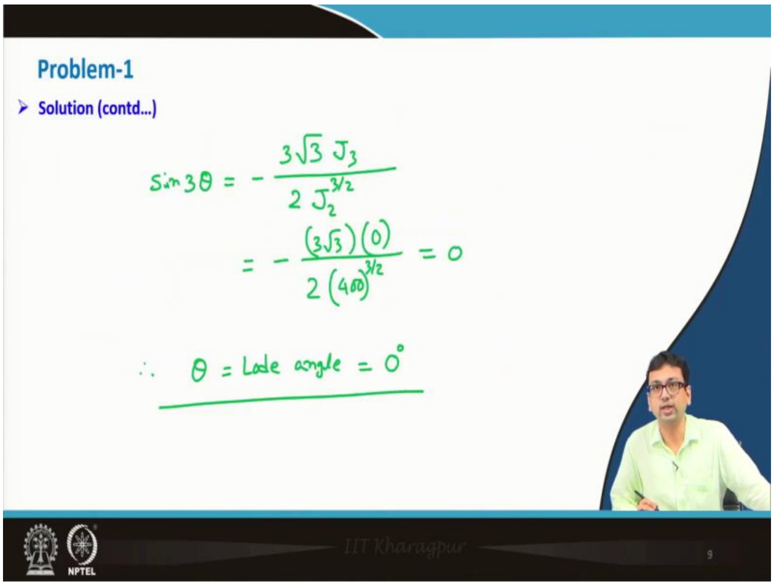

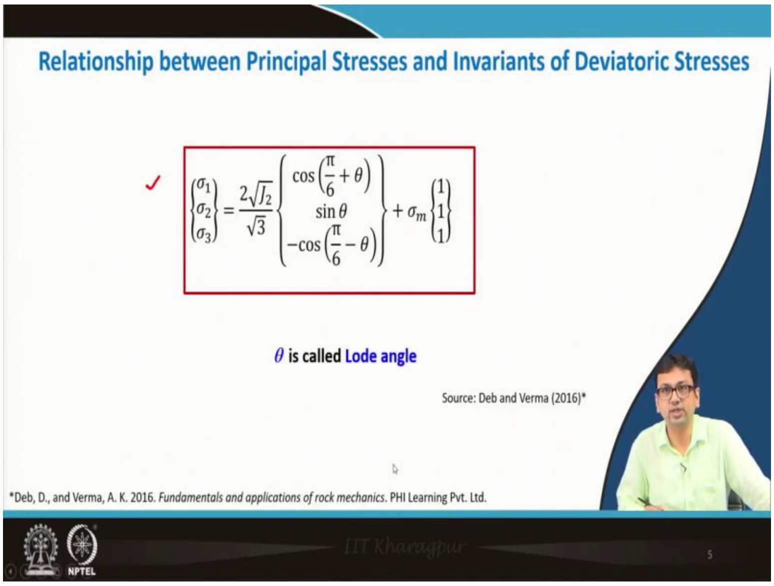

So, $\sin 3\theta = -\frac{3\sqrt{3}J_3}{2J_2^{3/2}} = -\frac{3\sqrt{3}\times 0}{2\times(400)^{3/2}} = 0$

So, the Lode angle (θ) = 0.

So, it is a very essential relationship between the principal stress and the invariant of deviatoric stresses and through which we could able to learn about the Lode angle also.

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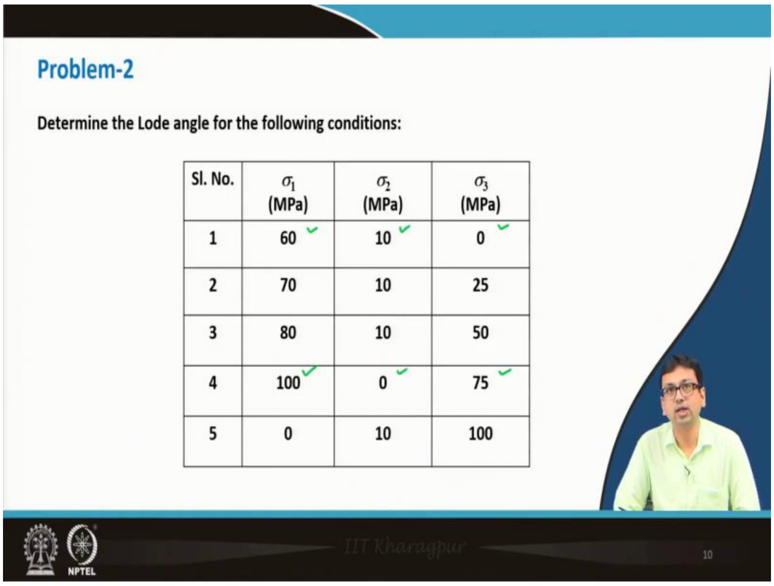

Now, let us take another problem. In this problem, we have to find the Lode angle where five sets of like values are provided. In set 1, $\sigma_1 = 60$ MPa, $\sigma_2 = 10$ MPa, $\sigma_3 = 0$ MPa. Likewise, in set 4, $\sigma_1 = 100$ MPa, $\sigma_2 = 0$ MPa, $\sigma_3 = 75$ MPa.

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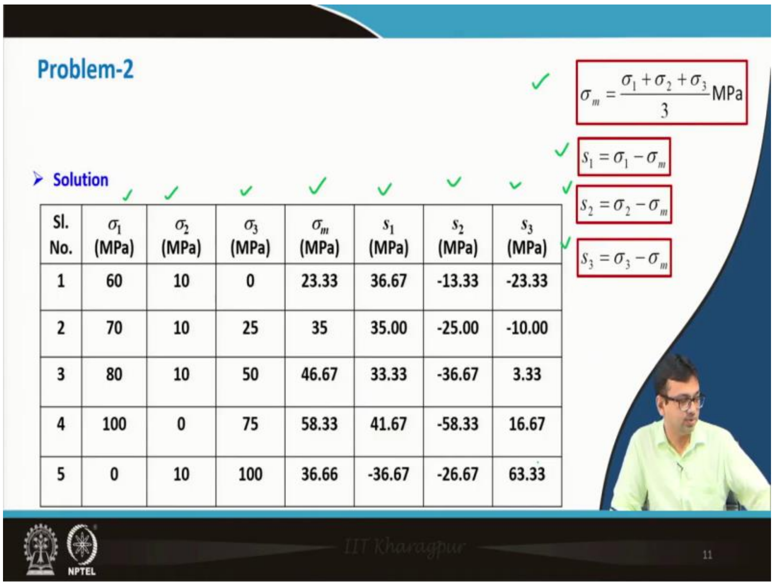

So, in the problem, $\sigma_1, \sigma_2, \sigma_3$ are provided. First, we have find the σ_m. Then, we have to determine the S_1, S_2, and S_3. So, we have presented the five sets of $\sigma_1, \sigma_2, \sigma_3$, σ_m, S_1, S_2, and S_3 in a tabular form. Thus, using the following equations, J_2 and J_3 are evaluated.

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$$

$$J_3 = \frac{1}{2}(S_1^3 + S_2^3 + S_3^3)$$

After that, the Load angle (θ) is calculated from J_2 and J_3.

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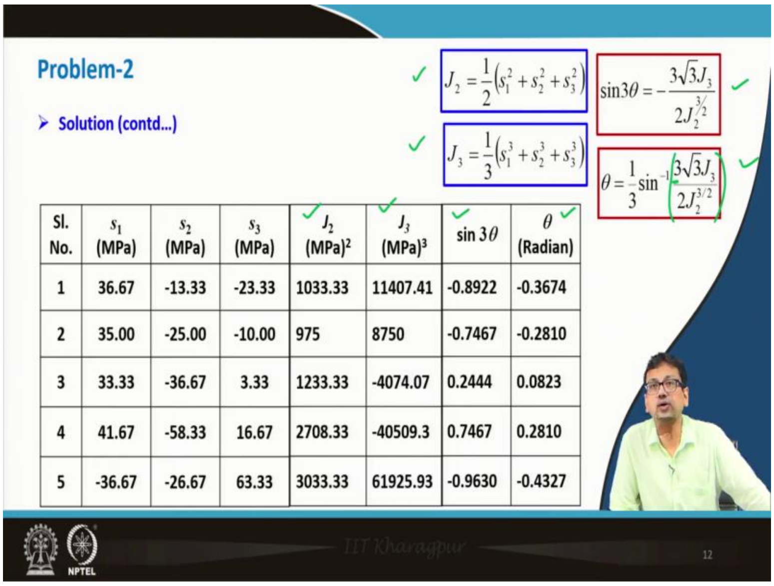

So, we can evaluate the five sets of J_2 from the five sets of S_1, S_2, and S_3. Similarly, we can obtain the five sets of J_3. Now, we know that $\sin 3\theta = -\frac{3\sqrt{3}J_3}{2J_2^{3/2}}$, which will give us the θ.

So, the angle θ can be written as follows:

$$\theta = \sin^{-1}\left(-\frac{3\sqrt{3}J_3}{2J_2^{3/2}}\right)$$

Thus, using the above equation, we will get θ in in radian. After that, the angle θ is presented in degree.

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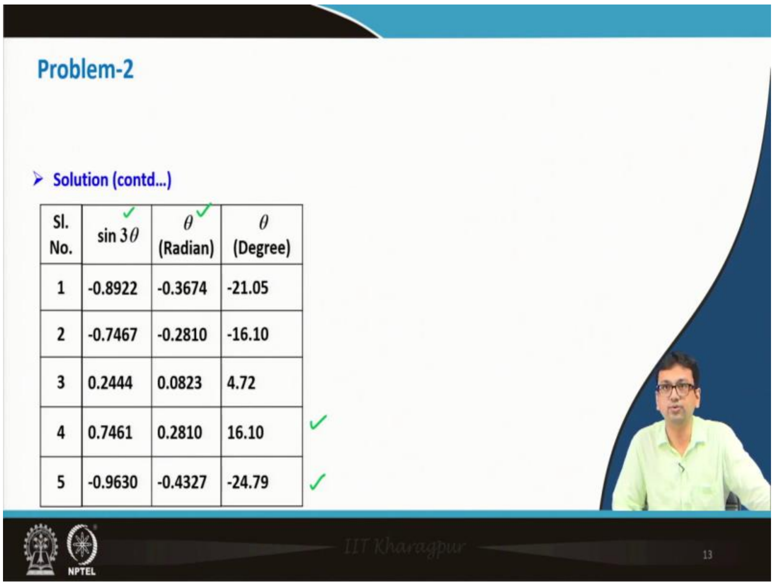

So, the $\sin 3\theta$, θ in radian, and θ in degree are given in the tabular form. We can notice over here that the magnitude of Lode angle (θ) is within the range of -30º to +30º. So, the maximum positive of Lode angle (θ) is 16.1º and the maximum negative value of Lode angle (θ) is -24.79º. So, the range of Lode angle (θ) is within the range of -30º to +30º. Thus, in this way, we could able to obtain the Lode angles for a given state of stresses.

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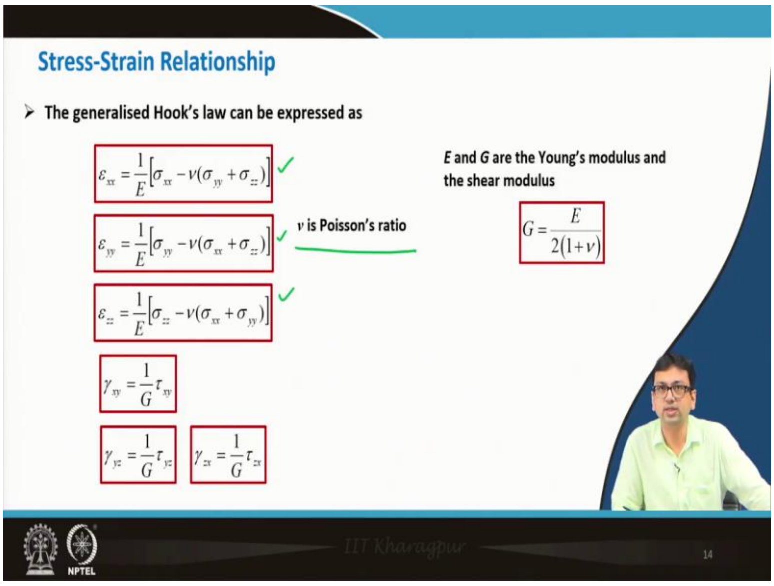

Now, let us discuss about the stress–strain relationship. We have discussed the generalized Hook's law in detail in our previous classes where we have derived these equations. So, the generalized Hook's law is written over here.

So, we can write down the stress strain relationship in a simple matrix form. We already know that the E and G are the Young's modulus and the shear modulus, respectively. ν is the Poisson's ratio. So, the shear modulus (G) can be written as $G = \dfrac{E}{2(1+\nu)}$.

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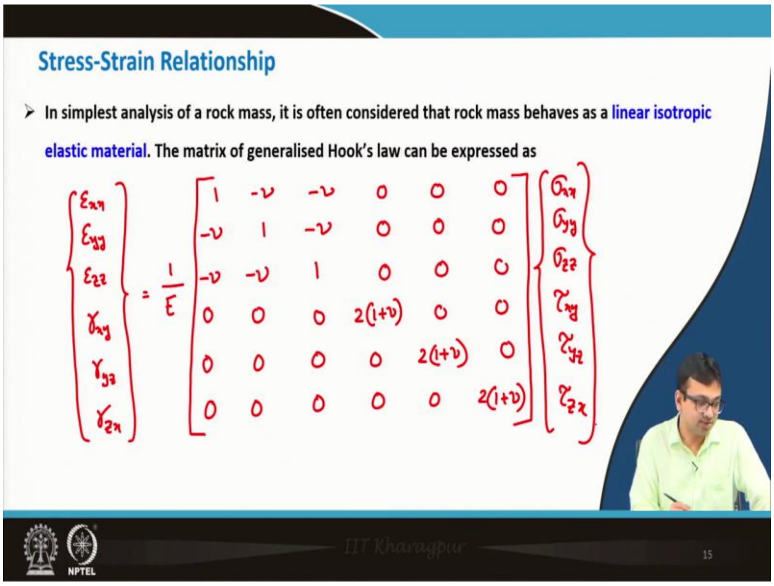

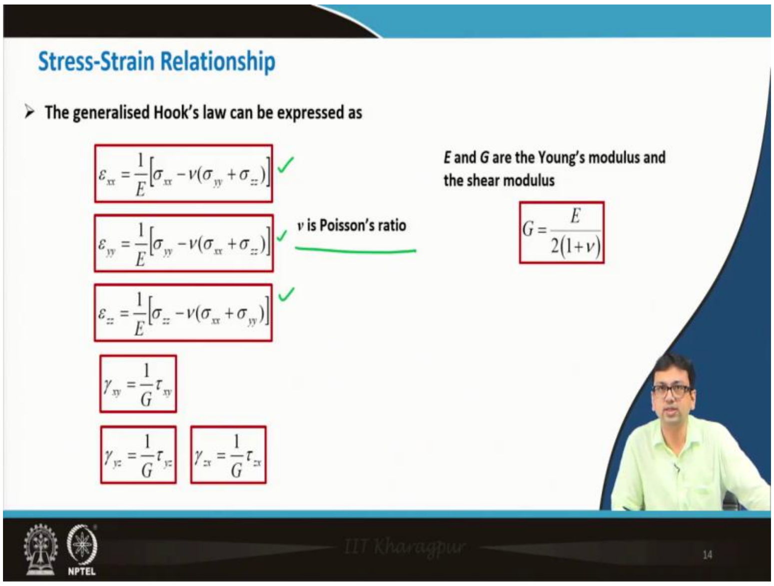

So, in simplest analysis of rock mass, it is often considered that the rock mass behaves as a linear isotropic elastic material. So the matrix of the generalized Hook's law can be expressed as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

So, we can represent the generalized Hook's law in the above matrix form. So, just for the cross-checking, $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$. Likewise, we can find out the other two normal strains ε_{yy} and ε_{zz}, and the shear strains (γ_{xy}, γ_{yz}, and γ_{zx}).

So, $\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$ is known as the compliance matrix.

So, here, the strains are represented in terms of stress. Similarly, the stress vector can be represented by the strain vector using the constitutive matrix.

Thus, we can get the stress vector by multiplying the constitutive matrix with the strain vector. Similarly, the stain vector can be achieved by multiplying the compliance matrix with the stress vector.

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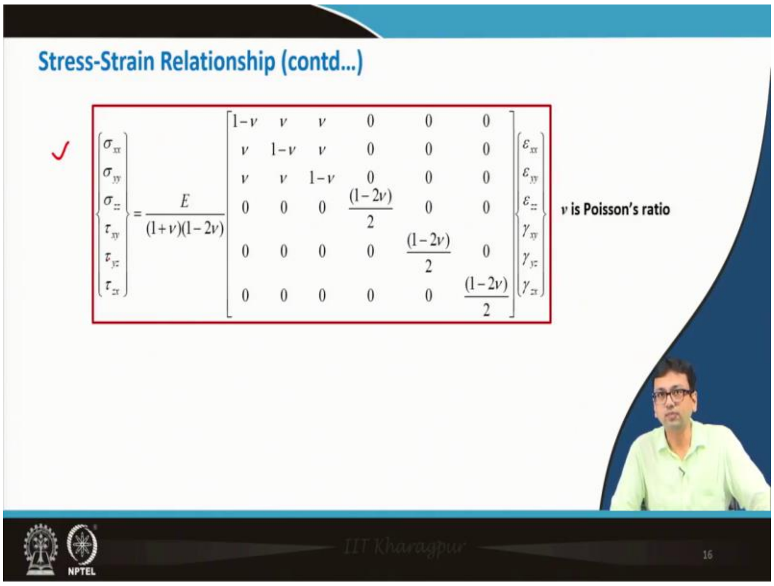

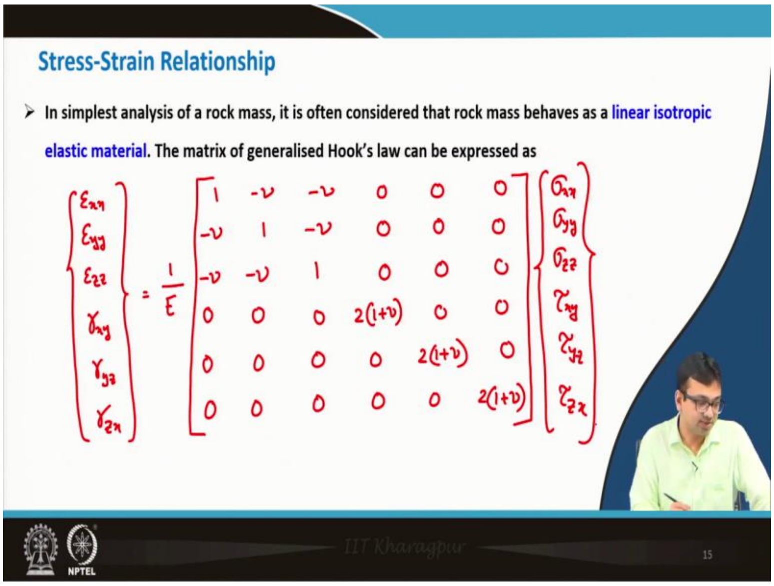

So, the stress vector can be represented by strain vector given as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

So, this is the stress strain relationship for the 3D case.

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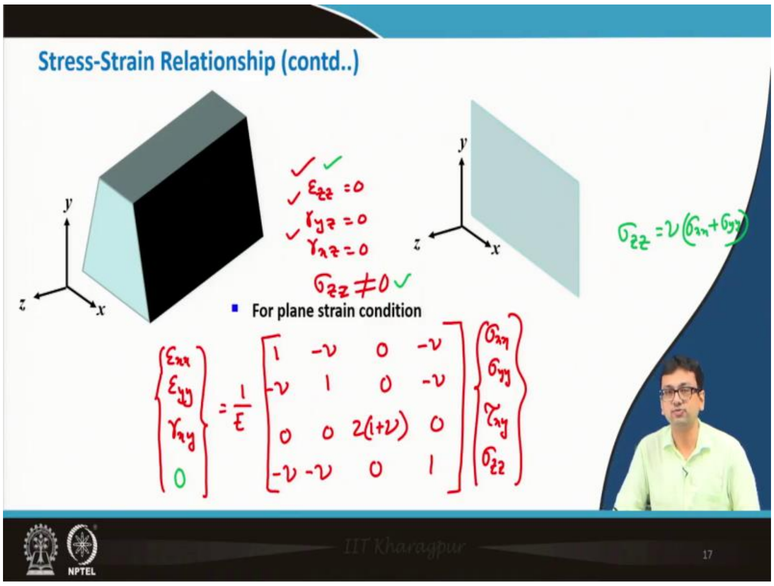

Now, for the long structures, like embankment, tunnel, strip footing or a dam, the length in out of plane direction (i.e., in the *z* direction as per the figure) is too long.

Thus, if in one direction, the dimension is very large and the loading also remains same throughout that direction (i.e., in the out of plane direction), the change in dimension in that direction is very negligible. If we refer to the figure, we can see that the dimension in the *z* direction is very large as compared to the dimensions in *x* and *y* directions.

Thus, we can say that the strains in z direction (i.e., ε_{zz}, γ_{yz}, and γ_{xz}) are equal to zero as the dimension in *z* direction is very large and the load is not changing throughout the length.

So, this condition is generally known as the plane strain condition. In the geotechnical engineering (i.e., in soil and rock mechanics), we generally deal with these type of structures like embankment, tunnel, strip footing.

Since in the out of plane direction (i.e., in *z* direction), the strain is zero, the analysis can be done in the *x*-*y* plane only. Thus, the problem domain becomes very simple. So, we can convert the 3D problem into a 2D plane strain problem. Hence, we can write, $\varepsilon_{zz} = 0$, $\gamma_{yz} = 0$, and $\gamma_{xz} = 0$, but $\sigma_{zz} \neq 0$.

Hence, the stress-strain relationship will be $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 & -\nu \\ -\nu & 1 & 0 & -\nu \\ 0 & 0 & 2(1+\nu) & 0 \\ -\nu & -\nu & 0 & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \sigma_{zz} \end{Bmatrix}$

Now, for the plane strain problem, $\varepsilon_{zz} = 0$, but $\sigma_{zz} \neq 0$.

So, the aforesaid stress-strain relationship can be written as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ 0 \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 & -\nu \\ -\nu & 1 & 0 & -\nu \\ 0 & 0 & 2(1+\nu) & 0 \\ -\nu & -\nu & 0 & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \sigma_{zz} \end{Bmatrix}$$

From, the above relationship, we can get $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$.