Structural Analysis 1
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Lecture 17
Deflection of Beams and Frames

Hello welcome to the first lecture of week 4. What we will be doing in this week and again we will continue to the next week as well, we will learn different method to determine deflections in statically determinate beams and frames. So our topic for week 4 and week 5, analysis of statically determinate structures, deflection of statically determinate beams and frames.

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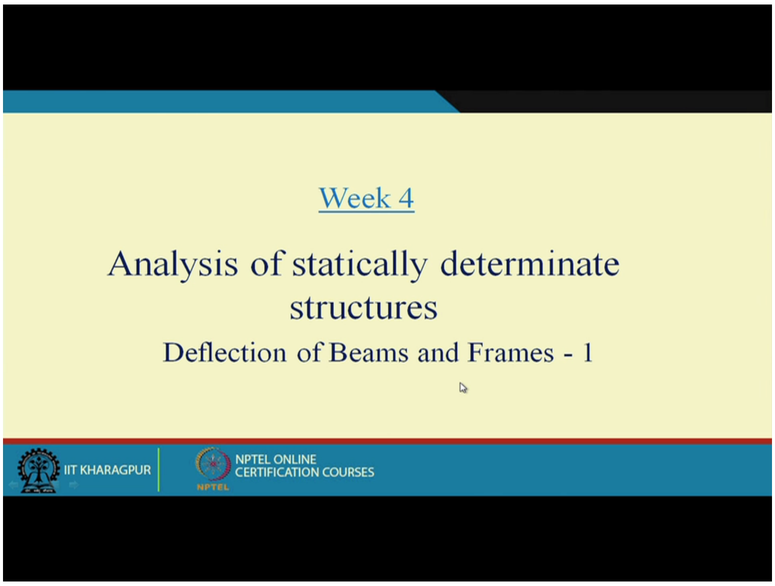

In this week we will learn different methods to determine deflection for beams and the next week we will extend some of the method to frame and also see principle of virtual work to determine the deflection. If you remember principle of virtual work was introduced last week, okay. Now as far as beam is concerned we will learn three methods here but we need to do all three methods as we move on. But there is a common thread between all these three methods and the common thread is equation for elastic curve for beams.

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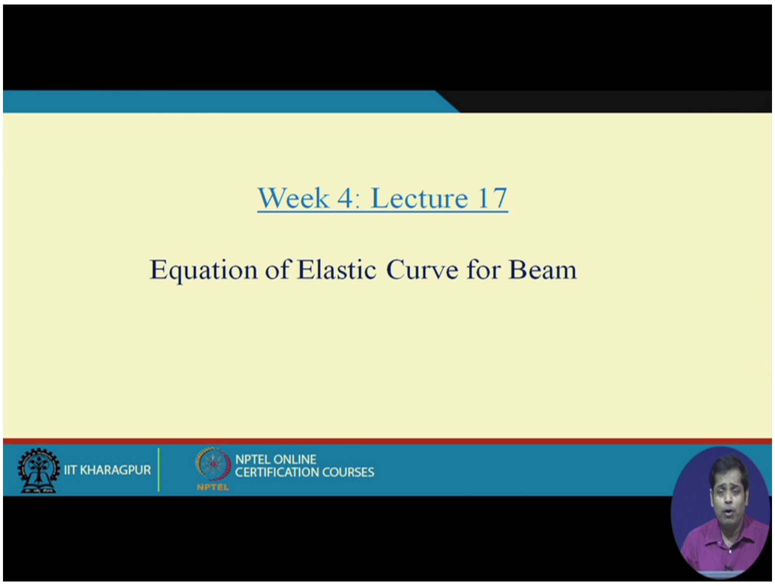

What we will do today is we will derive the equation for elastic curve. What is elastic curve? How is the equation? We derive that equation today and then next classes onward we will see the applications of that equation in different methods for determining deflection, okay. Now let us start with this. You see, take a beam, it is for demonstration purpose.

We are considering simply supported beam subjected to UDL but the equation that we develop that is applicable to any support condition.

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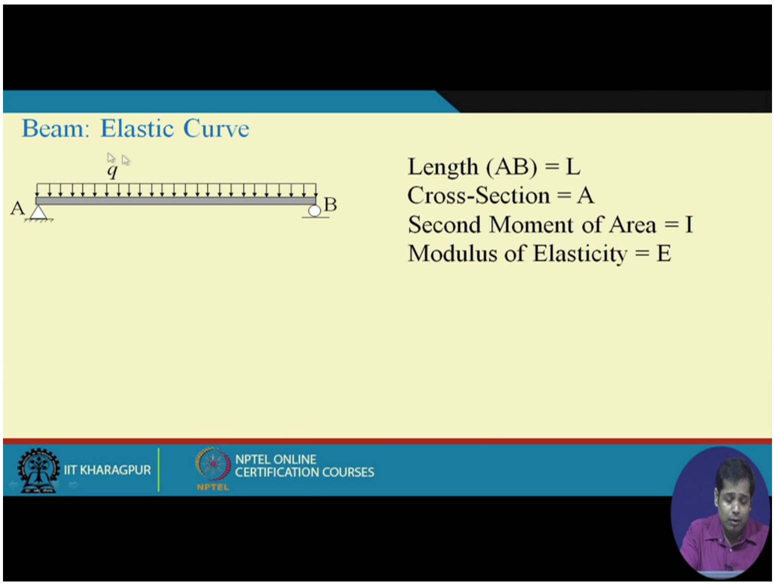

Now consider a beam which is subject to uniformly distributed load, intensity q. the length of the beam is L. Cross section cross section the beam is A. I believe that you are familiar with second moment of area. You have learnt it in your mechanics course. And modulus of elasticity or Youngs modulus of the material is E.

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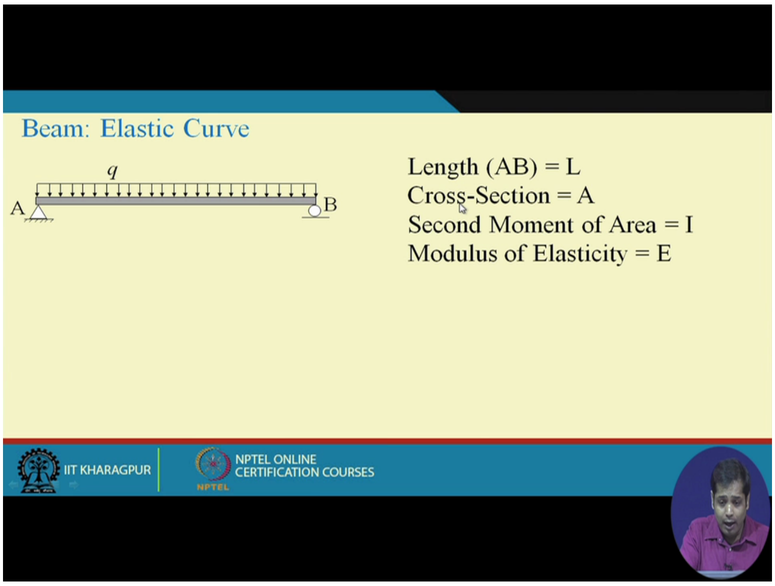

Now so this is the physical process. Now this deflects like this, right? Now this is an idealization of this physical process where the beam is idealized as line, okay, because this length is very small as compared to A.

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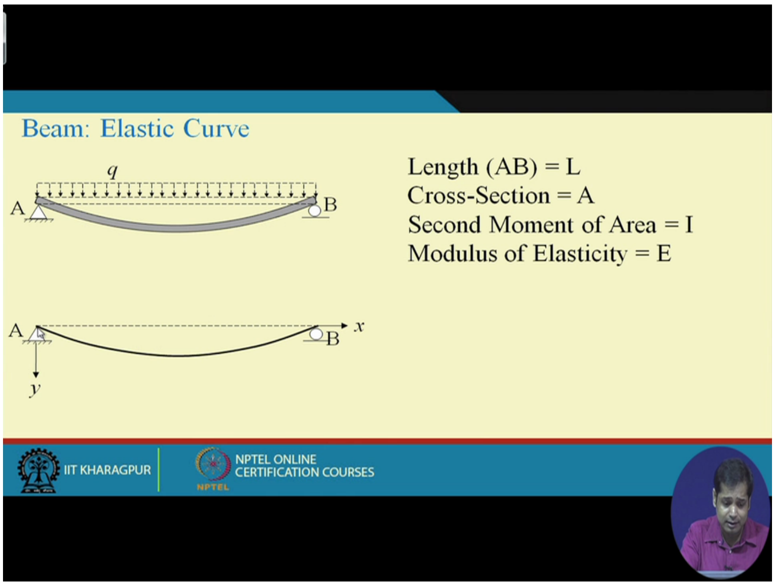

Now if we take any section m-n at a (dist) distance x from A then deflection of this beam is y. Now what we want is we want an equation that relates the force acting on the structure, deflection, material and geometric property. Means we want the force, the deflection as a function of x because as we can see that deflection changes with x which is obvious if we change q the deflection is also changed. If we increase q deflection will be more.

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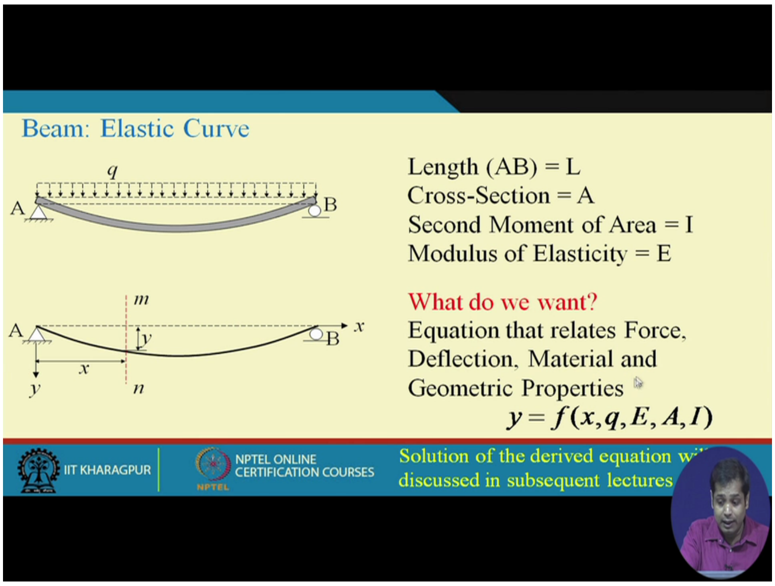

The Youngs modulus is the material strength has influence on the deflection and then of course the geometry of the section also has some influence on this section, okay.

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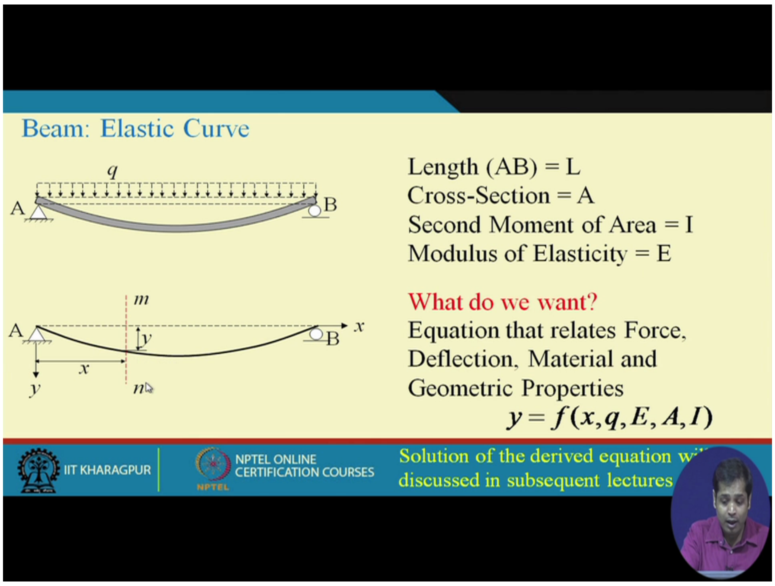

Now what we want is this. We want to derive an equation which gives you that relation. Now if you remember in the first lecture the process we followed is physical process, then idealization, then derivation of the equation representing this idealized physical system through some equation and finally solving those equations. What we will do today is we will derive this equation and then how to solve this equation in what form that we will see in the subsequent classes, okay.

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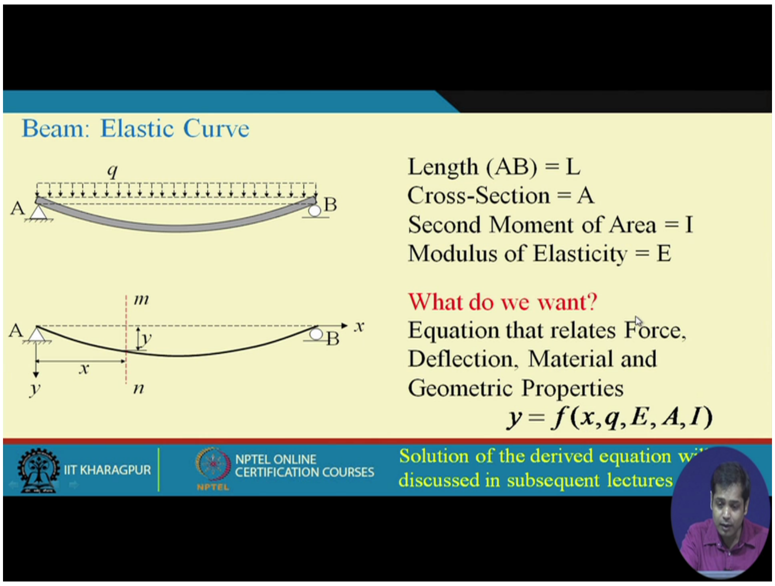

Now any model is based on certain assumptions. What are the assumptions that we have for the model that we are going to develop is this. The first is the material obeys the Hookes Law. You are familiar with Hookes Law. What it says?

Implication of Hookes law that if you have a spring which is subjected to a force P and undergoes deformation x and the spring stiffness is K then we say then P can be determined from Kx. When K is the deflection P is deformation applied load.

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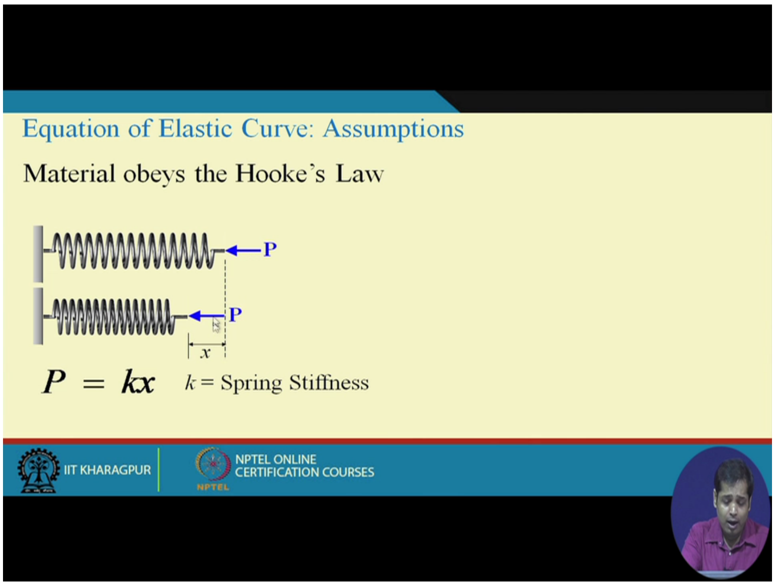

Now it says that the applied load and the deformation is linearly related and the slope of this relation gives you the spring stiffness. And if you have to write it in a stress and strain form then this is the typical stress strain relation of any material.

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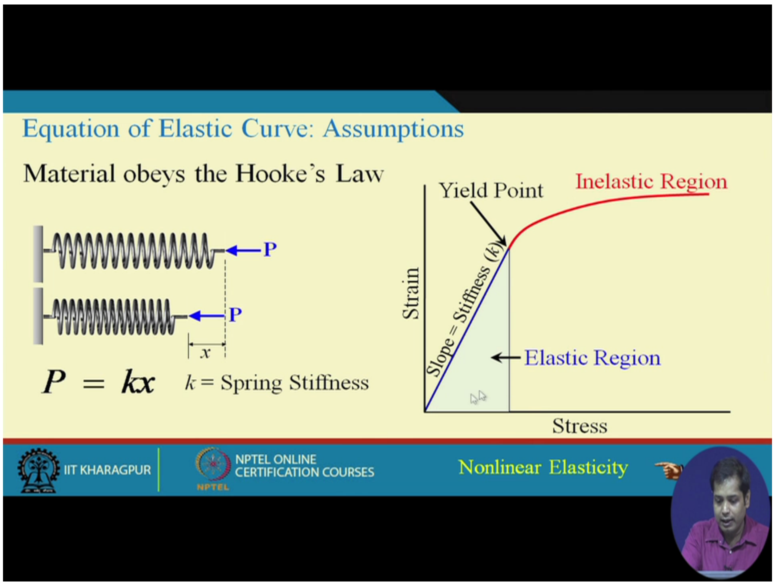

It is initially linear elastic. It obeys Hookes Law and then it becomes non linear and goes into inelastic region. Now slope of this elastic region will give you the stiffness. The point when it changes from elastic to inelastic is called yield point, okay. Now assumption is another important part you can note is here is, elasticity does not always means that the relation has to be linear. There is nonlinear elasticity as well.

But that you can Google or look into different books what non linear elasticity exactly means. Now this assumption is the theory the model that we are going to develop that is applicable when the material is in this elastic region. If the material goes below or outside this region then this equation is not applicable, okay.

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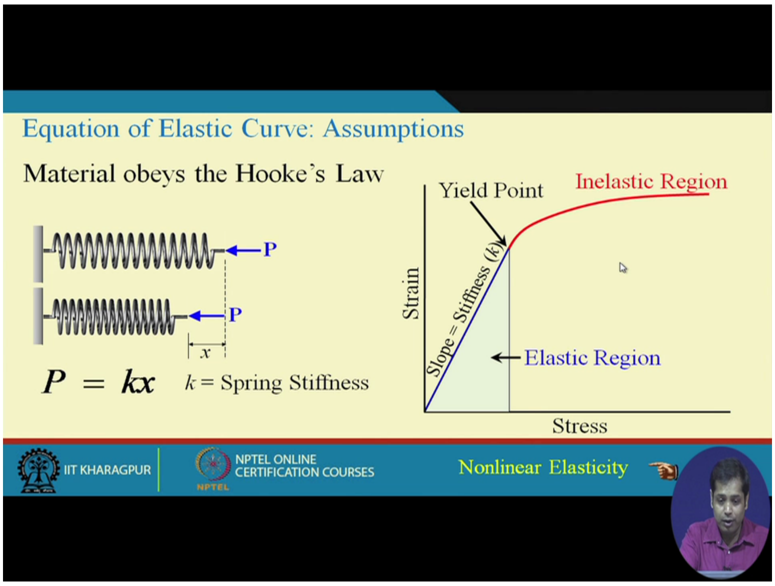

Okay, then the second assumption is curvature is small. This is the curvature of the beam. This is the deflected shape and rho is the radius of curvature and the curvature gives you 1 by rho. And it is assumed that this is very small.

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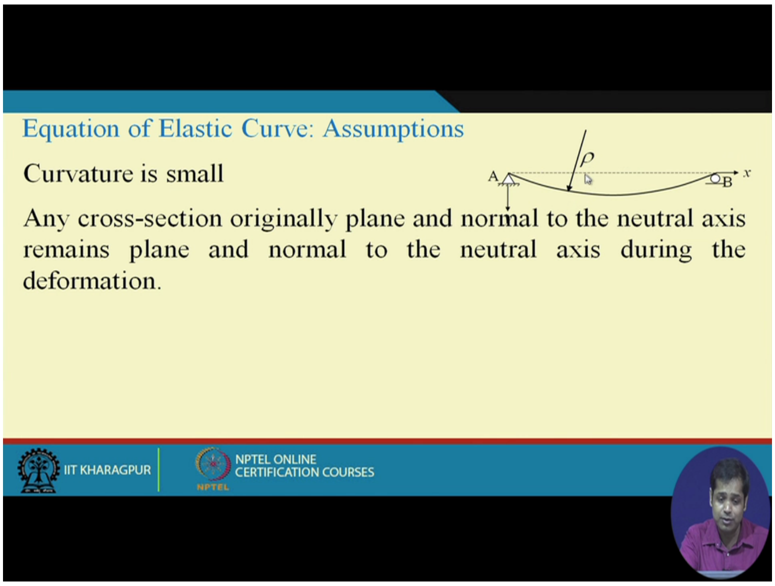

Now the third assumption is it is a very important assumption. These two assumptions are in some way related. Third assumption is very important assumption. What it says? Please note, any cross section originally plane and normal to the neutral axis remains plane and normal to the neutral axis during the deformation. Let us see what it exactly means. Now suppose this is a beam in undeformed configuration and if we apply some moment then it goes deformed configuration like this.

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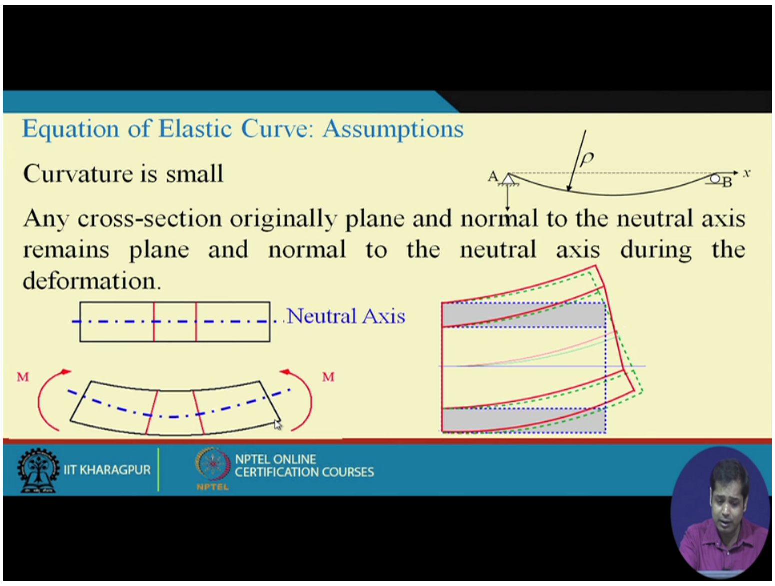

Now this is the centroidal axis or the neutral axis. Now if we take suppose two sections here and here which is normal to the neutral axis. So this angle is 90 degree and this is deflected shape of the neutral axis then also this section becomes this in deformed configuration. And if this is 90 degree, initially this 90 degree angle preserves. So it remains normal to the neutral axis at this point, okay.

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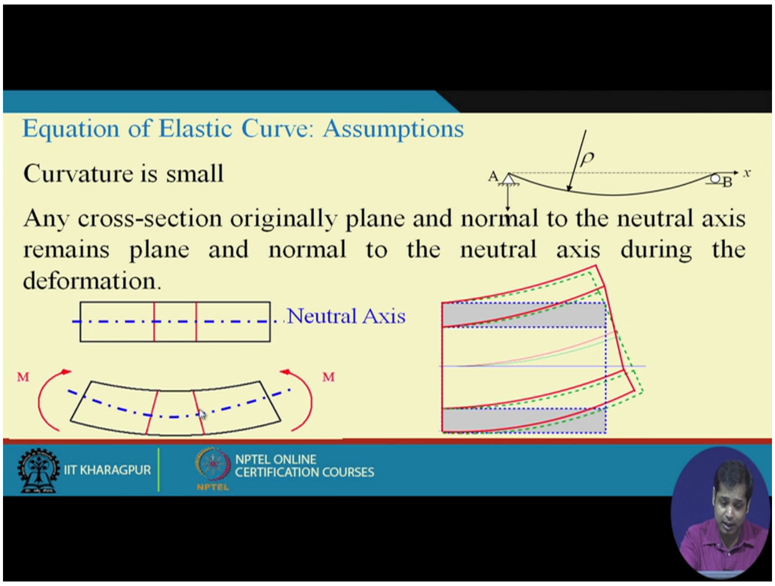

Now you see another implication of this assumption is suppose this is a beam and it is subjected to some load and it bends like this. Now two deformed configuration are shown. One is with bold red line and another one is dotted green line.

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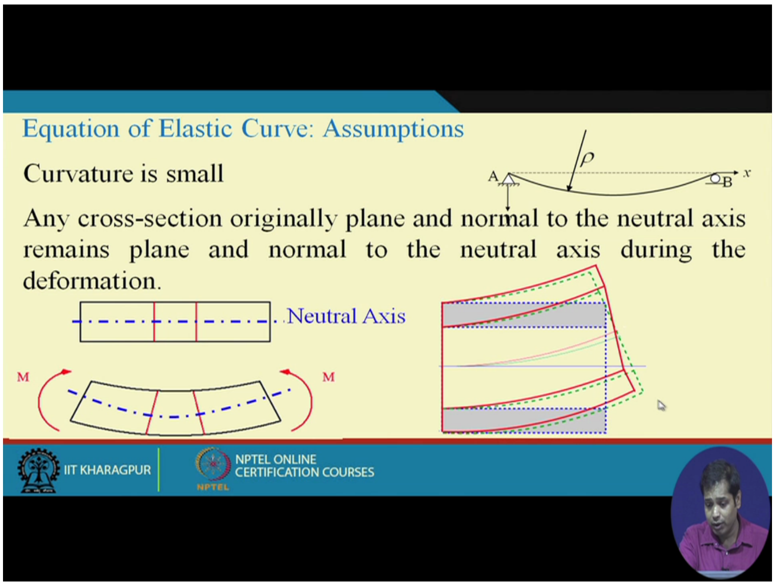

Now bold red line is probably the actual deformation that the beam may undergo. But this line is the blue dot line which is the cross section which is normal to the neutral axis. Now what happens when it undergoes deformation? In deform configuration it was initially undeformed configuration. This line is a plane but in deformed configuration the deformation which is shown in red line, this plane does not remain plane, right?

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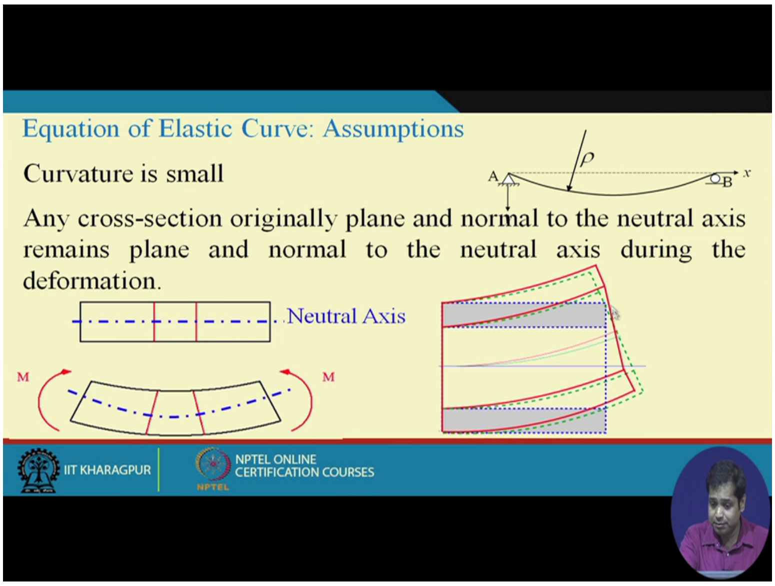

But our assumption is the green dotted line that if it is plane, it remains plane. So there are two important things here. One is a section which is plane, it remains plane. This is first thing. And the second is if the section is normal to the neutral axis in undeformed configuration it remains normal to the neutral axis even in deformed configuration. Now this is very important. The reason is my experience is whenever I have asked students to say the assumption the student says the plane section remains plane.

Plane section remains plane this is true but this is not the complete statement. This is an incomplete statement. What we have to say is the plane section remains plane of course but originally for undeformed configuration if it is normal to the neutral axis then it remains normal to the neutral axis even in the deformed configuration, okay.

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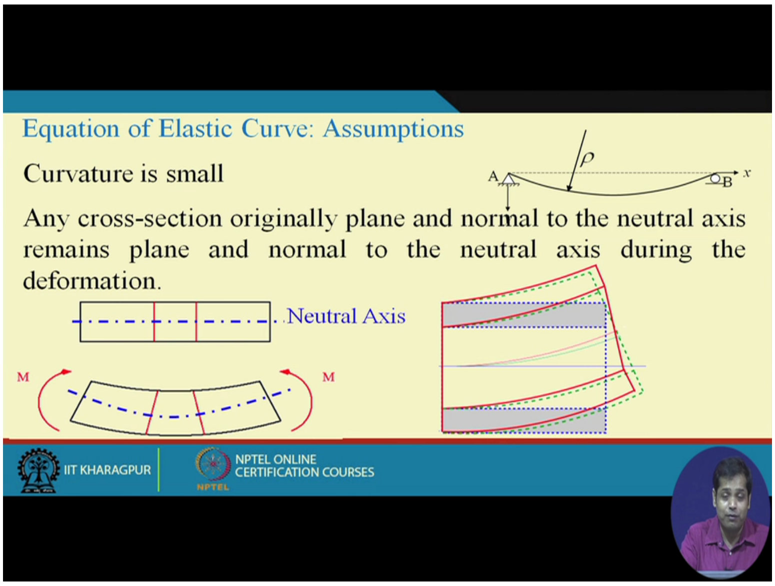

Now so these are the assumptions. As I said earlier assumptions are limitation of any model. We will see in some of the cases where these assumptions are not valid and therefore we may need a different theory to represent those physical processes, okay. But for our purpose these assumptions is reasonably enough because if we look it from civil engineering point of view if you remember there are two criteria for design.

Mainly two criteria, one criteria is the safety criteria another one is serviceability criteria. Safety criteria say that the internal forces in members they should not be more than the maximum capacity of those members. This is the safety criteria. One of the serviceability criteria is deflection should not be very high. Now that design criteria is very consistent with this assumption because anyway at the time of design we impose this restriction that your deflection should not be very large.

Therefore since the deflection should not be very large we are not going to allow deflection to be more. Therefore on those cases these assumptions are applicable. But remember beam not just belong to only civil engineering. Beam is a very important structural idealizing. It has applications in various fields. The concept of beam has application in various fields.

So there are cases there are (si) situations when the idealization is beam but the behaviour of that beam is not consistent with these assumptions and in those cases we need to go for different theories. But for us this is absolutely fine, okay. Now once the assumptions are made let us go with the derivations, okay. Now consider this is idealized model.

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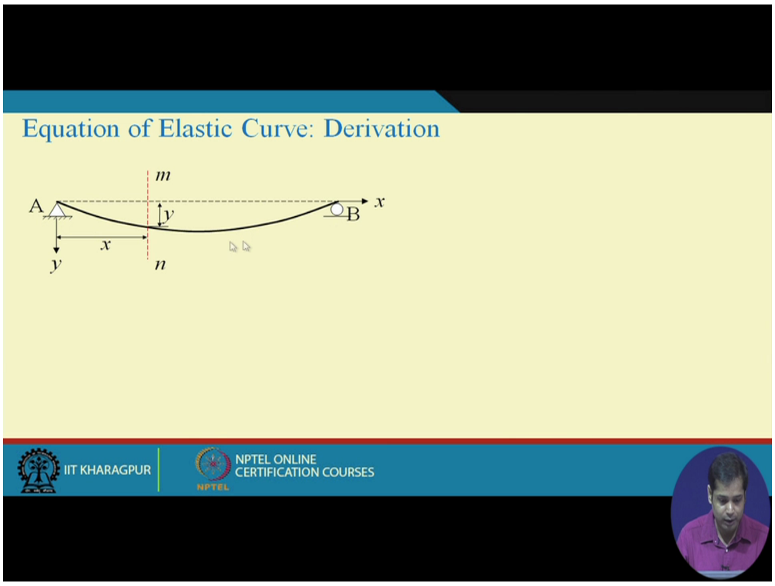

Now what we need to find out? We need to find out y as a function of different parameters, right? Okay, this is our objective, okay. Now you see the radius of curvature you can determine this is a very standard expression. If y is a function of x then the radius of any curve can be represented by this, okay.

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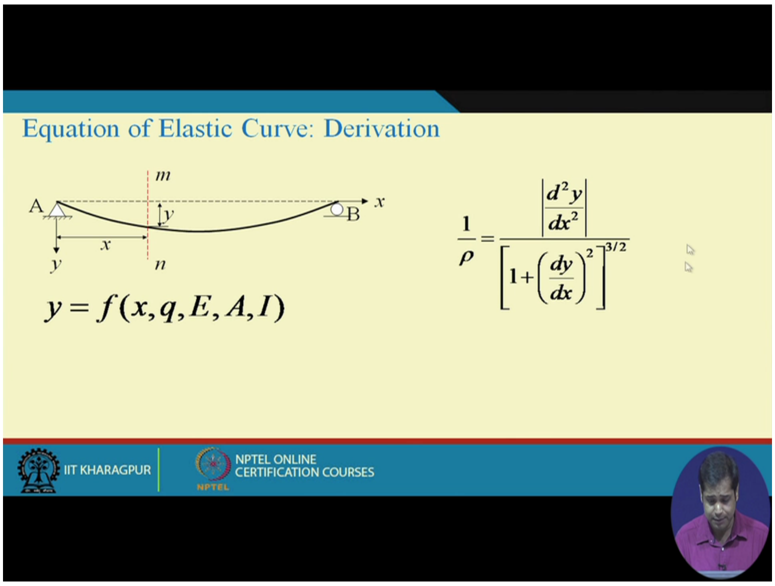

This is a very standard result. But again we assume our deflection is small. That is what just now we discussed the assumptions. Deflection is very small and if the deflection is a very small then this radius of curvature this expression can be idealized as deflection at the slopes are very small. So this part gives you the slope is very small and the deflection is very small so this radius of curvature can be approximated as this, okay.

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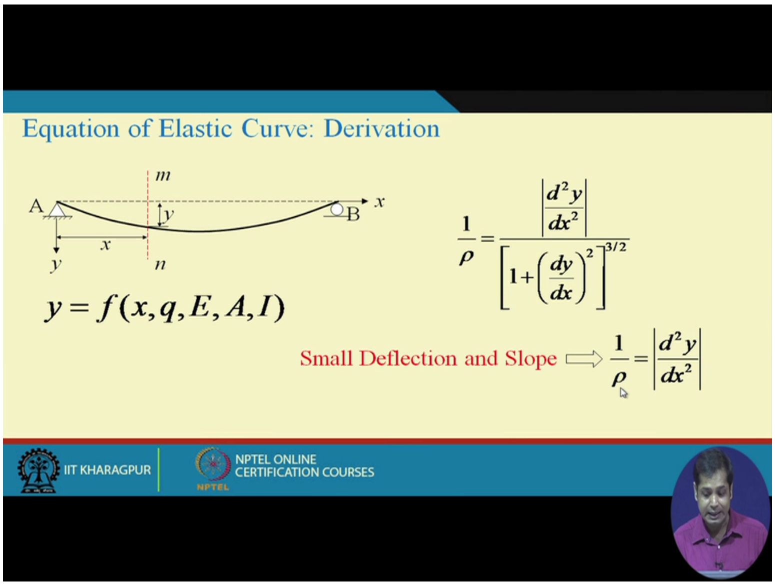

You put a star mark here, this absolute value. We will discuss what will be the sign of this radius curvature as we move, okay. So now radius of curvature is this, okay. Okay, now considered this is a beam which is subjected to pure bending, okay.

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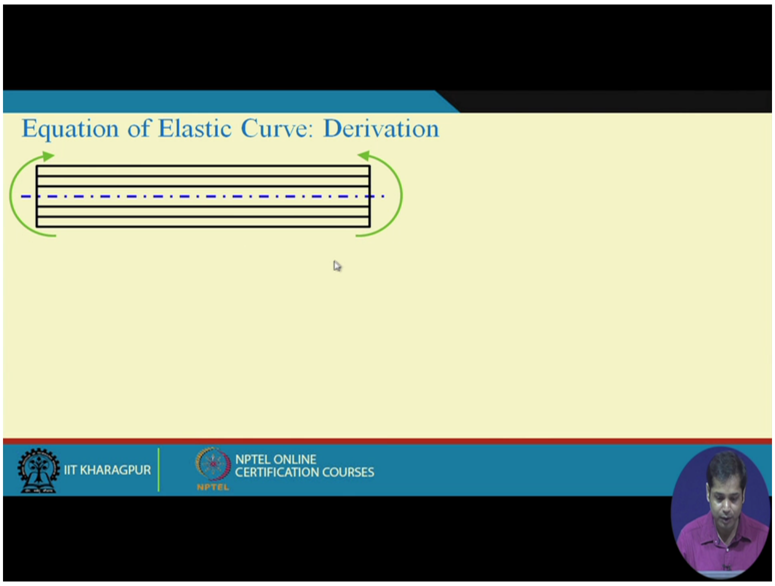

Means there is no shear force, only bending moment. This is the neutral axis the blue line that you can see. Now we take some lines you assume this beams are essentially a collection of certain fibres (ac) across the across the depth. So we will take some fibres above the neutral axis and then some fibres below the neutral axis. And the planes are in this direction along the length of the beam.

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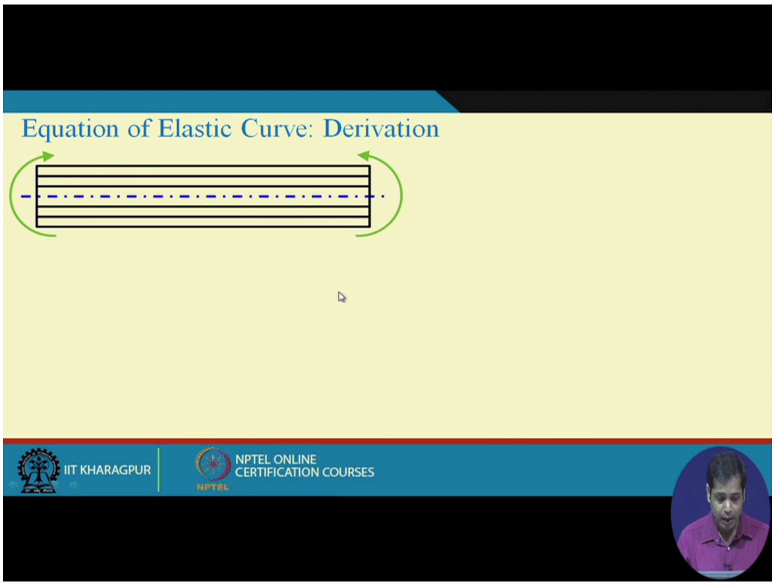

Now if it undergoes deformation. This is undeformed configuration and this is the deformed configuration, okay. This is the deformed configuration.

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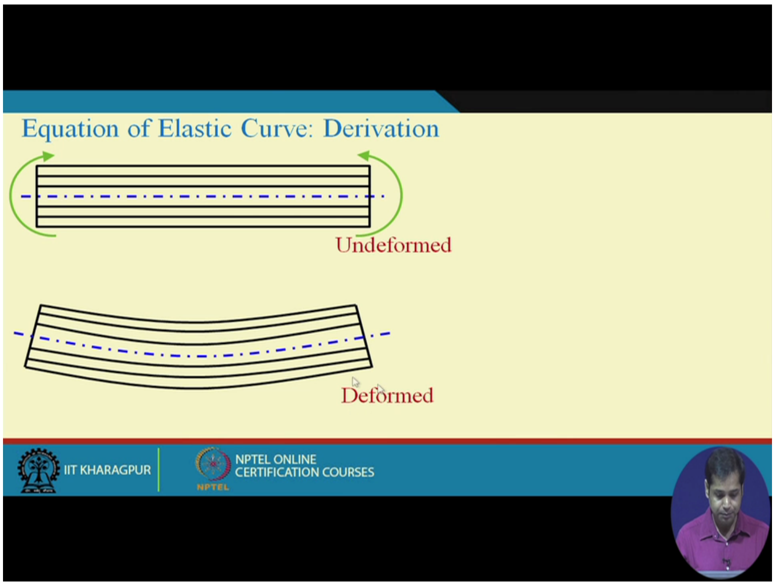

Now you know this but still in order to appreciate theory in a better way we are again discussing all these things. Now you see for this kind of bending the sign convention that we follow that sagging moment is positive and hogging is negative. Above the neutral axis all these planes all these fibres are under compression, okay.

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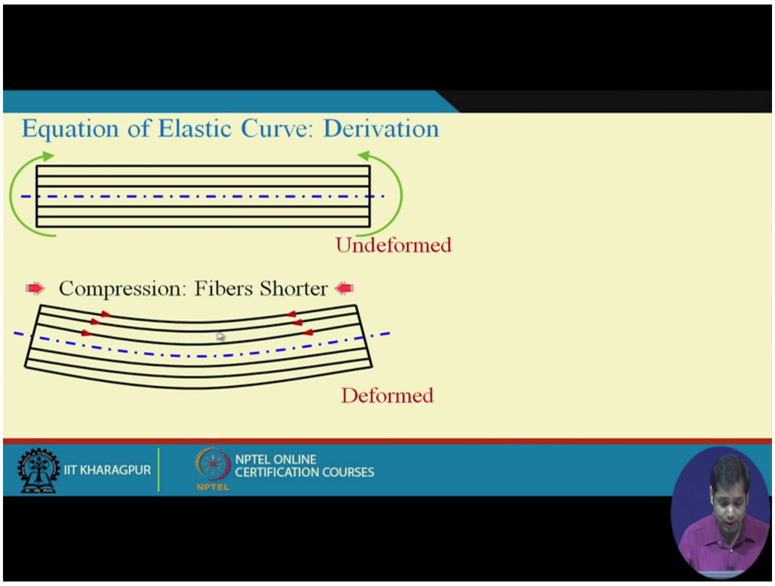

So their length decreases. Whereas all these fibres below the neutral axis they are under tension. This is not shorter this is longer, okay. There is a typo here please note. This should be longer, okay.

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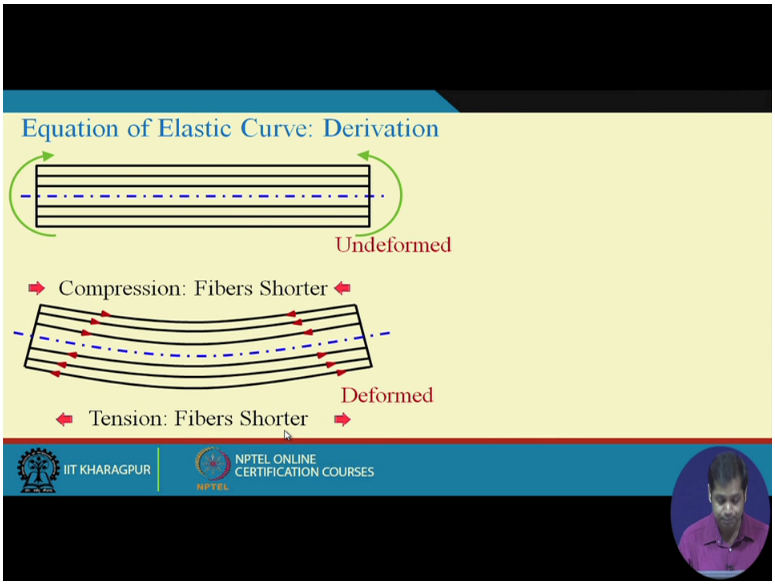

So this is in deformed configuration. Now what happens this is neutral axis, okay. Now we take two sections, a section m-n and p-q. Or we take a small element in this beam, m-n q dash and p dash. And correspondingly that element becomes this, okay.

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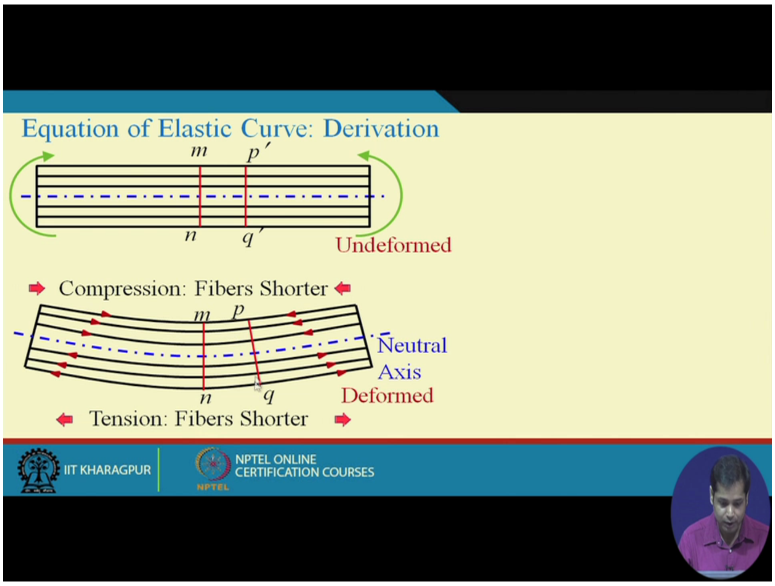

Our assumption was these are normal to the neutral axis. These remains plane and again it normal to the neutral axis. And suppose the p dash q dash which is undeformed configuration, in deformed configuration suppose this is p-q. And p dash q is undeformed position as shown in dotted lines. So p dash q dash now becomes p-q in deformed configuration, okay.

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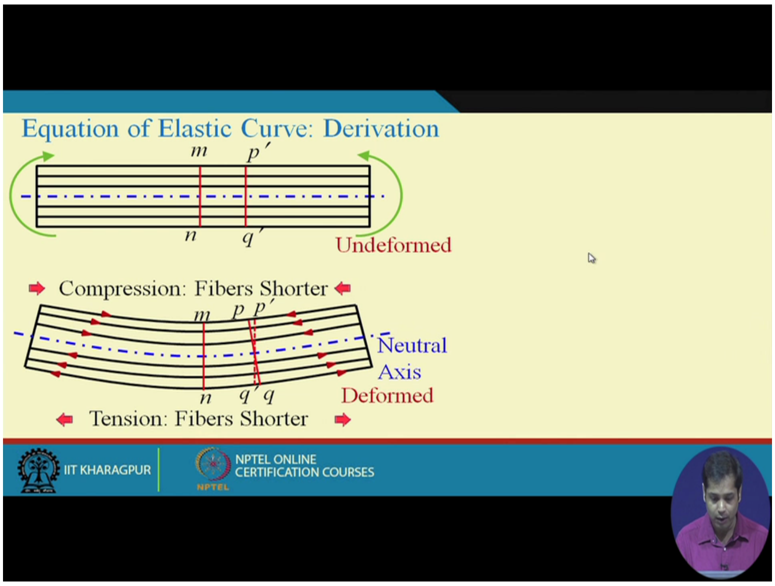

Now suppose the distance between these two points A and B is delta x, okay, or dx, okay. Now what we do zoom this part and draw it here. What are the things we have?

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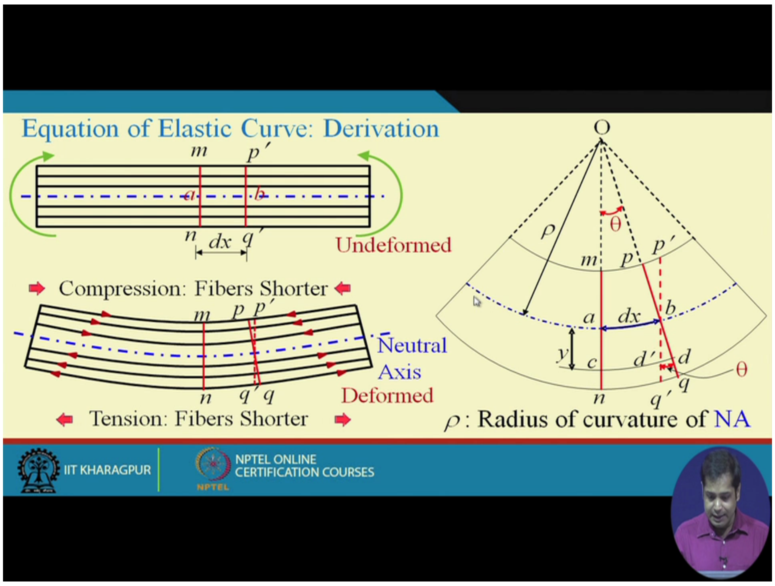

This is the neutral axis, this is larger view of this part, okay. This is the neutral axis in our top fibre which is under compression and bottom fibre which is under tension. The top fibre the length decreases and the bottom fibre length increases or for that matter all the fibres below the neutral axis their length increases and the above the neutral axis they decreases.

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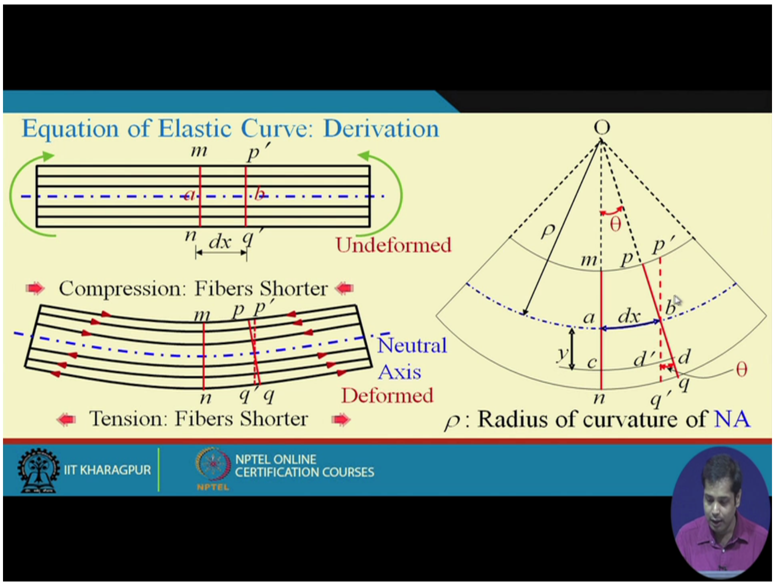

M-n and p dash q dash are these two planes. And p dash q dash they take a form p-q, okay. Now once they take a form p-q suppose this angle between this m and p-q is theta, okay. Or it is d theta. That is better way of expressing this. This is d theta.

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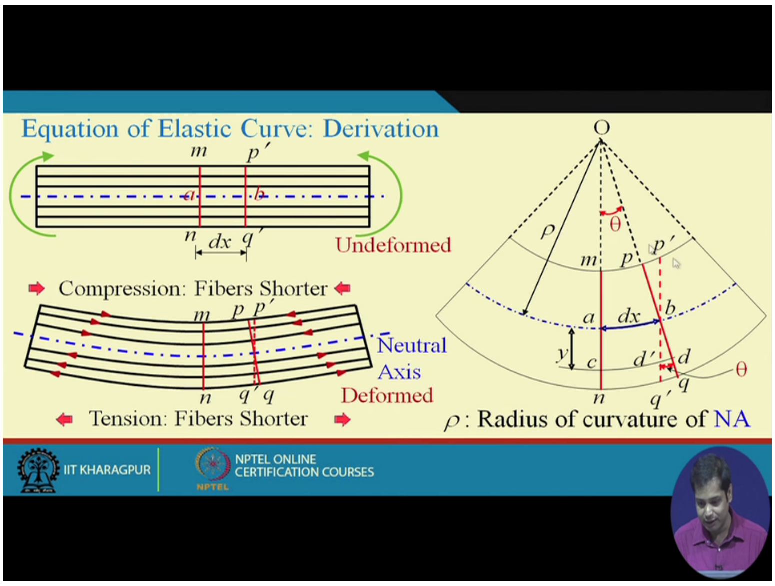

Therefore this angle is also d theta, okay.

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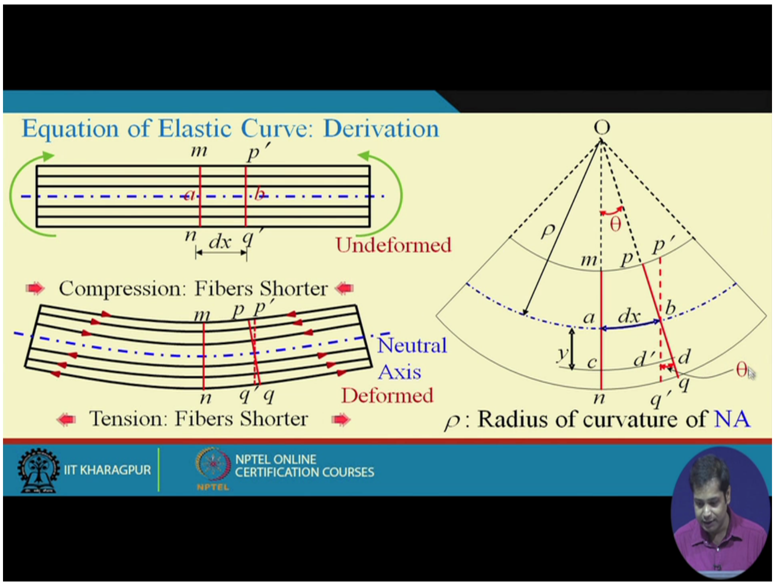

Now this is the position of p-q in undeformed configuration and their initial distance was dx. This is A and this is B, okay. And suppose this rho is the radius of curvature of the neutral axis, okay. Now we take a fibre at a distance y from the neutral axis, okay. Now since this fibre is below the neutral axis what happens is the initial length of the fibre was c-d dash. This length which is dx.

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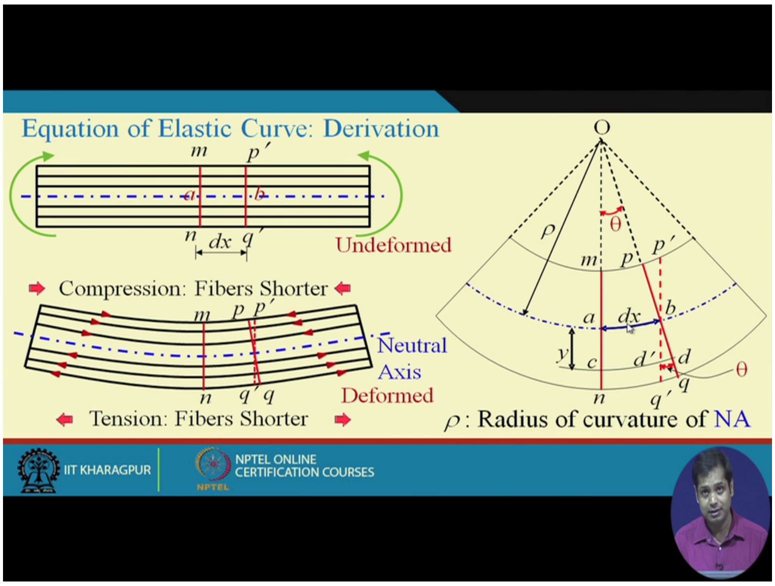

Now this c-d dash takes a form c-d. So c-d dash now elongated to c-d and the elongation of c-d dash is d-d dash, okay. So c-d dash is under tension and because of this tension the elongation takes place is d-d dash, okay.

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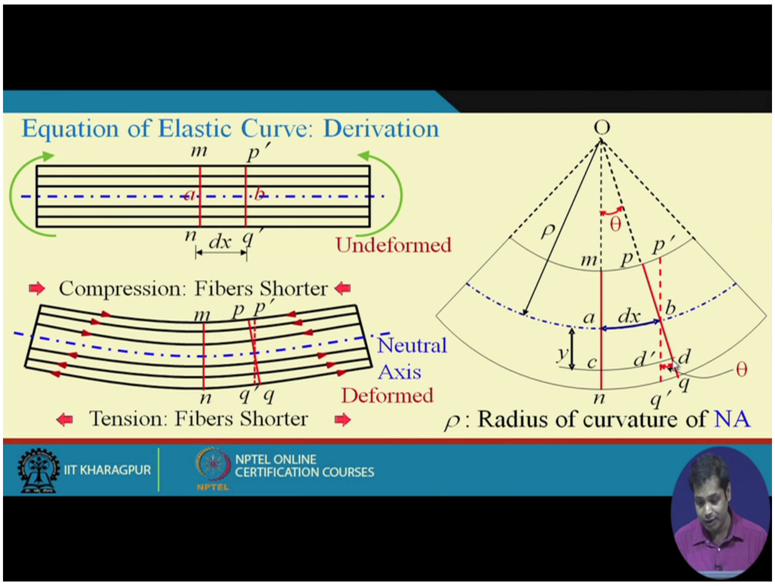

Now so this is the d theta, okay. Now if this is d theta and this is dx and rho is the radius of curvature so naturally we can write dx is equal to rho d theta and which gives you d theta dx is equal to 1 by rho. 1 by rho is the curvature.

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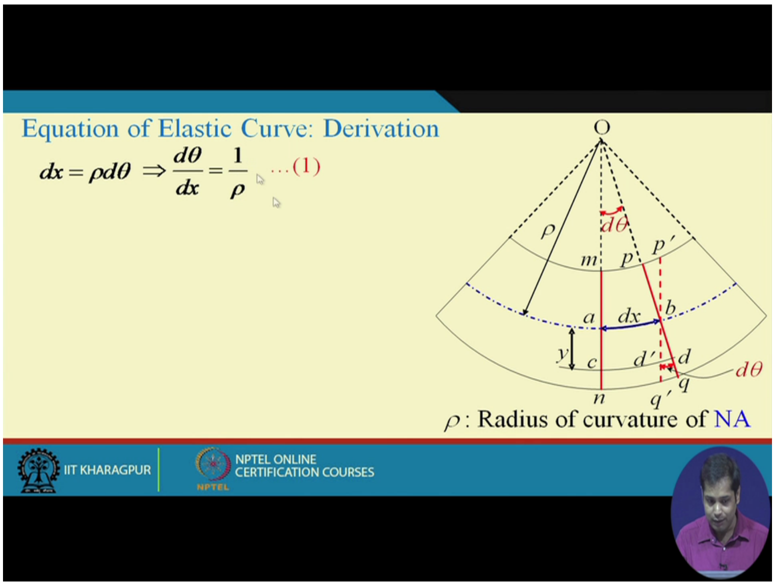

Now as I said the fibre c-d dash is elongated to c-d which is essentially c-d dash plus d dash d and since this angle d theta, d dash d can be written as y into d theta, okay.

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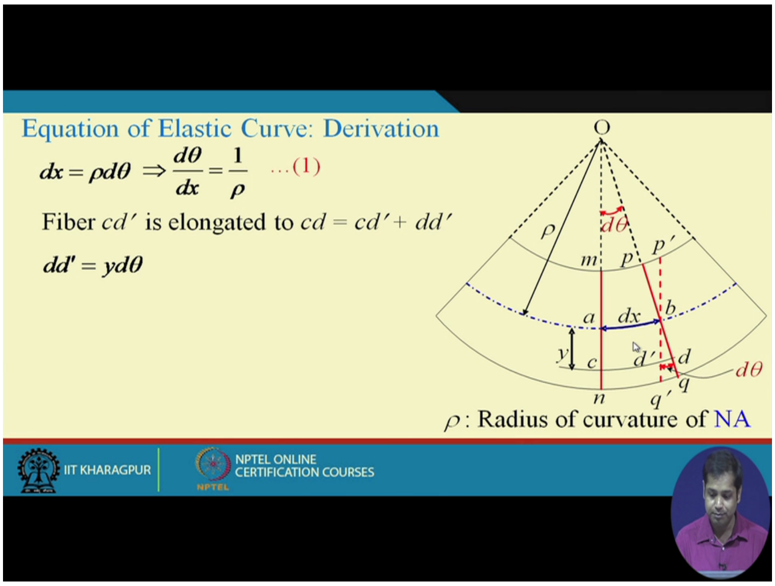

Now then what is the longitudinal strain in this fibre? This is under tension and the longitudinal strain in the fibre will be change in length divided by the original length. The change in length is d-d dash and original length is (del) dx so this is the longitudinal strain.

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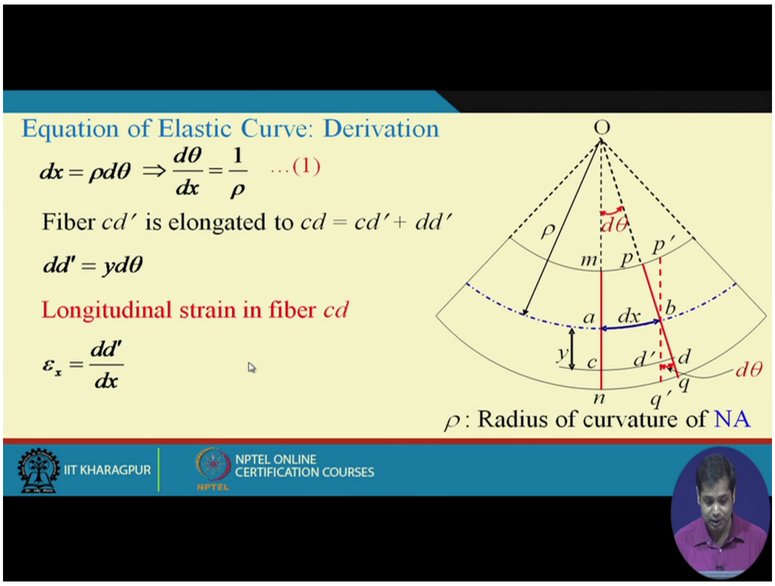

And d-d dash we just determined y d theta and this becomes y d theta. And from (ex) one we can write this elongation is equal to y by rho.

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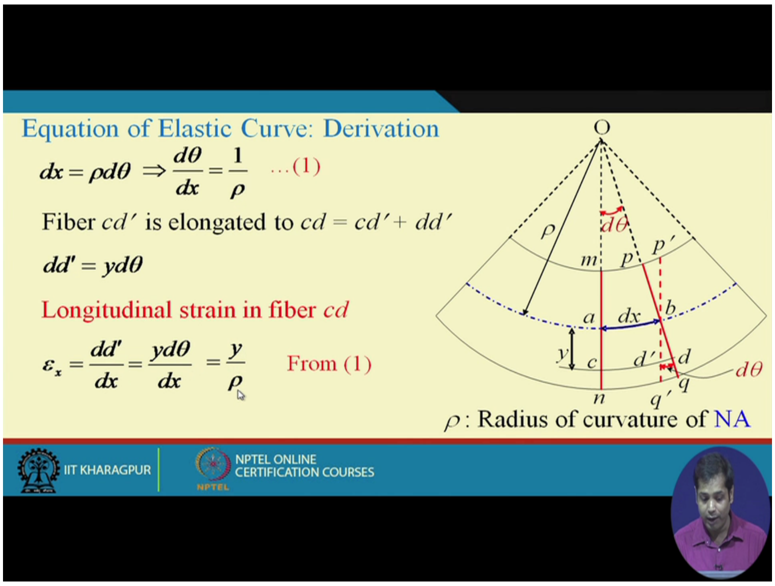

So one important observation here is when y is equal to zero then epsilon x is equal to zero. It says that at the neutral axis your strain is zero and y is equal to positive epsilon x positive it is tension, y upward direction negative (conv) in this case it becomes negative. So it is under compression and between top fibre to bottom fibre how do strain depend on y? The strain depends linearly on y. So the variation of the strain will be linear.

And the strain at the neutral axis is zero that is why it is called neutral axis. It remains neutral when the beam undergoes (deform) deformation, okay.

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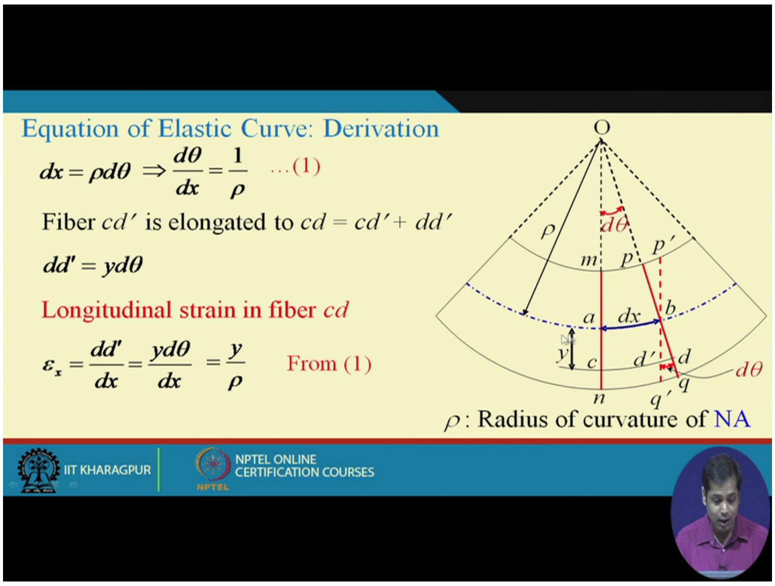

Now longitudinal strain in fibre is this. What will be the longitudinal stress in fibre? Longitudinal stress in fibre is this. This is Hookes Law that stress is equal to Youngs modulus into strain. So epsilon x if we substitute then stress is equal to this.

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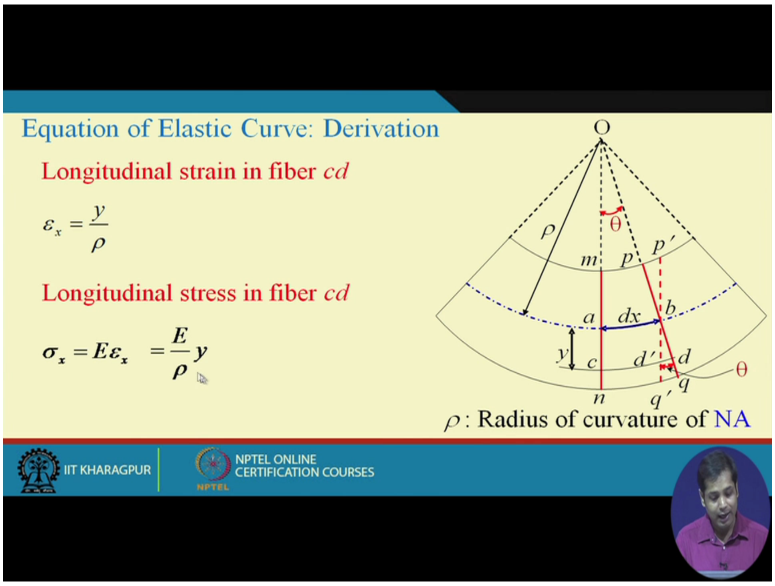

Again the same observation for stress. Stress is linearly varying across the depth and at y is equal to zero at the neutral axis stress is zero it remains neutral, okay. Now longitudinal strain and longitudinal stress we have longitudinal stress is this, okay. Now consider this is the beam and suppose this is the cross section of the beam, right?

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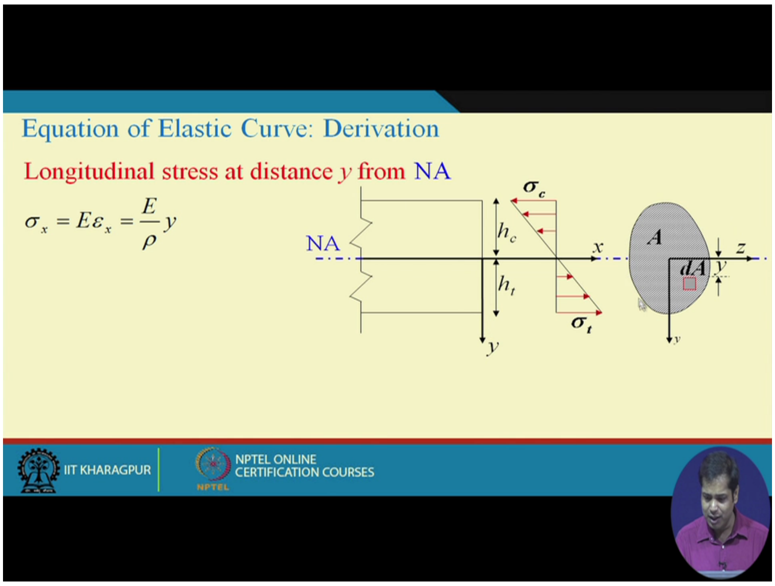

And this is the distribution of stress which is varying linearly across the depth, at the neutral axis it is zero. Sigma c is the maximum compression at the top most fibre. Sigma (tens) t is the maximum tension at the bottom most fibre. This is x direction and downward y for deflection is downward we say it is positive which is consistent with our sign convention in moment because we say the sagging moment is positive.

Sagging moment means which causes sagging in the beam. Sagging in the beam means it deflects downward. That is why when it deflects downward the deflection is considered as positive, okay.

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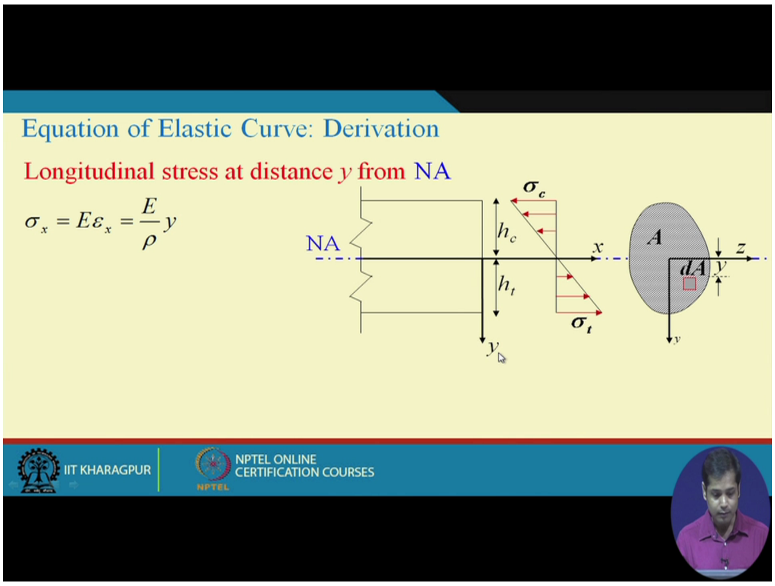

Now sigma x is this. Now the body is in equilibrium, right? When the body is in equilibrium then it should satisfy certain equilibrium condition. Summation of Fx is equal to zero, summation of Fy and summation of moments are is equal to zero. Summation of Fx is equal to zero says that total compression will be equal to total tension. You can verify this.

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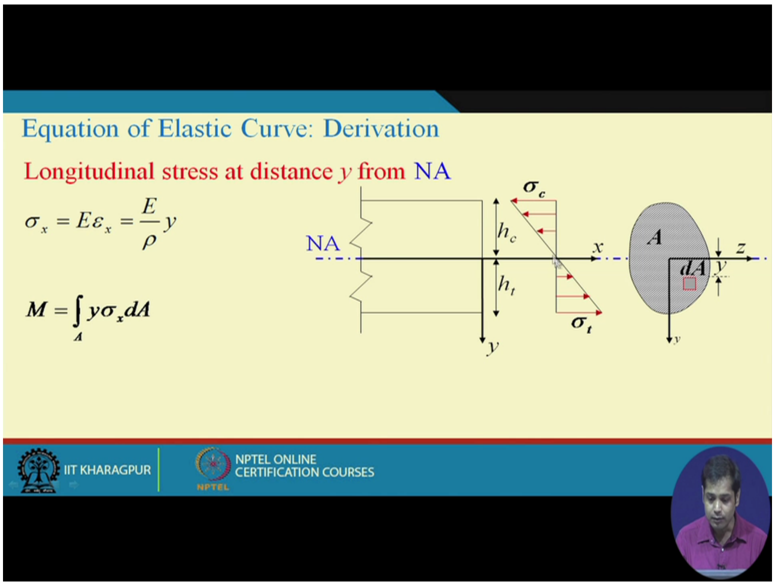

Now and then what is the summation of moment? Summation of moment is we have a (haggin) sagging moment M here and the induced stresses this sigma.

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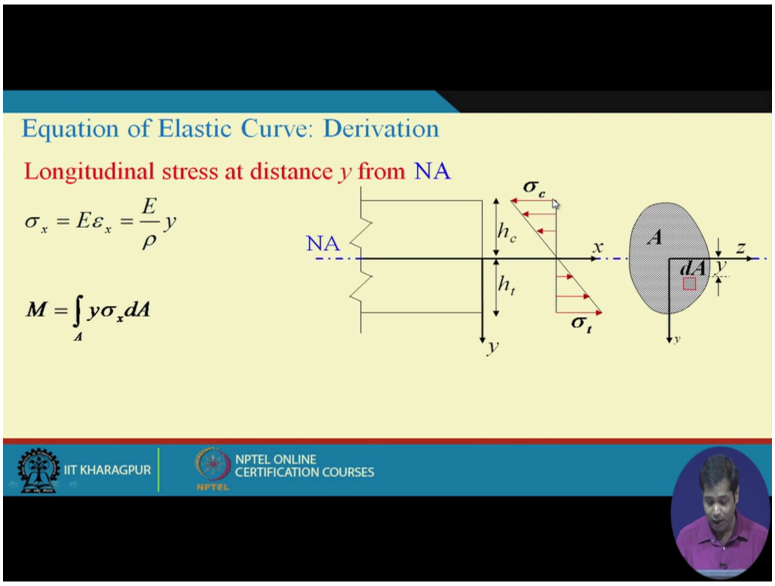

So whatever moment generated due to the stress that should be the moment at this section, okay. So, how to calculate that moment? Moment will be if you take any small section delta A then what will be the force in this section delta A? Force will be delta x into dA. DA is the cross section. This would be the force.

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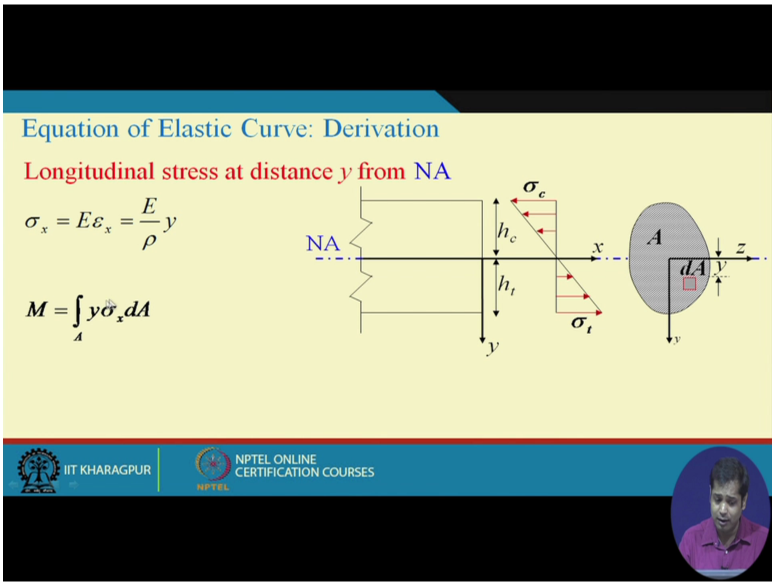

And what will be the moment of that force? Cross section is at a distance y from the neutral axis, the force will be y into this force and that will be the moment due to force on this area delta A.

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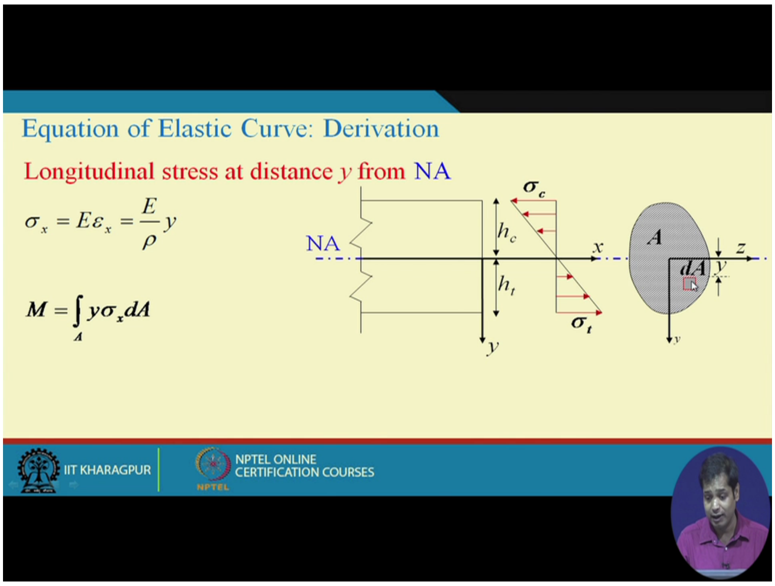

If we integrate over the entire area that will give the total moment. Sometime that is why this moment, shear force they are also called stress resultant because they are resulted due to these stresses. So this will be the moment, right?

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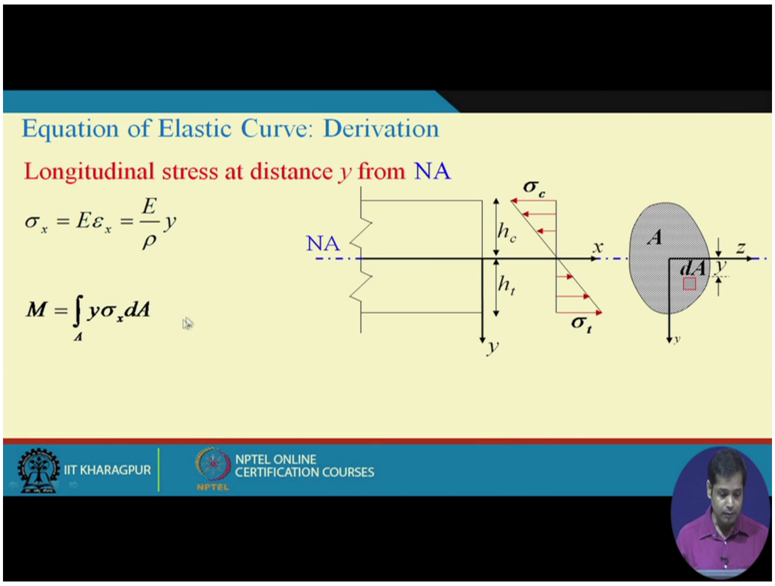

Now if we substitute the expression for sigma x from this equation then this gives you E by rho y square dA. Now you are familiar with this term. Integration over area y square dA, this term is called second moment of area, okay. In this case the second moment of area about this axis, okay.

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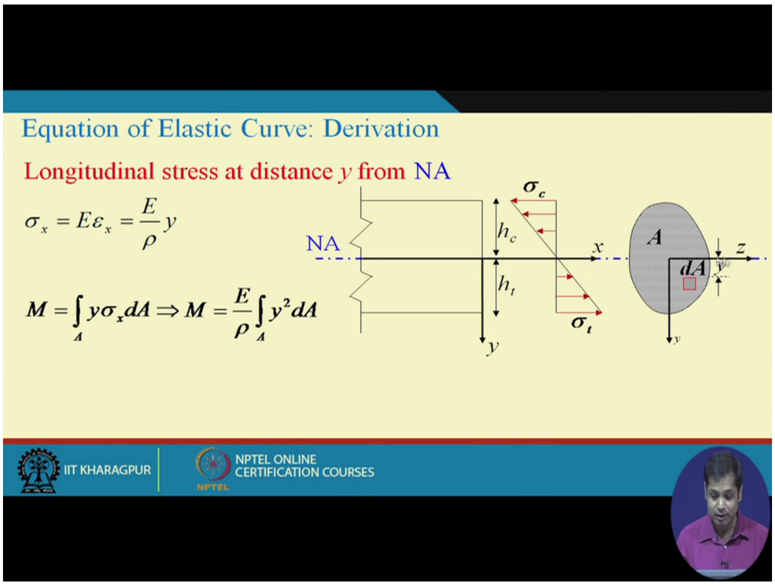

So generally second moment of area is represented by I. So it becomes that if we replace integration of y square dA by I and then this equation becomes 1 by rho is equal to M by EI, okay. And we already know 1 by rho is equal to this for small deflection and small slope 1 by rho the curvature can be approximated at this.

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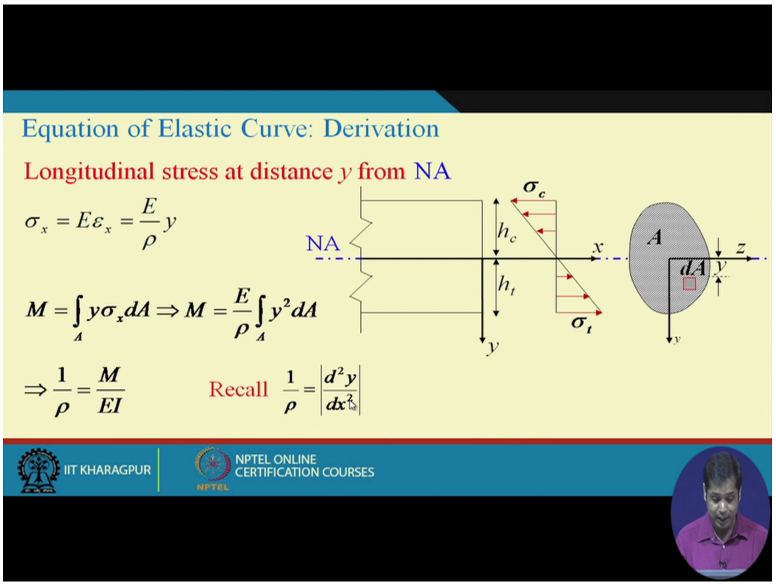

Now if you substitute this in this equation finally we get d2y dx2 is equal to this, okay.

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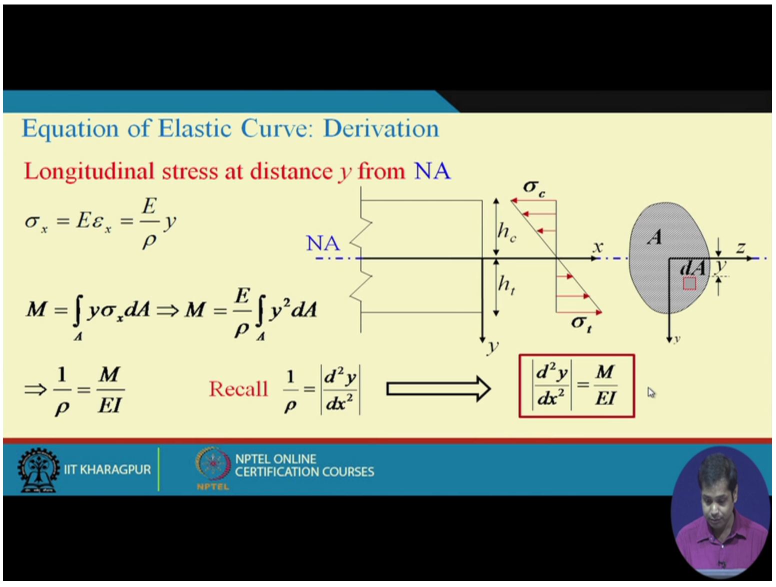

Now this is the relation but still it is not the final form. We will see the final form. So we get M by EI is equal to absolute value of d2y dx2, okay. Now let us see what would be the sign of this, okay. Now this is our deflected shape.

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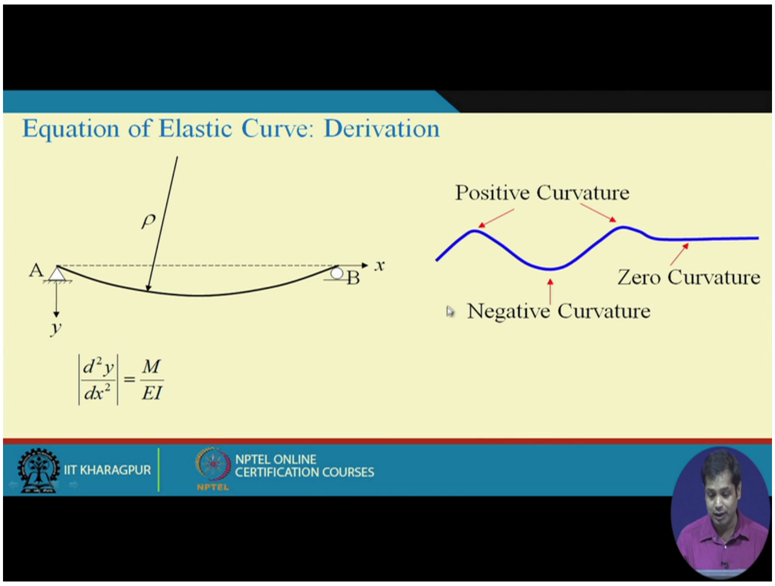

Now how we define this sign of curvature? For instance if this line shows you different curvature, if the curve is like this, the curvature is positive. If the curve is like this the curvature is negative and if it is a straight line then the curvature is zero.

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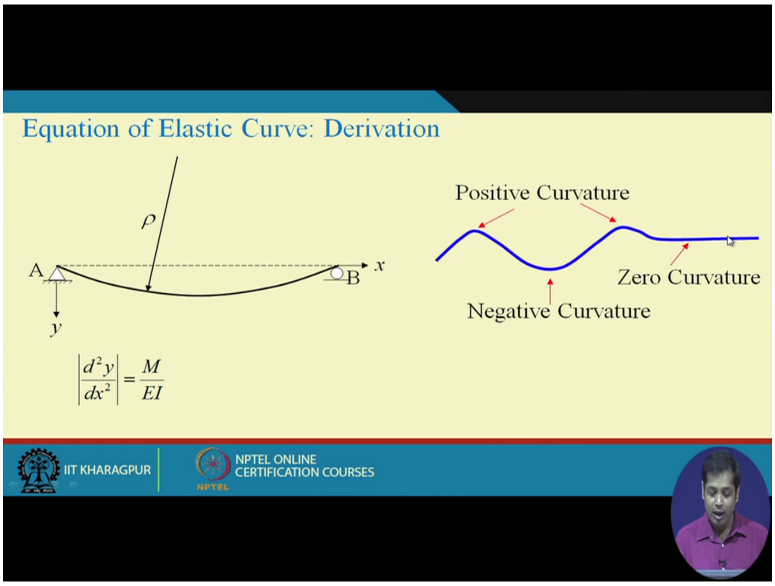

Now we assumed when moment is positive then your deflected shape is this, right? So when moment is positive deflected shape is this means curvature is negative. So moment is positive means curvature is negative and similarly if you take hogging moment, hogging moment will cause this kind of deflection which is positive curvature. And hogging moment is negative moment. So when the moment is negative, curvature is again positive, right?

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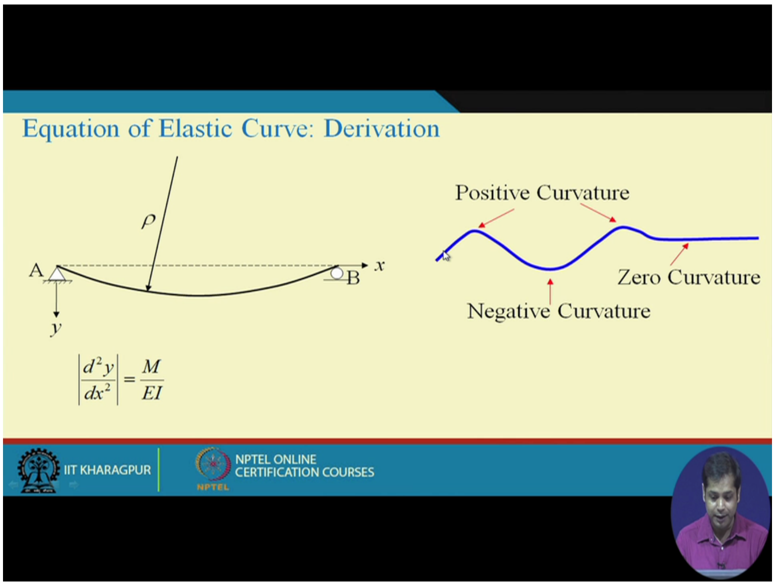

Now what happens therefore moment in curvature they are always in opposite sign. So if we know that if M is positive, they are in opposite sign so this can be written as this. The sign of this will be minus of this, okay.

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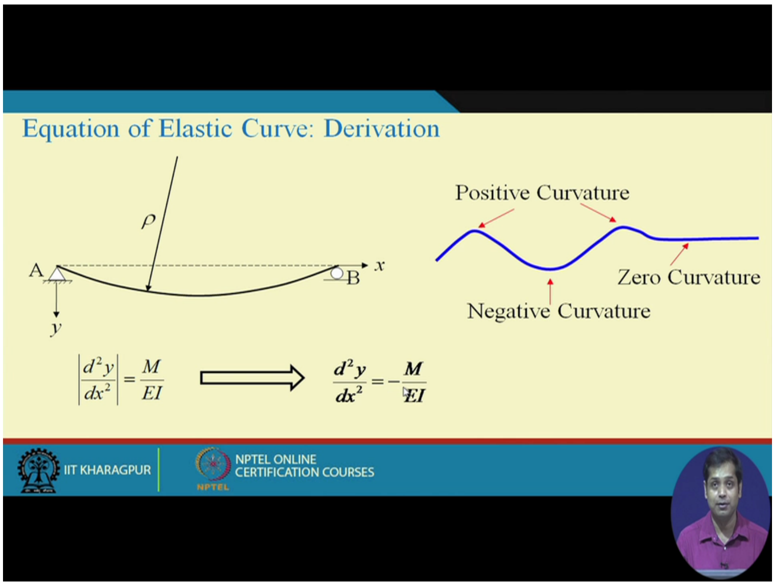

This is a very well known equation. This is called Euler Bernoulli Beam Equation, okay.

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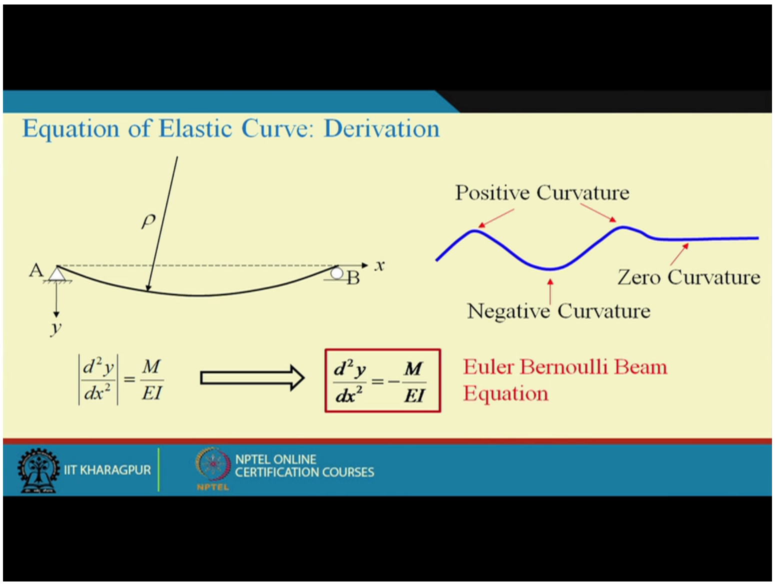

Now there are other forms of the same equation as well. Let us derive those equations. Suppose consider this is the small segment of the beam and which is subjected to (mo) (hog) sagging moment here M at A and then the moment is continuously changing so at B suppose the moment is M plus dM.

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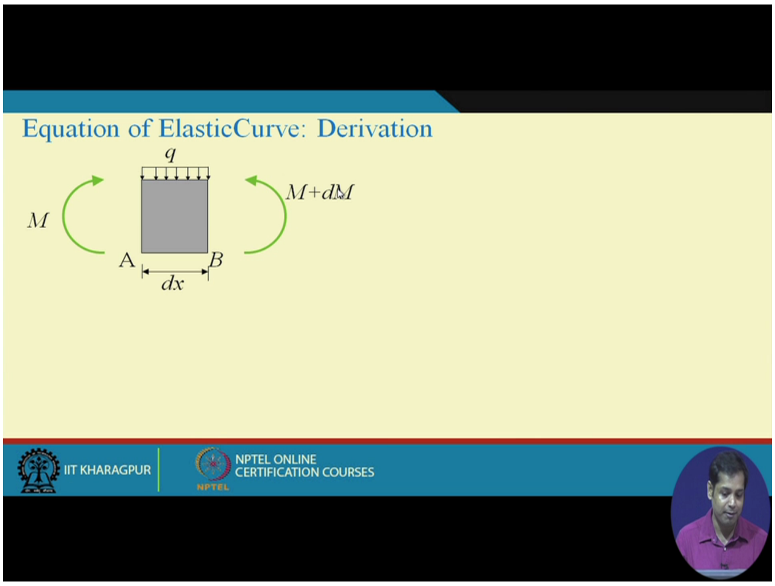
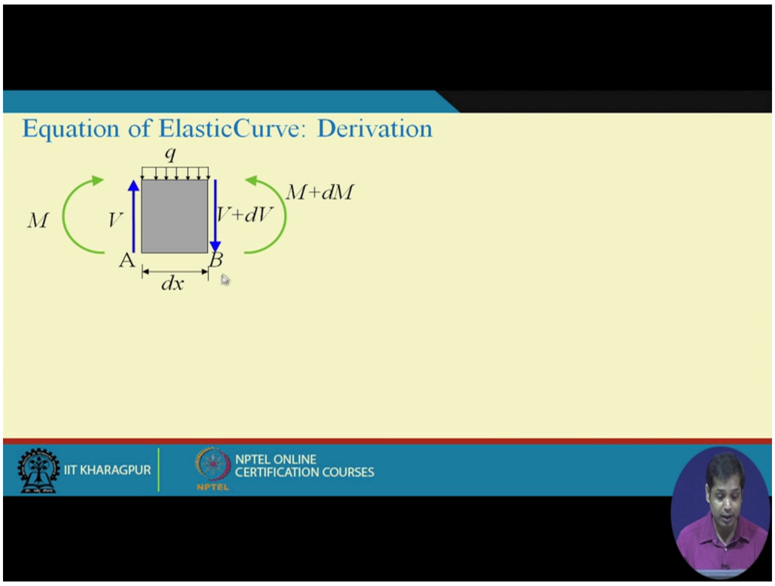


Similarly at A shear force is V and at B shear force is V plus dV. It is continuously changing. Now this is the free body diagram of any segment in a beam. Apply the equilibrium conditions on this free body diagram.



Summation of MA is equal to zero. If we take summation of moment about A is equal to zero then it gives you this moment which is clockwise, then anticlockwise M plus dM minus, then this force will not contribute any moment, okay.

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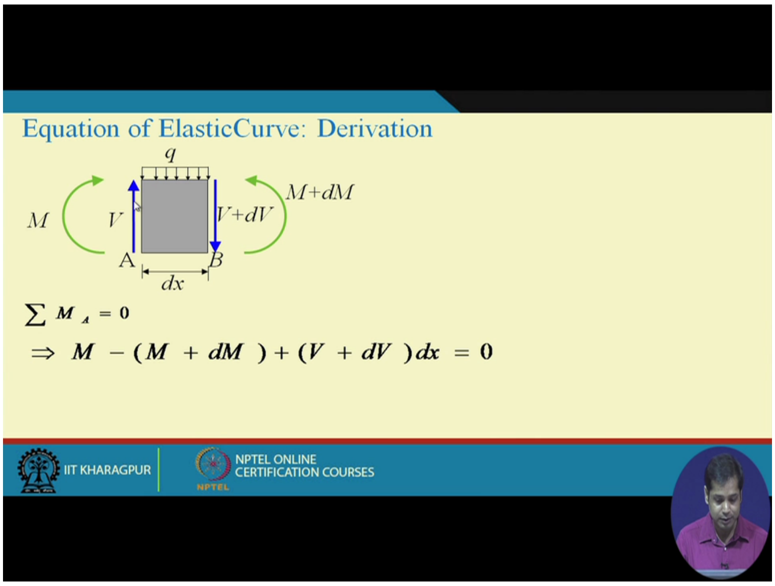

And then this force will contribute is V plus dV into dx. In this equation you consider this is under (bend) bending only. So there is no force like this, okay.

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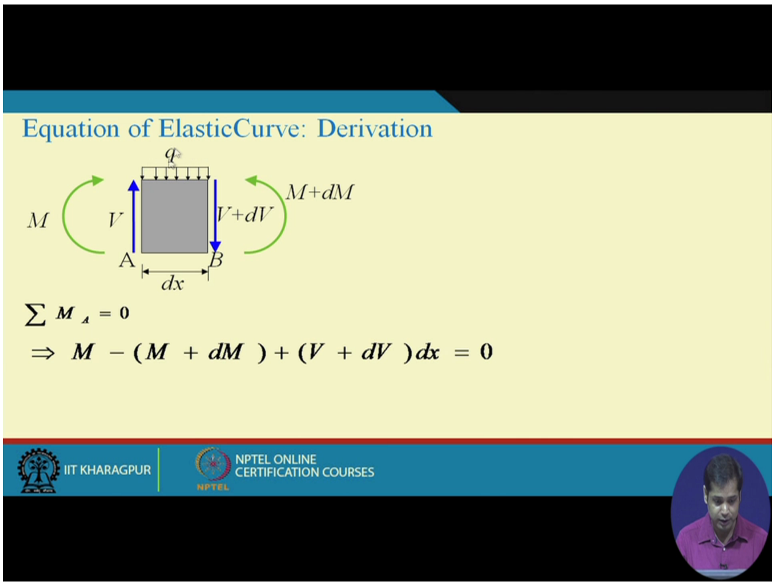

If we substitute this now what happens is see dV is very small quantity and dx is also very small quantity. So when dV is multiplied with dx then it becomes even smaller. And in that case we can neglect it and if we neglect it then only non zero part becomes this and this gives you V is equal to dM dx.

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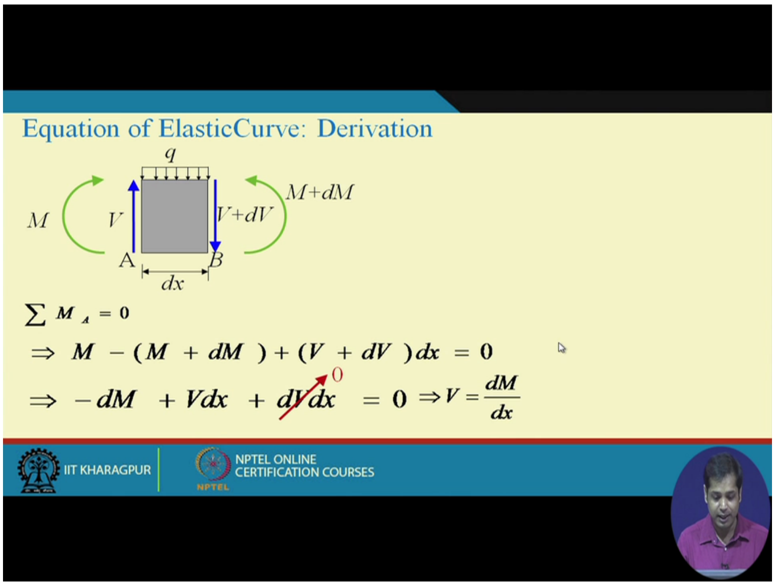

And similarly if we apply Fy is equal to zero and actually instead of taking moment about A better you take moment about the midpoint, okay. Then probably it is better because if we take moment about midpoint the contribution of this will go.

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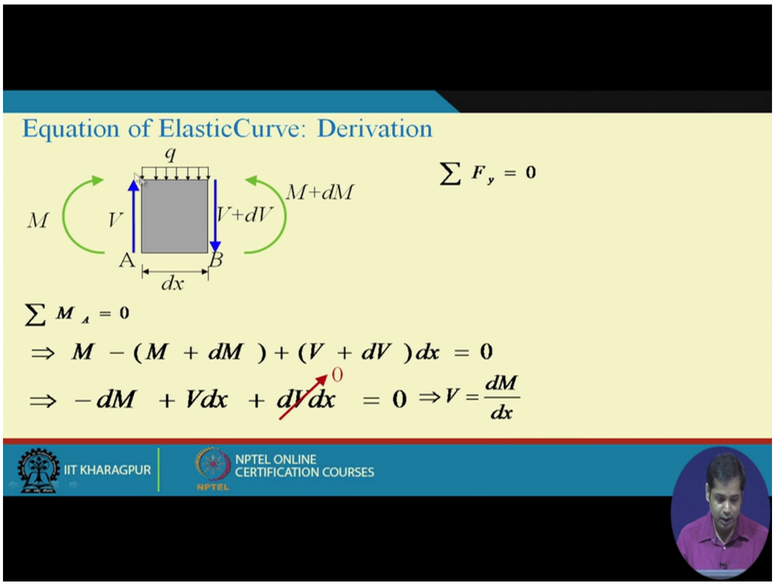

And then you can arrive at this (ex) expression. Take Fy is equal to zero and this is all the forces in Fy direction and this gives you q is equal to minus dV dx. It is just the brief review of the equation of elastic curve. You might have derived this expression in strength of material course or mechanics course.

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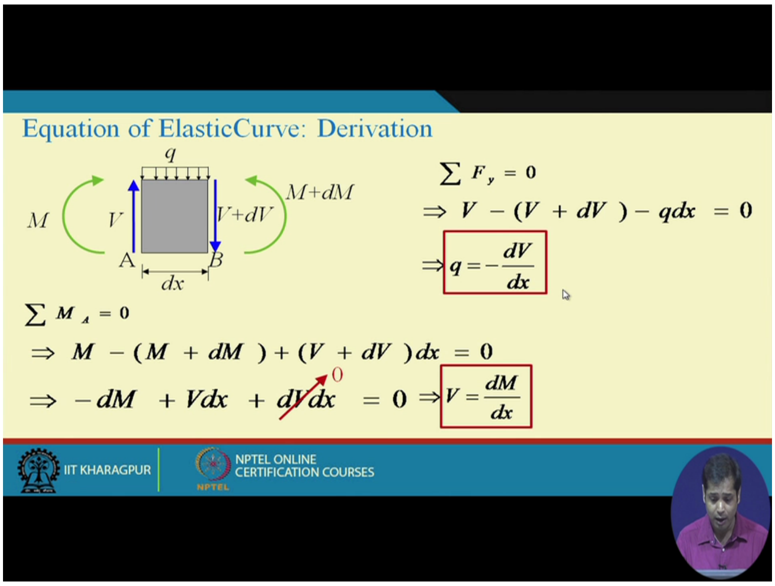

Now this is very important. If you remember when we were drawing bending moment and shear force diagram we observed that the bending moment is always one order higher than the shear force and this expression explains that, okay. Now these are the expression just now we derived and this is the expression that also we derived.

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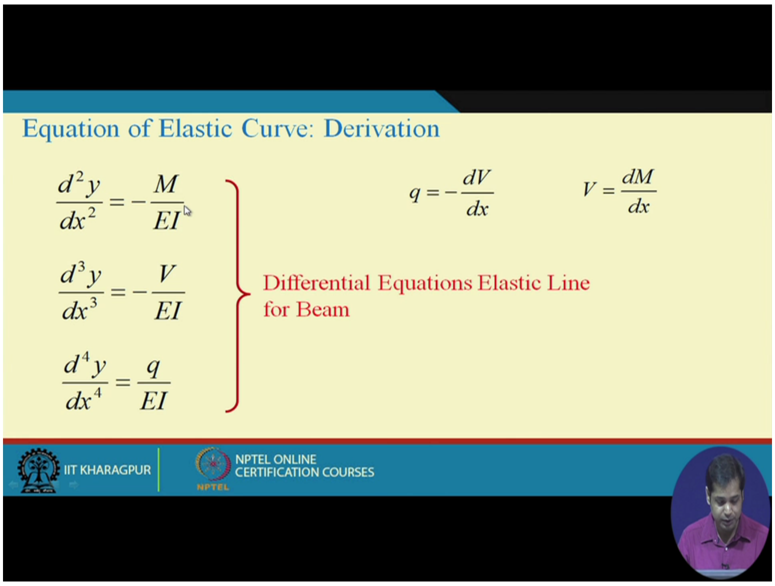

Now substitute this into this equation we get this and then substitute this into this equation will get this and again substitute this into this equation we get this.

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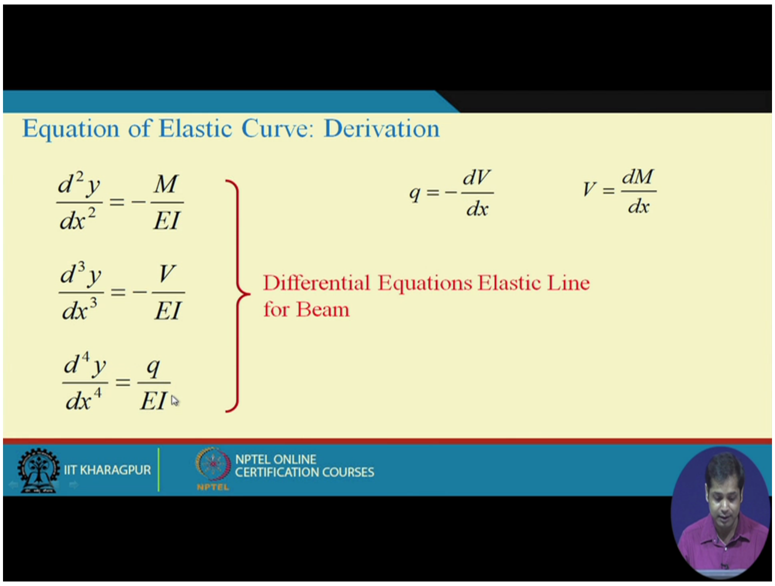

Now these are the differential equation of elastic line for beam or differential equation for elastic curve. Now so these equations are essentially the mathematical model which represents the deflection of a beam subjected to transverse loading. Now any form can be used and which form to be used that again depends on what information you have about the problem, okay.

Now you see so what we do is next class we will use these equations to find out deflection for different beams.

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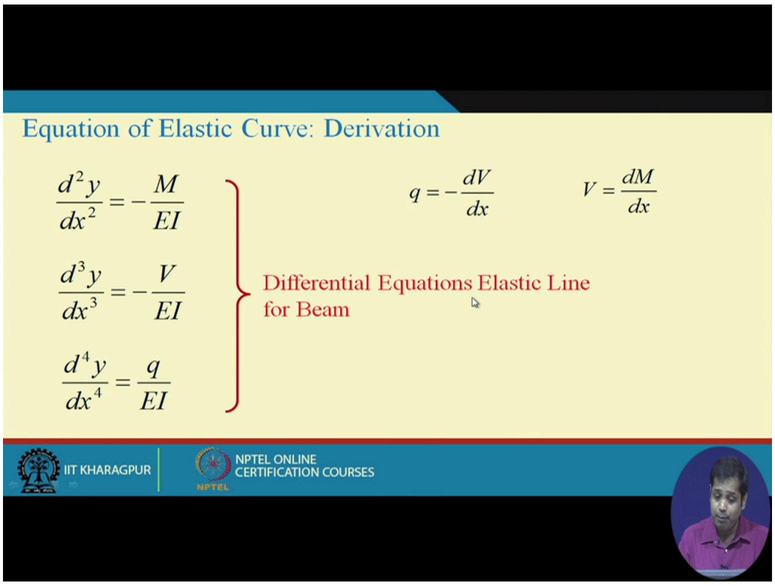

But before we stop today one important point see, for any theory assumptions are essential limitations, okay. Let us see what are the limitations? This theory is based on certain assumptions and many cases those assumptions are not applicable and there this Euler Bernoulli beam equation cannot be applied. We need to go for some other beam theory which can release those assumptions in Euler Bernoulli expression.

Now you will learn those different theory in different subjects but the idea what I am going to show you just one case where these assumptions may not be valid, okay. Suppose this is a representative figure, okay. Suppose you have a pipe network like this, okay. And the water flows through this pipe with very high speed for instance, okay.

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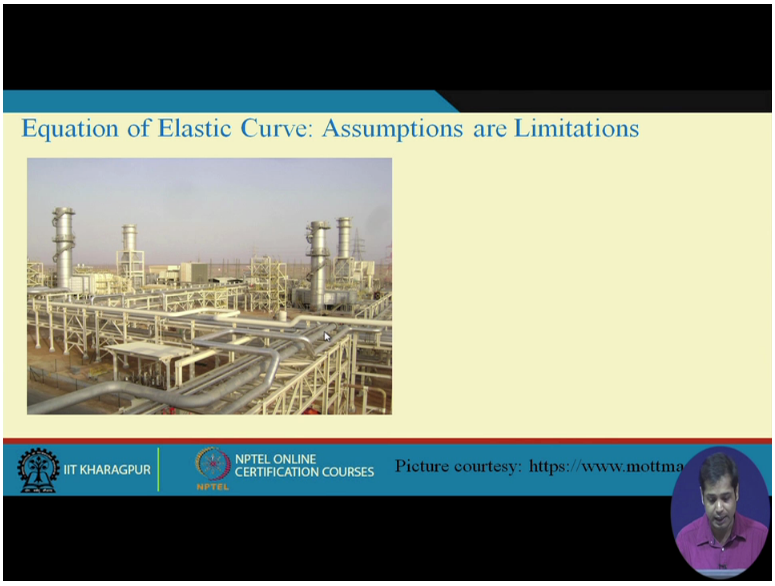

Now what happens for some reason if a pipe breaks. Suppose you consider this segment. If a pipe breaks here then what happens?

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Now if the pipe breaks here then fluid flows through this pipe and if it breaks then fluid will come out from this (fif) pipe. Now when fluid comes out from this (fli) pipe then it will exert an (equi) equivalent force as per Newtons third law. It will exert a force on this. So this can be treated as a beam which is subjected to a force by this, the reaction coming from this fluid. So this is the problem and idealization of (pro) this problem is this. You have a (ca) cantilever beam which is subjected to some force like this, okay.

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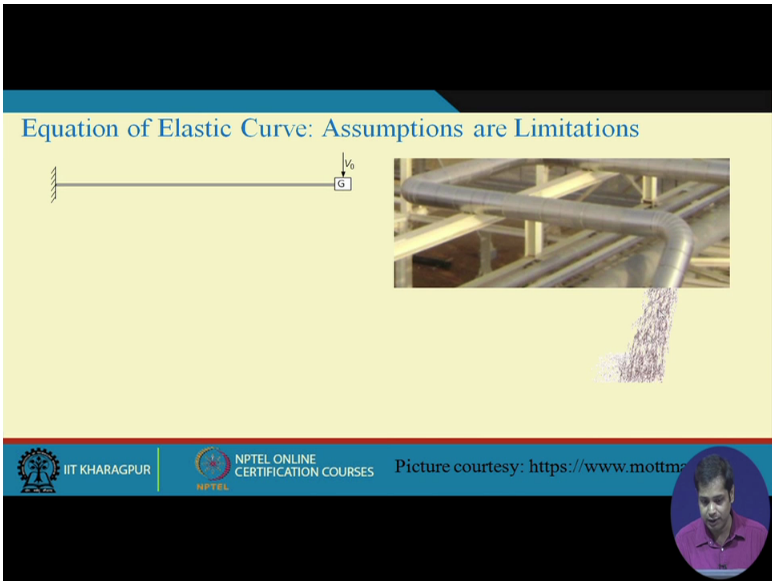

Now depending on the size of the diameter of the pipe, length of the pipe and the how much is the speed of the fluid through this pipe, this pipe may undergo severe deformation. For instance this is one case where if you see the kind of deformation. This is a cantilever beam, right? Any civil engineers structure probably does not experience this kind of deformation.

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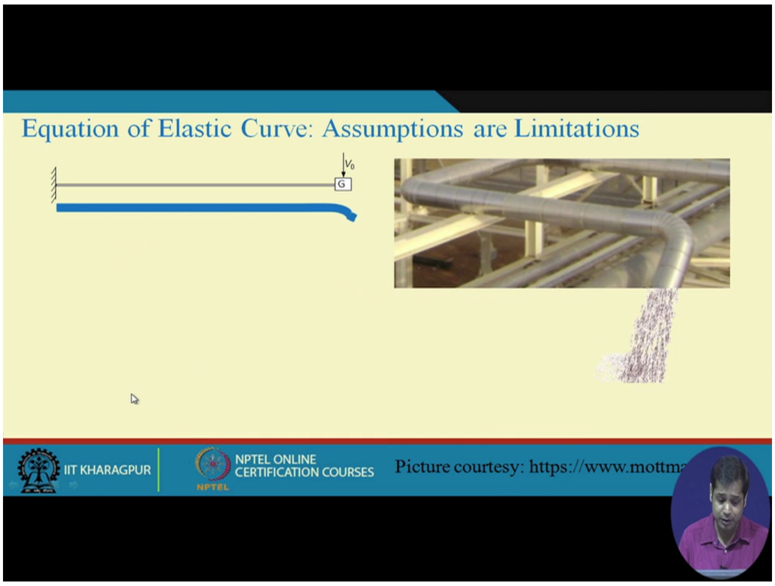

But as I said beam just not only a civil engineering structure. The beam idealization is used in many places, okay, many applications of mechanics. Now this is a very severe deformation as you can see. Now if we zoom this part, if you see at several cross sections the cross section gets distorted like this.

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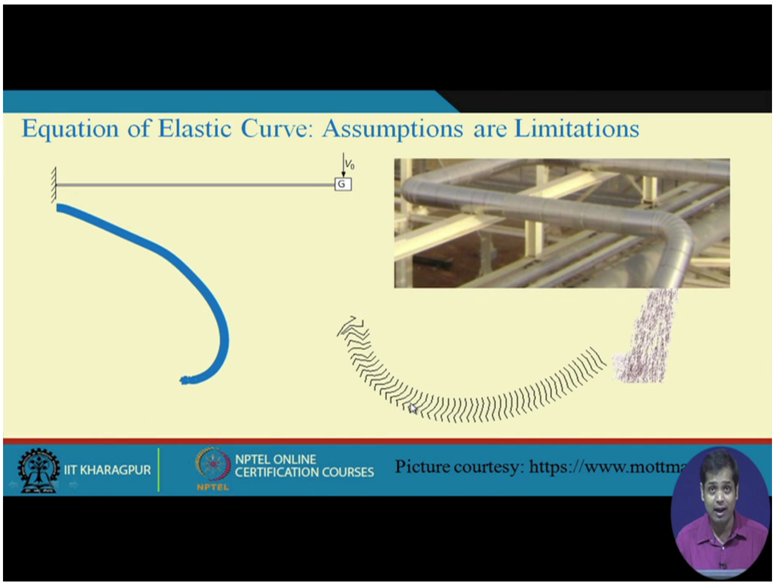

So they were plane but they do not remain plane, okay. So this is just one example where these assumptions are not valid. You have many such examples where these assumptions are not valid. And if you come across those situations you have to work on different theories which take care of all these different processes. But for all the civil engineering structures generally these assumptions are valid.

So we will go with these equations. What we do next is we will take few examples and then see how these equations can be used to find out deflection. Generally three methods we will study. One is just directly integrating this differential equation, this is one method and the second method is moment area method and the third method is conjugate beam method. So next class we will find out deflection of the beams by directly integrating the differential equation that we derived today.

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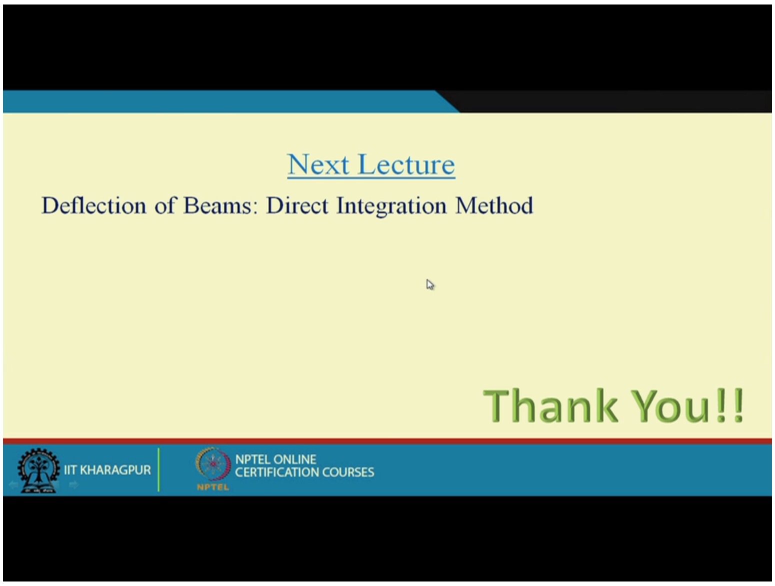

Thank you.