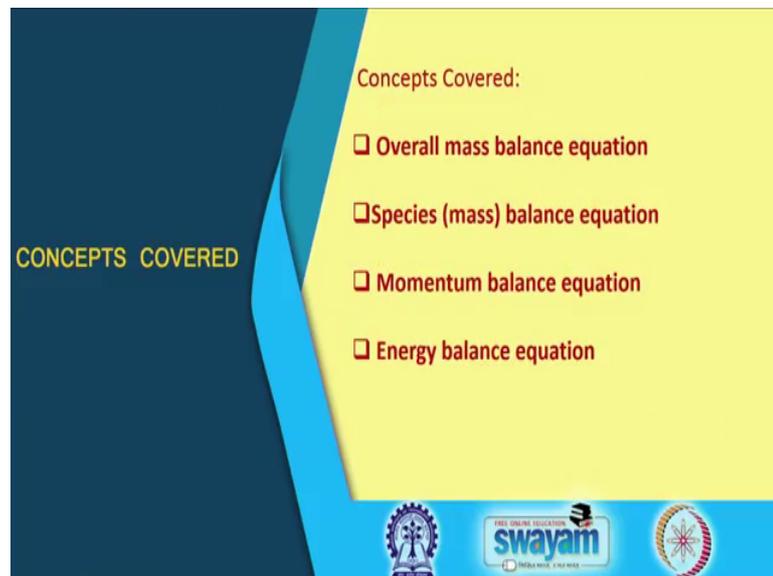


**Mass Momentum and Energy Balances in Engineering Analysis**  
**Prof. Pavitra Sandilya**  
**Cryogenics Engineering Center**  
**Indian Institute of Technology, Kharagpur**

**Lecture - 34**  
**Microscopic Balance Illustrations – IV**

Welcome, so far we have learnt how to deduce the model equations from a full fledged Navier-Stokes equation and with some assumptions we have made. And also we have seen some of the non-dimensional numbers and their significance. In this particular lecture, we shall continue with the Microscopic Balance.

(Refer Slide Time: 00:43)



Now in this, we shall be taking a one illustration to demonstrate the similarity between the momentum transfer, the energy transfer and the mass transfer so that, you can understand that if you know how to solve for the momentum balance equation you can extend your solution for the energy balance as well as for the mass balance or vice versa.

(Refer Slide Time: 01:08)

### Flow near a surface that is suddenly set to linear motion

- Consider a semi-infinite fluid with constant density and viscosity bounded by a horizontal surface (in  $xz$ -plane). Initially the fluid and the surface are at rest. At time  $t = 0$ , the surface is set in motion in the positive  $x$ -direction. The velocity  $v_x$  is a function of  $y$  and  $t$ . Assume the flow to be laminar, no pressure gradient or gravity force in the  $x$ -direction, and fluid to be Newtonian.

$t < 0$ , Fluid at rest     $t = 0$ , Surface set in motion     $t > 0$ , Fluid in unsteady flow

swayam  
FREE ONLINE EDUCATION  
INDIA 2020, 2021, 2022

So, let us first start with the momentum balance problem. Here we are given a flow near a surface that is suddenly set to linear motion. So, what we have been stated that a considered a semi infinite fluid with constant density and viscosity bounded by a horizontal surface in exit plane; it means semi infinite means that on the fluid is confined by one surface on one side and the other end of the fluid extends to infinity ok.

And this, there may be the case when you talk of a say on this particular table you can see that the air is there on the table and this is moving up ok. Or on the earth surface you can say that the air is there on the earth and on the as we go above it is just extending to a long distance.

Similarly, when we talk of a suppose a vehicle is there whether it is a car whether it is a train, you see that the air which is in touch with the train or the car it is also bound, on the one side it is bound by the particular object and on the other end it is going to infinity; that means, a long distance. So, all these are the examples of such kind of system of a semi infinite fluid. Initially the fluid and the surface are at rest; that means, suppose you can imagine that as if a car is standing there and in touch with the car is the air, so both the fluid and the car are at rest. At a time  $t$  equal to 0, the surface is set to motion in the positive  $x$  direction; that means, you can imagine as if the you have started the car and the car has started moving, it can extend for some same thing about a train also as your train has started moving.

The velocity  $v_x$  is a function of  $y$  and  $t$ ; that means, you know see that the velocity of the fluid will now become a function of the  $y$ ; that means the distance from the surface and time. Assume the flow to be laminar, no pressure gradient or gravity force in the  $x$  direction and the fluid to be Newtonian. So, these are some assumptions, but these are not restrictive these are just for demonstration purpose we have made the assumption. But you can extend the analysis for any kind of fluid for any kind of flow and for any other kind of forces that might be acting on the particular system.

So, here pictorially we have shown you these situations, here you can see that at  $t$  less than 0 fluid is at rest. So, here we have the surface and the  $y$  direction is away from the perpendicular to this binding surface and this is the fluid. And a time  $t$  equal to 0 this surface is said to motion. And what happens, at time  $t$  more than 0 we find that there is some kind of a velocity gradient that is  $v_x$ , the  $v_x$   $y$   $t$  means  $v_x$  means the fluid is also going to move in the same  $x$  direction, but the magnitude of the velocity we will keep on changing as we move away from the surface. And you can see the magnitude is 0 at the surface and it keeps increasing as we move away and ultimately again it becomes a constant value; that means, it attains the free stream velocity of the fluid ok, away from the surface.

And this change of the velocity occurs due to the finite viscosity, the viscous effect. If the ideal flow is there, then the viscous effect will not be experienced so it will be all the same velocity will be there ok, there no viscous effect.

(Refer Slide Time: 05:13)

**Flow near a surface that is suddenly set to linear motion**

- As given  $v_x \equiv v_x(y, t)$ ,  $v_y = v_z = 0$ . Thus we need to consider only the x-component of momentum equation.

$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P_x}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + g_x$$

- Thus we have

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

Or

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}, \quad \nu = \frac{\mu}{\rho}$$

The slide also features the Swamyam logo and a small video inset of a man in a pink shirt.

Now as stated in the problem, we are saying that the  $v_x$  is a function of  $y$  and  $t$  and we are assuming the other two components of the velocity that is  $v_y$  and  $v_z$  to be 0. So, because  $v_y$   $v_z$  are 0, so we have 3 momentum balance equations out of these three we need to consider only one momentum balance equation that is in the  $x$  direction ok.

So, this is the  $x$  direction momentum balance equation. Here you can see that we have the unsteady term, this give rise to a dependency of the  $v_x$  on time these are the inertial terms, this is the pressure force, and this is the viscous force ok, and this is the body force and as given in the problem this body force is taken to be 0.

So, because this is  $v_y$  gives it a 0 so, these two terms are dropping off ok. And because, in the  $x$  direction there is no change in the velocity so this term also goes out. So, what we are left with is  $\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$  ok. And as stated in the problem, there is no pressure gradient in the  $x$  direction so this term is also 0. Now please understand this, in this case, the flow is happening due to the motion of the surface. Flow can happen for by many ways that a pressure gradient also causes a flow. But in this case, the pressure gradient is 0 but the motion of the surface is causing the flow.

And in this case, we are finding that this  $\rho \frac{\partial v_x}{\partial t}$  terms are 0  $\rho \frac{\partial v_x}{\partial x}$  terms are 0  $\rho \frac{\partial v_x}{\partial z}$  terms are 0 so, only we are left with is the  $\rho \frac{\partial^2 v_x}{\partial y^2}$  so, we are writing this and the body force term is also taken to be 0 as per the question.

So, with this we are landing up with this equation and then we take the fluid properties that is, density and the viscosity to be constant. So, because they are constant there have been taken out of the differential. So now, we take the density on the right hand side and we have this particular equation; that means,  $\rho \frac{dv_x}{dt} = \mu \frac{d^2 v_x}{dy^2}$ . So, this is the particular equation if solved we will get the actual velocity profile provided we put the initial condition and the boundary conditions.

And you can see from this equation, this is a degree 1 with respect to time. So, we need only one initial condition and this is of degree 2 with respect to  $y$  so we need two boundary conditions. And let us see, how we prescribe this initial and boundary conditions.

(Refer Slide Time: 08:13)

**Flow near a surface that is suddenly set to linear motion**

- Initial condition: At  $t \leq 0, v_x = 0 \quad \forall y$
- Boundary conditions: At  $y = 0, v_x = v_0$ , and  $y = \infty, v_x = 0$  for  $t > 0$
- Non-dimensionalizing  $v_x$ , we get *no-slip*  $v_\infty$   

$$\bar{v}_x = \frac{v_x}{v_0}$$

We have

$$\frac{\partial \bar{v}_x}{\partial t} = \nu \frac{\partial^2 \bar{v}_x}{\partial y^2}$$

- So the initial and boundary conditions are given as  
At  $t \leq 0, \bar{v}_x = 0$   
At  $y = 0, \bar{v}_x = 1$ , and  $y = \infty, \bar{v}_x = 0$  for  $t > 0$

The slide also features a Swayam logo and a presenter in the bottom right corner.

So, here we have the initial condition, we said that at  $t$  less than equal to 0  $v_x$  is equal to 0 as stated in the problem for all  $y$  ok. And a boundary conditions is that on the surface  $y$  is equal to 0; that means, it is on the surface we have the  $v_x$  equal to  $v_0$  and  $v_0$  is the velocity with which the surface has been moving so, this is the  $v_0$ . So,  $v_x$  equal to  $v_0$ ; that means, we are assuming that there is a no slip boundary condition. We call it no slip that is no relative velocity between the surface and the fluid.

So, this is the no slip boundary condition, no relative motion between the fluid and the surface at the surface and then at  $y$  equal to infinity. Infinity here means, that a long distance from the surface where we do not feel the effect of the drag by the particular

surface. So,  $y$  equal to infinity the  $v_x$  is taken to be 0. In this case, 0 because we are saying that the fluid was at rest initially, but it need not be 0, it could have some other non 0 value suppose  $v$  infinity which call the free stream velocity ok. So, in this case as per the question we are taking this  $v_x$  to be 0 at  $y$  equal to infinity for all the time  $t$  more than 0.

Now, what we do we go for the non-dimensionalization of the particular equation. So, we define the non-dimensional  $v_x$  as  $\bar{v}_x$  and define it as the  $v_x$  divided by  $v$  naught. So,  $v$  naught is taken to be the characteristic velocity that is equal to the velocity with which the surface is moving. So, with this definition we get this particular equation for the momentum balance with in terms of the non dimensional velocity.

Now we see that the initial condition is now that  $\bar{v}_x$  equal to 0 and now, at for a time  $t$  less than and equal to 0 and for the boundary conditions you see that for  $y$  equal to 0  $\bar{v}_x$  equal to 1 and  $y$  equal to infinity  $\bar{v}_x$  is equal to 0 ok. So, we find that this is also a normalized velocity. Next what we do that.

(Refer Slide Time: 10:54)

**Flow near a surface that is suddenly set to linear motion**

- Since both the boundary conditions are of type 1 (Dirichlet),  $\bar{v}_x \equiv \bar{v}_x(y, t; v)$ .
- Since  $\bar{v}_x$  is dimensionless,  $y, t$  and  $v$  should appear in dimensionless form. This may be obtained by defining a dimensionless coordinate direction  $\bar{y}$  as  $\bar{y} \equiv y/\sqrt{4\nu t}$  - This is the "Method of combination of (independent) variables". The factor "4" has been included in view of the type of solution obtained.
- Thus we have
 
$$\frac{\partial \bar{v}_x}{\partial t} = \frac{d\bar{v}_x}{d\bar{y}} \frac{\partial \bar{y}}{\partial t} = -\frac{1}{2t} \frac{d\bar{v}_x}{d\bar{y}}$$
- Also
 
$$\frac{\partial \bar{v}_x}{\partial y} = \frac{d\bar{v}_x}{d\bar{y}} \frac{\partial \bar{y}}{\partial y} = \frac{1}{\sqrt{4\nu t}} \frac{d\bar{v}_x}{d\bar{y}}$$

$$\frac{\partial^2 \bar{v}_x}{\partial y^2} = \frac{1}{4\nu t} \frac{d^2 \bar{v}_x}{d\bar{y}^2}$$

The slide also features the Swayam logo and a small image of a presenter in the bottom right corner.

We see that both the boundary conditions have been taken as the boundary condition of type one or the Dirichlet boundary condition. So, we find that  $v_x$  is a function of  $y, t$  and  $\nu$  only these three ok. So, if it is not of type 1; that means, it could be a Neumann boundary condition. Then you will find that, it will also be a function of the flux term,

some kind of momentum flux term. So, because of this particular situation, we find that this  $v_x$  is dimensionless and then  $y$ ,  $t$  and  $\nu$  should appear in a dimensionless form.

And you can see that if we define a new coordinate  $\bar{y}$  and this  $\bar{y}$  we can put in this way that  $\bar{y}$  divided by  $\sqrt{4\nu t}$ . Now this particular way of defining a non dimensional numbers in terms of the variables which are appearing within the equation is called method of combination of variables. And in this case, all the variables happen to be independent. So, we are getting a non dimensional distance with respect to the variables we which are there already in the particular equation ok.

And this factor four has been taken as some kind of a factor which we will see that how the inclusion of the 4 helps in our the equation ok. So, this 4 is not sacrosanct you may take it or you may not take it is not mandatory to have this 4. But, looking at the particular equation we obtain ultimately we find that if we put the 4 then we will be having a the equation non-dimensional equation which will be quite compact. So, that way we take the 4 inside.

So, now you will see that  $\frac{dv_x}{dt}$  may be written this is purely calculus. So, we find that this will be  $\frac{dv_x}{d\bar{y}}$ , now why you see that  $\frac{dv_x}{d\bar{y}}$  we are putting  $d$  not  $dv_x$  because now you see that with this definition of  $\bar{y}$   $v$  becomes this  $\bar{v}$  becomes a function of only  $\bar{y}$  ok. So, that is why the partial differential is gone we are replacing the partial with a total differential. So, it is  $\frac{dv_x}{d\bar{y}}$  and, but  $\frac{d\bar{y}}{dt}$  because  $\bar{y}$  is a function of both  $y$  and  $t$  ok.

So, with this understanding we find that this is coming as  $-\frac{1}{2}\frac{d\bar{y}}{dt}$ . And then similarly,  $\frac{dv_x}{d\bar{y}}$  is equal to  $\frac{dv_x}{d\bar{y}}$  into  $\frac{d\bar{y}}{dy}$  and this is this. And  $\frac{d^2v_x}{d\bar{y}^2}$  is equal to coming like this.

Now, what have we done by doing this, we find that we have converted a partial differential equation into an or any differential equation. And how does it help, it helps in simplifying the solution of this particular equation.

(Refer Slide Time: 14:19)

**Flow near a surface that is suddenly set to linear motion**

- Combining the above expressions we get
$$\frac{d^2 \bar{v}_x}{d\bar{y}^2} + 2\bar{y} \frac{d\bar{v}_x}{d\bar{y}} = 0$$
- With
$$\bar{y} = 0, \bar{v}_x = 1,$$
$$\bar{y} = \infty, \bar{v}_x = 0$$

The slide features a yellow background with a dark blue curved shape on the right side. At the bottom, there is a blue banner with logos for Swamyam and other educational institutions, and a small inset image of a man in a pink shirt speaking.

So, now after getting this terms we put it in our original equation and we find that this is the equation we are getting and this is an ordinary differential equation ok. Now you see, the boundary conditions are now getting modified. Now we need not talk of initial condition that has in the absorbed already in  $t$  so this is now forming a boundary recondition.

Now what we find that  $\bar{y}$  equal to 0 we get that  $\bar{v}_x$  as 1 and  $\bar{y}$  over bar equal to infinity, we get  $\bar{v}_x$  over bar equal to 0. So, this is how we are able to model the fluid motion which is there a semi infinite fluid which is bounded by a surface and the surface is given a linear motion.

(Refer Slide Time: 15:07)

**Flow near a surface that is suddenly set to sinusoidal motion**

- At time  $t = 0$ , the surface is made to oscillate sinusoidally in the  $x$ - direction with an amplitude of  $X_0$  and frequency of  $\omega$ .
- The displacement  $X$  of the surface from its rest position is  $X(t) = X_0 \sin \omega t$
- Initial condition: At  $t \leq 0, v_x = 0 \quad \forall y$
- Boundary conditions: At  $y = 0, v_x = \frac{dX}{dt} = X_0 \omega \cos \omega t$ , and  $y = \infty, v_x = 0$  for  $t > 0$
- Let us put the amplitude of velocity oscillation as  $v_0 = X_0 \omega$ , so that  

$$v = v_0 \cos \omega t = v_0 \mathcal{R} \exp(i\omega t)$$

Where  $\mathcal{R}(z)$  means the real part of  $z$

*Handwritten note:  $e^{it} = \cos t + i \sin t$*

The slide also features logos for Swamyam and other educational institutions, and a video feed of a lecturer in a pink shirt.

If we extend this particular problem now we can extend it for a sinusoidal motion; that means, it need not be a linear motion it away sinusoidal. And this sinusoidal motion it is oscillating like this. So, with this oscillation will be in the  $x$  direction only. And we say that the displacement from the mean value is taken to be capital  $X$  and this capital  $X$  is a function of time and is given by the mean value  $x$  naught that is the amplitude and the omega is the frequency ok.

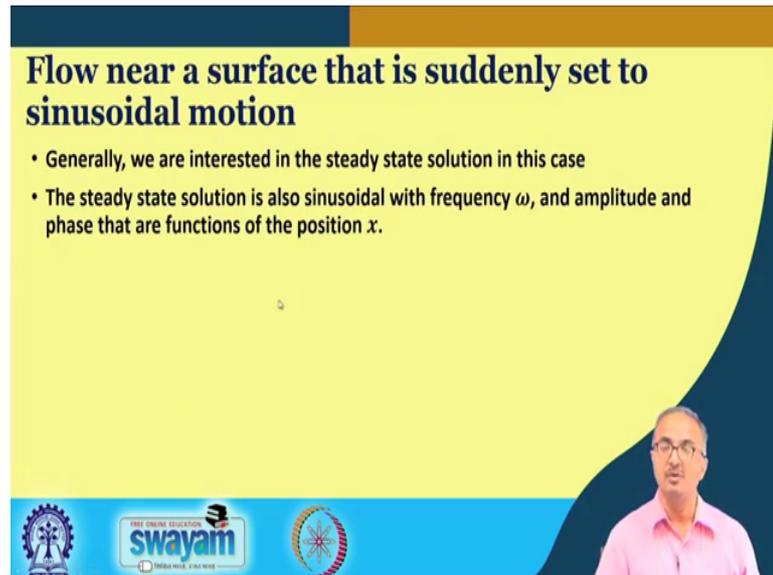
Now we will find that initial condition is the same as before that  $v_x$  equal to 0 for all  $y$  and the boundary condition is now like this, that at  $y$  equal to 0 we are giving  $dx$  by  $dt$  because  $x$  is a position.

So,  $dx$  by  $dt$  is the velocity. So, we are having  $dx$  by  $dt$  equal to  $x$  naught omega  $\cos$  omega  $t$  from this particular expression we are differentiating with respect to time and  $y$  infinity equal to  $v_x$  equal to 0 as we have done it earlier for all  $t$  more than 0. Now let us put this amplitude of the velocity like  $v$  naught equal to  $x$  naught into omega because this is going to be constant.

So, we replace this  $x$  naught omega by this  $v$  naught omega, this a amplitude of the velocity. Whereas,  $x$  naught is the amplitude of the particular displacement of the wave from the mean value that is the amplitude ok. So, this is now the  $v$  velocity of the fluid of the surface is also coming like an sinusoidal motion and this is in terms of the amplitude  $v$  naught into  $\cos$  omega  $t$ .

Now, you know that  $\cos \omega t$  may also be written in terms of the exponential form and exponential form you know that  $e^{i t}$  is equal to  $\cos t$  plus  $i \sin t$ , this we know ok. So, this particular  $\cos$  is the real part and  $\sin$  is the imaginary part. So, we are writing this  $\cos \omega t$  as this  $r$  is signifying the real part of exponential of  $i \omega t$  ok. So, we are writing it in a still more concise manner.

(Refer Slide Time: 17:42)



**Flow near a surface that is suddenly set to sinusoidal motion**

- Generally, we are interested in the steady state solution in this case
- The steady state solution is also sinusoidal with frequency  $\omega$ , and amplitude and phase that are functions of the position  $x$ .

The slide features a yellow background with a dark blue curved shape on the right side. At the bottom, there are logos for 'swayam' and 'INDIA'S EDUCATION RENEWAL' along with a portrait of a man in a pink shirt.

Now, generally in this case situation we are interested to know the steady state solution rather than the transient solution ok. And this and we find that even if you do not solve it, but we find that the steady state solution will be a sinusoidal with frequency  $\omega$  and the amplitude and the phase will be functions of the  $x$ ; that means, that is how it will change ok.

The velocity of the fluid due to this change in the sinusoidal motion of the surface will also have a sinusoidal nature; that means, the  $v_x$  will also be sinusoidal in nature, but it will have a phase and frequency that will be changing with the position of the surface ok. So, I will not be going into the solution of it we are just learning how to deduce the model equations.

(Refer Slide Time: 18:39)

**Unsteady heat conduction**  
**Temperature distribution in a semi infinite solid**

Consider a solid material occupying the space from  $y = 0$  to  $y = \infty$  to be at an initial temperature of  $T_0$ . At time  $t = 0$ , the surface at  $y = 0$  is suddenly raised to temperature  $T_1$  and maintained at that temperature for  $t > 0$ .

Assume the material to be homogeneous and isotropic, and no heat generation within the solid

$y = \infty$   $T_0$   
 $y = 0$   $T_0$   
 $t < 0$

$y = \infty$   $T_0$   
 $y = 0$   $T_1$   
 $t \geq 0$

swamyam  
FREE ONLINE EDUCATION  
INDIA WISE LEARNING

Now, we come to a similar situation, but in a energy transfer problem; energy transfer problem. In this what we find that, we have again a semi infinite solid material you can see the solid material occupying the space from  $y$  equal to 0 to  $y$  equal to infinity ok.

So, instead of the fluid now we have the solid material and that the initial temperature is taken to be  $T_0$  ok. At time  $t$  equal to 0, the surface at  $y$  naught 0; that means, this if this is taken to be the  $y$  equal to 0 and this taken to a  $y$  infinity ok. So, we find that at  $y$  equal to 0 v at time  $t$  more than equal to 0 this surface is given the temperature of  $T_1$  ok. And that and it is saying that it is raised to a temperature  $T_1$ , it is not mandate we have to raise it; it maybe also lowered ok. So, it can go either way ok. So, if it is in this problem it is raised to  $T_1$  and maintained at the temperature for time more than 0.

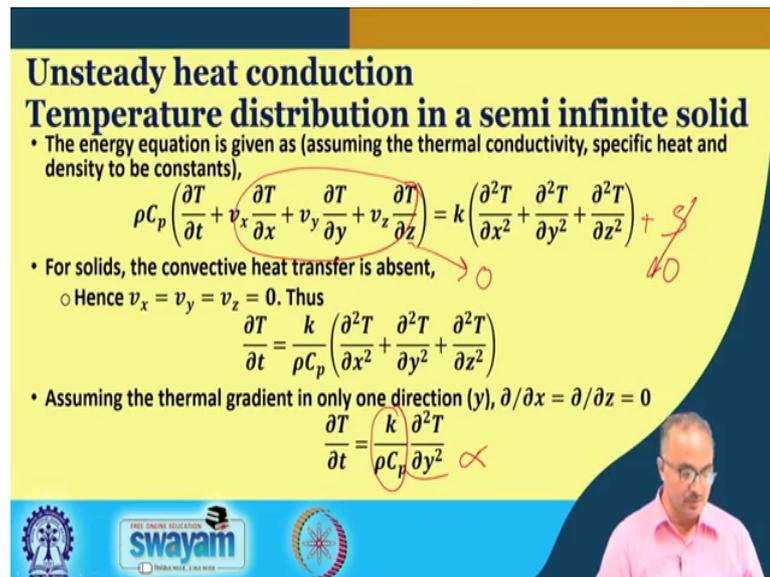
We have to assume that the metal to be homogenous an isotropic so, that we can assume a single value of the solid property in the particular phase and no heat generation within the solid.

(Refer Slide Time: 20:07)

**Unsteady heat conduction**  
**Temperature distribution in a semi infinite solid**

- The energy equation is given as (assuming the thermal conductivity, specific heat and density to be constants),
 
$$\rho C_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \dot{q}$$
- For solids, the convective heat transfer is absent,  $\rightarrow 0$ 
  - Hence  $v_x = v_y = v_z = 0$ . Thus
 
$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
- Assuming the thermal gradient in only one direction (y),  $\partial/\partial x = \partial/\partial z = 0$ 

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} \propto \alpha$$



So with these assumptions, now again we consider the particular energy balance equation and we know the temperature is a scalar. So, we have only one energy balance equation and in this energy balance equation these are the unsteady state term, these are the convective term, or the inertial term, and this is the conductive term and we are neglecting any source term. If there is any kind of generation, so we are neglecting the source terms.

So, with this assumption and in the solid we know there will not be any convective heat transfer so that all the velocity terms, all the velocity terms will be taken to be 0 in a solid ok. So, we are having all the unsteady state part and we are also assuming that the properties of the solid that is k that in the thermal conductivity the density and the specific heat are remaining constant so that they have been taken out of the differential and we are getting this particular equation. And now, we are assuming that the thermal gradient or the temperature gradient occurs only in the y direction so, that we are taking the  $\frac{\partial T}{\partial x}$  and  $\frac{\partial T}{\partial z}$  to be 0. So, with this assumptions we find that we are having this  $\frac{\partial T}{\partial t}$  in terms of the  $\frac{\partial^2 T}{\partial y^2}$ .

Now, if you look at this particular equation now you see that this is the thermal diffusivity  $\alpha$  ok. So, you see that the mathematical form of this equation is exactly the same as we have derived earlier for the semi infinite fluid with a surface which is

being set to motion at time  $t$  equal to 0. So because the mathematical forms are the same, so the solutions will also be the same.

(Refer Slide Time: 22:08)

**Boundary conditions**

- Defining the non dimensional temperature difference as  $\theta = \frac{T-T_0}{T_1-T_0}$

The non-dimensional energy equation becomes,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

- Boundary and initial conditions become
  - ✓ Initial condition:  $\theta = 0$  at  $t \leq 0 \quad \forall y$
  - ✓ Boundary condition:  $\theta = 1$  at  $y = 0 \quad \forall t > 0$
  - ✓ Boundary condition:  $\theta = 0$  at  $y = \infty \quad \forall t > 0$

The slide also features logos for Swamyam and other educational institutions, and a small portrait of a man in a pink shirt.

Now, in this case we non-dimensionalize the temperature like this the  $t$  equal to the  $\theta$  equal to  $T$  minus  $T$  naught divided by  $T_1$  minus  $T$  naught. And as we know that, in this case the temperature will be varying between  $T$  naught and  $T_1$  so in this case we find that,  $\theta$  is also normalized. So, it will be varying between 0 and 1.

So with this definition, we again put it in our parent equation and we have this particular equation and here to we are writing the initial condition boundary equation like this that initially  $\theta$  equal to 0 because  $T$  equal to  $T$  naught. So,  $\theta$  is equal to 0 initial condition and boundary condition is that initially the temperature is at  $T$  equal to at the boundary or that  $y$  equal to 0 the temperature is taken to  $T_1$ . So we put,  $T$  equal to  $T_1$  here so  $\theta$  is 1 at  $y$  equal to 0. And at infinity, we find that  $T$  is equal to  $T$  naught so  $\theta$  is 0 at  $y$  equal to infinity for all time  $t$  more than 0.

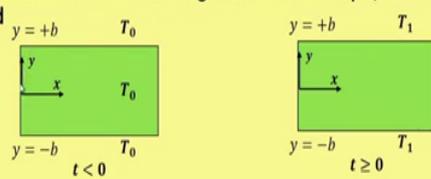
(Refer Slide Time: 23:20)

### Unsteady heat conduction

#### Temperature distribution in a finite solid

Consider a solid slab occupying the space from  $y = -b$  to  $y = +b$  to be at an initial temperature of  $T_0$ . At time  $t = 0$ , the surfaces at  $y = \pm b$  are suddenly raised to temperature  $T_1$  and maintained at that temperature for  $t > 0$ .

Assume the material to be homogeneous and isotropic, and no heat generation within the solid



The diagrams illustrate the initial and boundary conditions for the unsteady heat conduction problem. The left diagram shows the slab at time  $t < 0$  with both surfaces at temperature  $T_0$ . The right diagram shows the slab at time  $t \geq 0$  with both surfaces at temperature  $T_1$ . The coordinate system is defined with  $y$  as the vertical axis and  $x$  as the horizontal axis.



Now, this particular problem may be extended in a semi, we have taken to be infinite bound solid semi-infinite bound solid now it need not in semi-infinite, it may be a finite solid. Now in this case what we are saying that here you can see the pictorial description that here we have a finite width of the solid and in this case we now place the origin at the midpoint in the  $y$  direction ok.

So, this is the  $x$  direction the  $y$  direction. So, because the midpoint we take it is plus  $b$  on this side and minus  $b$  on this side; that means, the total length is taken to be  $2b$  ok. So, plus  $b$  and minus  $b$ , in a same situation that initially the whole system is at  $T_0$  and suddenly at  $t$  equal to  $0$  we give a temperature change to that is lower surface to  $T_1$  ok. And again, we make the same kind of assumptions that isotropic homogenous no heat generation.

(Refer Slide Time: 24:20)

## Unsteady heat conduction

### Temperature distribution in a finite solid

- The energy equation is given as (assuming the thermal conductivity, specific heat and density to be constants),
 
$$\rho C_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
- For solids, the convective heat transfer is absent,
  - Hence  $v_x = v_y = v_z = 0$ 

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
- Assuming the thermal gradient in only one direction (y),  $\partial/\partial x = \partial/\partial z = 0$ 

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2}$$

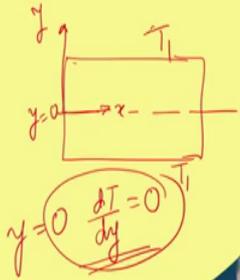


And with that, we get from this particular equation energy balance equation we deduce that this is the equation. Till now, the equations remain the same and also after making these assumptions that there is no temperature gradient in the x and z directions so we get this particular equation. So, till now we have the same equation as before.

(Refer Slide Time: 24:43)

## Boundary conditions

- Defining the non dimensional:
  - Temperature difference as  $\theta = \frac{T_1 - T}{T_1 - T_0}$
  - Coordinate or length as  $\eta = \frac{y}{b}$
  - Time as  $\tau = \frac{at}{b^2}$
 The non-dimensional energy equation becomes,
 
$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2}$$
- Boundary and initial conditions become
  - Initial condition:  $\theta = 1$  at  $\tau = 0 \forall \eta$
  - Boundary condition:  $\theta = 0$  at  $\eta = +1 \forall \tau > 0$
  - Boundary condition:  $\theta = 0$  at  $\eta = -1 \forall \tau > 0$




Now, comes the thing that now we start non-dimensionalizing. In the non-dimensionalization we go with the theta first, theta is taken to be  $T_1 - T$  by  $T_1 - T_0$  and the coordinate is taken to be  $y$  by  $b$ . Now here what we are

doing that we are not solving the, or we are not writing the model equation for the entire domain  $2b$ . We are assuming that across the axis at the  $0$  there is a symmetry. So, we are going to solve only for one half in the width direction. So, in the solve for one half so we are putting in terms of  $b$  only not in terms of  $2b$ . And we are defining this non-dimensional time  $\tau$  as  $\alpha t$  by  $b^2$ .

Now, with this we what we now find that the non-dimensional equation will come like this. And once we have found this non dimensional form, you can see that at  $\tau$  equal to  $0$  if you see that this  $\theta$  is equal to  $T_{naught}$  so this will be  $T_{naught}$  here, so you find that  $\theta$  equal to  $1$ . And that  $\tau$  at in this  $\eta$  equal to  $1$  that is you put the value of  $b$  ok.

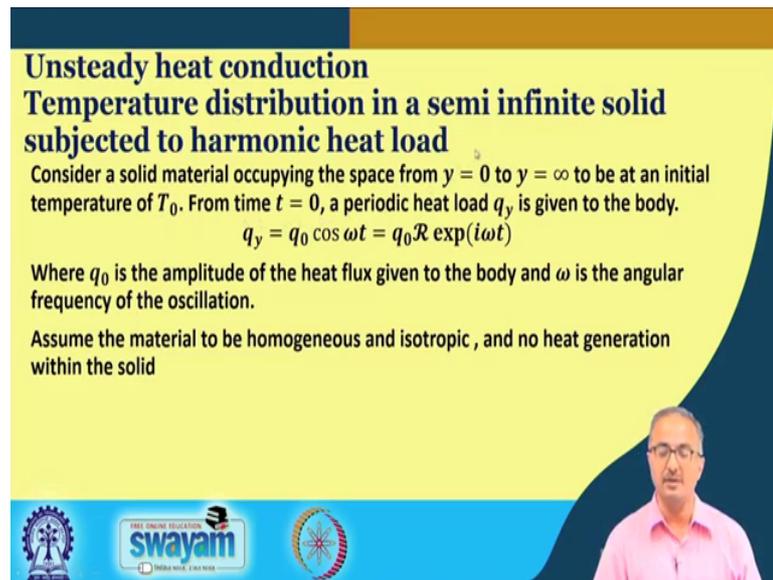
You find that there is the temperature is still  $T_1$  you have given the temperature  $T_1$  at the two surfaces at this two surfaces you have given  $T_1$  temperature ok. So, at  $b$  and plus  $b$  and minus  $b$  have given  $T_1$  temperature. So, you put the  $T_1$  temperature over here so you find  $\theta$  reduces to  $0$ . And at the midpoint this is shows the, this is not the midpoint this is the other end that is at minus  $b$  and plus  $b$  you find that  $\theta$  equal to  $0$ .

So, either you can solve for the whole domain or you can also, in this case if I want to take the one half of the solid; that means, we have if am I given system is like this that here we have taken our origin. So, if I am going to solve for this one half of this then and if I take them this is symmetrical then if this is  $y$  equal to  $0$  so at  $y$  equal to  $0$  i can take due to symmetry I can take  $dT$  by  $dy$  equal to  $0$ .

So, you can see that depending on the geometry we are considering my boundary condition would change. For the problem on the both the sides I have  $T_1$   $T_1$ , it is becoming the Dirichlet condition. But if I consider only one half of this we find that on one side we shall be having the Neumann boundary condition and this boundary condition is telling about the symmetry effect ok. So, it is the continuity, so symmetry effect, so that is how it is coming.

So, you can choose either this formulation or this formulation, does not matter whichever formulation you choose you will obtain the same result. After this, we shall go for another situation as we have done for the momentum balance.

(Refer Slide Time: 28:10)



**Unsteady heat conduction**  
**Temperature distribution in a semi infinite solid subjected to harmonic heat load**

Consider a solid material occupying the space from  $y = 0$  to  $y = \infty$  to be at an initial temperature of  $T_0$ . From time  $t = 0$ , a periodic heat load  $q_y$  is given to the body.

$$q_y = q_0 \cos \omega t = q_0 \mathcal{R} \exp(i\omega t)$$

Where  $q_0$  is the amplitude of the heat flux given to the body and  $\omega$  is the angular frequency of the oscillation.

Assume the material to be homogeneous and isotropic, and no heat generation within the solid

The slide also features a small inset image of a man in a pink shirt in the bottom right corner and logos for Swamyam and other institutions at the bottom.

Here, we are having now the same situation as earlier for the energy transfer, heat transfer, but now we have the semi-infinite solid subject to a harmonic heat load ok. Now harmonic heat load, we are giving in this way that the  $q_y$  that is the heat load equal to  $q_0 \cos \omega t$  that is the heat amplitude then into  $\cos \omega t$  and again we are writing  $\cos \omega t$  in terms of the exponential form we are writing with that radial part of the exponential  $i\omega t$  ok.

And again the space varies from  $y$  equal to 0 on one end of the solid to  $y$  equal to infinity that is a semi- infinite solid initial temperature is taken to be  $T_0$  ok. And at time  $t$  equal to 0 this kind of a heat flux is given to the particular surface at  $y$  equal to 0. Again, we are assuming the material to be homogenous and isotropic and there is no heat generation within the solid.

(Refer Slide Time: 29:17)

**Unsteady heat conduction**  
**Temperature distribution in a semi infinite solid**  
**subjected to harmonic heat load**

- The energy equation is given as (assuming the thermal conductivity, specific heat and density to be constants),
 
$$\rho C_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
- For solids, the convective heat transfer is absent,
  - Hence  $v_x = v_y = v_z = 0$ 

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
- Assuming the thermal gradient in only one direction (y),  $\partial/\partial x = \partial/\partial z = 0$ 

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$$



So, with these assumptions we go back to the energy balance equation and we deduce that this is the energy balance equation, this is the after this is assumptions of dou by dou x dou by dou z setting to 0 we get this equation. The up to this we find that we have the same formulation as before.

(Refer Slide Time: 29:40)

**Unsteady heat conduction**  
**Temperature distribution in a semi infinite solid**  
**subjected to harmonic heat load**

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Multiplying by  $-k$  and operating on the entire equation with  $\frac{\partial}{\partial y}$  gives,

$$\frac{\partial}{\partial t} \left( -k \frac{\partial T}{\partial y} \right) = \alpha \frac{\partial^2}{\partial y^2} \left( -k \frac{\partial T}{\partial y} \right)$$

But,

$$q_y = -k \frac{\partial T}{\partial y}$$

$$\Rightarrow \frac{\partial q_y}{\partial t} = \alpha \frac{\partial^2 q_y}{\partial y^2}$$


Now, what we do that this particular thing we put in terms of the k because this q is given in terms of the k. The our in that initial that that is boundary condition is given in

terms of the heat flux and heat flux we are representing in terms of the k. So, we put in terms of the k ok.

Now, we find that if you do this in terms of the k we get this particular equation for this. Now you see that we are on this side we have put the q in terms of the k and because the within the solid there the heat transfer will be due to the conductive heat transfer ok. So, that is how we are able to write in terms of the conductive heat flux.

So, you will get the same expression as before on this ok. So, now, we find that this is the kind of expression you are getting in terms of the q and not in terms of the t why we put in terms of q because our initial the boundary condition is given in terms of the q.

(Refer Slide Time: 31:02)

**Boundary conditions**

$$\frac{\partial q_y}{\partial t} = \alpha \frac{\partial^2 q_y}{\partial y^2}$$

- Boundary and initial conditions are
  - ✓ Initial condition:  $q_y = q_0$  at  $t = 0 \forall y$
  - ✓ Boundary condition:  $q_y = q_0 \mathcal{R} \exp(i\omega t)$  at  $y = 0 \forall t > 0$
  - ✓ Boundary condition:  $q_y = 0$  at  $y = \infty \forall t > 0$

The slide also features logos for Swamyam and other educational institutions, and a small video inset of a speaker in the bottom right corner.

So, with this if you do not do this then we have to other way is that we have to convert this q in terms of t, that is also possible that you can put the q in terms of t. So, what were you, where you do you will get the same solution.

So, for convenience sake we put in terms of the q and we find that the initial condition that it is q y equal to q naught at t equal to 0 and one into the boundary condition is given like this particular sinusoidal heat load and on the other side we have no heat load so we are putting that to be 0 ok. So, that is how we are able to reduce the initial energy balance equation in terms of temperature to in terms of the in heat.

(Refer Slide Time: 31:53).



References

- ❑ Bird, R.B., Lightfoot, E.N. and Stewart, E.W., 2007. Transport phenomenon. John Wiley and Sons.
- ❑ Brodkey, R.S. and Hershey, H.C., 2003. Transport phenomena: a unified approach. Brodkey publishing.
- ❑ Fox, R.W. and McDonald, A.T., 1994. Introduction to Fluid Mechanics, John Wiley & Sons.

The slide features a dark blue background on the left with the word 'References' in a yellow, cursive font. The right side has a yellow background with a list of references. At the bottom, there are three logos: the IIT Bombay logo, the SWAYAM logo (with the text 'FREE ONLINE EDUCATION swayam' and 'INDIA'S COURSE'), and a circular logo with a gear and a star.

So, these are the references which you can refer to for more details about these problems.

Thank you.