Theory of Elasticity

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Lecture - 49
Thermoelasticity

Welcome, this is module 10. Actually, from module 10 onwards we will be discussing some special topics on elasticity. For instance in this module we will be discussing Thermo-elasticity and in the next module we will be discussing probably photoelasticity and then, some elements of non-linear elasticity in the next module. So, in these three modules essentially these, theories are very it is not it will not be fully covered in this module. So, we will give a brief introduction of these three theories for instance for the current module we will be talking about thermo-elasticity. So, thermo-elasticity or thermo-mechanics itself is a subject. So, what we will try to discuss here, is that the basics of thermo-elastic deformation here and with some given assumptions that is helpful for our systems.

So, in general, we all know: what is the effect of heat in a structure. So, for instance, if I heat a steel bar or steel rod it is we know that it expands and this knowledge is coming from our strength of material knowledge. So, we have seen that, the strain is essentially if the length of the bar is l then coefficient of thermal expansion that is alpha then, the change in temperature that is delta t is related to the strain of the bar. So, this strain is essentially can be converted into the change in length which is delta l for instance if the bar is of a length l. So, delta l by l is the length and how it is related with the coefficient of thermal expansion and the temperature rise or decrease. So, this was our initial concept of thermal effect on a elastic structure.

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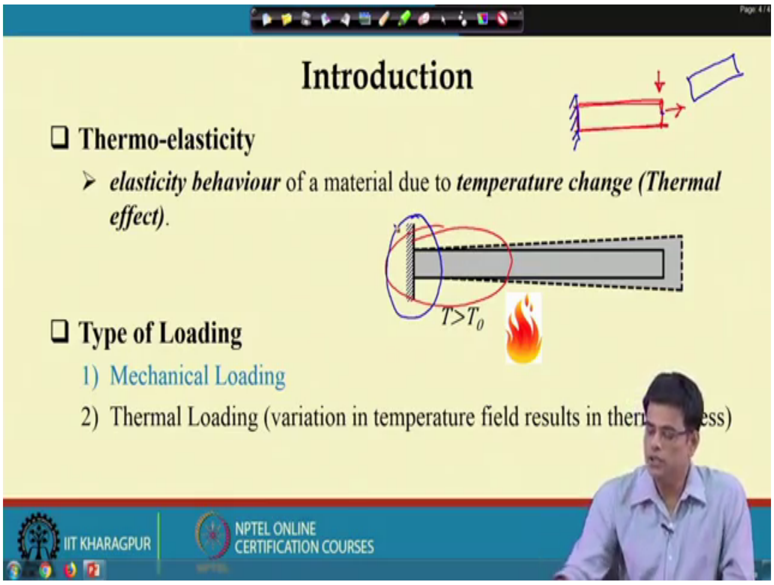

So, here, we will start with that and then slowly, define what is the thermo-elasticity or what kind of thermo-elasticity will be considering here. For instance, in our introductory lecture, possibly I have told you that will be mostly interested in uncoupled thermo-elasticity; where, the heat and elastic behaviour are not communicating each other; that means, there is the two types of differential equation for instance heat if I heat a body. So, heat is propagate from one end to the body to another end of the body and these physical phenomena can be modelled by the heat equation.

Similarly, if I give a load to a structure then, this structure is deformed and this deformation is modelled by the elasticity equation which we have discussed till module 9; and now, if there is a heat then also there is a stress generated. For instance so, basically thermo-elasticity is nothing, but elasticity behaviour of a material due to temperature change. So, that is the thermal effect on elastic body. So, for instance we have seen the mechanical loading what is the response of a elastic body if I give a mechanical loading to the body. Now, we are discussing what is the response of the elastic body if, there is a change in temperature or the thermal loading specifically. For instance, if I this is a bar for instance this is a bar and this bar if I want to heat it uniformly.

So, now there are some assumptions that this is this uniform heating and all those things. So, if I heat it will expand. So, what will happen if I heat this bar to a uniformly to everywhere in the bar if I heat. So, this bar will expand like this. So, now, the question is if there is no mechanical loading will there be thermal stress generated. So, the answer is no because it is freely expanding is so, there will be no thermal stress generated, but for instance if I restrict the bar at one end then there will be thermal stress because, the bar is now not freely expendable in these region. So, that is why, this uniform heating will generate thermal stress in the body.

So, these phenomena actually we are trying to model. So, essentially what is this why this thermal stress is generated because there is a the bar was in a normal say room temperature and then you heat through a heating source then, this heating source will create the change in temperature in the body compared to the outside temperature. Now, this temperature change will generates stress and also we know that if, there is a constraint or there is a restriction of expansion, for instance this case if there is a restriction of expansion then only it will generate the stress. But, if there is no restriction of expansion for instance the if I remove the constraint here that is the built in them in here then the bar is free to expand and there is no thermal stress generated.

Now, this is very similar fact rigid body motion for instance in a rigid body motion, if there is a in case of a elasticity purely mechanical loading if they if I if this bar is for instance this is a bar and then, you give a load here. So, this bar or you beam or bar whatever it is it give some load here. So, what will happen to this bar? This bar will move freely in space. So, this bar will be somewhere here. So, for instance this bar will be somewhere here. So, with this so, the each point in the bar is translated and rotated. So, it is not a deformable motion; it is a rigid body motion, but if I give a constraint to the bar for instance similar to this, then only the deformation is generated.

So, the similar the concept the thermal concept of thermal stress here is very similar to that. So, here only the constraint I am introducing and then only the thermal stress is generated. So, this effect we will see, how we can mathematically model this thermal effect on a elastic body. So, to start with, we will be discussing what are the basic things or the basic principles by which we will start our modelling.

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So, for instance this example of thermo-elasticity is in our everyday life. So, I will not discuss much about these things. These things are very common to our life and we know what the thermo elastic effect is for instance the dual composite bar with a thermal heating effect and which is used in every system that we are using.

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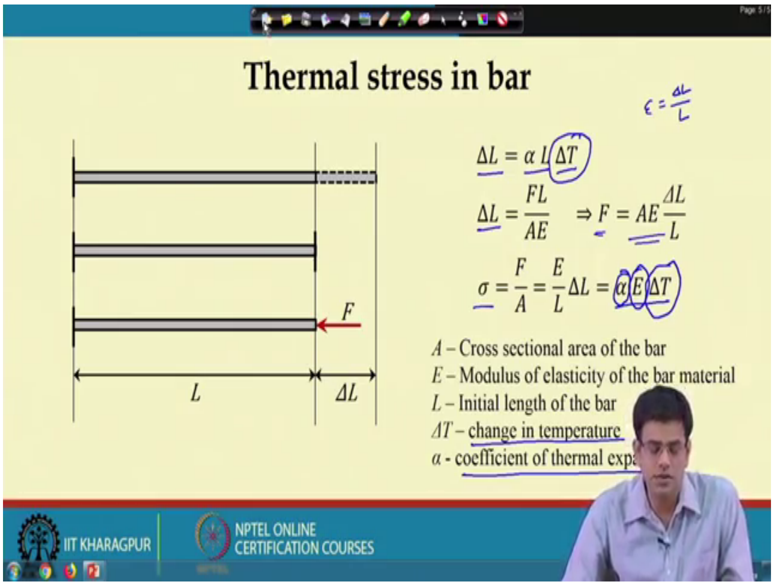

So, now let us also, review what is the thermal stress if I heat a bar uniformly. For instance, this is a bar where I have not fixed this bar at the one end. Then I try to heat this bar. Now, this is from our strength of material knowledge we know this delta L which is essentially the increase in length is alpha L delta T. So, the strain is essentially delta L by L. So, this we know. Now, from our if there is an equivalent loading of suppose there is an equivalent loading equivalent force which is F. So, also we know that this equivalent load if, I put here then the strain will be generated is F L by A E this is also we know from the strength of material knowledge.

Now, so I can write simplifying F I can write in this manner. Now, if I want to compute the stress due to this heating this thermal heating then I just simply know the one dimensional stress formula is force by area. So, force I know area, I know of the bar so which comes out to be E L delta L by L. So, which essentially again comes out to be alpha E delta T. So, you see the stress is actually involving the elastic constants also. So, that is how this response of a elastic body due to change in temperature what is this delta T? Delta T is the change in temperature and alpha is the coefficient of thermal expansion.

So, now it is important this simple example shows us, how stress is actually related with the thermal coefficients that thermal property of a material as well as elastic property of the material and why it is happening it is happening due to the change of delta T. So, change of temperature that is delta T. So, this is the basic principle we need to understand before we proceed the more complicated 3 D behaviour of a elastic body under thermal stress.

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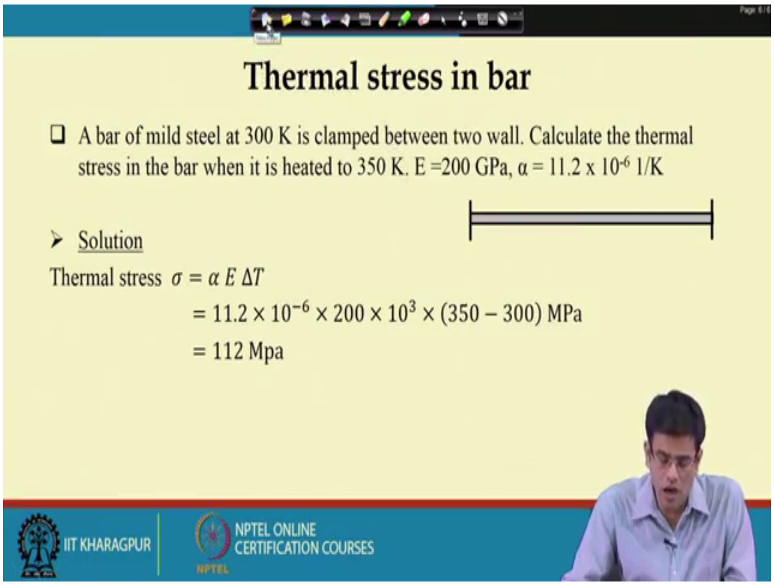

So, now let us see, what kind of this is the simple example for instance, if a mild steel bar at 300 degree Kelvin is clamped between two wall. Calculate the thermal stress of the bar when is it is heated to 350 degree Kelvin. Now, important fact is that that there is we observe thermal they we observe temperature changes, but if this temperature changes significant or it depends on the material also. So, then the Thermal steel stress is not negligible; we need to compute the thermal stress to account for the failures of the to prevent the failures of the structural body or the material.

Now, if the Young's modulus is 200 GPa and alpha is the coefficient of thermal expansion which is expressed in per degree temperature that is, per degree Kelvin here or per degree Centigrade or Fahrenheit any unit you can use. Now, if I want to calculate the thermal stress I will just simply use that this E alpha delta T which comes out to be 112 Mpa.

So, this example shows that even if, you use if for a small mild steel bar it depends on the how the alpha and delta T as well as the property of the material that is the Young's modulus of the material. So, the stress that will generate by heat change in temperature is dependent on the elastic body itself not only that elastic body the elastic properties of the body as well as, its thermal property. So, this is the main thing that we are looking for here.

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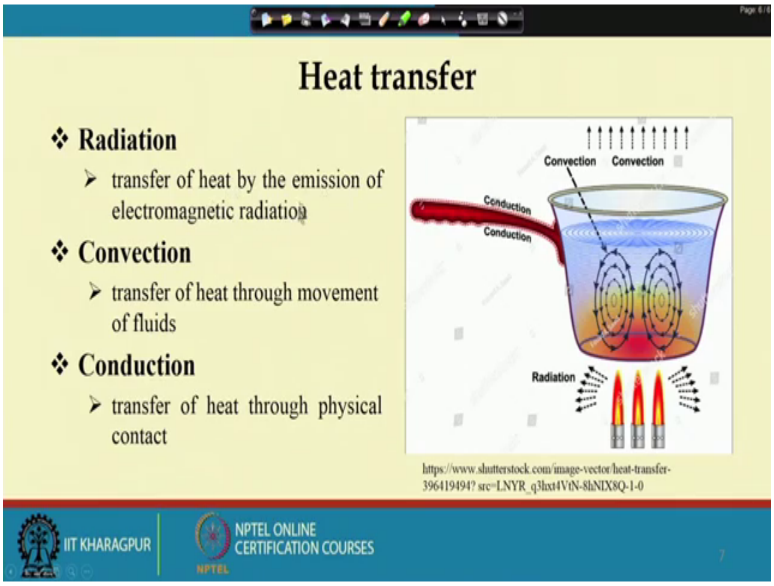

Now, for instance what is Heat transfer? For instance, if you boil a pan of water then, what kind of heat we get. So, for instance if I heat it. So, there is a source of heat is going to the surrounding through radiation. So, radiation is transfer of heat by emission of the electromagnetic radiation. So, this we call Radiation. So, we are not interested here right now the radiation heat transfer. So, now, there is another thing which we observe if there is a there is if I start heating this pan with a water and then, the water molecule will go up and down and. So, this will also release some heat. So, where this heat will this heat will go through the go to the surrounding atmosphere. So, this process is also known as the Convection process.

So, the transfer of heat through movement of fluids this is the convection process. Now, another observation we can made it from here that, if I touch this bar. So, we will feel will this handle of this pan then we feel the it is heated because this heat is actually propagated here. So, this is known as the Conduction, the heat conduction. So, transfer of heat when transfer of heat happens to a physical contact then we say it is a conduction. Now, we are mainly interested in the conduction heat conduction.

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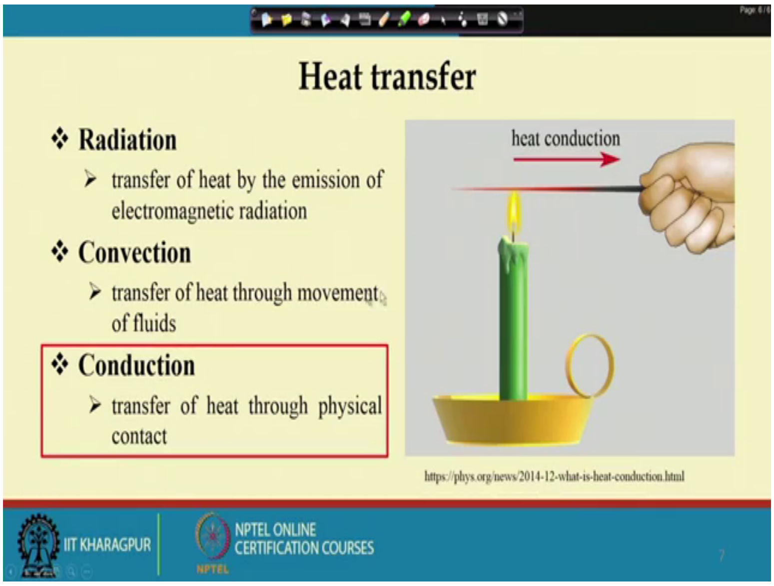

So, for instance this another example simple example of heat if you in a candle if you put a metallic rod or metallic bar and touch it. So, heat flows from this side to this side and this actually happens through this metallic bar and. So, you experience the temperature change of your finger. So, now this property or this is the basic physical phenomena that we are interested in.

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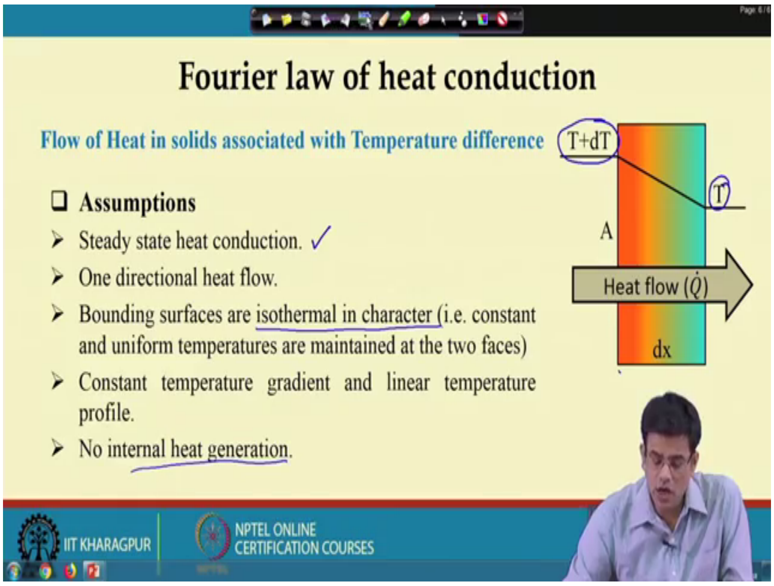

For instance, how heat flows? So, this is governed by the basic Fourier law of heat conduction. So, what it states it is it the main purpose of this heat conduction is that how Fourier law is the how heat will flow? Flow of heat in solids associated with the temperature difference.

That means, that how in a solid heat will flow from one body to another body we one part to another part. So, we all know that the higher temperature side it they heat will flow to the lower temperature side. So, now, if consider a simple two dimensional body where there is a these boundary these boundary there is a temperature is T plus delta T and in this side the temperature is T. So, it is natural from our physical observation as our intuition that heat will flow from this side to this side. So, heat flow is module as Q dot. So, Q dot is the heat flow. Now, naturally Q dot is the, if I if the temperature is more here. So, this side the opposite direction the heat will flow.

Now, what are the assumptions on it? So, the assumption on it that, that there is a steady state heat conduction. What this steady state means, that is there is no fluctuation involve. So, heat is flowing in a steady state; so that means, there is no transient or there is no change of heat flow. So, and it is one direction of flow here; now, another important thing is that we compared to the realistic situation that this side the always temperature T plus delta T is maintained and this side always temperature T is maintained this is known as the isothermal character or isothermal property. So, all of this things we know; now so these Constant temperature gradient; that means, linear profile all this things we will see and there is no heat source into it. So, in this body there is no heat source or there is no externally supplied heat in to the body.

Now, when we valid these conditions when you valid this conditions then, this governing Fourier law of heat conduction does not apply even. So, we will first assume this quantity or these assumptions and then, we will derive what is the or we will see what it looks like Fourier law of heat conduction.

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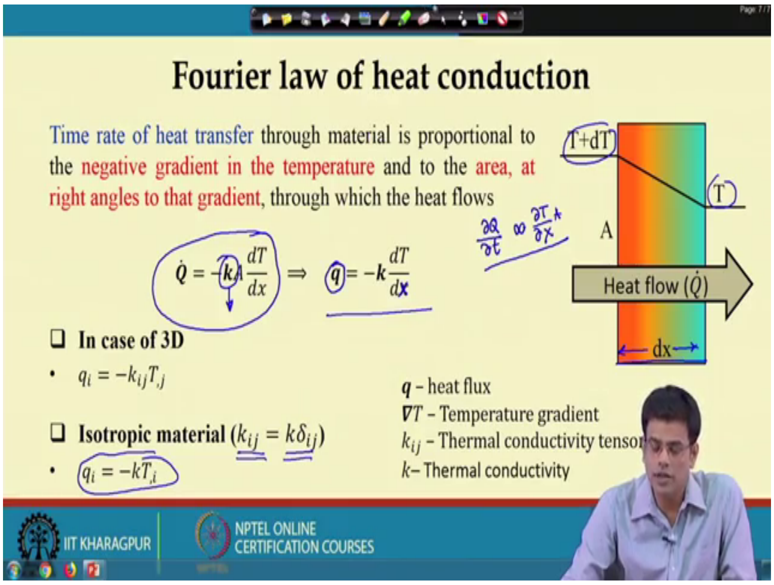

For instance, the Fourier law states that, Time rate of heat transfer through material. So, time rate means, that heat transfer over time and its rate. So, the basically the heat transfer is if it is Q then dQ by dt. So, is the heat transfer. So, time rate of heat transfer through material is proportional to the negative gradient in the temperature; that means, why negative gradient?

Because, you see the heat the temperature is higher this side and lower in this side so; obviously, it is going in the opposite direction. So, that is why it is negative. So, that means, dQ or if I write in a differential form dQ by dt is proportional to the gradient of the temperature; that means, what is the gradient of the temperature? Gradient of the temperature if one side it is T plus delta T another side is T; so, T plus delta T minus T by delta x. So, why did delta x because the path by which it is flowing. So, this is proportional to del T by del x and then, it is also dependent on the area of the section; so, that means, it is also proportional with the area of the section.

Now, as this is a proportional quantity so, to make it equality we need to have a constant k. So, this constant k is known as the conductivity or thermal conductivity of the material. So, that here the material property comes in to where the heat flows from one side to another side. So, material property of the comes into picture here, they are depending on the thermal conductivity of a material the heat flow will be different; for instance, if I use the wood and if I use the steel the thermal conductivity from one side to one side; that means, the thermal conductivity of wood and thermal conductivity of the steel is different. So, heat flow will be different.

So, this relation is actually known as the Fourier law of heat transfer. Now, if I see now, that the q by A is a quantity which I considered as a heat flux. So, that q dot by A if I consider; that means, heat flow per unit area is known as the heat flux. So, these heat flux if I write from here. So, it is the k into dT by dx. So, in a one dimensional case I am using this. Now, in a 3D obviously, I can extend it for 3D which is then q will be your a three dimensional quantity and k the conductivity; obviously, as we have seen for elasticity case conductivity will be a conductivity tensor for instance, it is a second order tensor now conduct similar to the elasticity case.

Now, dT by dx will be the gradient will be in terms of three special coordinates dT dx, dT dy, dt dz all those quantities will come. Now, this conductivity again for instance in a isotropic material where, the material properties are same in all directions. So, there these conductivity does not depend on whether it is x direction or y the direction or the z direction. So, these conductivity will be same in the x direction y direction and z direction. So, I can write k ij that is the conductivity tensor is a diagonal tensor which is k k into delta ij right. So, these things we know. So, if I now, put it in one dimensional setting indicial notations it looks like this. Now, so this will be bold. So, now, this is the basics of Fourier law that that we are interested in now let us see, how these things works.

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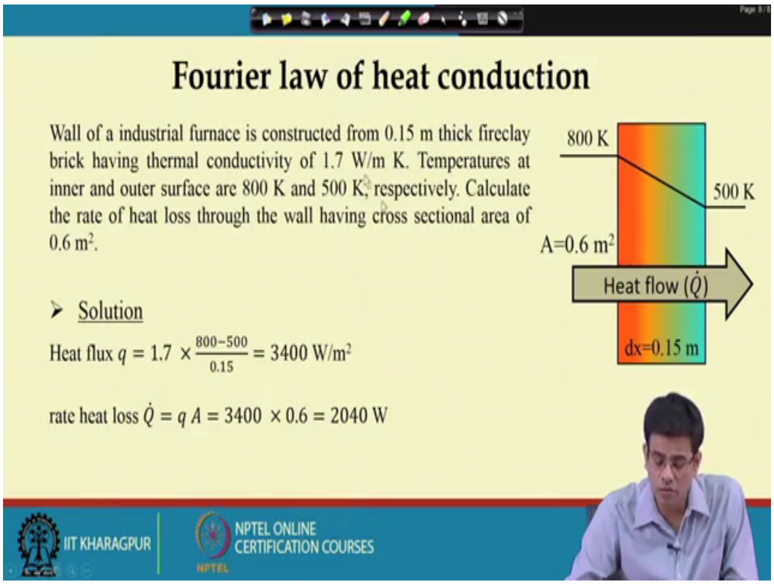

So, for instance Wall of a industrial furnace is constructed from 0.1 meter thick fireclay brick having thermal conductivity of 1.7 watt meter watt per meter Kelvin and temperature at the inner and outer surfaces is 800 and 500 degree Kelvin. So, calculate the rate of heat loss through the wall having cross sectional area 0.06 metre square; now. So, what is dx; dx is 0.5. Now, this is 800 and this is 500 and area is 0.6 meter square. So, if I just can compute the Heat flux. Heat flux is essentially, conductivity which is 1.7 and then del T by del X. So, del T is your 800 minus 500, 300 degree temperature change and del x is 0.15.

So, this gives me the Heat flux 3400 Watt per metre square then, rate it of heat loss. So, heat is from this surface to going to the surface. So, this surface will have a heat loss so, which is q into A. So, that is 3400 which we have got from the heat flux you may multiply by the area. So, 2040 here, you see you have not use sign. So, we know that heat loss heat flow that is in this site. So, that is why you have not changed the sign.

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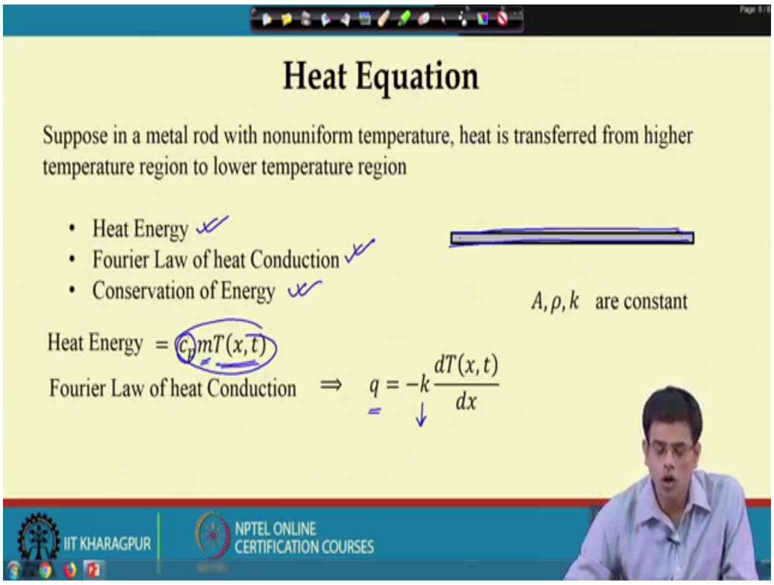

So, now, let us see, how this happens if there is a differential temperature setting. What does these differential temperature setting means in that means, I have in the beginning as I have told that if I have a simple body and then, these simple body starts which starts heating then at a position of a body for instance this is a bar and this bar I am trying to heat.

So, if I heat from here then the heat will propagate from this side to the side and then this will give rise to the temperature also. So, this is actual phenomena the real phenomena is occurring in the, physical body. So, how do you model that? We model via heat equation. So, heat equation probably you have learned it. So, here what we will do we will do how what are the principles that is required to derive this heat conduction from one side to another side and what are the laws involved in it and we will derive it for a one dimensional simple case and then you can extend it to for three dimensional case very easily. For instance suppose, a metallic rod with non uniform temperature here non with non uniform temperature heat is transferred from higher temperature region to the colder the lower temperature region.

Since then, it to derive heat equation; what are the principles we need? We need the heat energy. What is the heat energy expression and we also need the just we have learned the Fourier law of heat conduction and the conservation principle similar to the elasticity case. So, now heat energy is, can be written as the specific heat c m mass and T is the temperature. So, the energy into is the specific heat we know, the specific heat is today is the temperature of a unit mass what is the energy required that is specific heat and if mass is m. So, we can compute the energy and there is a temperature. So, this is essentially by the heat energy. And now, similar to that Fourier heat conduction I also know that q my heat flux is conductivity k and then dT dx the temperature gradient.

So, now what is left what is the conservation principle? What is the conservation of heat energy? Now, here before discussing the conservation principle we also discuss we also have to have the assumptions in invoked into it; what is the assumptions? (Refer time: 27:22) Assumptions are for instance if I start heating this bar from this point. So, these there is a insulator this side. So, these insulators will not allow me to lose the heat from the boundary of the bar.

So, these are essentially, assumptions because otherwise we cannot derive the equation because then we have to consider the radiation, all the heat transferred, all phenomena all phenomena involved into it we have to consider and then, we can consider slowly we can realize these assumptions and then consider the actual case. So, we will not discuss in detail all those actual cases here, but rather we will more discuss more simplified way off how to derive the heat equation which is very commonly used.

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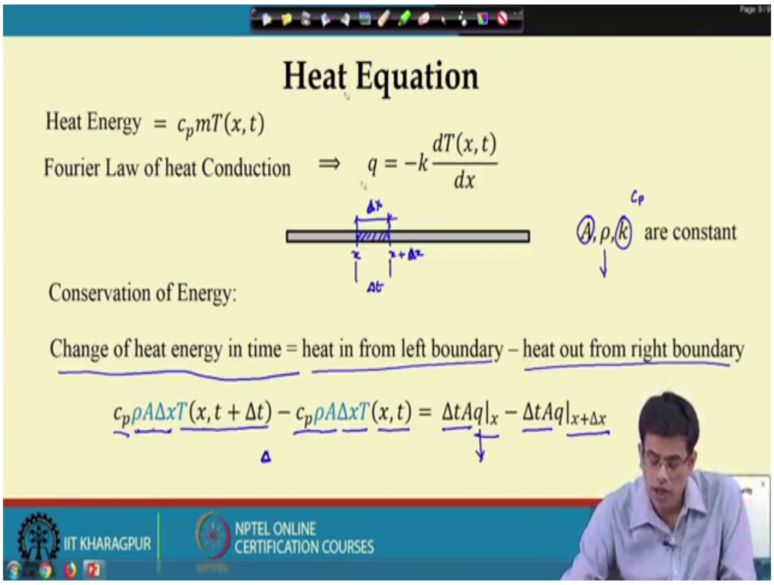

Now, the heat energy we know we know Fourier heat conduction now conservation of energy. For instance, if I now considered a point here, which is say, which is x and then a at a certain distance del x which is x plus delta x. Now, this is this is delta x. So, now, here this delta x or this is delta x. Now, what I assume based on our assumption is that temperature in these field is constant because its a very small element.

So, like we have derived all our physical laws. So, this temperature here is constant; now. So, specially, constant I mean constant means it does not depend on x. So, the what change of heat energy; What conservation principle I have to follow is it change of heat energy; that means, here in this zone what is the change of heat energy will be heat from the left boundary that is from this boundary and heat from the right boundary. So that means, heat which is entering from the left boundary and which is going through the which is going out from the right boundary. So, this is the basic conservation principle that we need to follow.

Now, similar to our previous heat equation, heat energy expression; so, what is the heat for instance the here the time lapse is delta t from here to hear the time is delta T; now. So, that means, t at x plus delta t x comma t plus delta t should be multiplied with the mass that is mass is rho A into delta x. So, what is the, this is the density and this is the length and this is the area which are all constant I assume k is also constant c is also constant or specific it is also constant. So, now so, my heat energy at time t plus delta t will be the c p rho a delta x T in t of x t plus delta t minus c p rho A delta x T A of x comma t.

Now, this will be my total increment of heat energy delta increment of the heat energy; now, which should be the heat from the left boundary and minus heat from the right boundary. So, heat from the right left boundary is delta t A into q x. So, this is over time. So, this is q x q at x that is q is a heat flux. So, total heat is A into q we know from Fourier law; so, similarly delta t A q at x plus delta x which is again from the Fourier law.

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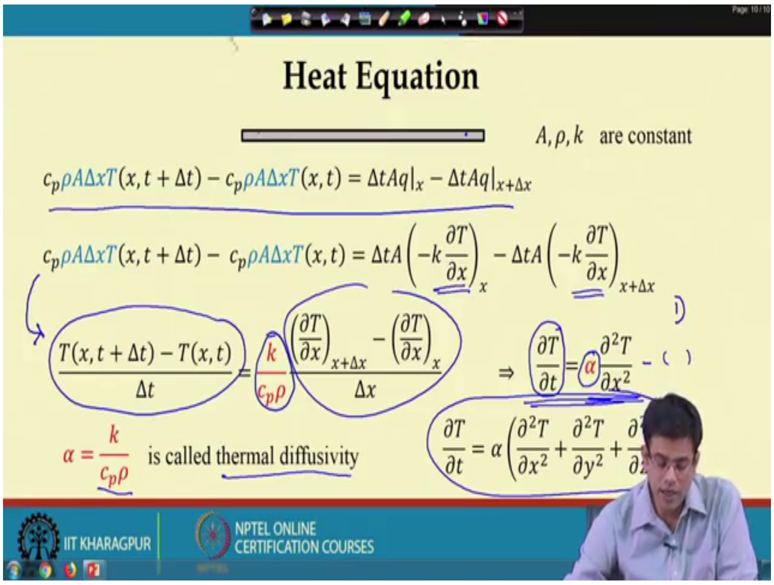

Now, if, I know expand it if I just use our differential form. Then if I after doing some modification we can get the heat equation. For instance, if I now, use this equation and then I substitute q is equals to minus k delta T delta x which is, the Fourier law and then, minus k delta T delta x at a x minus k delta T delta x at delta x plus delta x.

Now, after some manipulation we get coefficient and this is the quantity and this is the quantity I get. So, this is coming from here after some manipulation. So, just put delta t down and then delta x down here and then A is cancelled from the both side and this will give me actually, if I now make delta x tends to 0 and delta t tends to 0. Then, I can use the partials. So, this quantity becomes del T by del small t.

So, what is del T by del small t it is the change in temperature. So, now change in temperature over time; now, then this is this coefficient I am saying this is alpha which is known as the k thermal diffusivity or k by c p rho, this is known as the thermal diffusivity. Then, this is the special derivative or del square T by del x square. So, this is nothing, but the second derivative if, you know the final different the from the for difference at thing we can find out.

So, now here this is the one dimensional heat equation that we need to solve. Now, this is the governing differential equation; a differential equation is not complete unless you specify, the boundary conditions. For, instance this is a time dependent differential equation. So, you need to specify boundary condition as well as the initial condition; that means, time dependent conditions. So, this can be specified depending on your problem the specification first can be of Dirichlet boundary condition or the your temperature dependent boundary condition then, it can be flux dependent boundary condition and then, it can be mixed boundary condition also.

So, all those things we will not discuss here. So, it is just the purpose of this thing just to introduce you that there is a heat equation involved which actually capture the heat propagation from this side to this side. And, this equation actually tells you what is the temperature from one side of the body to the another side of the body. For instance if heat is propagated from here to here then what should be the temperature and this is governed by the heat equation.

Now, this can be also done extended from 3D and 3D heat equation looks like this. So, there is a the dt by dx, dt by dy. So, these terms will (Refer Time: 35:23). So, this equation this is the basic heat equation, now these can be modified for a radiation and all those things for instance if you want to use the radiation than some term will be deducted from this equation. So, all those things can be incorporated in the heat equation.

So, the basic objective of describing these things is that the heat equation is governed from governed in the material and this heat equation the purpose of heat equation is to know, from which side to from what is the temperature profile from this side to this side of the material. So because, in the elasticity when will solve the effect of thermal stress on elastic material we need to know the delta t. So, how do you find delta t; actually you need to find it from the heat equation. So, this was the objective. So, I stop here in this and the in the next class we will start the basically the general and uncoupled form of elasticity, thermo-elasticity.

Thank you.