Structural Analysis 1
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Lecture 18
Deflection of Beams and Frames (Contd.)

Welcome, what we discussed in the last class is we derived the equation of elastic curve which is essentially the mathematical representation of beam undergoing deformation. Elastic beam undergoing deformation and the deformation is small. That was the assumption, that model. Now once the mathematical model is derived our next step is to solve those equations, right?

What we will going to do today is we will going to show you one way of solving those equation that is direct integration method. Now before we do that let me show you something, okay. Suppose this is a piece of paper, consider this is beam. This is the length of the beam.

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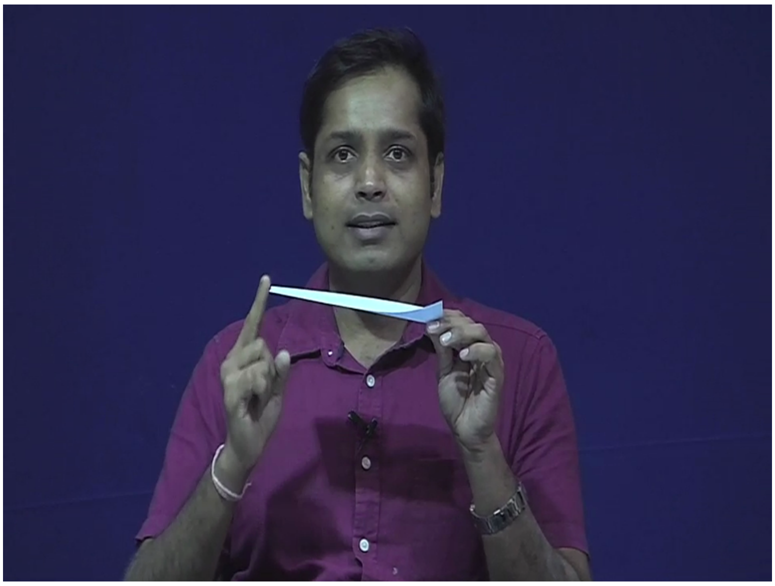

This is the cross section of the beam.

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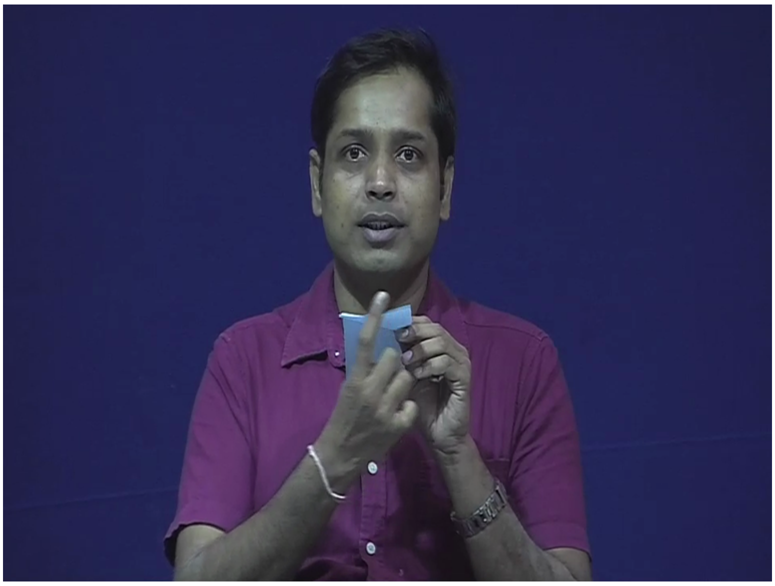

The thickness is very small. Now what I will do is I will put that beam on a support and apply some load on it. Let us see what happens to that, okay. Suppose these are two supports, right? Now these are two supports, okay. Now put this beam on this.

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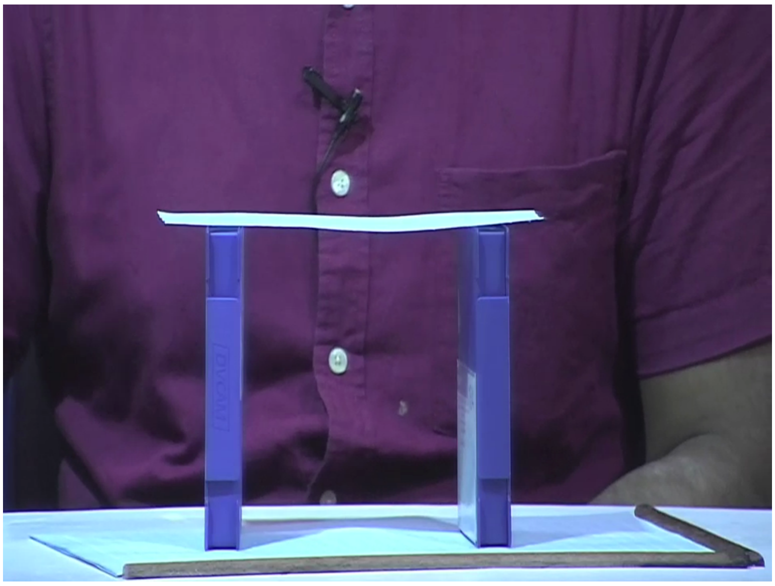

This is my mobile which is not very light. When I will keep my mobile here so essentially the self weight of the mobile, that will be exerted on the beam as uniformly distributed load.

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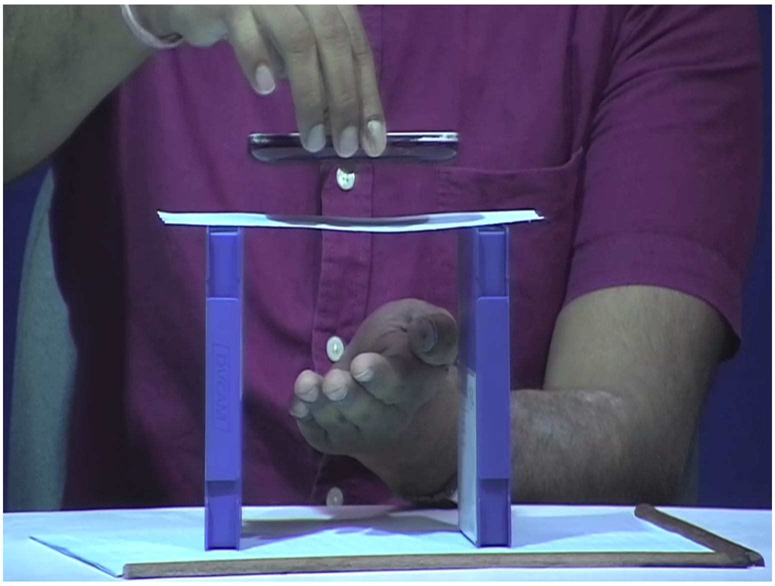

So I keep it and it fell down, right?

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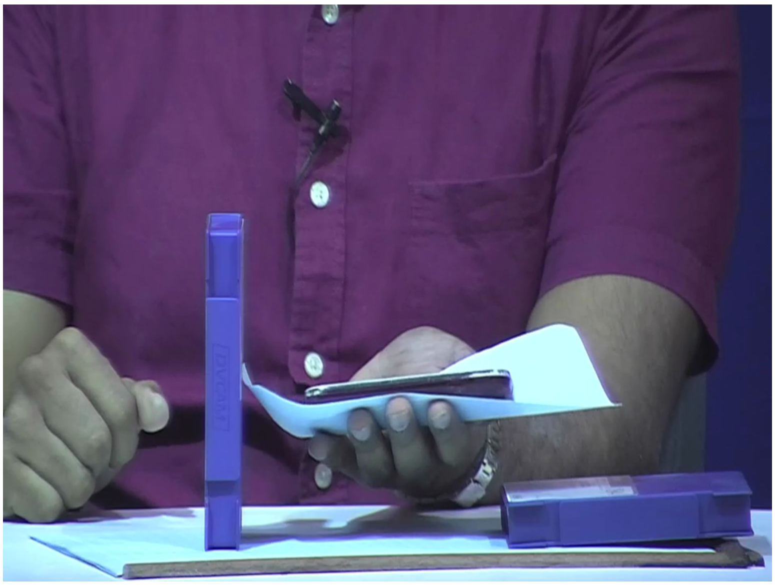

Next what I do is take the same beam but now it is folded like, okay. But it is same length, same dimension but only thing is it is folded like this.

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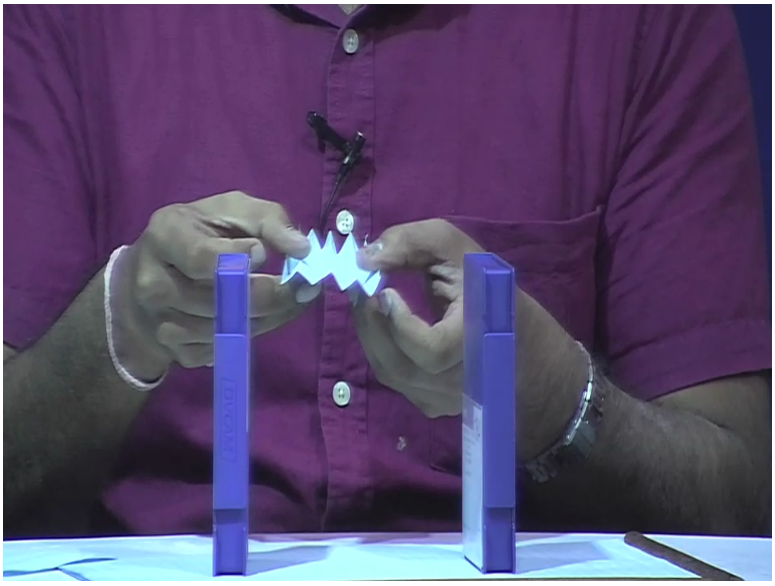

Let us keep it here, okay.

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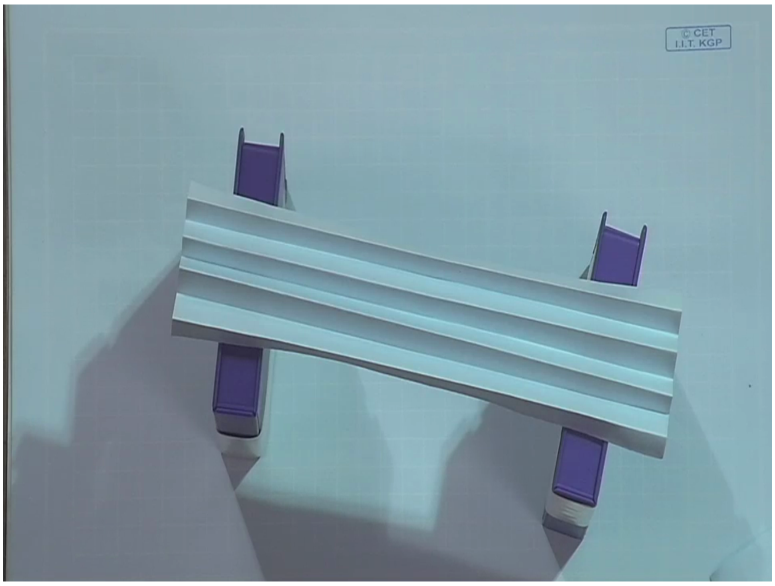

And then I keep my mobile on this. You see it is perfectly stable. No problem at all. Now it is the same beam, same piece of paper when it is used as it is then we found it was not stable but now when it is folded we found it is now enough to take care of the load exerted by the mobile, okay.

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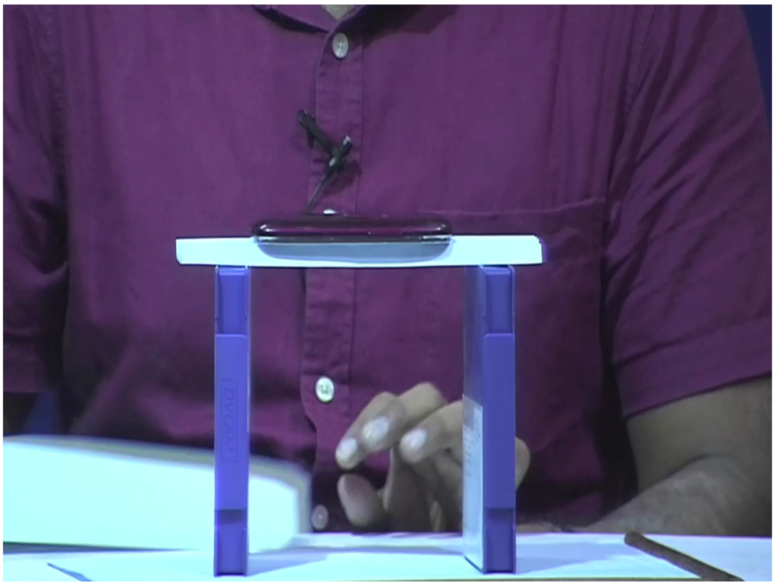

Now I am not going to tell you the reason behind this. By the end of this lecture you should be able to explain this phenomena, okay. Now let us take one example. This is one example, okay. Example that just now we shown. That is a simply supported beam. This was a simply supported beam and it is subjected to uniformly distributed load q from the self weight of the mobile and these are two supports, okay.

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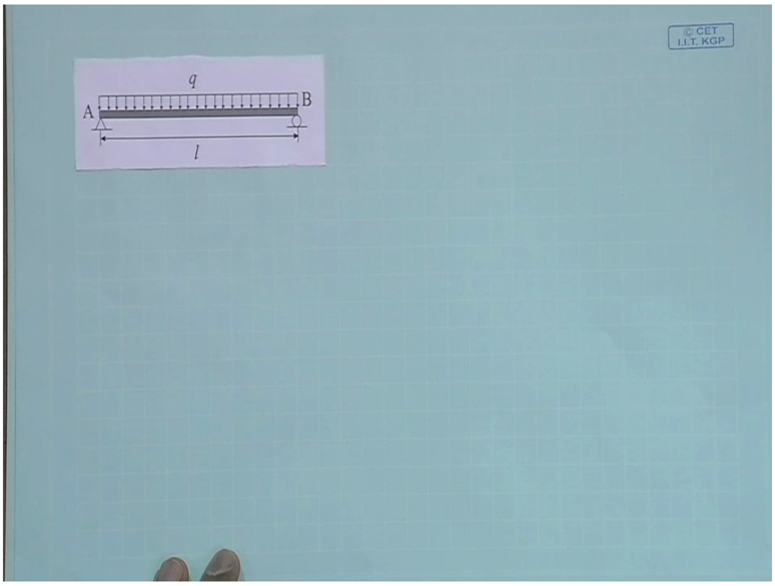

What we need to find out here is we need to determine what is the deflection of the beam at any distance x from support A using equation of elastic curve. The equation what we have is d2 minus M by EI. Since M is the function of x better write Mx as a function of x, okay. So curvature is equal to moment at distance x divided by Youngs modulus and divided by second moment of area, right?

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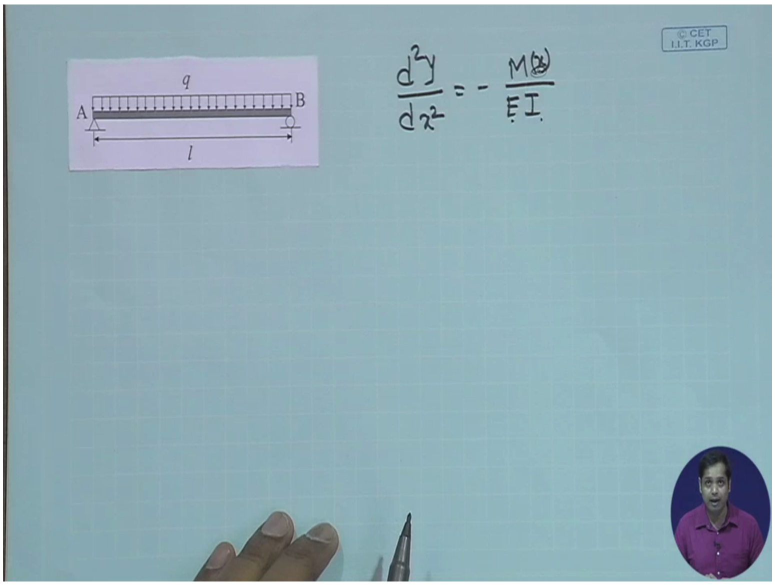

Okay, so first determine the support reaction and we have seen this problem many times in this course. So if we draw the free body diagram of the entire structure and apply the equilibrium condition of this then the (bound) support reactions will be Ax that becomes zero and Ay is equal to By is equal to q into L by 2. We know how to determine this.

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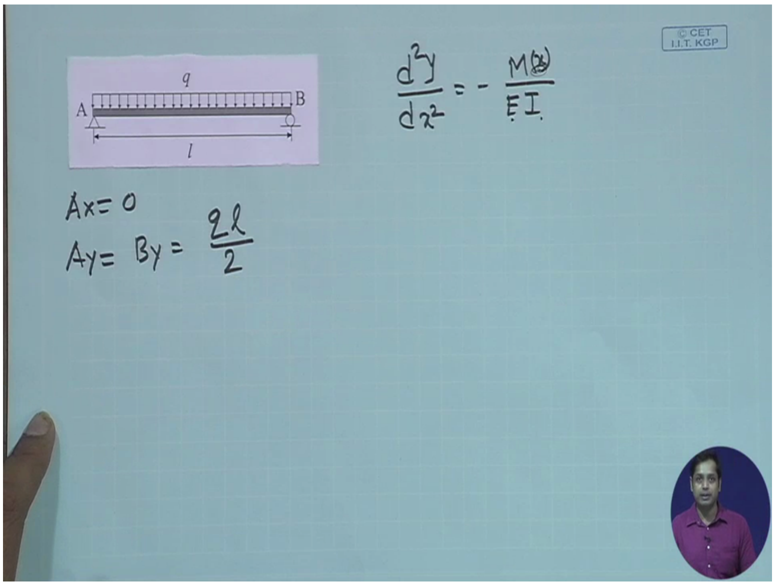

So I am not going to do it again here. Take a section here. Say 1-1 which is at a distance x from the support. And if we draw a free body diagram of this what we have? This is Ay, Ax is equal to zero so not writing explicitly. This is q, this distance is x and then this is moment x and this is shear force Vx. This is a free body diagram of this part, okay.

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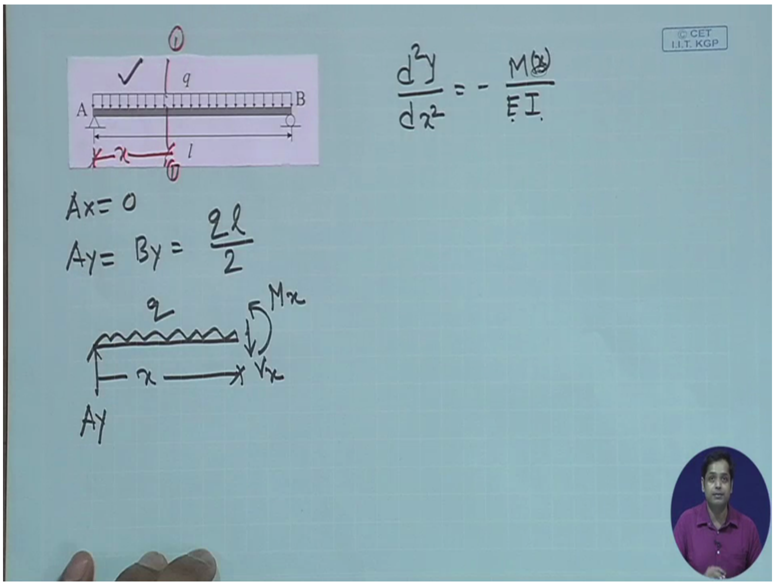

Now if we take moment about this point then what? Summation of Mx is equal to zero.

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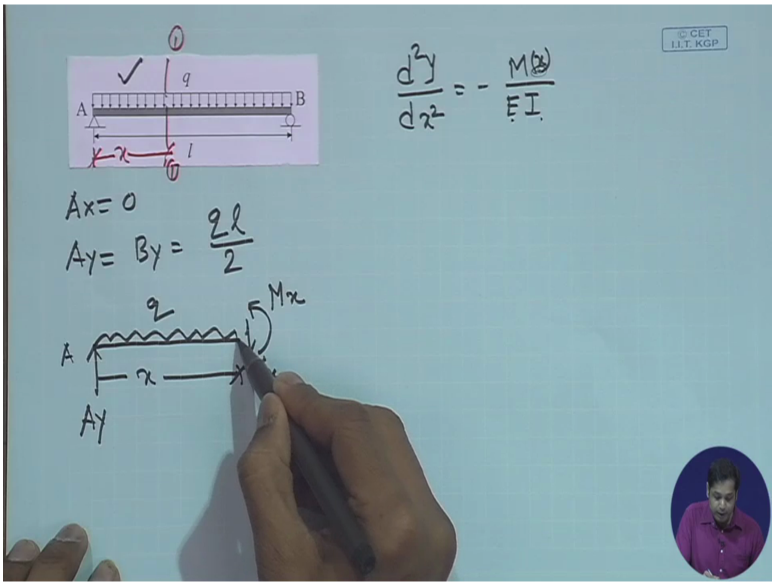

And this means this gives us Ay into x. Ay is clockwise and then minus qx which is the total load, into x by 2 that is equal to zero. And this becomes Ay is equal to qL by 2 plus minus

Mx. Mx is anticlockwise that is why it is minus. So this gives us Mx is equal to qL by 2 x minus q x square by 2. So that is how the moment changes with x. We also have derived this in the first week when we reviewed bending moment and shear force diagram. So this is Mx, okay.

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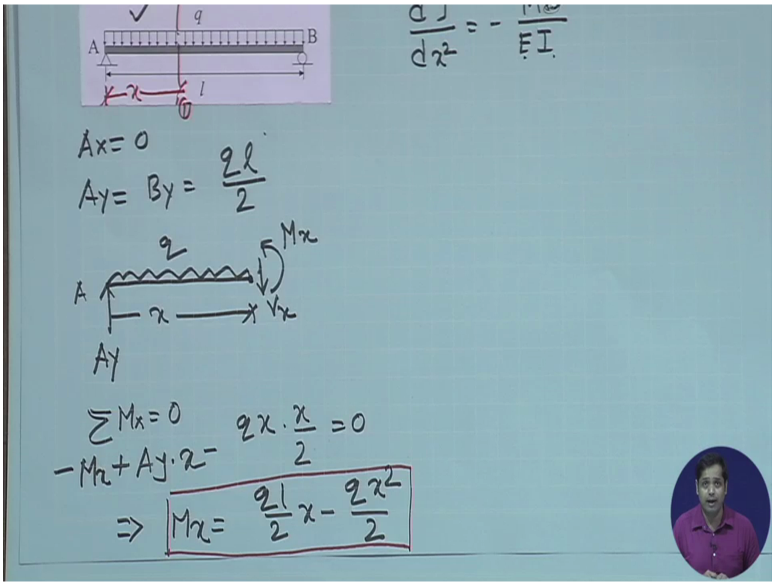

Now we assume in this problem that Youngs modulus is constant and the cross section of the beam is uniform across the length. So E and I both are constants. So only thing which is function of x is the moment, okay.

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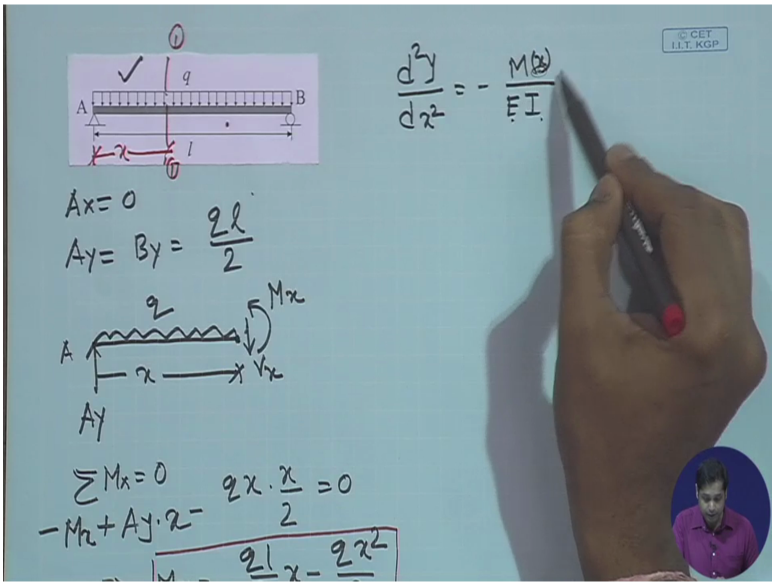

Now what we will do is this differential equation we will directly integrate. So if you substitute Mx here what we have is this. So d2y dx2 is equal to or you can write EI here, minus Mx. Minus Mx becomes minus qL by 2 x plus qx square by 2, okay.

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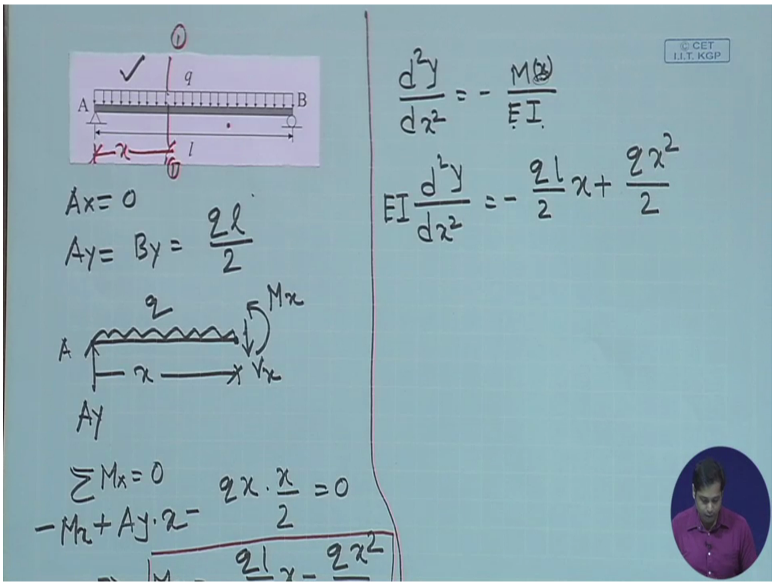

So if we integrate it then what we have is EI dy dx is equal to integration of entire part, qL q are all constants. So this becomes minus qL by 2 x square by 2 plus q by 2 into x cube by 3 and then plus constant C1 which is integration constant, okay.

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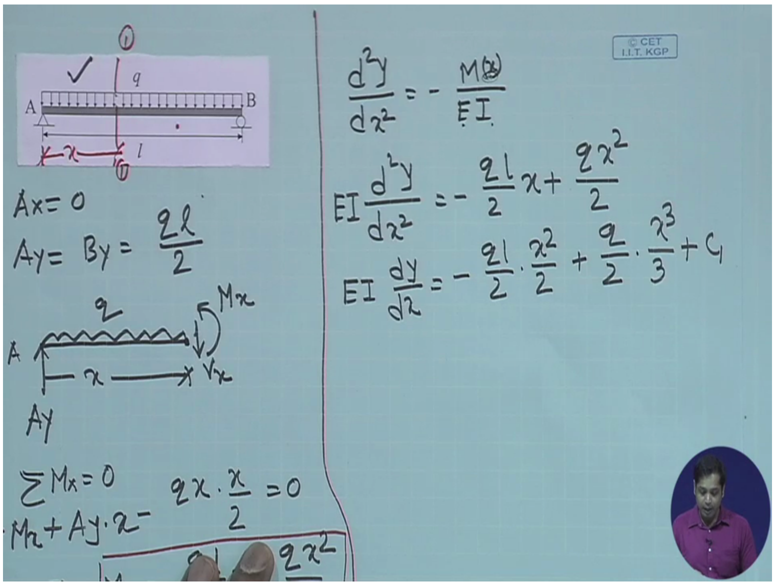

Now again if we differentiate, EI first time and the second time if we differentiate then EIy, y is the function of x. I am not writing explicitly in bracket. So this becomes minus qL by 4 x cube by 3 plus q by 6 into x 4 by 4 plus C1x plus C2. Again C2 are the another constant. C1x into C2, okay. Or let us write here C1x plus C2, okay.

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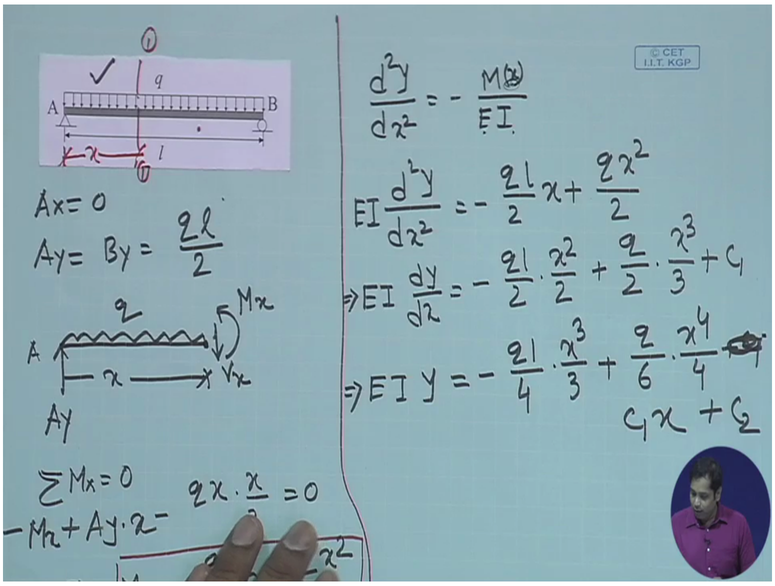

Now this is one constant and this is another constant. How to determine this constant? We need to apply boundary conditions to determine this constant, okay. Now here one point please note if you remember again in the first week we discussed when we were talking about the physical process then idealization and the mathematical model for the idealization, I mentioned there that (mat) when I say the mathematical model it means the equation along with the conditions that the variable should satisfy.

For instance this is the equation proof for any problem any beam transversely you can beam, this equation is true.

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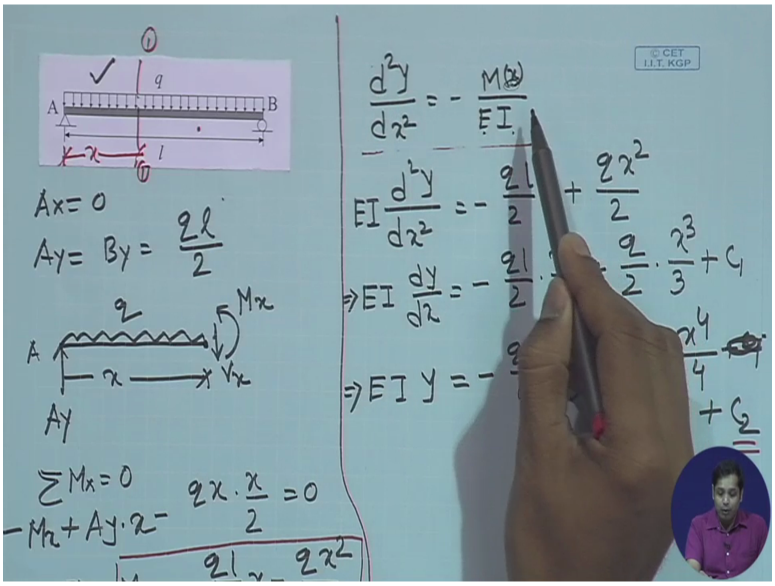

Now when you solve this equation, this is very general equation when you solve. But for this particular problem in order to get the unique solution we need to give some constraint. The constraint is that this boundary conditions will help us to determine these constants C1 and C2, okay. Otherwise if you do not have these boundary conditions then we cannot determine the constants C1 and C2. So we cannot have a unique solution, right?

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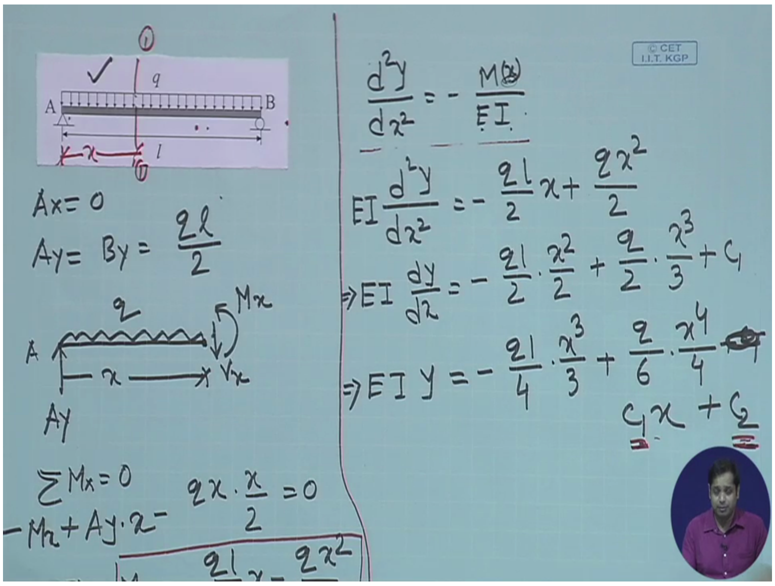

Now let us see what are the boundary conditions we have? This point is pin support and this is roller support.

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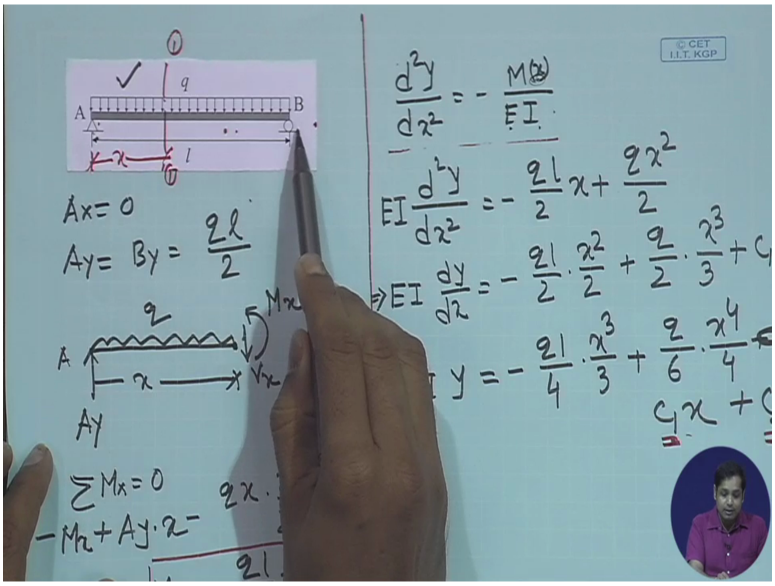

So slope is not zero here. Here also slope is not zero. We have y is equal to zero at this point and y is equal to zero at B.

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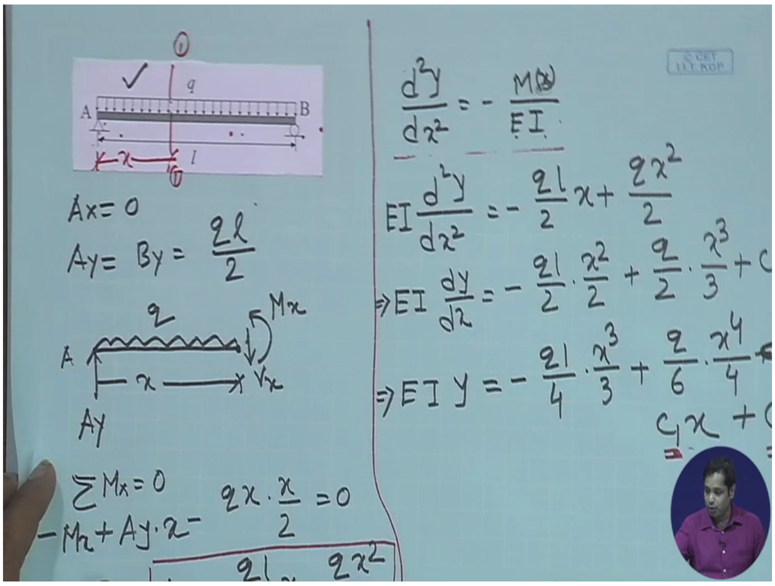

So boundary conditions for this problem we have y at zero is equal to zero and y at L is equal to zero. So what we will do is we will apply this boundary condition in this equation to determine these constants, okay?

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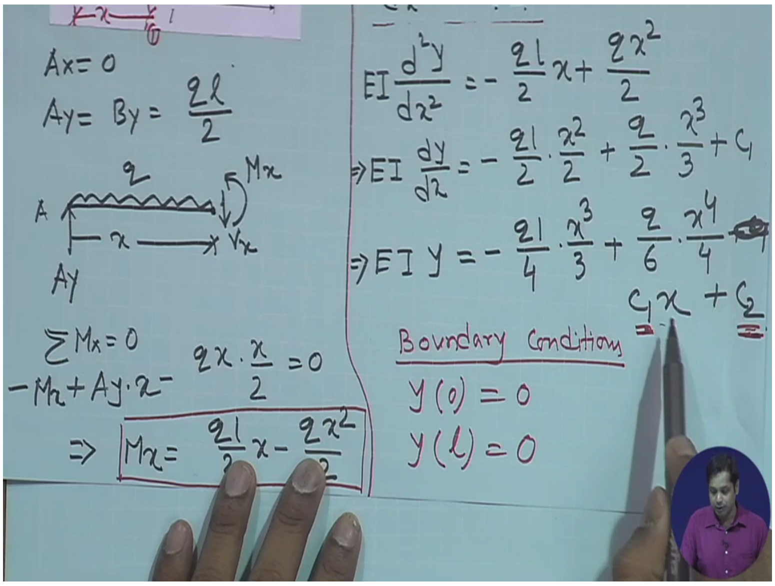

Let us do that, okay. Now the equation was EIy, okay. Now first boundary condition is y at x is equal to zero, is equal to zero. If you substitute x is equal to zero here then what we get is C2 is equal to zero directly, okay.

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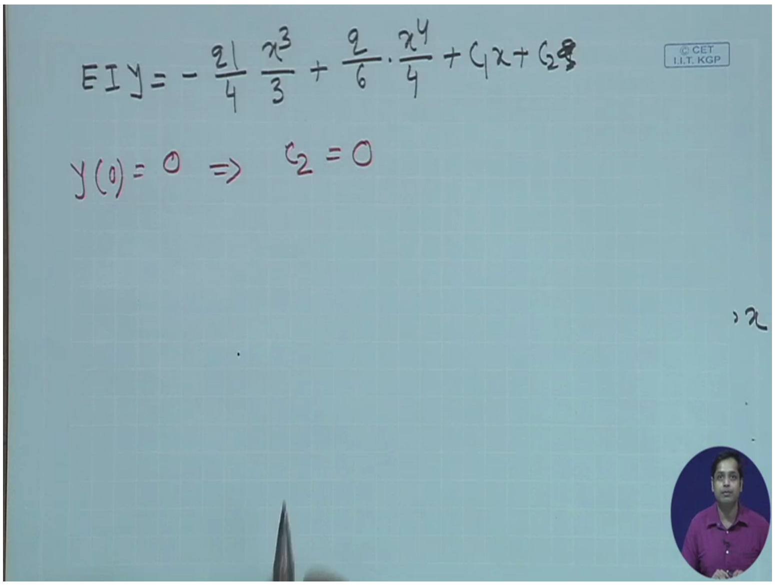

Now another boundary condition is at x is equal to L,* y at L is equal to zero. Means that x is equal to L, y is equal to zero. If you substitute this then this gives us the left hand side become zero that is equal to minus qL by 12 L cube plus q by 24 L to the power 4 plus C1 into L is equal to zero, okay.

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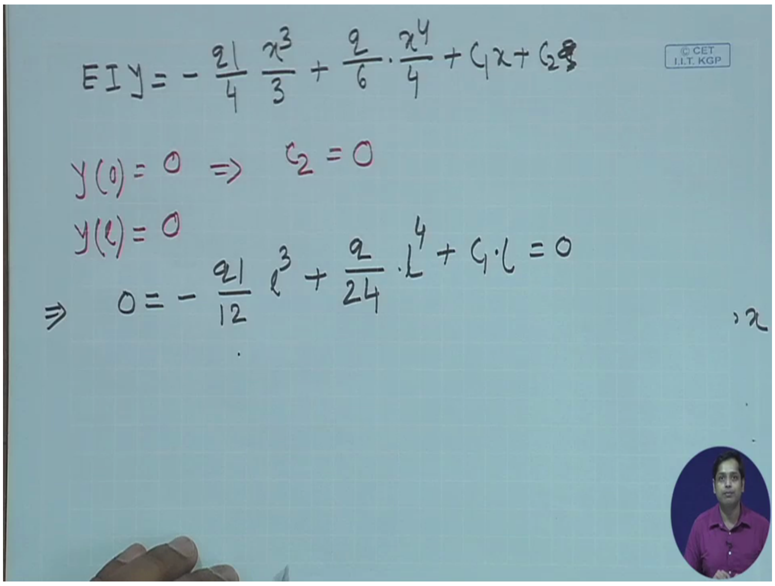

And then only unknown is C1 here. If you calculate C1 then C1 we will get qL cube by 24. C1 is equal to this, okay.

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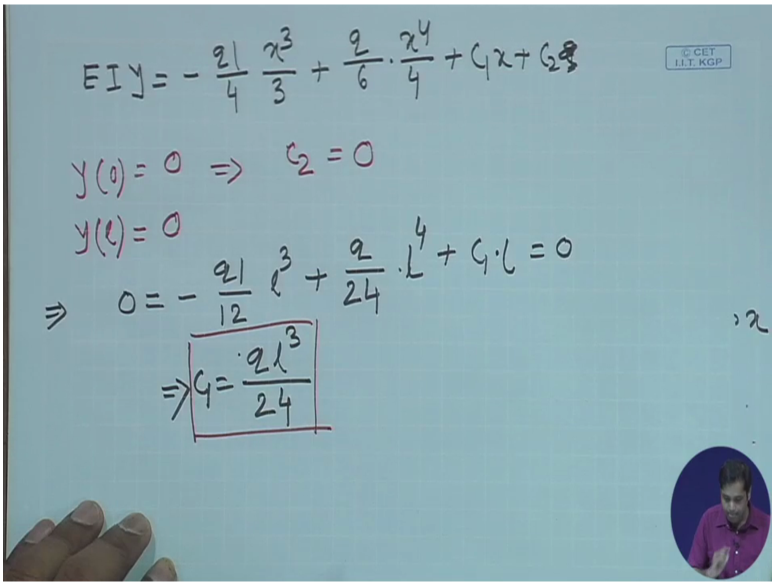

Now in this equation C2 is equal to zero and C1 is equal to this. Now substitute C1 in this equation. If you do that what we get is this. If you substitute that then we will get y is equal to, let us by EI here. So what we get is 1 by EI into 2 qL cube by 24 x. This is this term.

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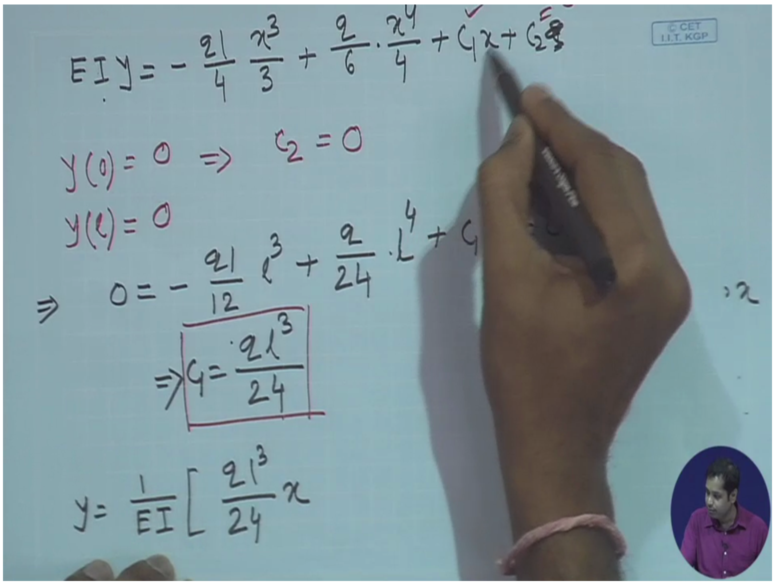

And then minus qL by 12 x cube. This is this term and then plus q by 24 x to the power 4. I have written in the increasing order of x, okay.

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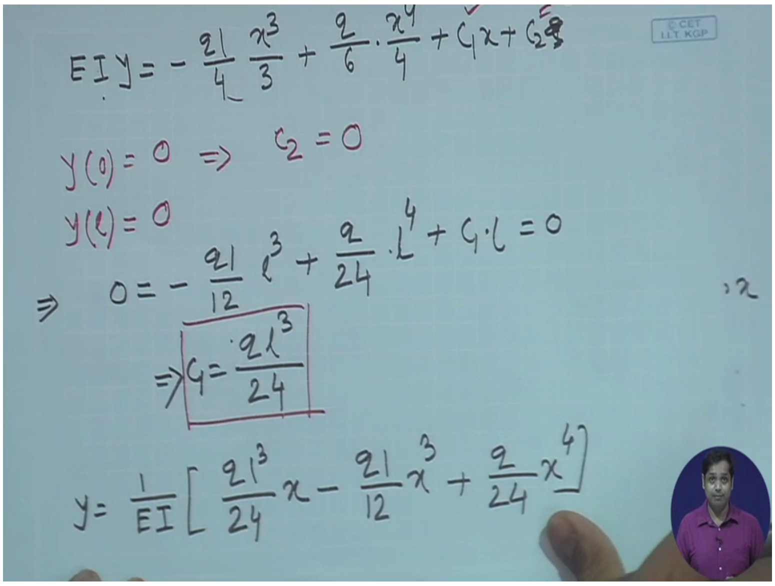

Now it can be slightly more modified. This becomes if you take q by, q is common, 24 EI it gives me L cube x minus 2 Lx cube plus x to the power 4. So this is the expression of y as a function of x, okay.

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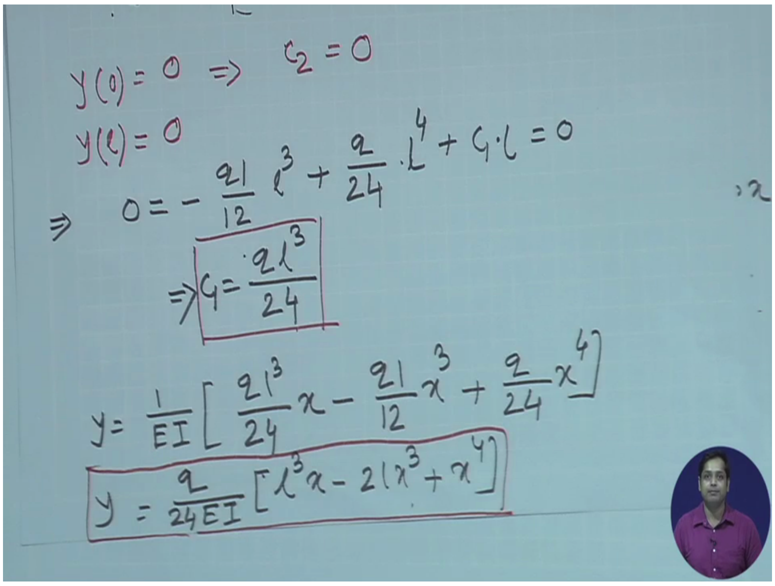

So if you look at this beam, this beam will deflect like this and that deflection at any point, that y can be obtained by this expression, okay.

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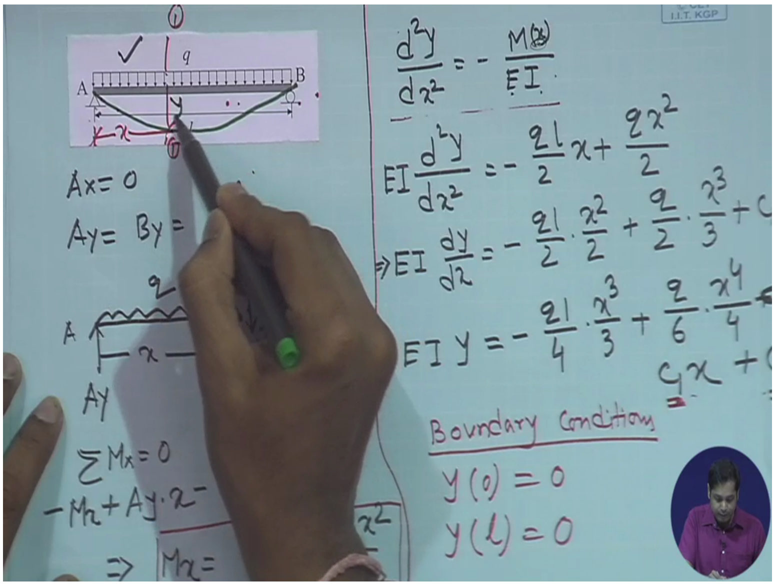

Now is it consistent with this, that x is equal to zero? Yes, if you substitute at x is equal to L then also y is equal to zero. Now since the beam is symmetric let us find out at what point the deflection is maximum. You can do expression of y is equal to this, so from x when deflection is maximum then dy dx has to be zero and at that when you substitute dy dx is equal to zero it gives you an expression and you get what is the value of x where y is maximum, okay.

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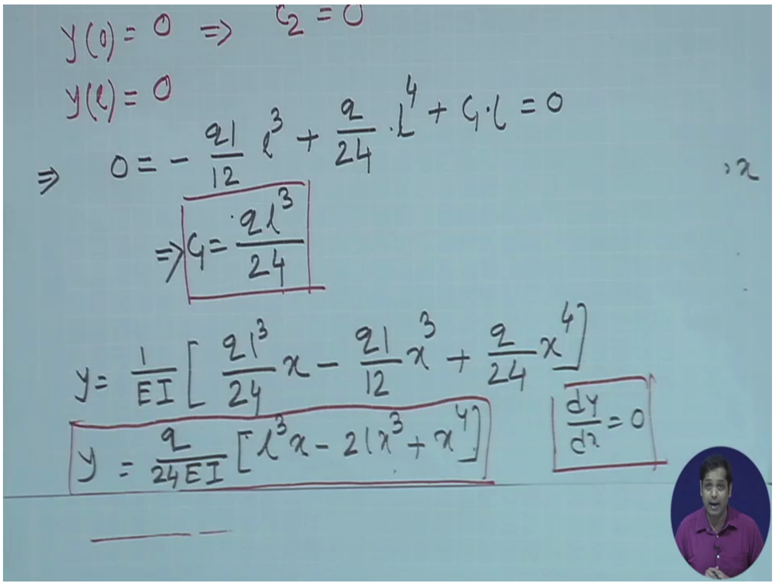

But at least for this problem since it is symmetric we can say that y is maximum at x is equal to L by 2. Now if we substitute x is equal to L by 2 then what we get is for this problem y max is equal to 5qL to the power 4 by 384 EI. This is very standard result so it is better to remember these expressions, okay.

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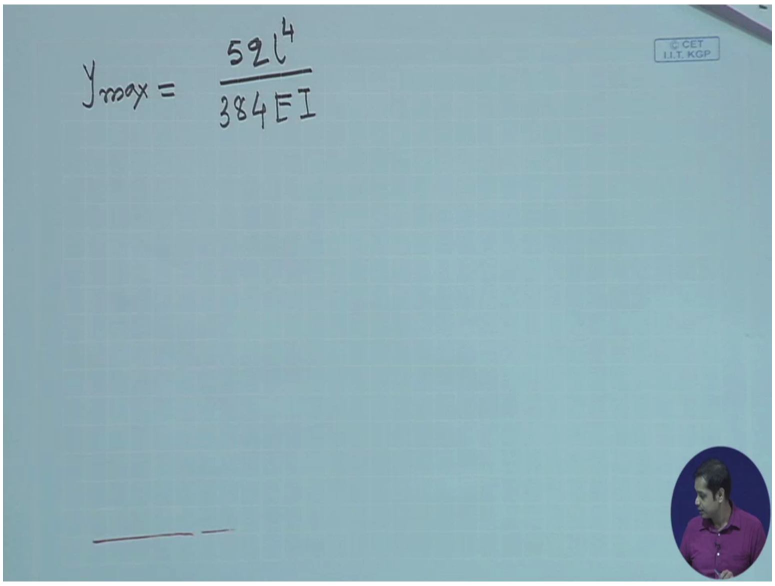

Now it is very handy if you remember this results, okay. So what it says? That this is the maximum deflection for simply supported beam subjected to (17:44) and the deflection (max) is maximum at x is equal to L by 2. Now let us see what are the expressions we have? What are the parameters which can influence the maximum deflection? Okay.

The first parameter is q which is the load acting on this beam. So by intuition we can say if q is more, then deflection will be more and this is consistent with this expression. That if q is more, deflection is more.

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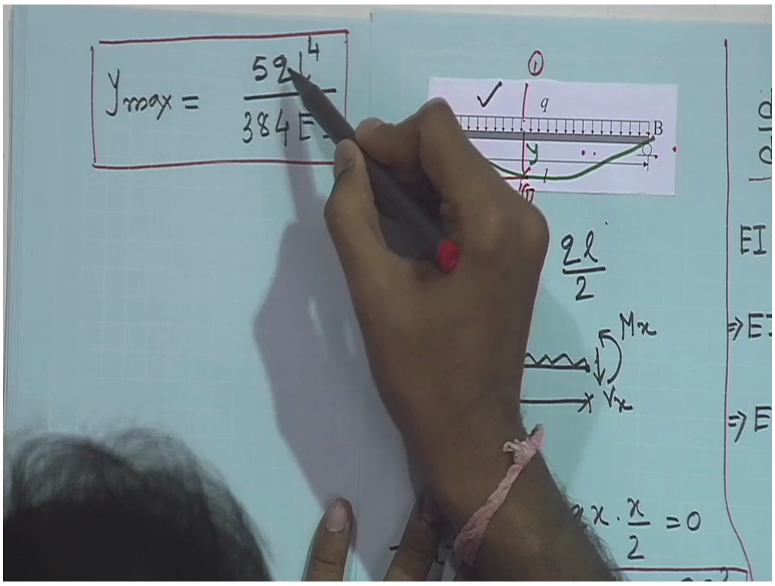

Now here one more another point to be noted is you see as I mentioned we assume the material is elastic. The structure behaves in such a way that the hooks law is applicable. So deformation is linearly dependent on the load. And if you see in this expression again this is evident. This is a deflection which is linearly depends on the load, okay.

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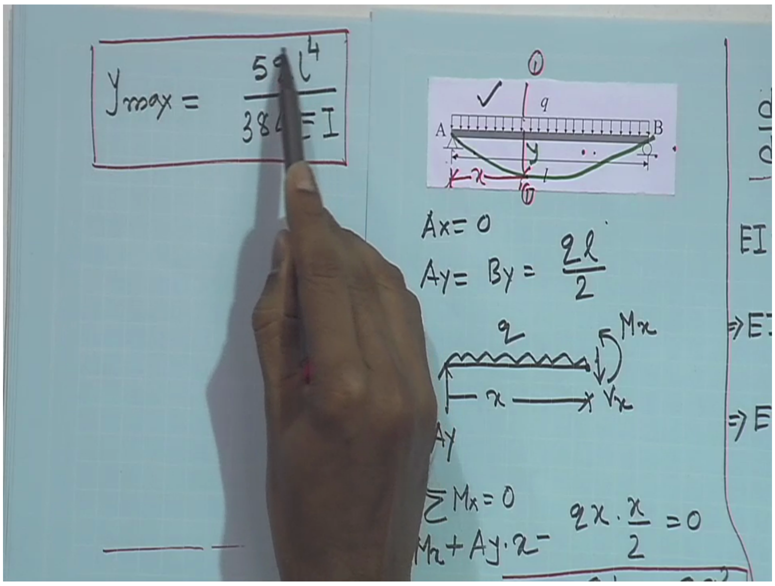

Now then it changes with L, if we have a shorter length L, the deflection will be less. If length is more, then deflection will be more. By intuition we can say that the deflection should increase with increase in length but how much at what proportion what order it will increase? That gives you this expression it will increase the fourth order, okay.

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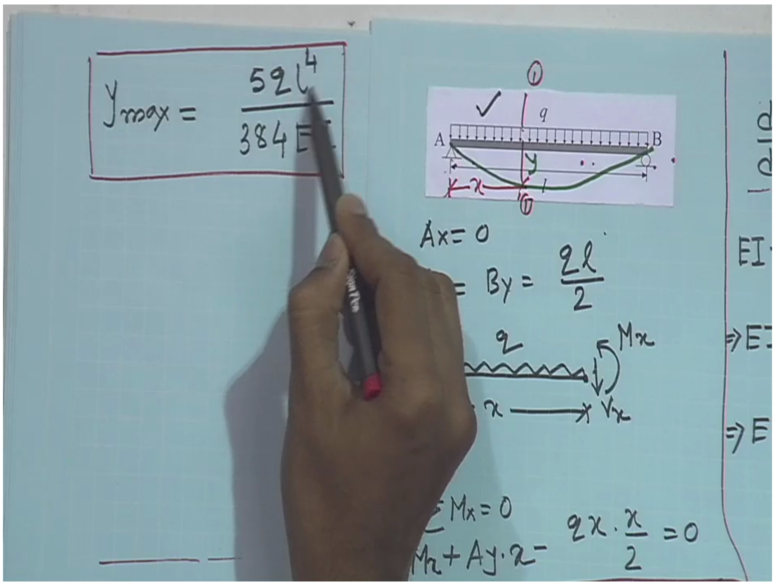

So then this is the Youngs modulus. So Youngs modulus is more. Means the structure is more stiff and if the beam is more stiff so naturally the deflection will be less which is also evident here that the deflection is inversely proportional to the Youngs modulus. So more stiffer beam, less the deflection.

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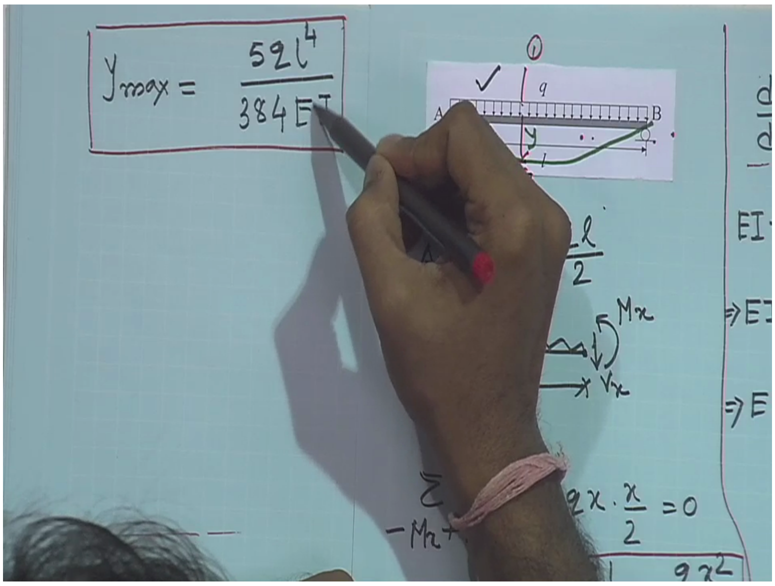

Then it is second moment of area, it says that it is also inversely proportional to the second moment of area. So if second moment of area is more, deflection is less, okay.

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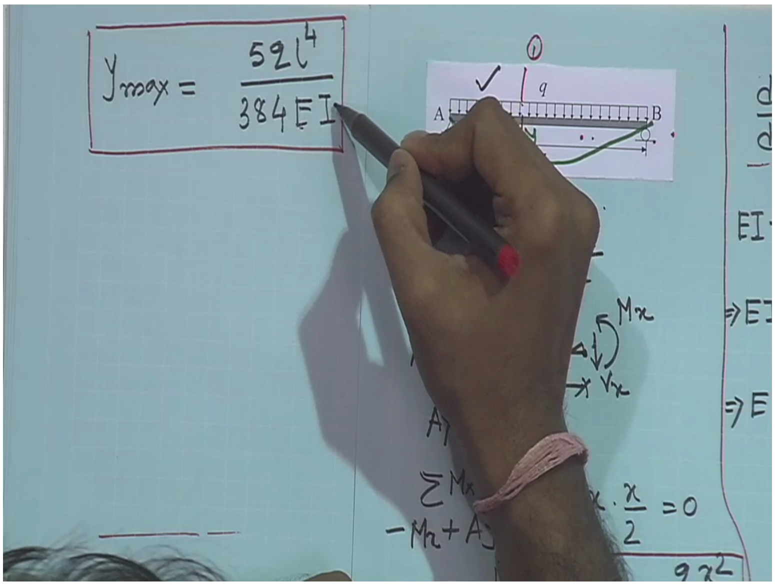

Now again I go back to the example that I showed. When the beam was like this, what happens the length is this. This is the first beam.

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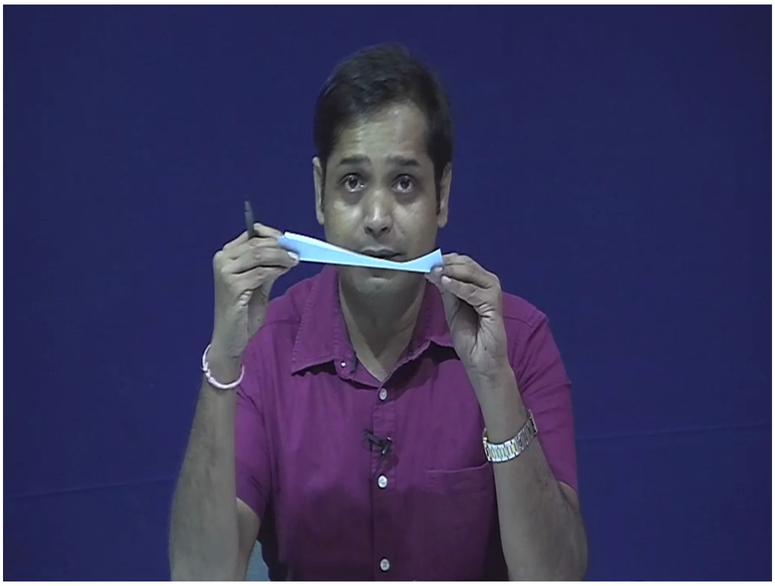

And this is the second beam. In both the cases length is same. So in both the cases the applied load on this beam was same.

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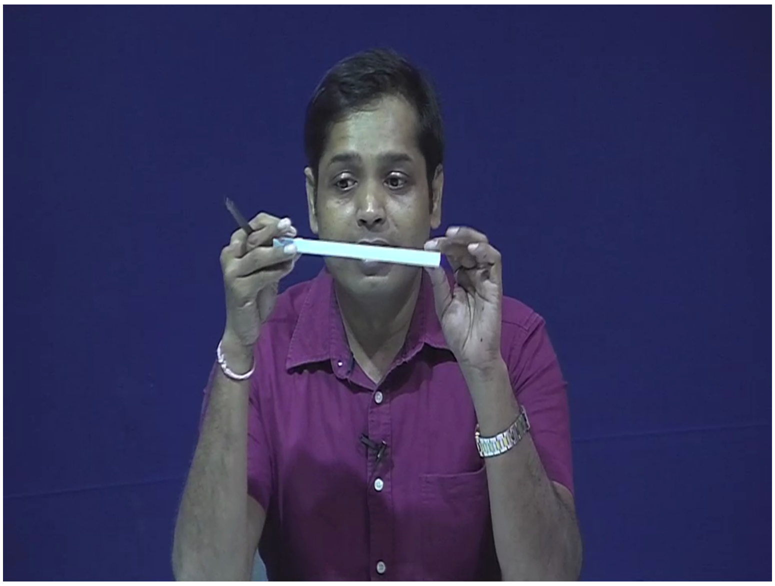

Both are made of same material, same paper, so Youngs modulus is also same. But what was not same in both the cases was the second moment of area. Now I believe you know how to compute second moment of area. For your beam your cross section was this, second moment of area was very less.

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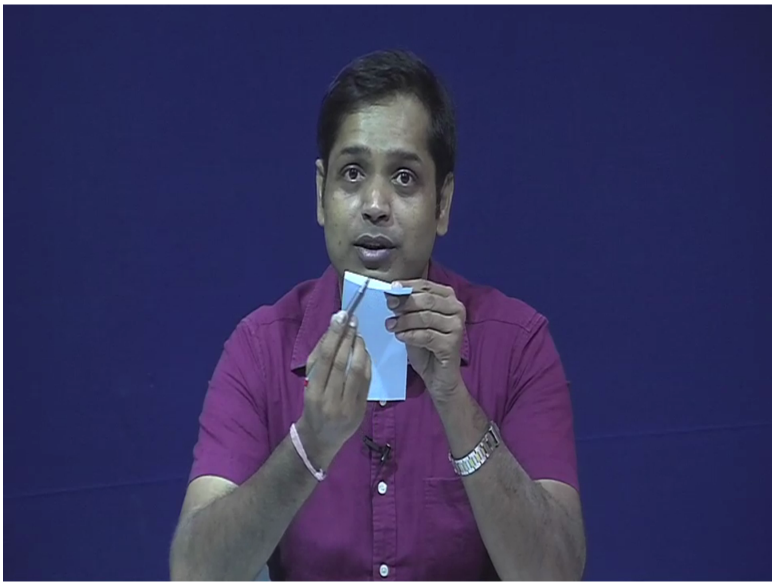

But when you make a cross section like this then your second moment of area becomes much more as compared to this. And second moment of area increases then naturally your deflection is less. And therefore this can carry whatever load we have applied on the structure, okay.

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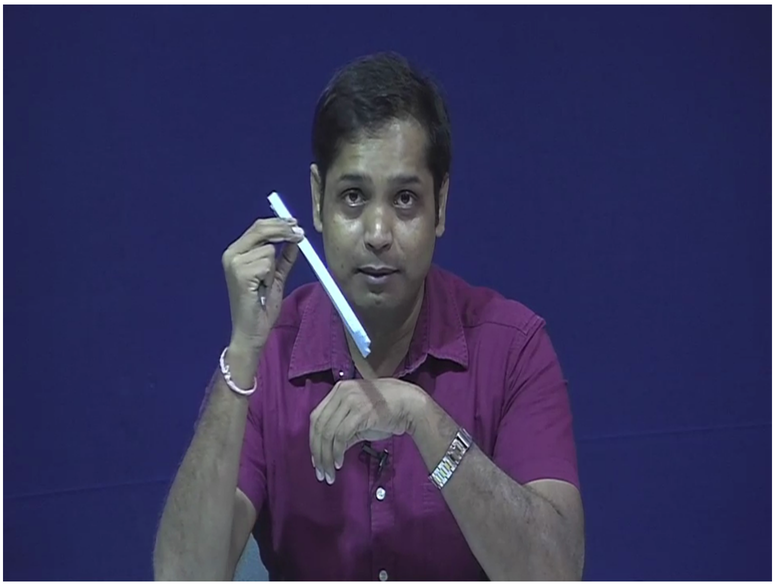

Now this observation you will see every problems that we follow, we will find out the same observation. Now suppose we have a problem like this. It is simply supported beam and then it is subjected to uniformly distributed load and then on top of this load you have some concentrated load like this, P1, P2, something like this, okay.

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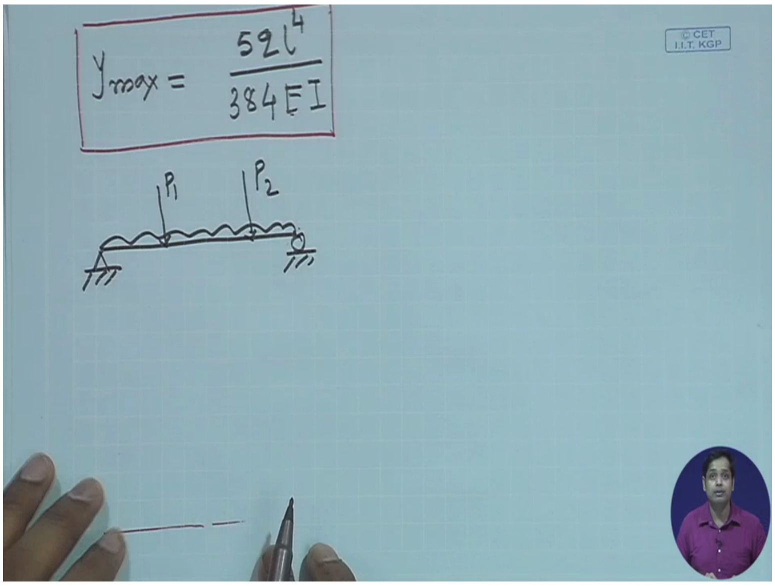

Now since the deflection is linearly proportional to the load what we can do is we can divide this problem into three sub problems are as follows. This can be taken as a beam which is subjected to only uniformly distributed load q and then plus beam which is subjected to P1 and then another plus subjected to another load P2.

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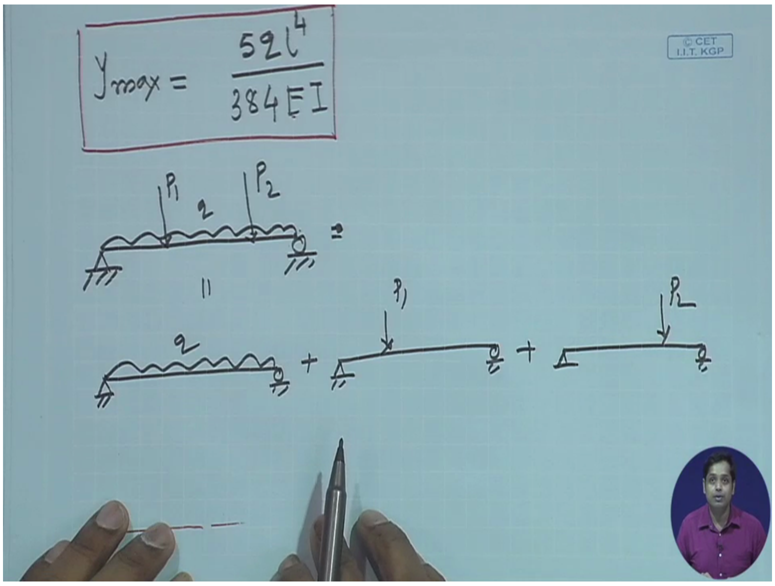

So get the deflection of this beam the delta 1, this is delta 2 and this is delta 3.

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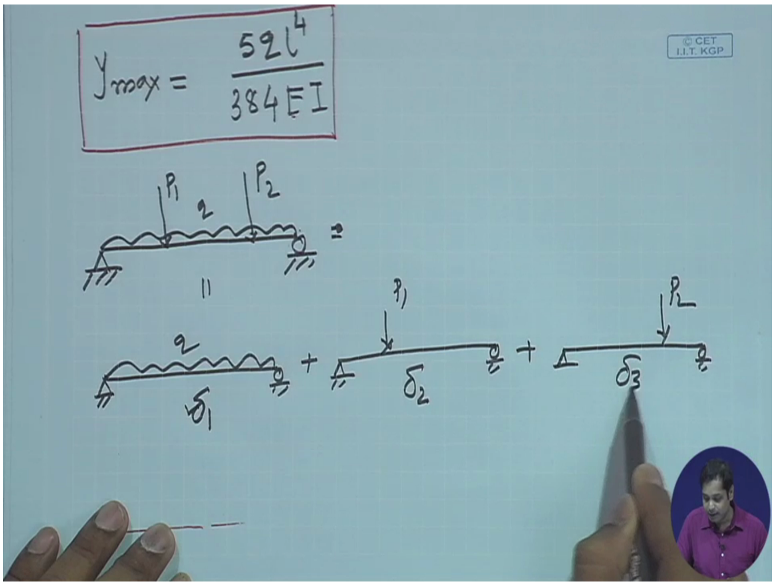

At any section the total delta will be delta 1 plus delta 2 plus delta 3. This is a total deflection. And it is not only for deflection you can apply the same thing for internal forces as well. For instance for this beam if you have to (con) compute the moment at any particular section you can decompose it into three parts, get the moment here, moment here, moment here or shear force and add them together to get the final moment and shear force for this member. This is called principle of superposition.

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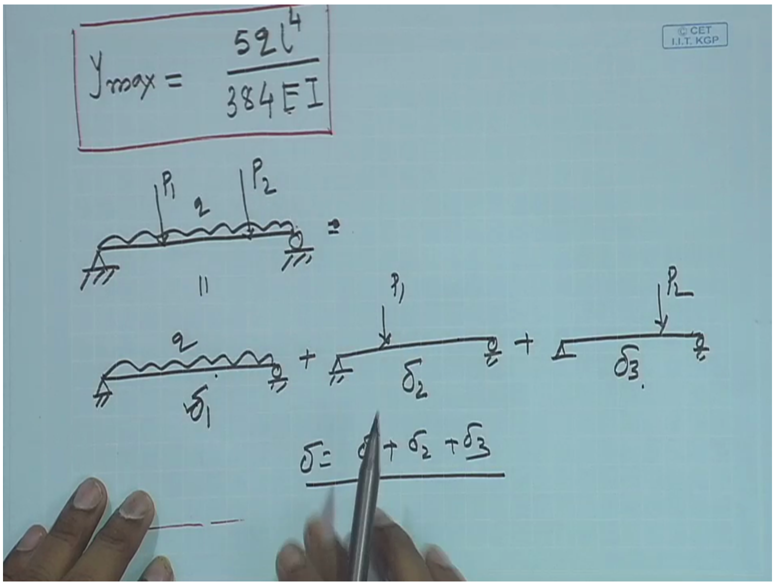

Means we are super posing different problems and get the solution of the actual problem, okay. So this principle of superposition is applicable here because our one of the major assumption is the linear behaviour, okay. If the behaviour is not linear then it is may not be applicable. Just to give you one, this approach is how divide into smaller part is make sure makes your life easier let us give you one example quickly.

Suppose this is the problem. This is L by 2, this is L by 2, okay. A, B, C and this is q. So it is simply supported beam which is subjected to uniformly distributed load over half length and then another half length BC has no load on it.

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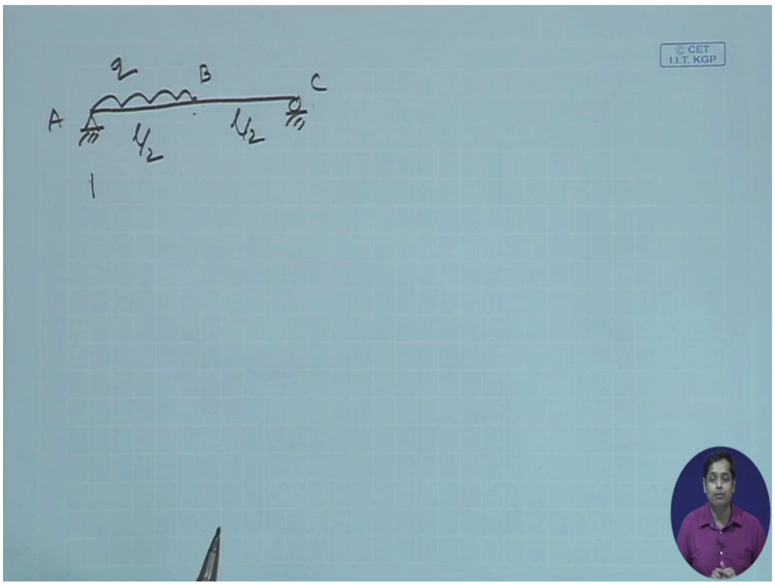

Now we need to find out what is a deflection of the beam. Now if you remember when we discussed bending moment and shear force diagram for this kind of problem we had two expressions for bending moment. One expression which is applicable which is true for when x between A to B and another expression when x between B to C. But these two expressions should have common value at B, okay.

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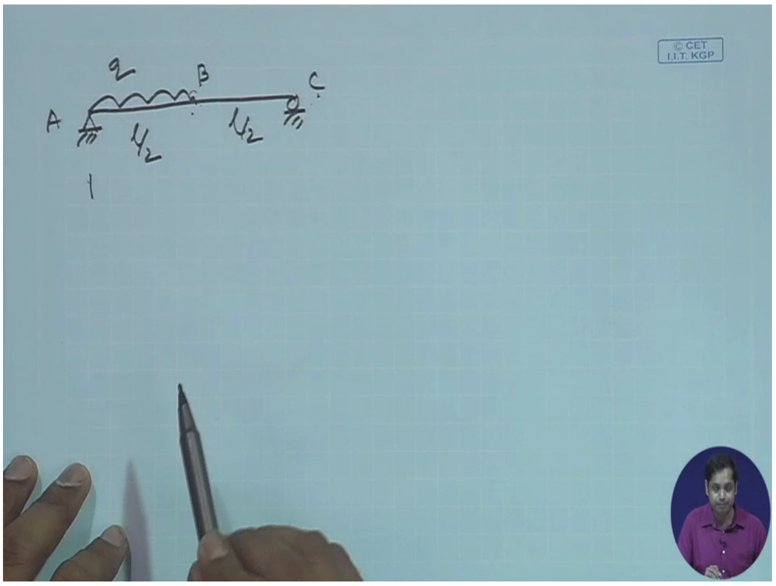

Now when you solve this equation d2y dx2 is equal to minus M by EI. Now expression for M is different for this part and expression for M is different for this part.

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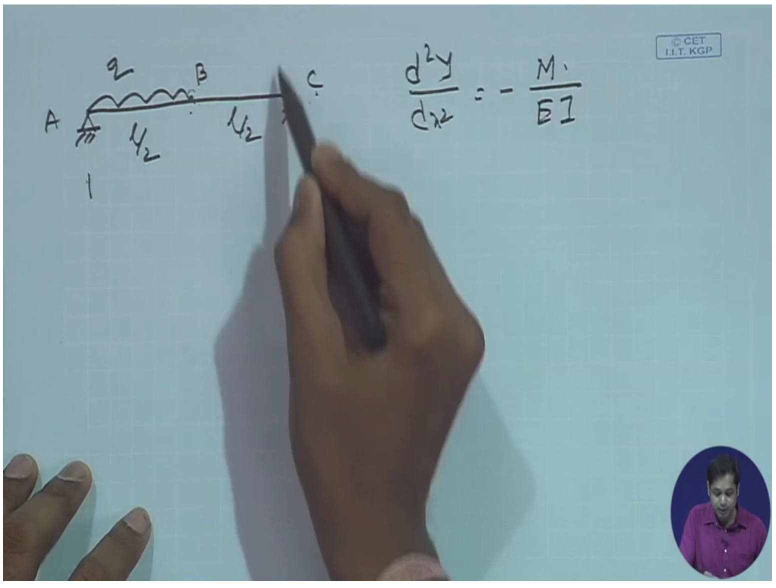

What you need to do is now we need to (con) (ap) apply this expression here with expression of moment at this point and again apply this equation on part BC with the expression of moment you can have for BC, okay.

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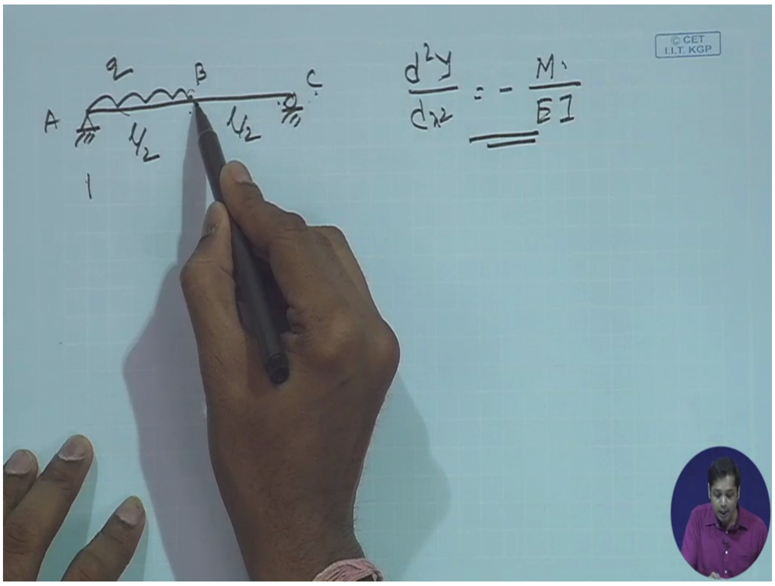

Now when you apply for this part, this will give you two constants because it is a second order equation. This will give you two constants C1 and C2, okay.

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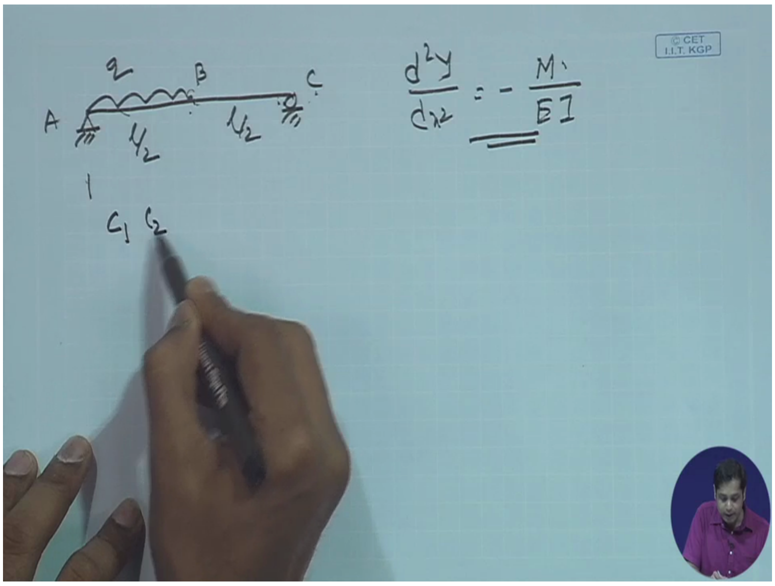

Like in a simply supported beam when we integrate it we have two constants and those constants we could determine from the boundary conditions, okay. Now when we apply this equation over AB, this will give us two constants and we need to determine these two constants.

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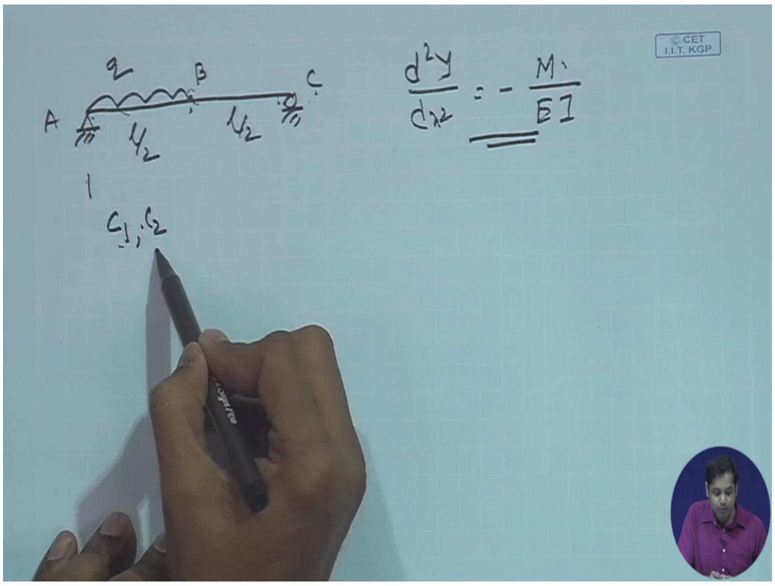

Now again when we apply this equation over BC that also give us two constants. So say this is C3 and C4. Now we have in the problem total four constants.

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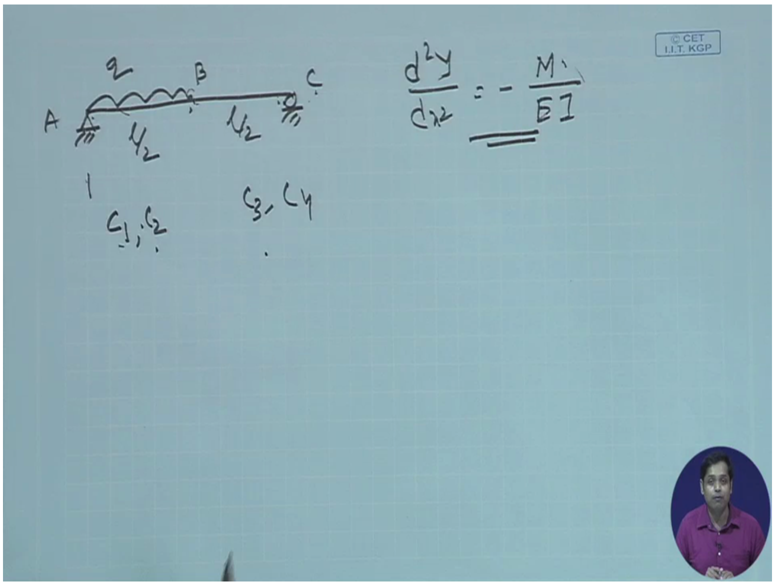

Then let us see how many boundary condition we have. Very straightforward boundary condition is, at this point deflection is zero and this point deflection is zero.

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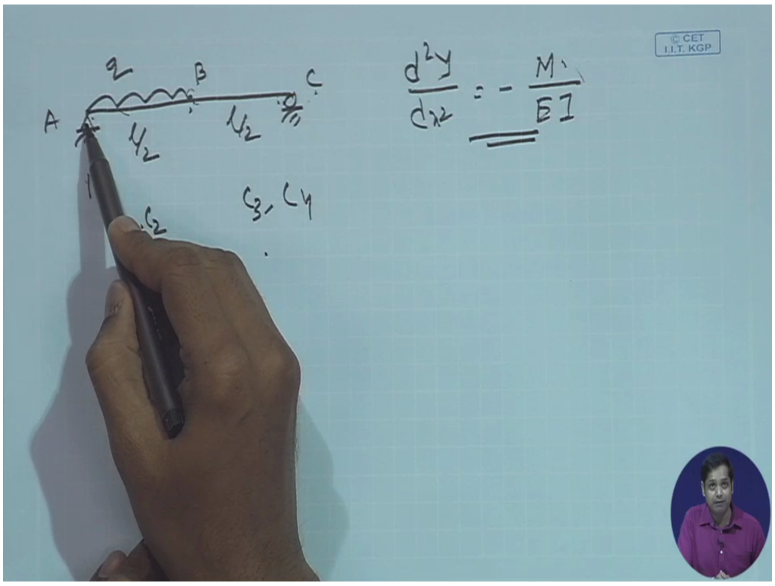

So y at x is equal to zero, y is zero is equal to zero and y L is equal to zero, okay. Now these are two boundary conditions.

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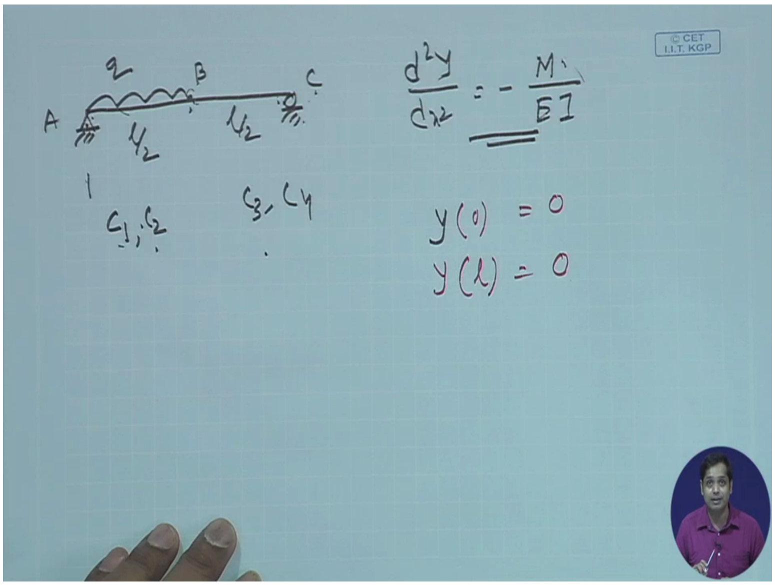

But we have four constants, we need four boundaries, four constraints, four equations to determine these four constants. Now what are the other two boundary conditions? You see when we write this equation that it is a differential equation where we have a second derivative of y is equal to M by EI. It is implicitly assumed that y is differentiable up to second order.

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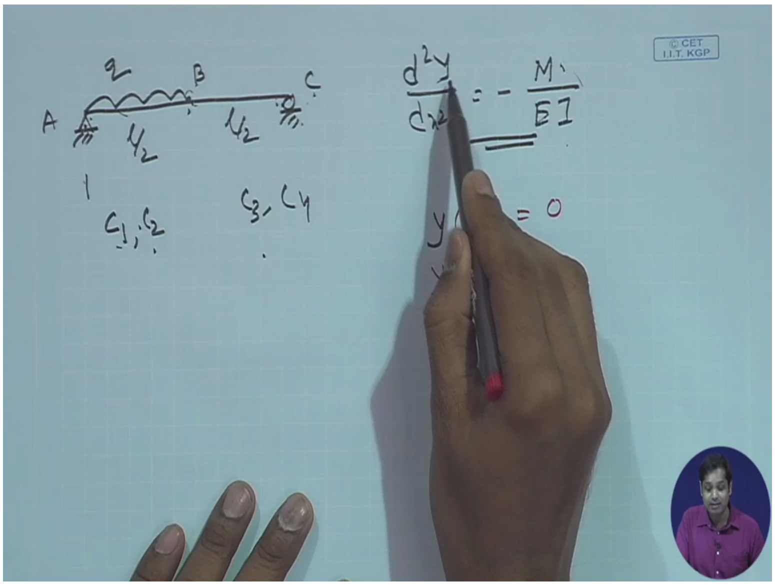

If y is not differentiable then what happens? This equation itself is not valid. So that we can use as criteria to get another two equations. You see in this (ex) beam maybe their bending moment expressions are different for AB and BC. But if we look at the deflected shape, the deflected shape should be continuous at B, okay. Deflection should be such that there is no jump in y at B.

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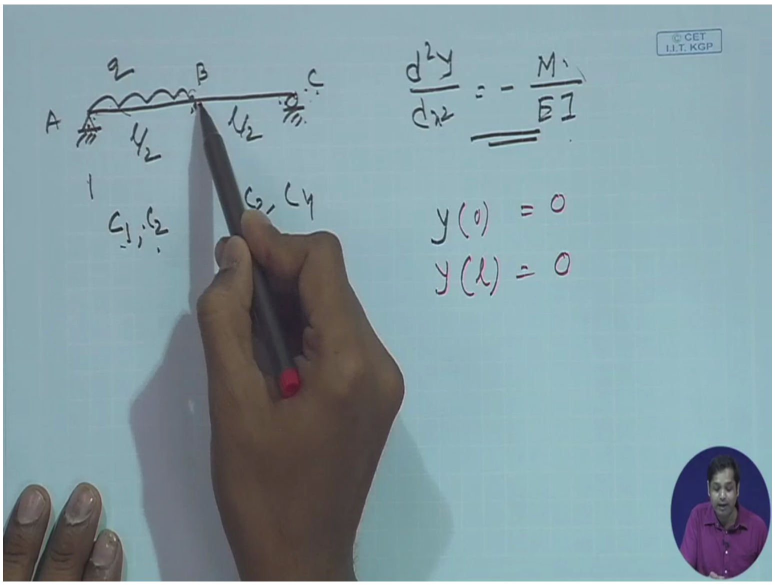

So y should be continuous at B because if y has a jump at B, suppose if your deflected shape cannot be like this because if your deflected shape is like this then y has a jump in this.

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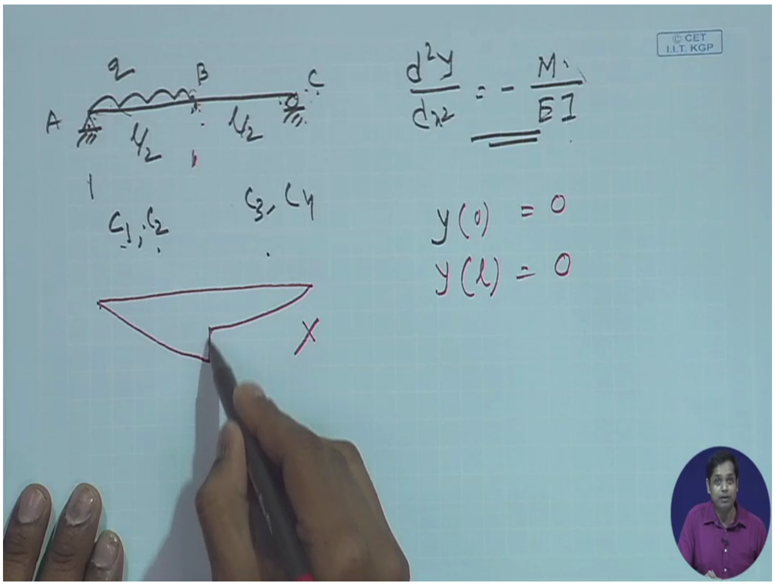

If y has a jump then this equation itself is not valid, okay.

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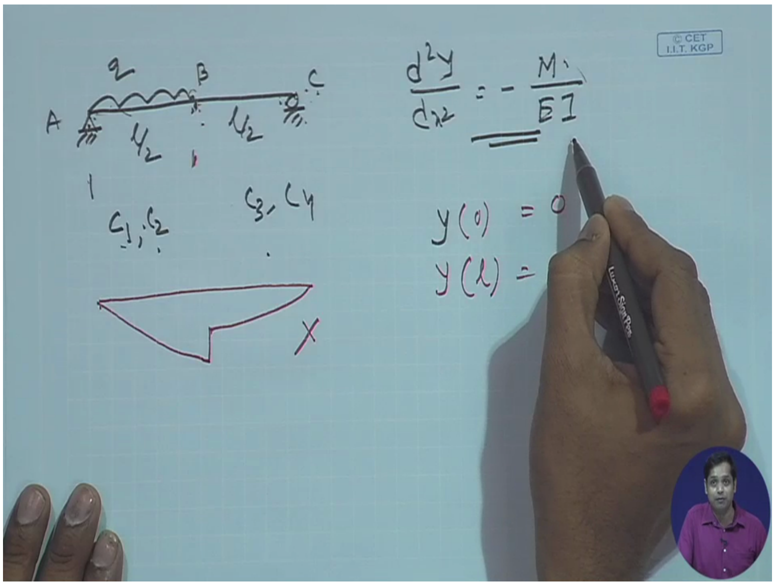

So your deflected shape will be continuously smooth deflection, is not it? So one condition is that at y is equal to B that y at L by 2, your deflection computed from this part and deflection computed from this part should be same. Means delta at B computed from AB should be equal to delta at B computed from part BC, okay.

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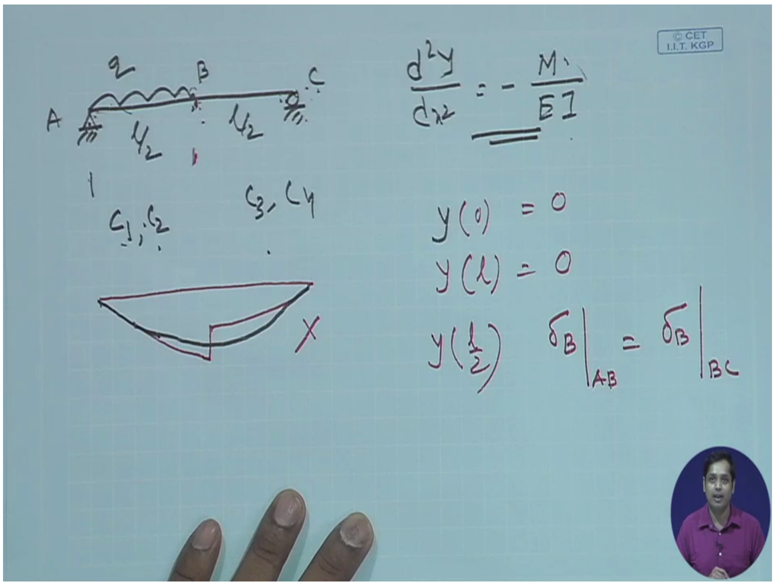

So this will give you one more equation. And similarly not only the deflection, slope also has to be continuous to have this equation valid. So this is slope say y dash at B computed from AB should be equal to y dash at B computed from part BC. So these are four equations and you apply these four equations to get the four unknown.

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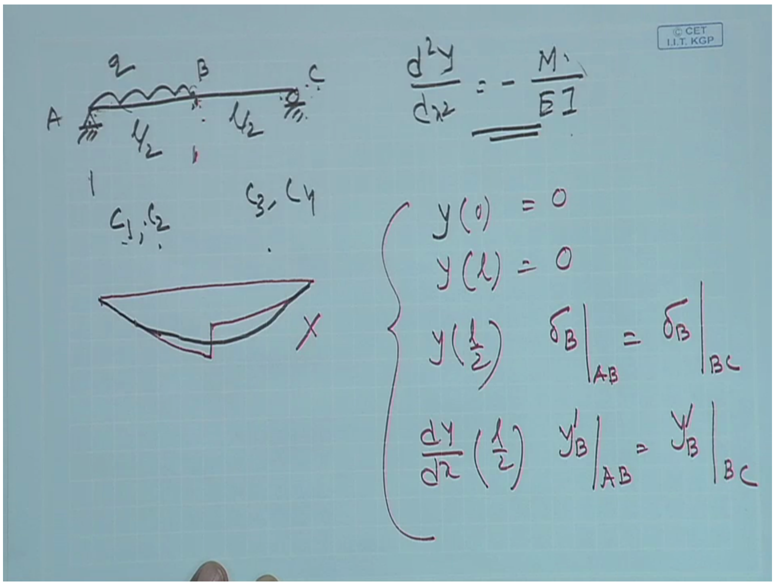

For instance if you have a problem like this, the similar concept can be used. These are very standard problems. If you take any book then the solutions of these problems are given. What I am telling you the concept behind the solution. So same in this problem you have an expression for bending moment for this part and you have another expression for bending moment for this part.

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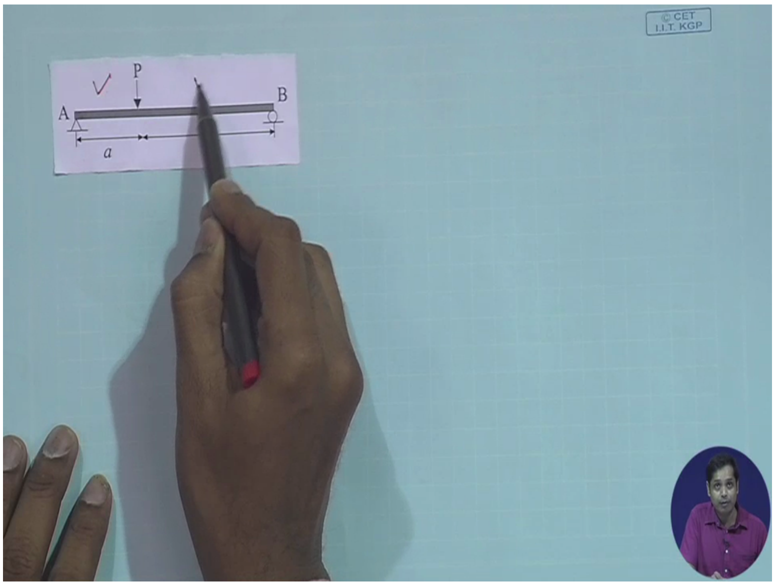

This part we have two constants, this part we have two constants. Boundary condition y is equal to zero here, y is equal to zero here, two boundary conditions. We need two more constraints, y is equal to y computed at this part, y is continuous at this point and slope is continuous at this point.

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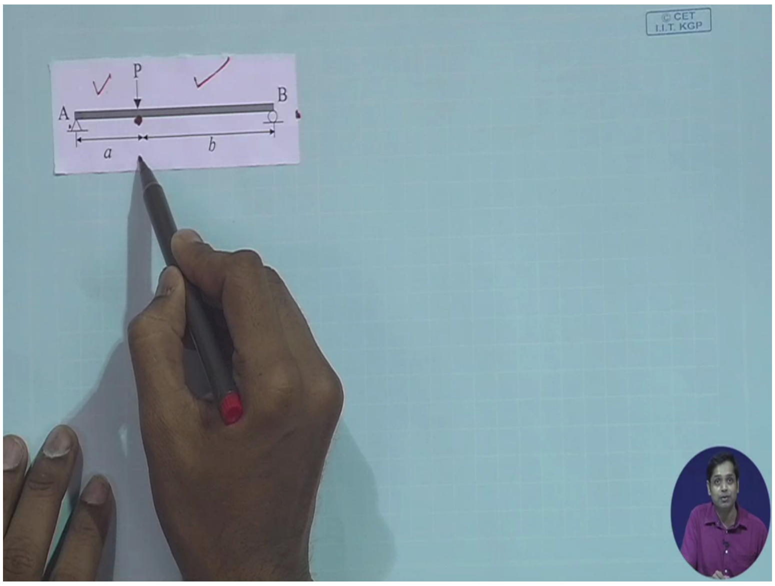

Another two constraints. All these four constraints will give you four boundary conditions. Now before we close one small you can think of a puzzle also. Consider this problem once again, this is L by 2, this is L by 2, this is q, A, B. Now if this is C.

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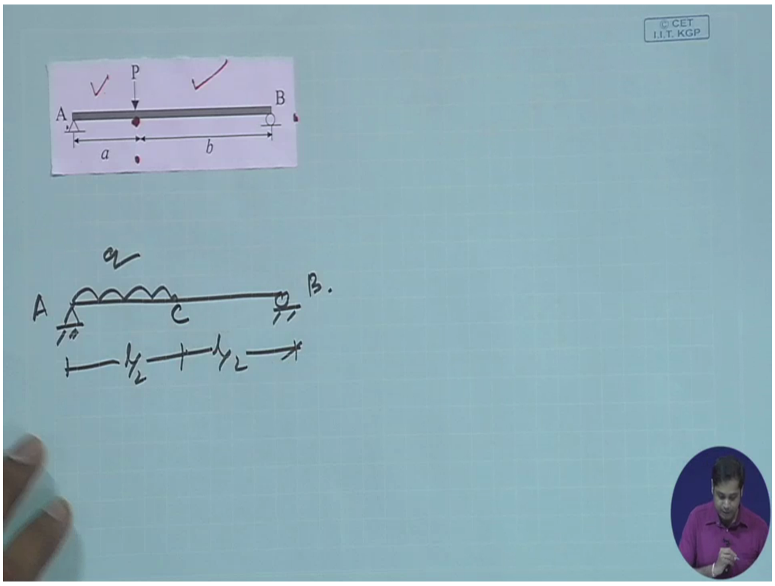

The problem is compute delta c, okay. What we need to do is how to compute delta C? Just now we discussed the procedure. We have two parts. One part here, another part here.

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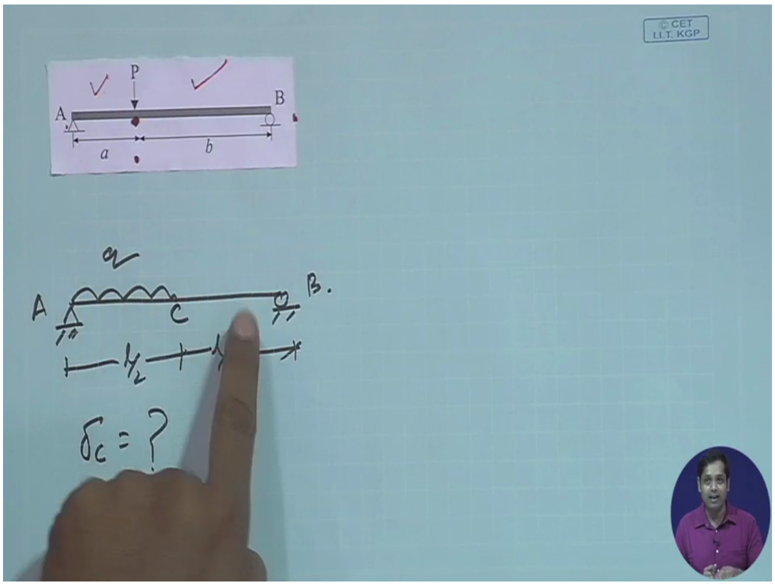

Four boundary condition, four constraints, four constants determine those four constants. We can get the solution of this part very quickly playing a trick. You see can we say that this problem and this problem there is no horizontal force so Ax is anyways zero. So can we say that this problem and this problem are same? A, B, C. This problem and this problem same. They are same because one is mirror image of another, okay.

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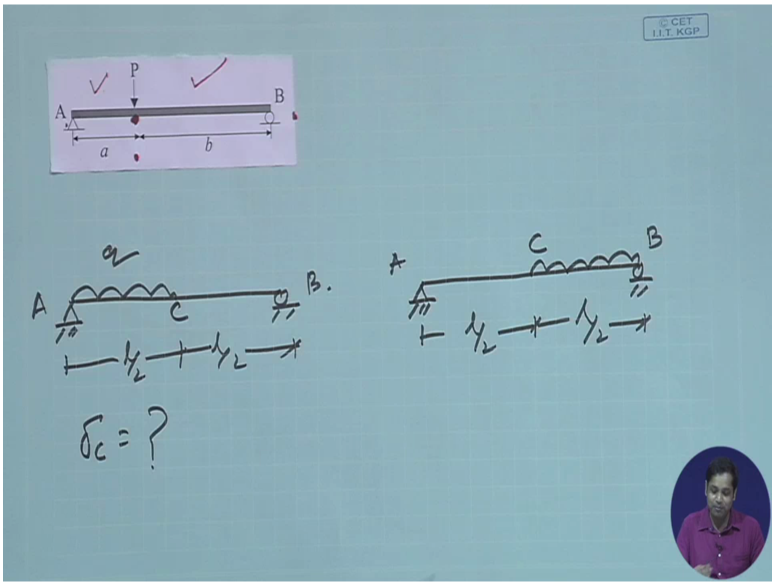

So these two problems are same, great. Now if these two problems are same then delta C here and delta C here, they should be same, is not it?

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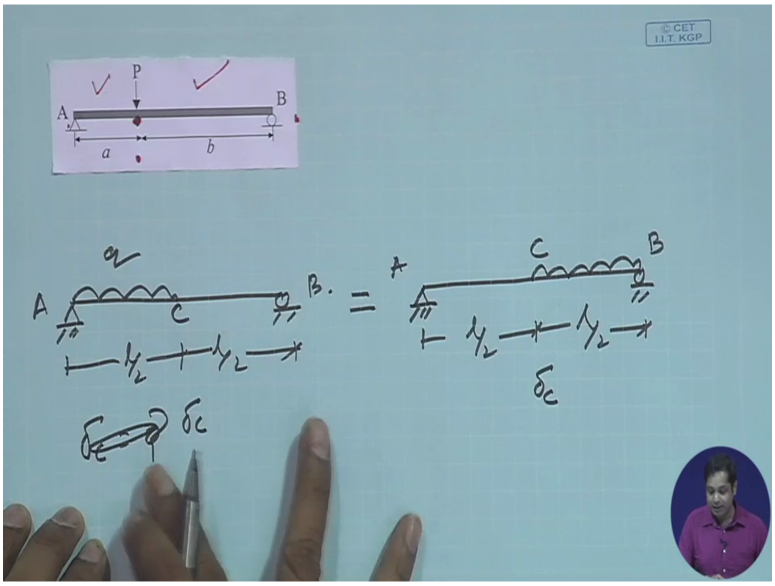

Suppose it is delta C1, (deta) delta C2. So we can say that delta C1 is equal to delta C2, okay.

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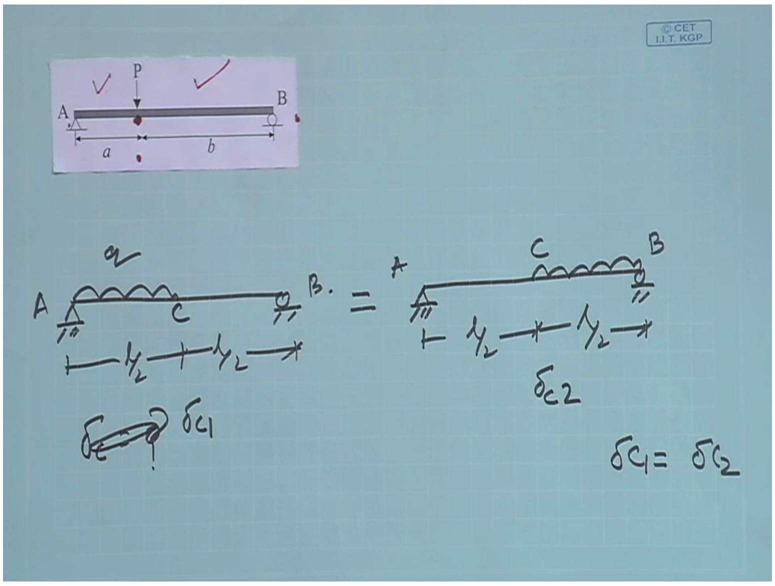

If we add these two problems so these are same. If we add these two problems then what happens? Then (tot) individually for this problem this is delta C1, individually for this problem it is delta C2. If we add these two problems using principle superposition then what problem will have? From this problem A, C, B and then from this problem it is L by 2.

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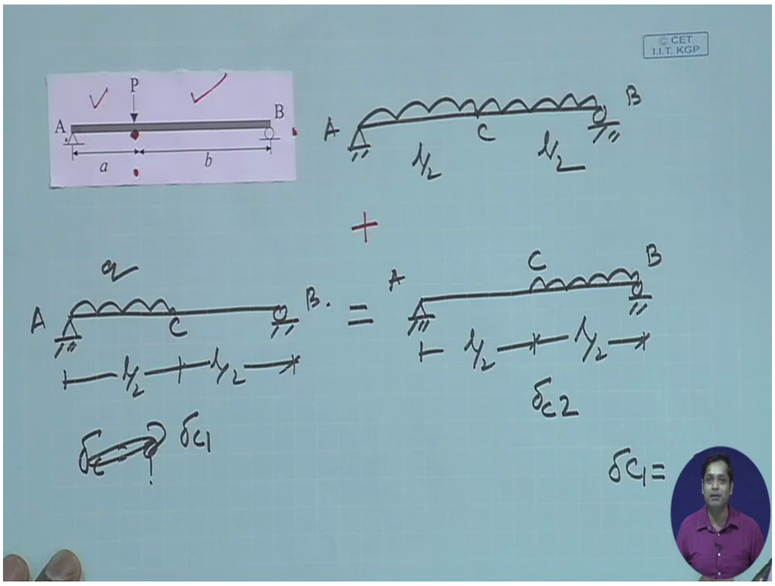

So it is a simply supported beam subjected to uniformly distributed load, right? So this plus this is equal to this.

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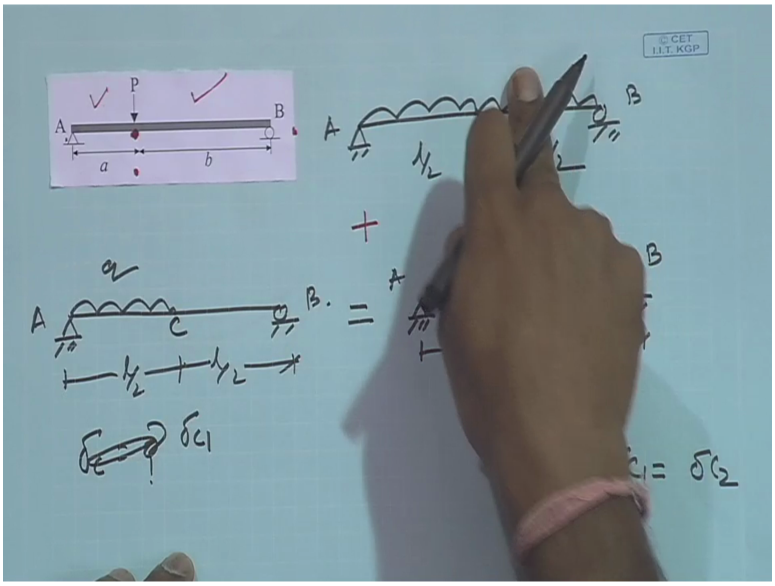

Suppose this deflection is delta C, okay. So we can write delta C is equal to delta C1 plus delta C2. Again delta C1 and delta C2 are same so we can write delta C is equal to 2 delta C1 and delta C1 is equal to delta C by 2, is not it?

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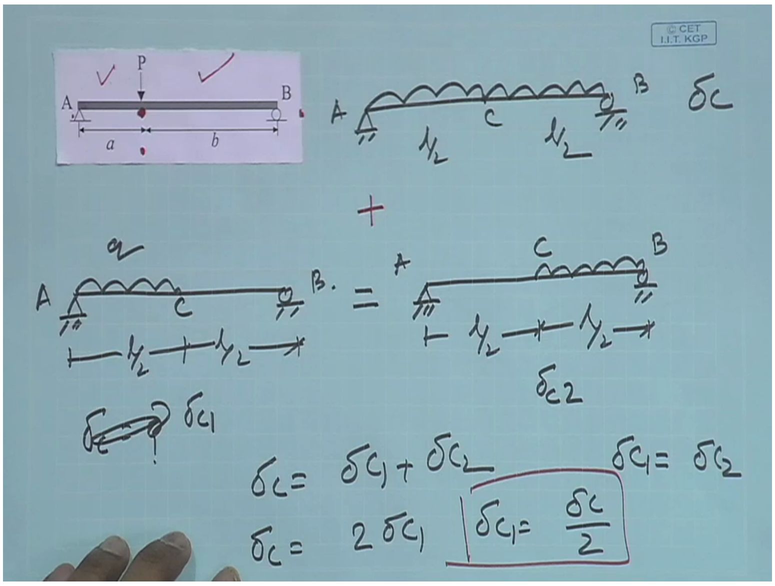

For this problem just now we computed delta C is equal to 5qL to the power 4 by 384 EI. So we substitute here, you will get delta C1, okay.

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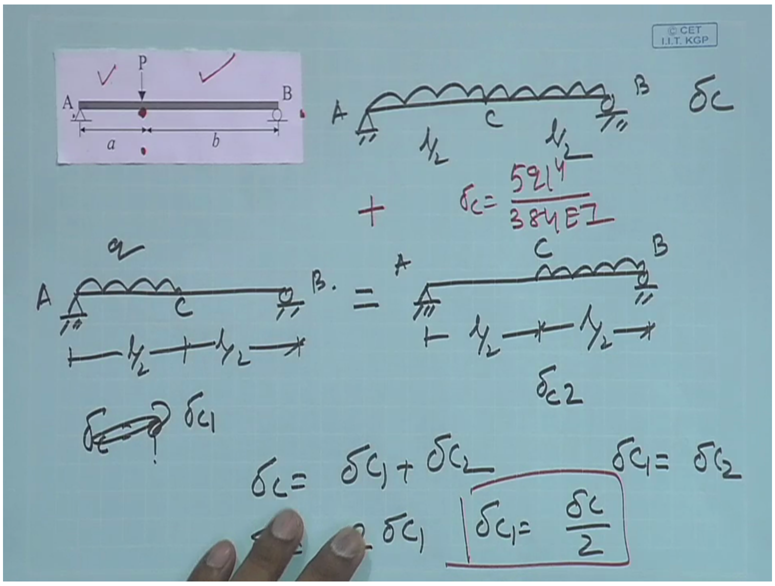

So what point I wanted to make here? You have a problem and do not jump to the problem and try to solve it. You see the problem and then plan your approach of solving the problem, okay. As per as concept, solution is very straight forward. The solution has itself a very minor role at least in this case. But what is important is understanding the problem and formation of the problem.

There are many cases where the same problem if you formulate in a different way, then your solution becomes easier, okay. So please you keep in your mind when you do some exercise, okay. That is all for today. Please go through any book and do some exercise and next class what you will do is next class we will see another method called moment area method for determining deflection of statically determinate beams. But again even if moment area method the underlying principle, the premise of that method is again same equation of elastic line, okay. Thank you.