Theory of Elasticity
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Lecture - 32
Solution of Boundary Value Problems (Contd.)

Hello everyone, this is the last class of this week. In the last week we discussed the equations we did not derive this equation the reason is the derivation of those equations are very similar to the derivation in Cartesian coordinate system, we just saw the final expression for the for the governing equations in polar coordinates system.

And then also we have seen what is the final expression, if we use the stress function approach then what is the final expression in terms of stress functions when the body force is 0. And then you also saw what is the general solution of that expression, we did not explicitly solve that equation we did not derive that solution, but the reference where the solution is given in detail that was also mentioned.

It was also mentioned that the one point that we that a segment that we made something back, that compatibility condition helps us to arrived at evening displacement field, but it may not be so it may not be guaranteed always specially when your problem becomes your domain becomes non simply connected or multiply connected domain. And these has you may find out you miss this has a very serious application that application we discuss when we when we talk about complex visual approach specially suppose you have problem where you have you have some cracks or defects in those cases your problems not from geometry. But because of the presence of defects and presence of small crack or crack like flows your domain becomes multiply connected or in the process it becomes multiply connected in that case, the uniqueness gets violated will discuss that in details later.

And then; and then what will do today is will take up one or two example in polar coordinates system and then see how the problems in polar coordinates can be solved, how the stress will can be determined, how the displacement can be obtained using airy stress function method. Will just take one or two example, but if you take any book there are many examples given many exercise given. So, please this is just for your reference and for understanding the steps, but you have to follow all the examples many examples given in the book ok.

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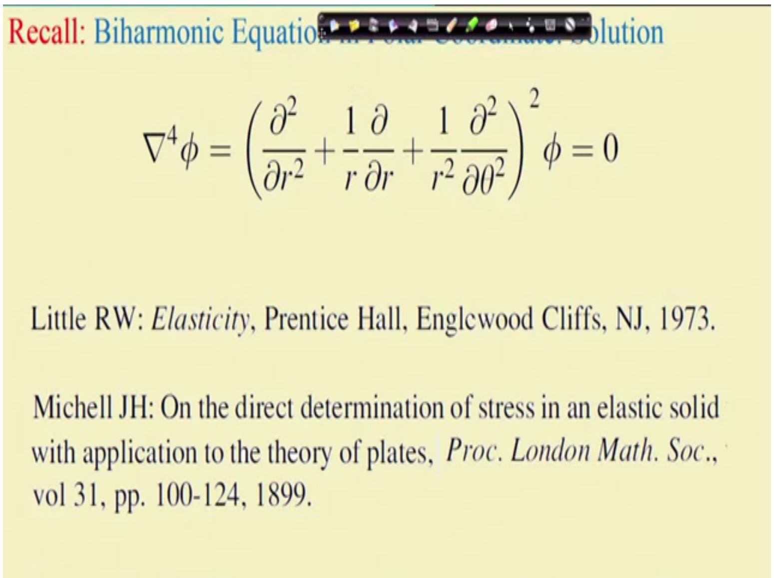

With this statement let us start recall; this was the this is the final expression biharmonic expression in the absence of body force solution of this expression is given in this two, these are the references and the general solution of this expression is this that we discussed.

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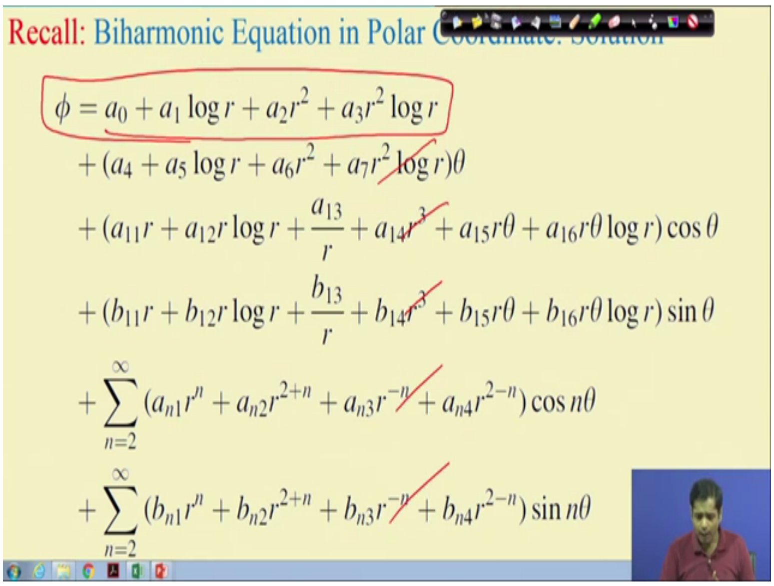

We also discuss for an axi-symmetric problem we can make all this theta dependent term in the stress function; means all this term like, all this term where we have theta explicitly we can remove them and take only this term as our stress function.

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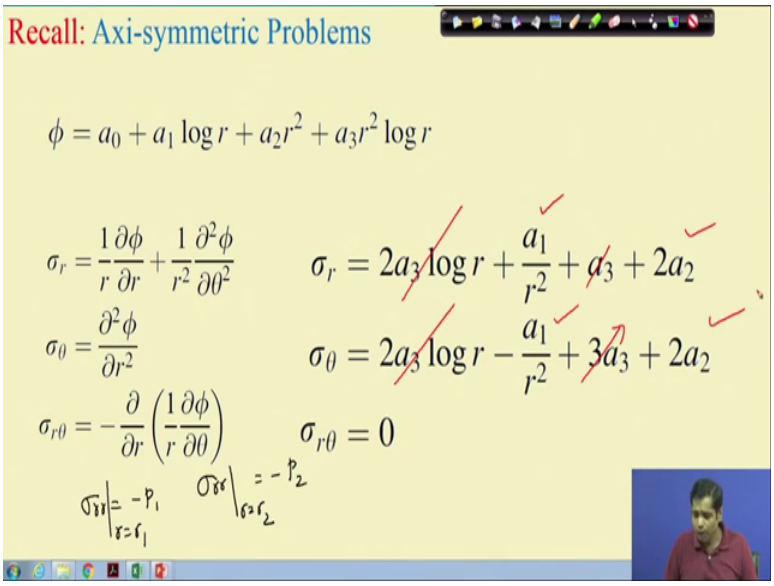

And then we also have seen that if we take this term as our stress function, then this is the definition of airy stress function and then if you substitute that definition we have expression for sigma r and sigma theta as this ok. We also seen that if your domain is non simply, domain is a simply connected domain, them then this stress function can only give you a constant stress field, you cannot get more than that if you take this as your stress function ok; now the problem so this is we know ok.

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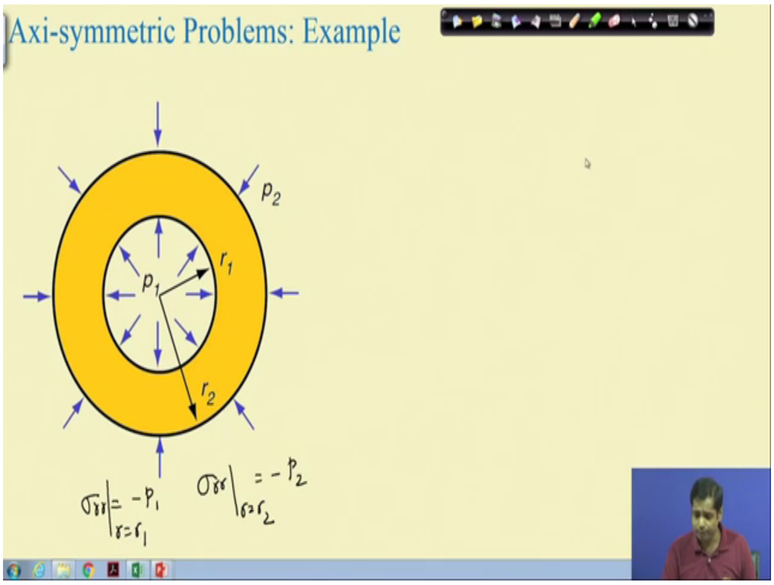

Now let us take one example, it is the non simply connected it is a multiply connected domain so therefore, that restriction we had that restriction means: only constant field can be represented through that stress function for a simply connected domain that restriction you do not have here. So, this is the problem let us complete the description of the problem so, domain is given the forces and everything is given.

So, the boundary conditions which is one of the very important part of problem description and the boundary condition if you look at this the boundary conditions are that sigma r r is equal to you write the boundary condition here; sigma r r is equal to say at r is equal to r 1 is equal to minus p 1 and sigma r r at r is equal to r 2 is equal to minus p 2 right this is the boundary conditions ok.

We can also have the boundary condition, for instance the sigma theta on sigma r theta on the surfaces r 0 so this is the boundary condition. Now, what we do is we have let us take our stress function if you go back to the previous light this is our stress function, now and corresponding sigma r is this, now suppose this log r term if you do not take the log r term do not take the log r term ok, means if you make a 3 is equal to 0. Now, we have to start with some assumptions of stress function let us take a simple assumption make a 3 is equal to 0 so this term will go so essentially we are left with this term and this term, this term and this term. So, which can be written as now I leave it to you please see, why I have ignored log r this a 3 term log r term, if I take log r term do this exercise please, if I take log r term the steps that will be following now follow the same steps with a 3 non 0 and then see what happens.

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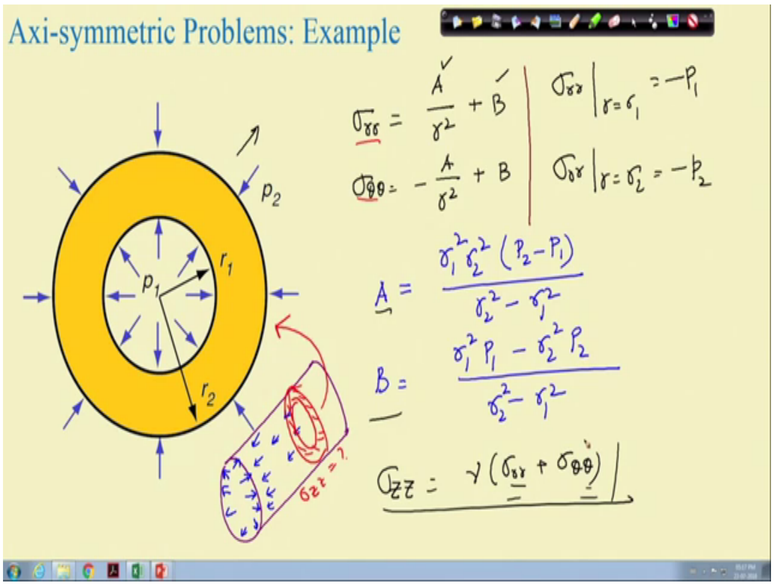

Now, so this is the problem description right, now so the boundary condition is given so if I remove the log r term what we have is sigma r is equal to say sigma r r is equal to say A by r square plus B which A is was actually A 1 and B was 2 A 2 and corresponding sigma theta theta is minus A by r square plus B, this is expression for sigma r r and sigma theta theta. And what did the boundary conditions are we have, the boundary condition we have is this that sigma r r at r is equal to r 1 minus p 1 and sigma r r at r is equal to r 2 is equal to minus p 2 that is the boundary condition, where p is minus p because this is compression had it been like this it would have been plus 2 plus p. Now, if you substitute this into this expression we have two constants here A and B we have two equations and if you substitute that; then we will get the expression for A and B as this please do this exercise, I am writing the final expression only r 1 square r 2 square into p 2 minus p 1 by please check that expression also do not take it for granted r 1 square. And then similarly we have B is equal to r 1 square p 1 minus r 2 square p 2 by r 2 square minus r 1 square so, this is the expression of A and B we have by if we apply the these boundary condition.

Now next is now: here one important point consider it is a planes let me first give you physical implication of this problem, this problem is essentially suppose you have a cylinder like this, you have a cylinder vessel like this and on the cylindrical vessel: we have some internal pressure throughout the length and also suppose we have some external pressure from outside ok, this is also over the entire surface, over the entire surface ok.

Now, this is a plane strain problem this dimension is very large so we can do the analysis by taking just only one plane and if you take a plane, just take a plane this plane is something like this and this is this problem ok. We have internal pressure and then external pressure, then we will see what happens when we do not have is a very general case you might have done cylindrical pressure vessel in your in solid mechanical strength of material, where we have the only the internal pressure, no external pressure in that case you can make p 2 is equal to 0 it is a very general case.

So, this is a plane strain problem, now plane strain problem means then we can have stress in z direction the sigma z z there will be some sigma z z along z direction and that sigma z z can be obtained by sigma r r and sigma theta theta. If you recall in that case sigma z z will be sigma z z will be nu into sigma r r plus sigma theta theta that is how ah; for plane strain problem the other component of stress can be obtained ok.

So, next is we substitute A and B into this equation get expression for sigma r r and sigma theta theta then substitute sigma r r and sigma theta theta here you get an expression for expression for sigma z ok.

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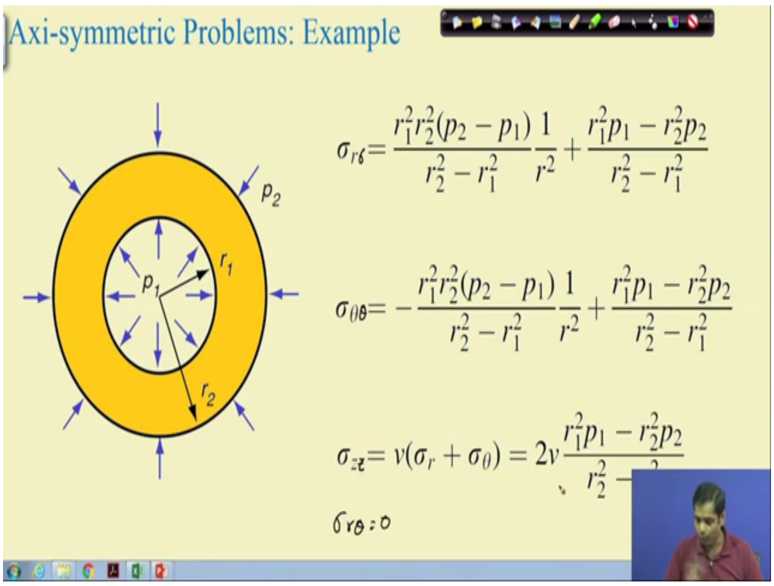

Now if you do that: what we have final expression of sigma r r these are all sigma r r is correct it sigma r r sigma theta theta and then sigma z z will be equal to, this sigma r theta will be 0; sigma r theta will be 0 ok, this is the expression for different stress components ok.

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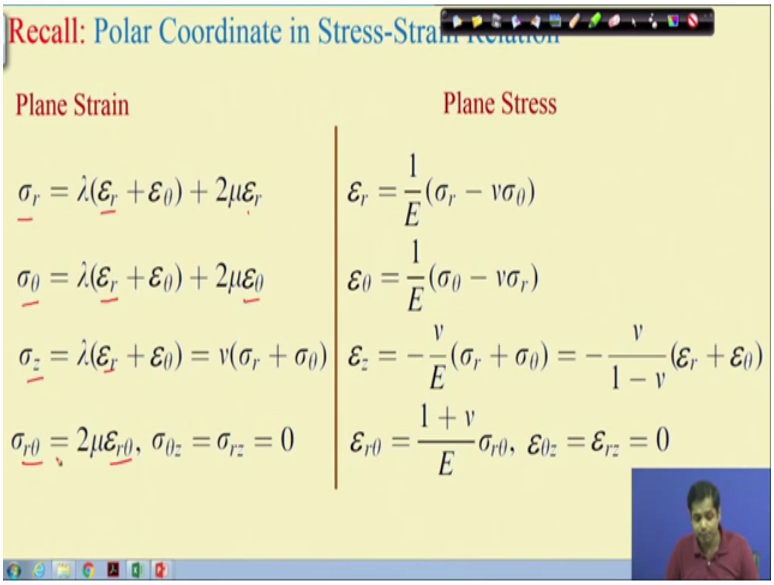

Now, so once we have the stress components now recall for plane strain problems so these are the stress components we have, now you can substitute this strain comp stress components here and get an expression for epsilon r r epsilon this substitute sigma theta, sigma r in this expression and get expression for sigma r r and sigma r theta, all this epsilon r r epsilon theta theta we can obtain.

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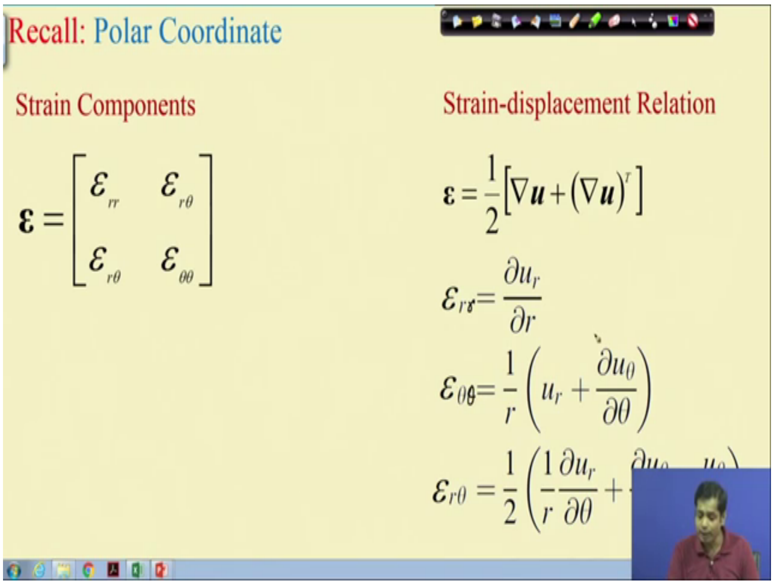

Once we have the strain component; then we know the strain displacement relation these are the strain displacement relation so, we know sigma r r now from the previous slide sigma r r known sigma theta theta known. Then you have to integrate it sigma r as epsilon r r epsilon theta theta and epsilon r theta, if you integrate it then we get displacement in that integration will have some constant that constant need to be determined based on the boundary conditions, what are the displacement boundary conditions given for this problem ok.

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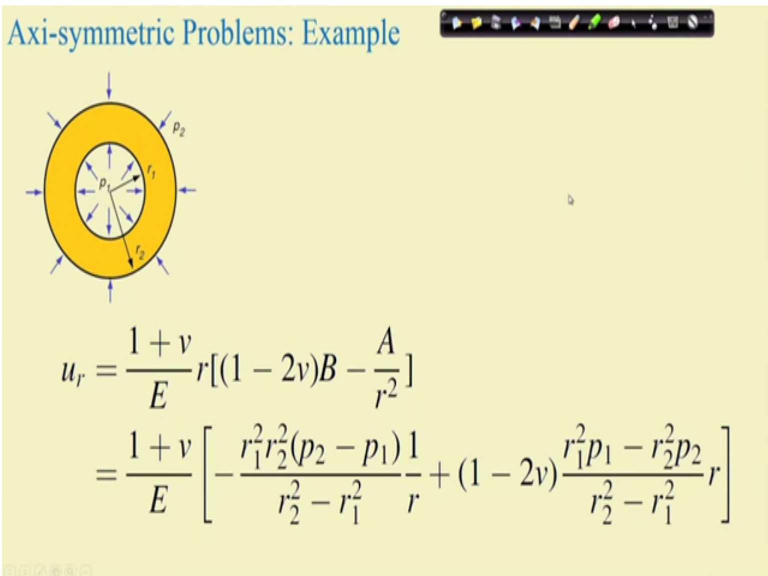

Now if you do that exercise; then the expression for u r will get, this is the expression for u r please do this exercise. Now, so this is the complete solution of this problem where is A and B are some are the constants ok.

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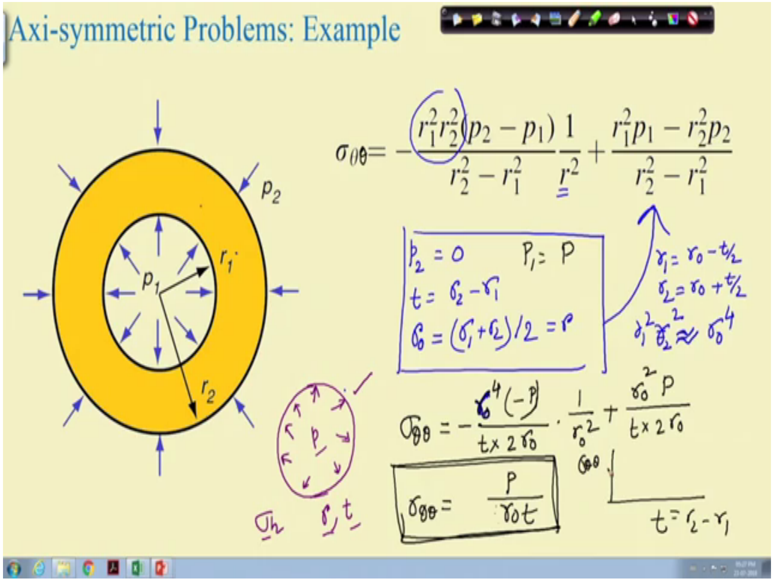

Now, so before moving on let us if you recall in the introduction to this course and also on various vocations it was mentioned you see whenever you use any theory every theory is based on certain assumption and assumptions are very important because assumptions are limitations for any theory.

You recall you might have you have studied cylindrical pressure vessels in your strength of material course or in solid mechanics course and you obtain some result for hoop stress right. Now this is also pressure vessels, but in this case it is we have not when you obtain that the vessels for hoop stress you make some assumptions like a let's say thickness is very small. Therefore it can be considered as thin wall pressure vessel, whether it is thick wall pressure vessel or thin wall pressure vessels so we had to assume before we started deriving the solution that we did in strength of material or solid mechanics.

I will just to that the very standard well known solution for thin wall pressure vessel that you are obtained instead the material course or in first year mechanics course, but here we have not made such assumption, here we have not made whether the we have not made any assumption whether the pressure vessel is thin or thick we make some assumptions, but that assumptions are different assumptions, linearity assumption stress and relation is how this stress and relation is those assumptions may be in isotropic those are the assumptions, but as far as the geometry of the problem is concerned, thickness of this in this case concerned we have not made such assumption, but these assumptions were made in the solution while obtain the solution in strength of material and mechanics course.

Now, suppose so this is the expression for hoop stress right we obtain from the elasticity theory is this sigma theta theta, in the previous slide if we recall this is the expression for hoop stress ok. Now let's see if I have to find out so this is a very general expression very general expression means this expression is applicable for thick walled and it should give me the result at the limiting case with the result for thin wall pressure vessel as well for instance.

You recall the problem this problem: we you have a pressure vessel very thin pressure vessels which is subjected to some internal pressure say p p and suppose the radius of the pressure vessel is r and thickness is t so, we and the hoop stress is say sigma hoop stress is this. So, we had we drive some relation between hoop stress r t and p ok, now this is the expression for sigma theta theta.

Now, suppose if the pressure is if this problem if I even if I consider this problem with this problem, as a limiting case of this problem, then what happens then we have we have p 2 is equal to 0 and then we also have t is equal to r 2 minus r 1. And we also have since, t is very small it is thin wall vessel then r 1 and r 2 is also very close to each other and therefore, the average r 0 is equal to r 1 plus r 2 divided by 2 which is r here so this is equal to r. So, if I in the limiting case, if I have to consider this case the limiting case this we have to impose this condition on this expression ok.

Now, let us substitute this let us do substitute this condition now, then sigma theta theta becomes what? Then sigma theta theta become; sigma theta theta becomes, minus you see if r 1 is equal to what r 1 is equal to r 1 is equal to in this case it will be r 0 minus t by 2 and r 2 will be r 0 plus t by 2. Now, similarly then r 1 square r 1 square there will be a quad non-linear term of t r 1 square and r 2 square if we take there will be non-linear term of t, since t is very small we can remove this non-linear term and we will see that this will all this will be almost r 0 to the power 4 ok.

So, this term will be r 0 to the power 4, in the limiting case when t is very small and then then sigma theta theta becomes r 0 to the power 4 r 0 to the power r 0 to the power 4 and then p 2 is equal to 0 this is minus p 1 suppose p 1 is equal to p 1 is equal to say P so, this is p. And then this becomes 1 by r 0 square and r 2 minus this become r 2 minus r 1 into r 2 plus r 1 r 2 minus r 1 becomes t and r 2 plus r 1 becomes 2 into r 0 ok.

Then plus this term P 2 is equal to 0 so, this become r 0 square this is P and this becomes t into 2 r 0 so, this becomes essentially finally, it becomes P by 2 then we have r 0 into t sigma theta theta. So, now, there will be no 2 there will be no t 2 term, because 2 will be this is the final expression r 0 into 2. So, you recall in strength of material are in your mechanics first year mechanics for thin wall pressure vessel you obtain the expression as this and we just know so these expression can be obtained from this expression as a special case. So, next what you can do is you can try one thing, you plot the you plot take a plot where you plot t here thickness which r 2 minus r 0 r 2 minus r 1 and then you plot sigma theta theta here and then see how the sigma theta theta; how sigma theta theta varies with thickness. And then you will see that when thickness is very small, then this will give you the result from the that is also you can show graphically results that you are obtained from obtained in mechanics first year mechanics of strength of material.

So, this is this is the beauty of this formulation that we do not have this is more general and with this as we apply this theory for say beam bending problem and also for torsion problem, then will also see the some of the assumptions that we that we that actually used while driving the methodology in our in your mechanics course or strength of material course and those assumption we do not have. And therefore we can get a solution which is more general and that solution give you bit more insight into the actual into the behavior of this behavior of the different problems.

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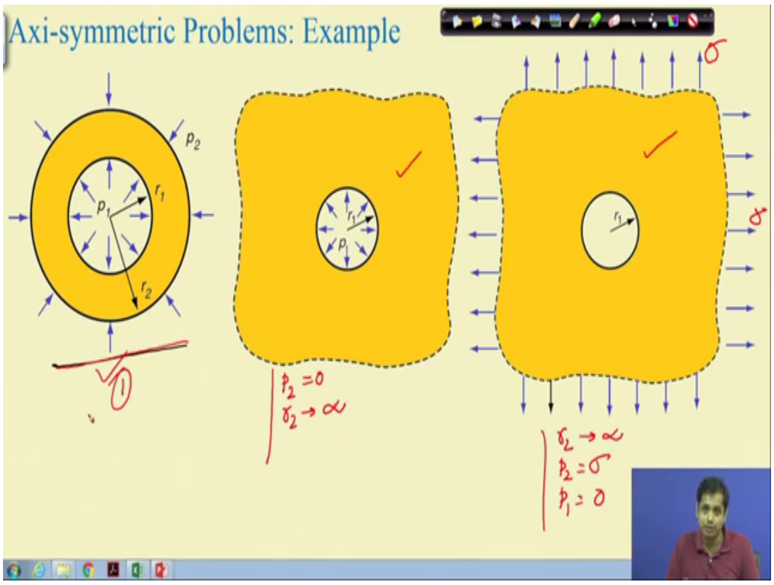

So, now, with this so next is suppose so, we have a solution for this problem; we have a solution for this problem, this problem just we have solve it now next suppose we have a problem like this: where we have an infinite plate, plate of infinite dimension and then we have a small hole which where you have some internal pressure. So, with this solution; solution from this problem we can obtain solution from this problem as well. We see one extreme case we have seen that the thin wall pressure vessel can be considered as a special case of this problem this also can be considered as a special case of this problem and in the in order to use the solution of this problem for this what you have to do is, we have to use it that p 2 is equal to 0; p 2 is equal to 0 and r 2 tends to infinity if this is infinite isn't it.

Now we can get a solution from this problem as well in that case, what will happen that in that case r 2 tends to infinite and then p 2 is equal to sum values say sigma sigma and p 1 is equal to 0. So, if we have a problem like this and then with the solution of this problem we can obtain the solution of this and this as a special case of this problem by substituting corresponding this.

So, what you can do is you have to do one exercise you first exercise is you find out the solution of the first problem this problem and if you know any if you know if you are familiar with finite element method or any other numerical methods you can find out the solution of all if you are familiar with any software see abacus or ansys or any other software you can model this you can get the solution of the same problem.

And then check the solution that you are obtained from elasticity theory and solution you obtained from numerical methods; is that are they same under all circumstances when irrespective of the thickness, irrespective of the forces is any discrepancy, if there is a discrepancy between the solution of elasticity theory and the solution finite element method that will give you an opportunity to understand what are the limitations of this solution ok, because this solution is also based on certain assumption.

Similarly, you can do all this exercise, not only this throughout this course whatever exercise problem, whatever example that we that we solve here that we all the examples for which the elasticity solutions are obtained in this course I suggest you to model this using finite element or any other numerical methods used in a software or we can write your own code. And find out the solution and compare the elasticity solution with the solution you get from those methods and you try yourself, find out yourself what are the reason for the discrepancy if there is any discrepancy at all ok.

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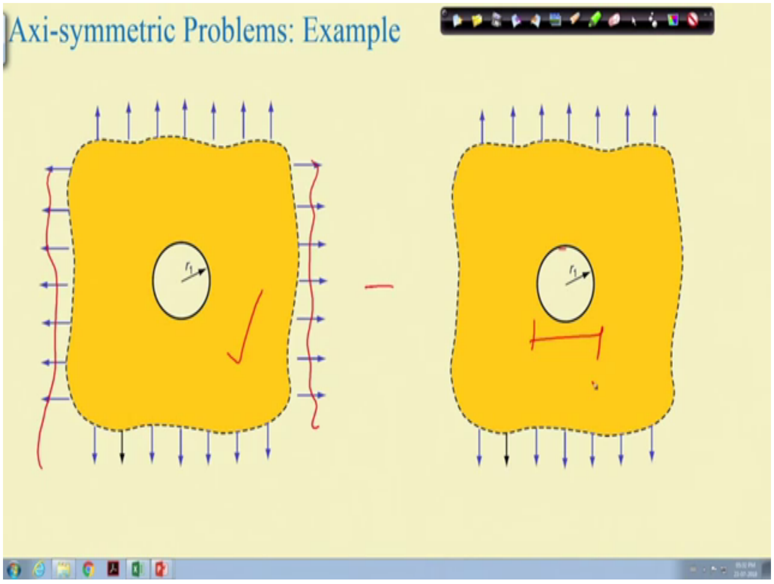

Now, before closing this before you stop today; this we have now have seen how to get the solution of this problem right, now suppose if you remove these forces remove this stress now, we have just a stress infinite plane, plate with infinite dimension and; we have a hole of the centre or anywhere and then this plane this plate is subjected to some uniaxial loading, uniaxial tension like this.

See whenever I say that infinite plate it does not literally mean infinite it means the dimension of the plate is very large as compared to the dimension of this hole. We will not find out the solution of this problem today, but the reason why I am stopping today's class with this problem is will come to this problem once again when we talk about complex variable approach. But the reason we stop in the next class onwards will be applying the airy stress function using for beam for beam problem, for torsion problem, this problem is important in many reason, the reason is suppose considered a problem like this.

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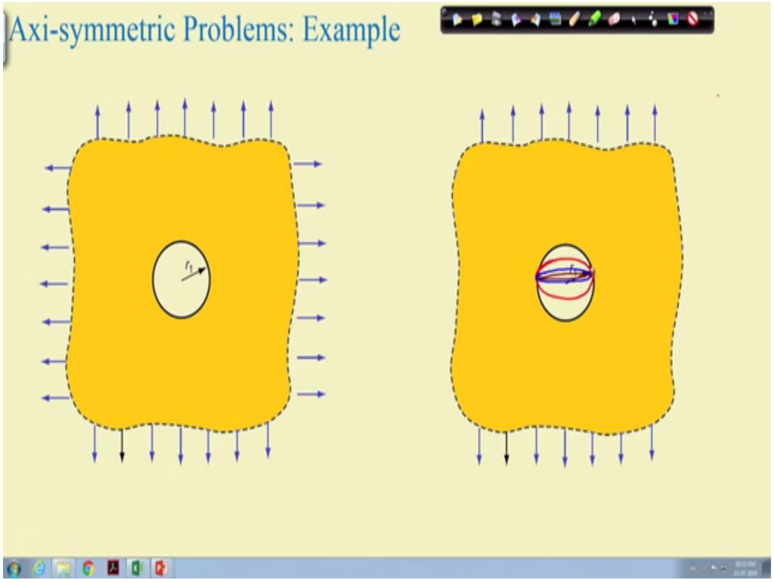

Just before stopping take the same problem suppose this problem is an infinite plate with a circular hole. Suppose this hole is not circular hole, what happens if suppose this hole is an elliptical hole. So, for a circular hole we will solve find out the solution, but suppose that hole is not circular hole is elliptical hole or suppose hole is elliptical, but then the minor axis is very small, minor axis is very small or even further smaller: even further smaller or consider a limiting case, where this minor axis becomes so small that it the hole becomes just a line ok.

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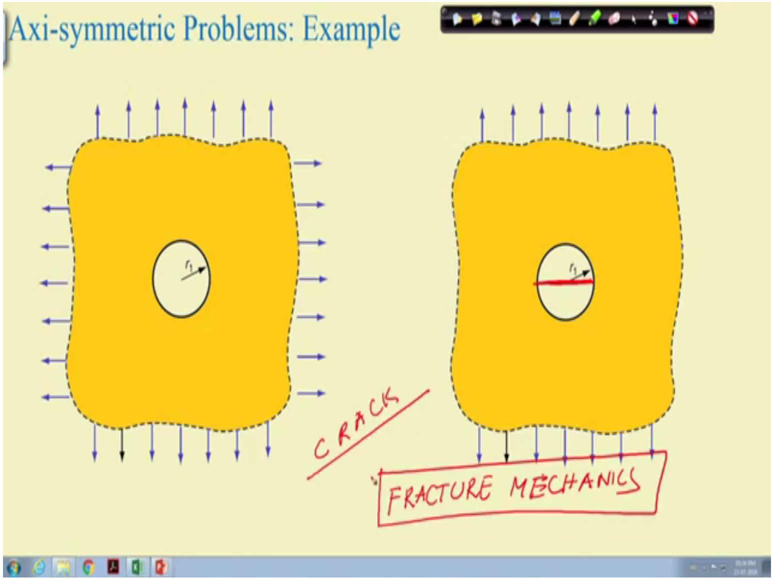

And this line corresponds to crack or defects or material flow, you see this problem we will solve when we talk about complex variable approach, a general solution will try to find out and we will see when this circle becomes ellipse then what happens to the solution, what happens to the stress field. We will also see if that minor axis of the ellipse continuously reduces and then what happens how these stress field continuously changes and the limiting case and the extreme case, when that limit becomes a crack becomes when the limit case when an ellipse becomes a line minor axis is so small then what happens to the stress field and that stress field will give us an opportunity to learn or to introduce rather a new subject called fracture mechanics ok.

We will have a very very brief introduction to fracture mechanics, linear elastic fracture mechanics, because you were talking about elasticity problem and that is the reason I am stopping today with these example then, I will stop today next class will apply these airy stress function for beams; see in the next class.

Thank you.