

**Structural Analysis 1**  
**Professor Amit Shaw**  
**Department of Civil Engineering**  
**Indian Institute of Technology Kharagpur**  
**Lecture 16**

**Analysis of Statically Determinate Structures: Method of Virtual Work (Contd.)**

Hello everyone, welcome. This is the last class of this week. What we have discussed so far in this week is we discussed what is virtual work method? Then we saw the principle of virtual work can be translated to a method which can be used to determine unknown displacements at any given point for statically determinate structures, okay. Now this method is unit load method.

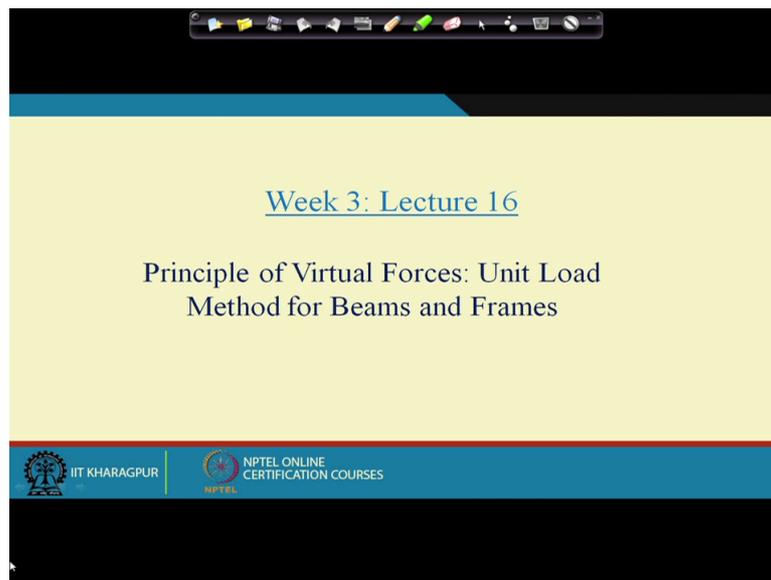
We demonstrated the unit load method in truss problems and then what we are going to do today is we will see what is the form of unit load method for beams and frames? This is the first thing and another important thing that we are going to do is this virtual work principle. We will see that from virtual work principle we will see some of the interesting behaviour of structure.

And this principle gives us some insight about the structural behaviour when a structure is subjected to a certain load then it undergoes deformation, it will develop some internal forces but how this internal forces are distributed (ho) and how this internal forces are related to this displacement and when in that relation do they follow certain interesting trend that we will try to understand through virtual work principle, okay.

Another important thing is the principal as of now what we have discussed is we have not made any assumptions about the material. Material has to be linear elastic. The principal is applicable to any material even if we have inelastic behaviour in the material. But since in this course we are not talking about inelasticity we are only concentrating on the linear elastic structures therefore the application of the virtual work is limited to only those problems.

But in principle this can be equally applicable to other material as well. So today what we are going to do is the principle of virtual force which will eventually gives us unit load method. We will see what is the unit load method for beams and frames?

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As far as the application of unit load method for beams and frames are concerned we will not discuss that in this week. When we talk about different methods to find out deflection in beams and frames there we will demonstrate the unit load method, okay. So let us quickly have some review of the method that we discussed so far. This is the gist of the method, okay. What it says that if you have two system, one is system 1 which essentially gives us the force field and system 2 which has displacement field.

Now the force and displacement field they are conjugate to each other but they do not have any cause effect relationship. It is not that the displacement field is caused due to the force field, okay.

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Principle of Virtual Work for Deformable Bodies

System I — Equilibrium

$$\sum_i F_i^{\text{ext}} \cdot D_i^{\text{ext}} = \sum_j F_j^{\text{int}} \cdot D_j^{\text{int}}$$

Compatibility — System II

Now but force field should be such that they should satisfy the equilibrium, they should be statically admissible and the displacement field which is in (sys) system 2, they should satisfy the internal compatibility. Now if we have two such systems then the principle of virtual work says that the external work done equal to the internal work done.

Means the work done by the external forces in the external displacement which are conjugate to the external forces is equal to the work done by the internal forces associated with the internal displacement, okay. Now then this is the general expression for principle of virtual work. Then we have two special cases depending on which system is real and which system is virtual.

Now if we take system 1 is real means the force field is real and the displacement field is virtual then the method says that it is principle of virtual displacement. And if we have the opposite means it is if the system 1 is virtual the force field is virtual and system 2 is real the displacement field is real then it is principle virtual force. It is the principle of virtual force which we have translated into a method which is unit load method, okay.

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**Principle of Virtual Work for Deformable Bodies**

$$\sum_i \mathbf{F}_i^{\text{ext}} \cdot \mathbf{D}_i^{\text{ext}} = \sum_j \mathbf{F}_j^{\text{int}} \cdot \mathbf{D}_j^{\text{int}}$$

System I is Real and System II is Virtual: Principle of Virtual Displacement  
System I is Virtual and System II is Real: Principle of Virtual Force

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Now this also we discussed in one of the class that if we take a beam which is subjected to a concentrated load say  $F$  at any given point and the corresponding displacement in the direction of the force is  $D$  then the external work done by this force is equal to  $F$  into  $D$ . And similarly if we have a moment at  $B$  and the associated displacement is in the rotation and then the external work done by this moment is equal to  $M_B$  into  $\theta_B$ .

Now if this displacement field is virtual then this work will be the virtual work. Similarly if the displacement field is real but the force field is virtual then this work will be virtual work, okay.

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**Virtual Work: Beams**

External Work ( $W^{\text{ext}}$ ) =  $FD$

External Work ( $W^{\text{ext}}$ ) =  $M_B \theta_B$

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But the computation of the virtual work that the force multiplied by the associated displacement and the moment multiplied that (asso) associated rotation remains same, okay. Now so this is external work. Let us understand what is internal work? If you remember in truss if we apply some load on truss then every member would be having some internal forces, okay.

And because of these internal forces the member may undergo axial deformation because it is the only deformation that we can have in each member of truss. Truss as a whole (dis) the joints may displace in x and y coordinate but if you take every member individually they will undergo only axial deformation. And the work done will be the internal force multiplied by the axial deformation. Now but in beam we have neglected the axial deformation. It is the main deformation mode in beam is bending like this, okay.

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Virtual Work: Beams

External Work ( $W^{ext}$ ) =  $FD$

External Work ( $W^{ext}$ ) =  $M_B\theta_B$

Curvature

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So suppose if we take the same beam AB which is subjected to some external load and for that load and it undergoes bending. And then this is the deflected shape of this beam, okay. Now rho is the radius of this deflected shape. Then curvature can be written as 1 by rho, okay.

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Virtual Work: Beams

External Work ( $W^{ext}$ ) =  $FD$

External Work ( $W^{ext}$ ) =  $M_B\theta_B$

Curvature =  $\frac{1}{\rho}$

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Now curvature you have done it probably in your mechanics course. That is if you have done it in your mechanics course how the moment and curvature are related? The curvature is equal to  $M$  by  $EI$ . You have (do) done this proof in your mechanics course but still we will be discussing this relation in subsequent weeks when we will talk about how to find out deflection for beams and when we derive the expression for this deflected shape.

For the time being you can take this expression for granted. We will prove this expression in subsequent lectures, okay.

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The slide, titled "Virtual Work: Beams", contains the following content:

- Top Left Diagram:** A beam of length  $L$  is supported at points  $A$  and  $B$ . A downward force  $F$  is applied at point  $D$ . The deflected shape is shown in red. Below it, the text reads: "External Work ( $W^{ext}$ ) =  $FD$ ".
- Bottom Left Diagram:** A beam of length  $L$  is supported at points  $A$  and  $B$ . A counter-clockwise moment  $M_B$  is applied at point  $B$ . The rotation at  $B$  is denoted as  $\theta_B$ . The deflected shape is shown in red. Below it, the text reads: "External Work ( $W^{ext}$ ) =  $M_B \theta_B$ ".
- Right Diagram:** A beam of length  $L$  is shown with a coordinate system  $(x, y)$  where  $x$  is along the beam and  $y$  is perpendicular to it. The deflected shape is shown in red. The radius of curvature is labeled as  $\rho$ . The relationship between curvature and moment is given as  $\frac{1}{\rho} = \frac{M(x)}{EI} = \Phi(x)$ . The curvature is also labeled as  $\text{Curvature} = \frac{1}{\rho}$ .
- Logos:** IIT KHARAGPUR and NPTEL ONLINE CERTIFICATION COURSES.
- Video:** A small circular video inset in the bottom right corner shows a man speaking.

So this is the curvature which is related to moment as this and  $EI$  is the flexural rigidity. If you remember in truss case we talk that because in truss the member undergoes axial deformation, it is the cross sectional area of the member which is which is important. And  $A$  into  $E$ ,  $A$  is the cross section in area,  $E$  is the Youngs modulus.

$A$  into  $E$  is called axial rigidity. For a given member if  $A$  is more the axial deformation in the member will be less. Similarly  $EI$  since in beam it is deflection or bending which dominates the deformation behaviour. This  $EI$  is called flexural rigidity, okay.

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Virtual Work: Beams

External Work ( $W^{\text{ext}}$ ) =  $FD$

External Work ( $W^{\text{ext}}$ ) =  $M_B\theta_B$

$\frac{1}{\rho} = \frac{M(x)}{EI} = \Phi(x)$

Curvature =  $\frac{1}{\rho}$

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Again we will be having detailed discussion on flexural rigidity and how the flexural rigidity may affect the deformation behaviour of beam that we will discuss in the subsequent lectures okay. So this is the curvature, right?

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Virtual Work: Beams

External Work ( $W^{\text{ext}}$ ) =  $FD$

External Work ( $W^{\text{ext}}$ ) =  $M_B\theta_B$

$\frac{1}{\rho} = \frac{M(x)}{EI} = \Phi(x)$

Curvature =  $\frac{1}{\rho}$

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Now then the internal work done will be the moment into curvature. Now in truss it was a discrete system. It was a system which is formed by joining some discrete members, right? But this is a continuous system. So the summation that we use in the case of truss instead of summation we will be having integration here. Now integration of moment which is the internal force and into the curvature that gives us the internal work done, okay.

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**Virtual Work: Beams**

External Work ( $W^{ext}$ ) =  $FD$

External Work ( $W^{ext}$ ) =  $M_B\theta_B$

$\frac{1}{\rho} = \frac{M(x)}{EI} = \Phi(x)$

Curvature =  $\frac{1}{\rho}$

$W^{int} = \int_{AB} M(x)\Phi(x)dx$

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Now again depending on whether it is virtual or real we can have the virtual work, okay. Now once we know what is external work? How to determine external work? And what is internal work? Let us see what is the unit load method in this case? Okay. Now take the beam AB which is subjected to some arbitrary loading and because of this loading it undergoes deformation like this, okay.

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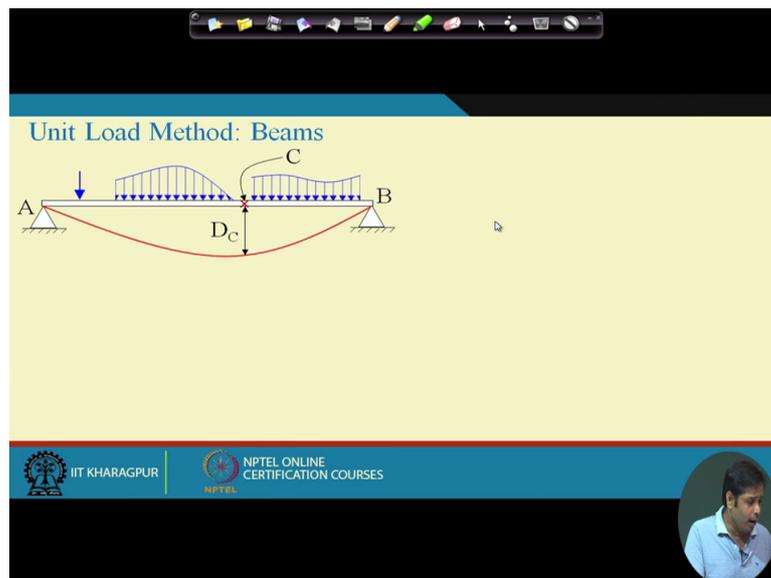
**Unit Load Method: Beams**

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Now take any intermediate point C and suppose at this point C the displacement is  $\delta_C$ , okay.

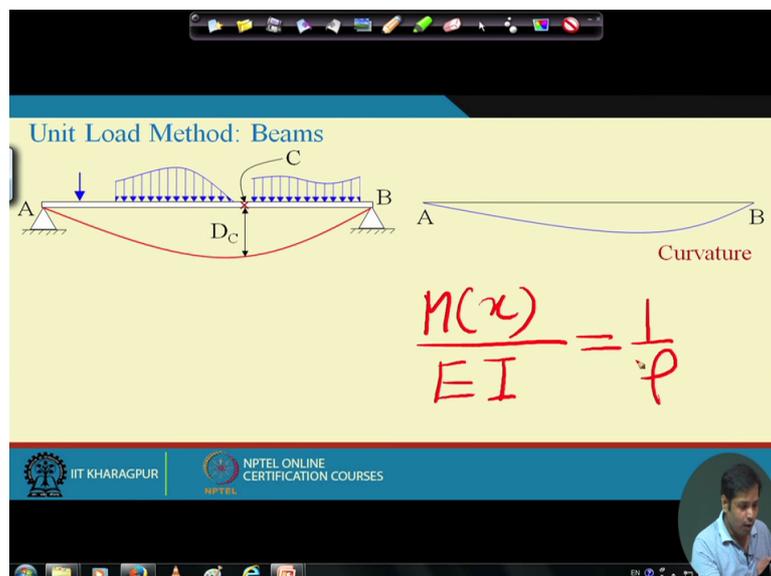


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Now if we draw the curvature diagram suppose this is the curvature diagram. Curvature diagram means just now we talked because of this load at any point of any section there will be a moment, suppose that moment is  $M$  which is function of  $x$  and then the flexural rigidity of the member is  $EI$  which could be also function of  $x$ , okay, and this is equal to curvature, just now we have seen, okay.

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Now if we plot this curvature, suppose this is the curvature. Now since it is a simply supported beam moment at A and moment at B is zero. So this point will be zero. So

curvature at point A, curvature at point B will be zero. And suppose this is a curvature of the beam, okay.

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Unit Load Method: Beams

The diagram shows a beam AB of length L. A unit load of 1 is applied at point C, which is at a distance  $D_C$  from point A. The beam is supported by a pin at A and a roller at B. The deflection curve is shown in red. The curvature diagram is shown in blue, with the label "Curvature".

$$\frac{M(x)}{EI} = \frac{1}{p}$$

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Now then what you can do is take any point x and at that point x suppose this value is  $M_x$  by  $EI$  that just now we have seen this, okay.

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Unit Load Method: Beams

The diagram shows a beam AB of length L. A unit load of 1 is applied at point C, which is at a distance  $D_C$  from point A. The beam is supported by a pin at A and a roller at B. The deflection curve is shown in red. The curvature diagram is shown in blue, with the label "Curvature". A point x is marked on the beam, and the curvature at that point is labeled  $\Phi(x) = \frac{M(x)}{EI}$ .

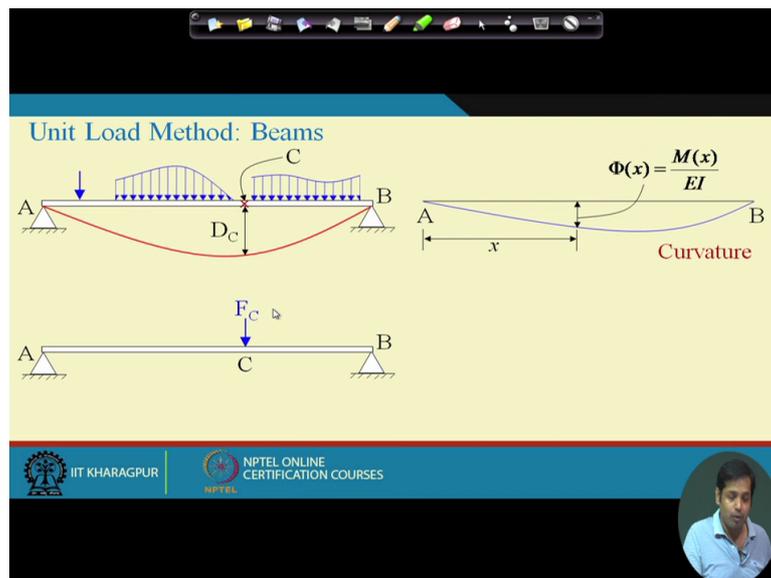
$$\Phi(x) = \frac{M(x)}{EI}$$

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So we have a beam which is subjected to some real load and because of this load the beam undergoes deformation and the corresponding curvature is this and at any given point x the

curvature is given moment by  $EI$ , okay. Now take the same beam and suppose at point C we apply a concentrated load.

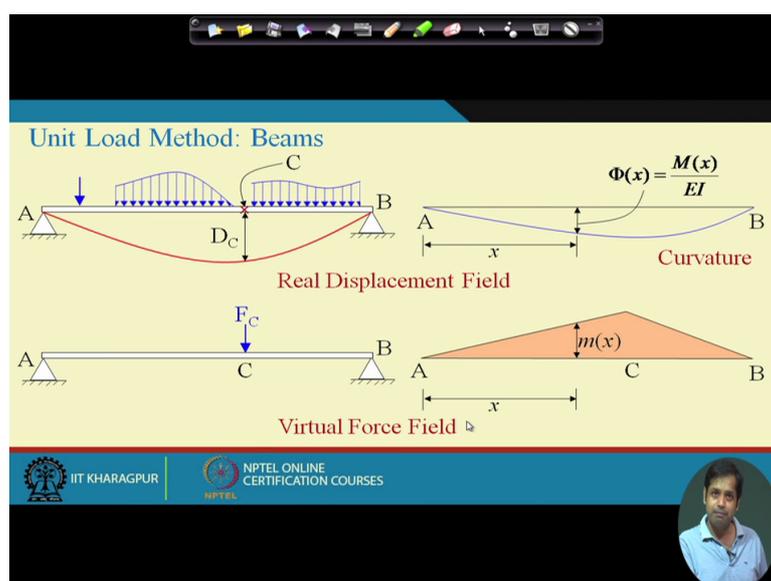
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You remember in truss also the point where we wanted to determine the displacement we applied one concentrated load in the direction in which we want to determine the displacement, okay. So we apply a force here and corresponding bending moment diagram will be C and suppose at any point the bending moment is denoted as small  $m$ , okay.

Now suppose (displace) this is the deformation in the beam due to the real the actual load, okay. And this is again the real curvature in the beam that is due to the actual load. So this is the real displacement field and suppose this is the virtual force field, okay.

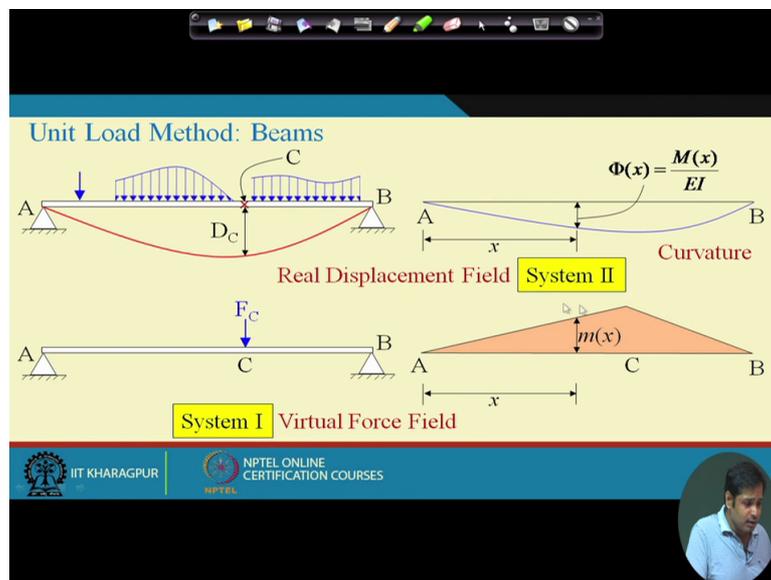
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We applied a virtual force at C and then corresponding moment diagram will be this. Now take this as system 1 and if we take this as system 2. Very similar to truss. The only difference is here is instead of the axial deformation here we will have moment. There the axial deformation and the internal force gave us the work, here the product of curvature and moment will give us the work and that work has to be integrated along the length of the beam because it is a continuous system.

Rest of this thing the (dis) real displacement field and the virtual force field as a concept they are very similar to truss. Now if we take this is system 1 and this is system 2.

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Now apply the principle of virtual work. This is the principle of virtual work. Now  $F$  external is  $F_C$  and  $F$  internal will be  $MC$ . So since this is our force field external force is  $F$  and internal force here is the moment. And corresponding internal forces is small  $mx$ . So external force is  $F$ , internal force is small  $mx$  and then external displacement is  $D_C$ .

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Unit Load Method: Beams

$$\int_{AB} F^{ext} \cdot D^{ext} dx = \int_{AB} F^{int} \cdot D^{int} dx$$

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This is the external displacement at DC which is the real displacement that we want to determine in the real structure when the structures subjected to real load and the corresponding internal displacement will be the curvature in the real beam, okay.

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Unit Load Method: Beams

Real Displacement Field System II

Curvature  $\Phi(x) = \frac{M(x)}{EI}$

Virtual Force Field System I

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So corresponding internal displacement will be the curvature. Now if we substitute that then this gives us  $F_C$  into  $D_C$  is equal to integration over  $AB$  length of the beam and  $m(x)$  into  $\Phi(x)$ . Now  $\Phi(x)$  is equal to  $M(x)$  by  $EI$  we know. This is the curvature expression and if we substitute  $F_C$  is equal to 1 the same that we did for the truss then this expression will become  $D_C$  is equal to this and this is unit load method for beam.

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Unit Load Method: Beams

$$\int_{AB} F_C^{\text{ext}} \cdot D_C^{\text{ext}} dx = \int_{AB} F_C^{\text{int}} \cdot D_C^{\text{int}} dx$$

$$F_C D_C = \int_{AB} m(x) \Phi(x) dx$$

$$\Phi(x) = \frac{M(x)}{EI} \quad F_C = 1$$

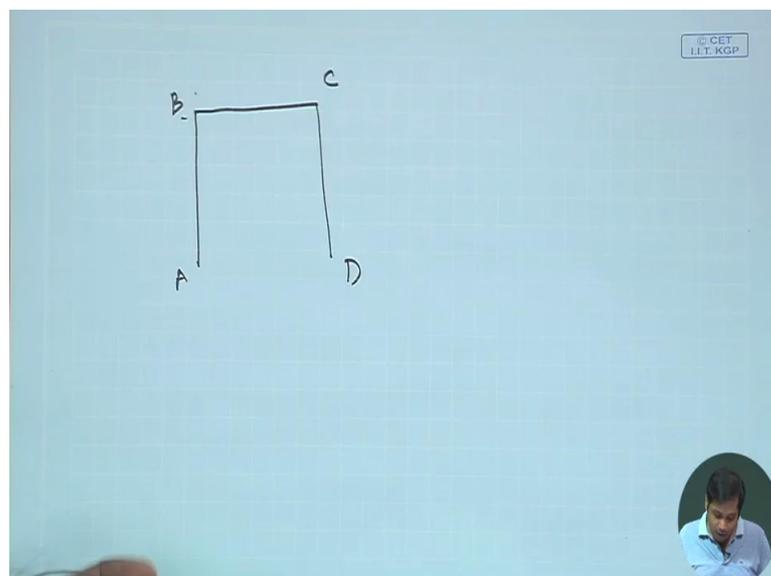
$$D_C = \int_{AB} \frac{M(x)}{EI} m(x) dx$$

Unit Load Method

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Now if we see a beam has several segments for instance a frame. Then for instance if we take a frame like this, this is A, B, C, D. That integration has to be performed over AB, over BC, over CD. So we need to get the expression for moment in AB, expression for moment for moment BC and expression for moment for CD. And then that integration has to be performed over each length.

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That things will be clear when we actually apply this principle to some problem. So this is unit load method. So we will not further discuss the unit load method for beams and frames.



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Unit Load Method: Beams

$$\int_{AB} F^{\text{ext}} \cdot D^{\text{ext}} dx = \int_{AB} F^{\text{int}} \cdot D^{\text{int}} dx$$

$$F_c D_c = \int_{AB} m(x) \Phi(x) dx$$

$$\Phi(x) = \frac{M(x)}{EI} \quad F_c = 1$$

$$D_c = \int_{AB} \frac{M(x)}{EI} m(x) dx$$

Unit Load Method

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The reason why we are not going to apply unit load method in this week because you see the formulation of this method based on certain expression. For instance one expression is here we assume that phi x is the curvature is equal to Mx by EI, right?

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Unit Load Method: Beams

$$\int_{AB} F^{\text{ext}} \cdot D^{\text{ext}} dx = \int_{AB} F^{\text{int}} \cdot D^{\text{int}} dx$$

$$F_c D_c = \int_{AB} m(x) \Phi(x) dx$$

$$\Phi(x) = \frac{M(x)}{EI} \quad F_c = 1$$

$$D_c = \int_{AB} \frac{M(x)}{EI} m(x) dx$$

Unit Load Method

*Handwritten:*  $\phi(x) = \frac{M(x)}{EI}$

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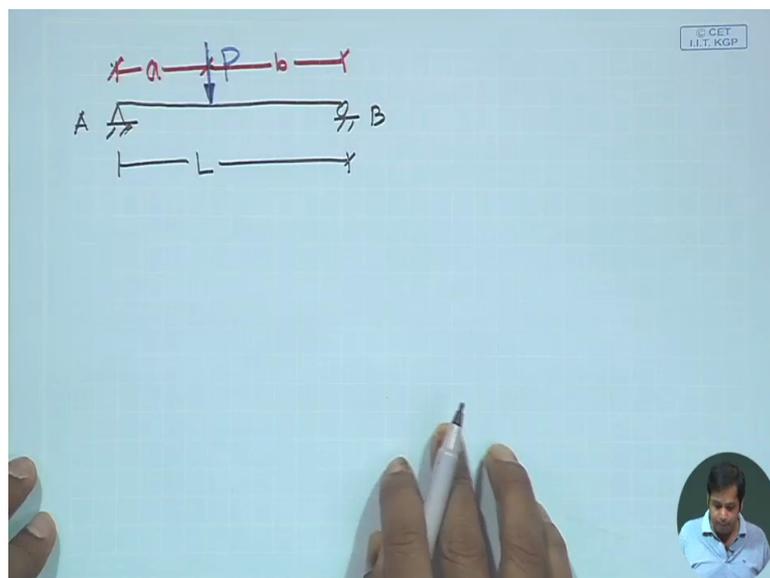
Now this we have not yet discussed. So what we will do is first we will discuss all this and then we will discuss about what is flexural rigidity and then we will apply the unit load method that will be done in subsequent weeks. For the time being this is unit load method, okay. Now you have to wait for two weeks to apply this unit load method for beam and frame problems, okay.

Now as I said this is the translation of principle of virtual work into a method which can be used to determine deflection of truss or deflection of beam at any direction or at any point, right? Now but the application of principle of virtual work is more wide. It is just not finding the displacement and the more you study the structural mechanics or advanced level of structural engineering probably better you can appreciate the principle of virtual work.

Now as I said principle of virtual work also gives some very interesting insight about the behaviour of the structure. Now before I formally tell you what is that behaviour and let us do one exercise and through that exercise let us try to understand that behaviour, okay. Now take a simply supported beam, okay. Now this length is  $L$  and suppose this is A, this is B this is A, this is B.

Now suppose this is subjected to a concentrated load at any arbitrary point which is  $P$  and this distance is  $a$  and this distance is  $b$ . So  $b$  is essentially  $L$  minus  $a$ , okay. This distance is  $b$  and this is the force, okay.

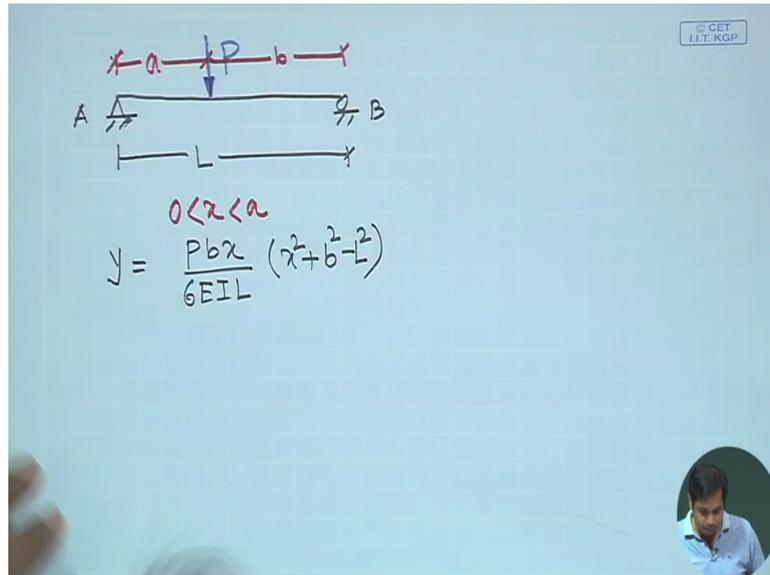
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Now some of the expression that I am going to write here for the time being you have to take them for granted. Next week when we actually start finding the deflection of beams then you can appreciate and you can understand those expressions. Now see this is a statically determinate structure. We can find out what is the bending moment, we can find out what is the support reaction, right? That is what we have done so far. We have not yet discussed how to determine the deflection of this beam.

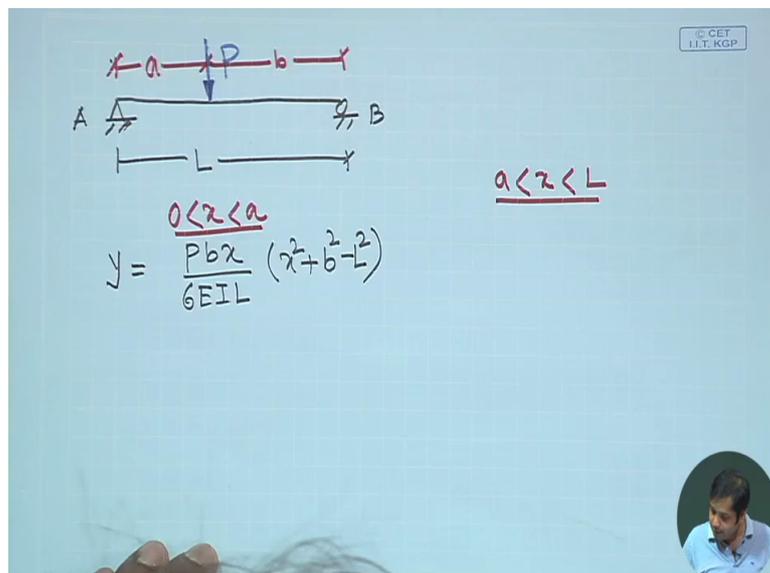
That we will be doing in subsequent lectures. Now if we do that there are many methods to determine the deflections. Now if we find the deflection then the expression will be you see between zero, x, a, okay, the deflection will be say , y is the deflection that is equal to P into b into x divided by 6 EIL then x square plus b square minus L square. You may be thinking suddenly out of nothing where we got this expression, right?

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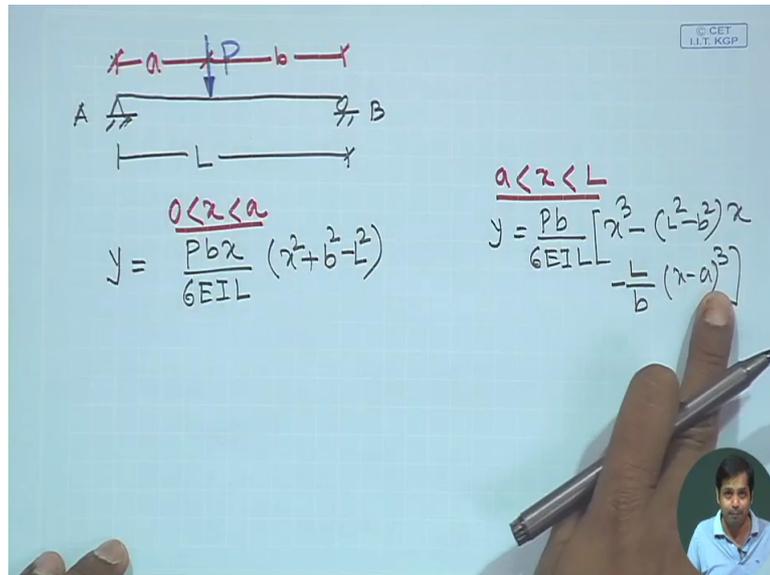
We got this expression if we apply any method to find out displacement of beam that could be conjugate beam theory, direct integration method or virtual or any other method. You can get this expression, okay. In order to appreciate the expression you have to wait one more week.

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Now similarly between this the expression of  $y$  will be  $Pb$  by  $6 EIL$  and then  $x$  cube minus  $L$  square minus  $b$  square  $x$ , minus  $L$  by  $b$   $x$  minus  $a$  whole cube, okay. This is the expression for displacement when you substitute  $x$  is equal to  $a$  here and  $x$  is equal to  $a$  here you will see both the displacements are same.

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It says that displacement is continuous. Not only that if you get the slope of this by differentiating  $y$  with respect to  $x$ , here also determine the slope and compute the slope at  $x$  is equal to  $a$ , you will see that slope is also continuous. So this is compatibility condition, okay. Now the point what we want to make? Suppose take now two points, okay. Take two cases, case 1 and this is true for any value of  $A$  and  $B$ , okay. Less than  $L$ . Case 1 we will take  $a$  is equal to  $L$  by 3 and  $x$  is equal to  $L$  by 2, okay.

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$0 < x < a$   
 $y = \frac{Pbx}{6EIL} (x^2 + b^2 - L^2)$

$a < x < L$   
 $y = \frac{Pb}{6EIL} \left[ x^3 - (L^2 - b^2)x - \frac{L}{b}(x-a)^3 \right]$

Case 1:  $a = \frac{L}{3}$      $x = \frac{L}{2}$

And now what it says? It says that a is equal to L by 3 and we want to find the deflection at x is equal to L by 2, okay. Now take case 2 where a is equal to L by 2 and we calculate the deflection at L by 3, okay.

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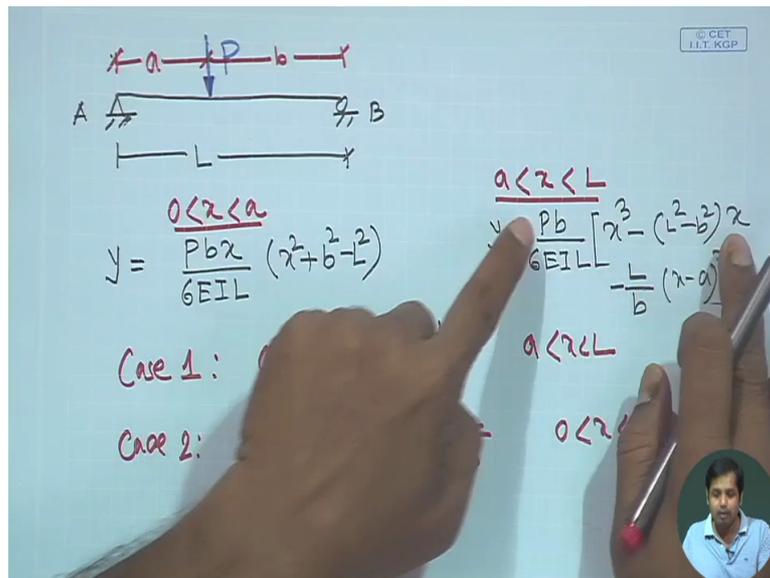
$0 < x < a$   
 $y = \frac{Pbx}{6EIL} (x^2 + b^2 - L^2)$

$a < x < L$   
 $y = \frac{Pb}{6EIL} \left[ x^3 - (L^2 - b^2)x - \frac{L}{b}(x-a)^3 \right]$

Case 1:  $a = \frac{L}{3}$      $x = \frac{L}{2}$   
 Case 2:  $a = \frac{L}{2}$      $x = \frac{L}{3}$

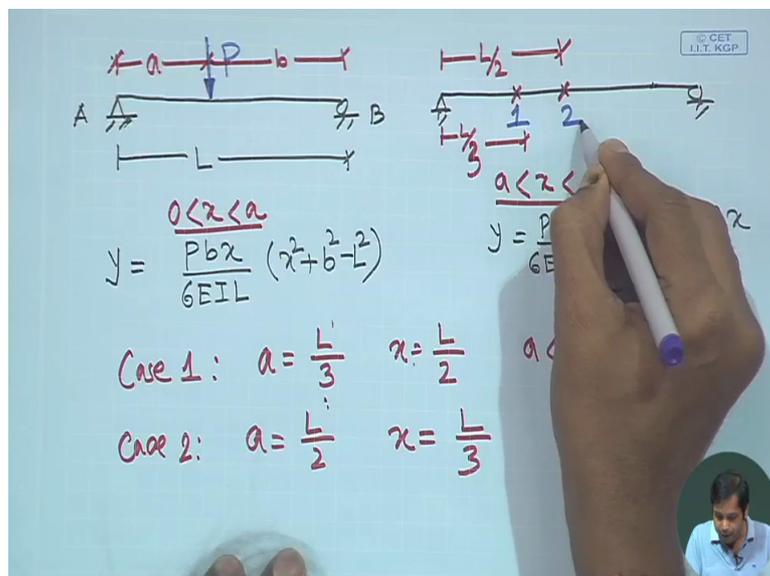
Means what it says, the case 1 is the load is applied at L by 3. We are computing the deflection at L by 2 and the case 2 is load is applied at L by 2 and we are calculating deflection at L by 3, okay. Now then for case 1 x is for case 1 and for case 2 it is a. So case 2 we need to find out the deflection from this expression, for case 1 this expression will be (va) valid, okay.

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Now so what essentially this means? We take the same beam once again. Now this is the midpoint of the beam, okay. And this is a point number and this is another point which is at a distance L by 3 and this is at a distance L by 2, okay. And this is point 1. This is point number 2, okay.

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Now what it says this here? The load is applied at point 1 and deflection is calculated at point 2. And in this case just the opposite, load is applied at point 2 and deflection is applied at point 1, okay. Now let us find out what is the deflection? Suppose this deflection is D21 and

this deflection is D12. D21 means the deflection at point 2 due to load at point 1 and this is deflection at point 1 due to load at point 2.

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$0 < x < a$   
 $y = \frac{Pbx}{6EIL} (x^2 + b^2 - L^2)$

$a < x < L$   
 $y = \frac{Pb}{6EIL} \left[ x^3 - (L-b)^2 x - \frac{L}{b} (x-a)^3 \right]$

Case 1:  $a = \frac{L}{3}$     $x = \frac{L}{2}$     $a < x < L$    D21.

Case 2:  $a = \frac{L}{2}$     $x = \frac{L}{3}$     $0 < x < a$    D12.

Let us find out what is D21 and D12. So D21 will be D21 which is case 1. In order to get D21 from this expression we need to substitute a is equal to L by 3 and naturally b will be L minus a, means 2L by 3 and x will be L by 2.

(Refer Slide Time: 24:51)

$0 < x < a$   
 $y = \frac{Pbx}{6EIL} (x^2 + b^2 - L^2)$

$a < x < L$   
 $y = \frac{Pb}{6EIL} \left[ x^3 - (L-b)^2 x - \frac{L}{b} (x-a)^3 \right]$

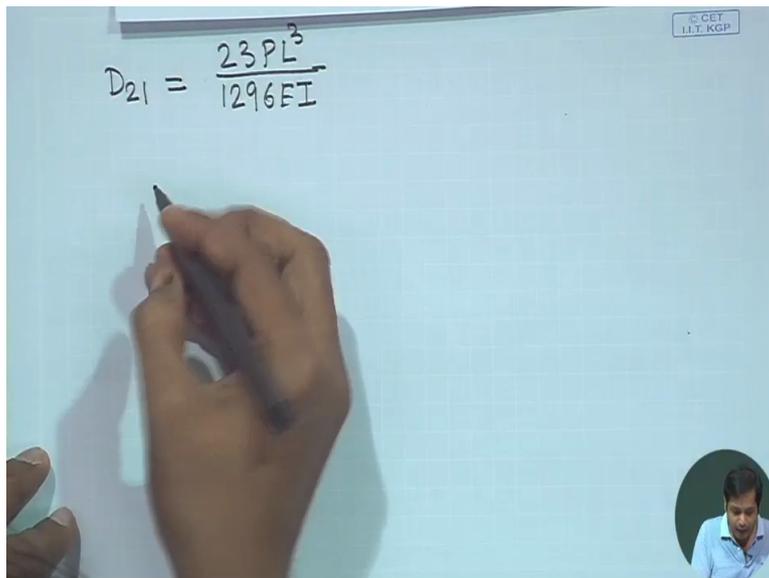
Case 1:  $a = \frac{L}{3}$     $x = \frac{L}{2}$     $a < x < L$    D21.

Case 2:  $a = \frac{L}{2}$     $x = \frac{L}{3}$     $0 < x < a$    D12.

And if we do that we get D21 is equal to 23 PL cube by 1296 EI, okay. You can determine this simple substitution in the expression and get the final result.



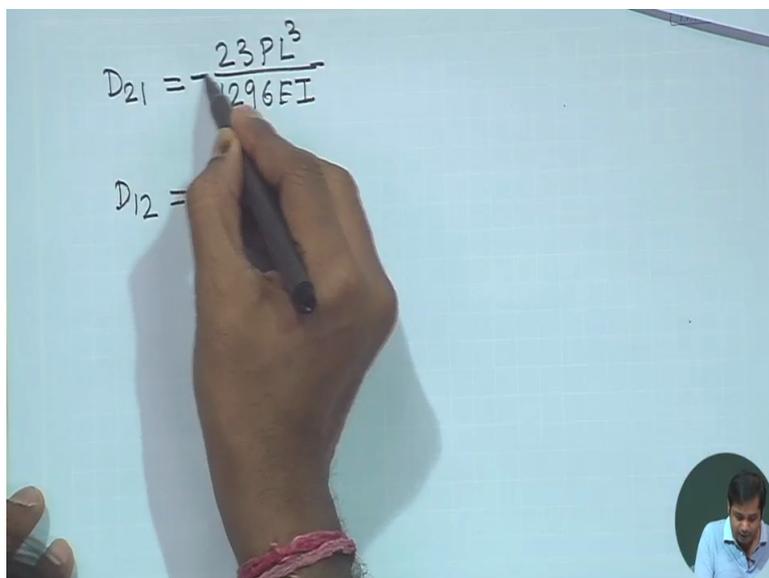
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A hand is shown writing the formula  $D_{21} = \frac{23PL^3}{1296EI}$  on a whiteboard. A small circular inset in the bottom right corner shows a man in a blue shirt looking at the board. In the top right corner of the whiteboard, there is a small logo that reads "CET I.I.T. KGP".

And similarly if you get  $D_{12}$  you need to substitute  $a$  is equal to  $L$  by 2 in this expression,  $b$  becomes  $L$  by 2 and  $x$  is equal to  $L$  by 3 in this expression. And if we get it there will be a minus sign here, okay.

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A hand is shown writing the formula  $D_{12} = -\frac{23PL^3}{1296EI}$  on a whiteboard. A small circular inset in the bottom right corner shows a man in a blue shirt looking at the board. In the top right corner of the whiteboard, there is a small logo that reads "CET I.I.T. KGP".

If we get it this will be minus 23 PL cube by 1296 EI, okay. Now you see they are same. Means that if I apply load at 1 and calculate deflection at 2, the deflection at 2 due to load at 1 will be same as deflection at 1 due to load at 2. If the applied force is same, okay.

(Refer Slide Time: 26:08)

$$D_{21} = -\frac{23PL^3}{1296EI}$$
$$D_{12} = -\frac{23PL^3}{1296EI}$$

Now this is one observation. But let us see whether we are getting the similar observation for other cases as well. Now take the same example which is subjected to a concentrated load again at the mid span. So this distance is  $L$  by  $2$  and this distance is  $L$  by  $2$ , okay.

(Refer Slide Time: 26:30)

$$D_{21} = -\frac{23PL^3}{1296EI}$$
$$D_{12} = -\frac{23PL^3}{1296EI}$$

And if you (ca) calculate theta at this point say this is A this is B. If you calculate theta B then expression of theta becomes  $P$  by  $16 EI$  into  $4 x$  square minus  $L$  square. And if you substitute  $x$  is equal to zero here then you get theta is equal to  $PL$  square by  $16 EI$ , okay.

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$$D_{21} = -\frac{23PL^3}{1296EI}$$

$$D_{12} = -\frac{23PL^3}{1296EI}$$

Diagram of a beam AB of length L with a point load P at the center. The deflection at the center is  $\delta = \frac{PL^2}{16EI}$ . The rotation at the ends is  $\theta = \frac{PL^2}{16EI}$ .

Now similarly if we apply a moment here M and (diff) if this is the deform shape, at the mid span delta becomes ML square by 16 EI, okay.

(Refer Slide Time: 27:42)

$$D_{21} = -\frac{23PL^3}{1296EI}$$

$$D_{12} = -\frac{23PL^3}{1296EI}$$

Diagram of a beam AB of length L with a moment M at the center. The deflection at the center is  $\delta = \frac{ML^2}{16EI}$ .

Now if we take P is equal to M then what it becomes? It becomes theta is equal to P is equal to M is equal to say Q then this becomes QL square by 16 EI and it is also become QL square by 16 EI, okay.

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$D_{21} = -\frac{23PL^3}{1296EI}$   
 $D_{12} = -\frac{23PL^3}{1296EI}$

$\theta = \frac{PL^2}{16EI}$

$\delta = \frac{ML^2}{16EI}$

Means again what it said? Now here this is the point 1, this is point 2. So if we apply a load at point 1, calculate the rotation at point 2 and then apply moment at point 2, calculate deflection at point 1, they are same. And this is not specific to this case, okay. Generally you take any structure and this is valid.

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$D_{21} = -\frac{23PL^3}{1296EI}$   
 $D_{12} = -\frac{23PL^3}{1296EI}$

$\theta = \frac{PL^2}{16EI}$

$\delta = \frac{ML^2}{16EI}$

$P = M = Q$

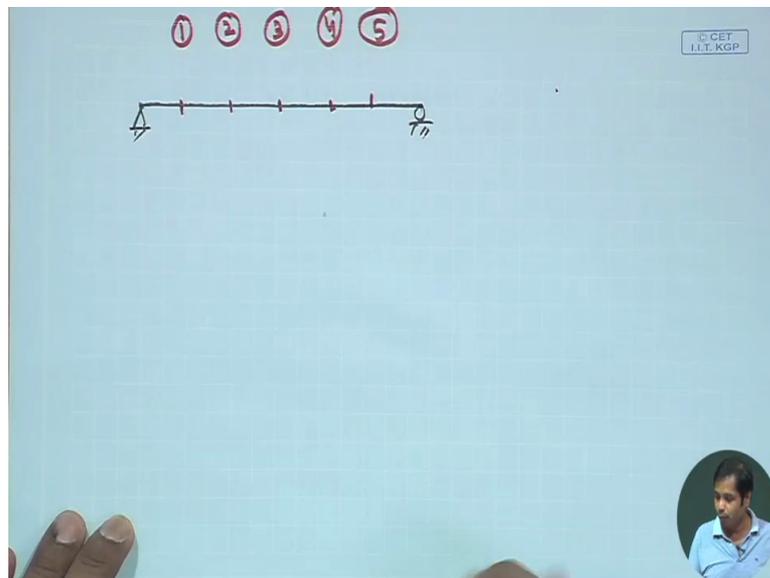
And this theorem is called Maxwells Reciprocal Theorem. But you know later it was observed that this observation is not only valid for one force. This observation (alid) is valid for if you have several forces then also this observation is valid. And then the generalized

Maxwells theorem and proposed a very general reciprocal theorem and that reciprocal theorem we are going to discuss now.

Now see I could have told you the theorem and proved it but before formally introducing you theorem if you actually experience that really it is happening then probably you can appreciate the theorem in a better way. And that is what I wanted to do. Now let us quickly see what is reciprocal theorem?

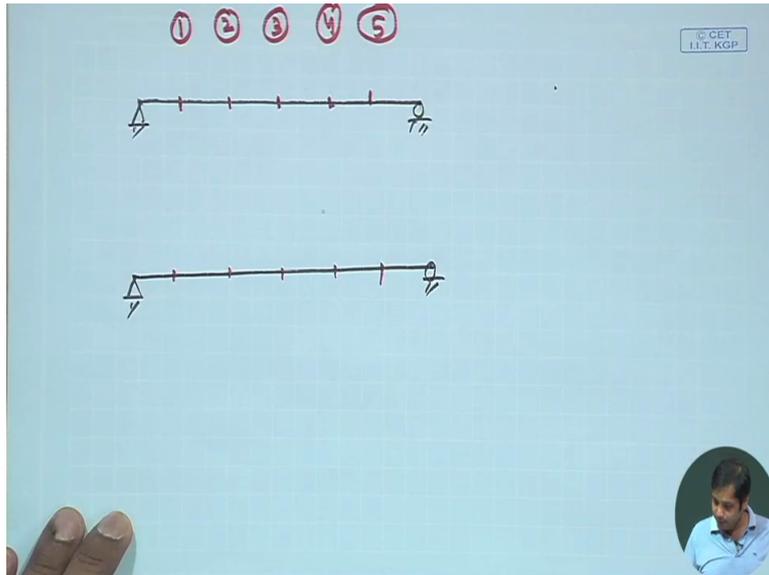
Now suppose again take simply supported beam, okay. And take five points in it 1, 2, 3, 4, 5. So this is (p) 1, 2, 3, 4, and point 5.

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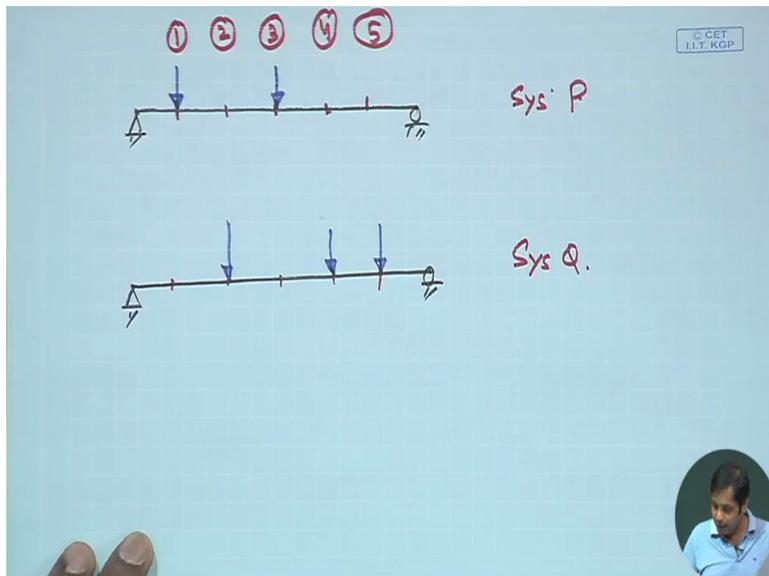
And again consider the same beam and similarly these are five points, okay, these are five points.

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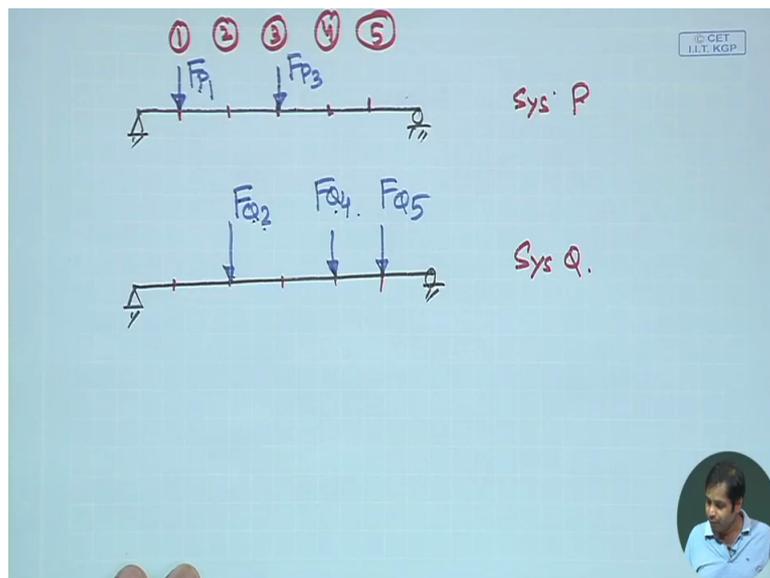
Now suppose this system is P and this system is Q, okay. How they are different? Suppose in system 1 we apply a force here at point number 1 and we apply force at point number 3. And in system 2 we applied force in point number 2, point number 4 and point number 5.

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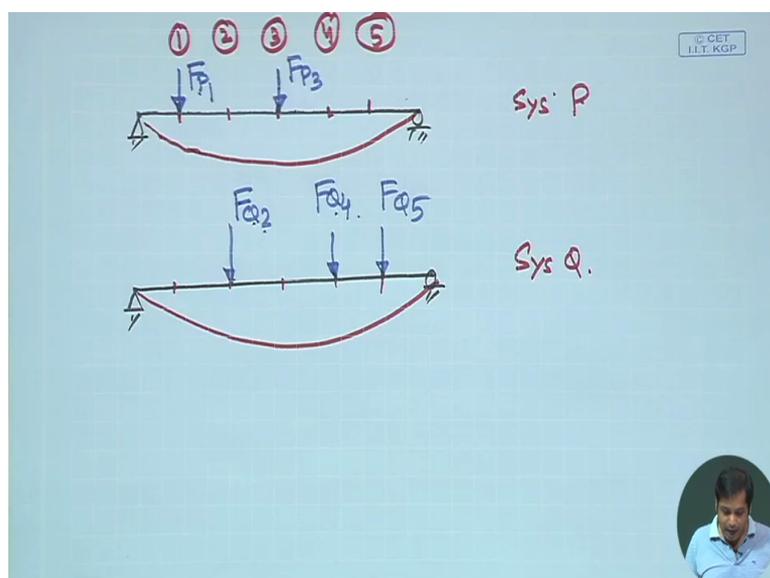
And suppose this is force applied at point number 1 in system P and this is force applied at point number 3 in system 3. Similarly this is force applied at point number 2 in system 2 and this is force Q4 and this is force Q5. So in this system the same length same beam the only thing is we have applied force at point number 1 and 3 and these forces are denoted as force in system 1 at point number 1. And in system Q we applied the forces other points 2, 4 and 5 is denoted as F in system 2 point number 2. F in system Q point number 4, okay.

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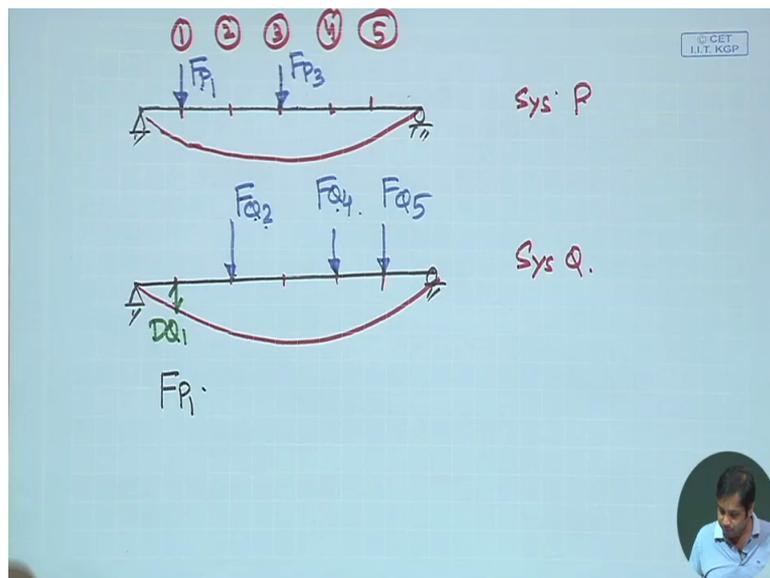
Now so in both the cases this beam will undergo some deformation and this beam will undergo some deformation, right? Undergo some deformation.

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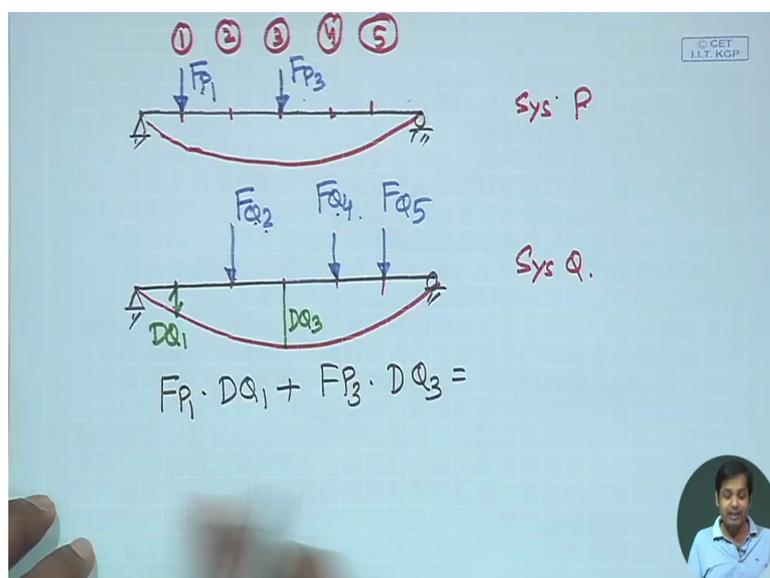
Now if we take the force in system P and if we assume the force in system P and the displacement in system Q they are conjugate to each other then we can apply the principle of virtual force and what is the principle of virtual force there? Now what are the forces we have? The force  $F_{p1}$  into this is the force in system 1, corresponding displacement in system 2 that is the conjugate, means suppose this displacement is  $D_{q1}$ .

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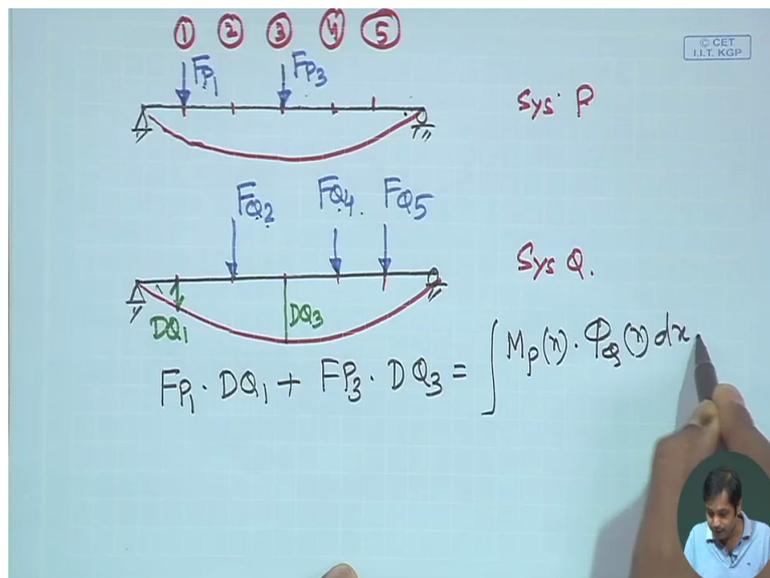
So this become into  $D_{Q1}$  plus force in system  $F_{p3}$  into corresponding displacement. Corresponding displacement is this, this is  $D_{Q3}$ . So this is the external work done.

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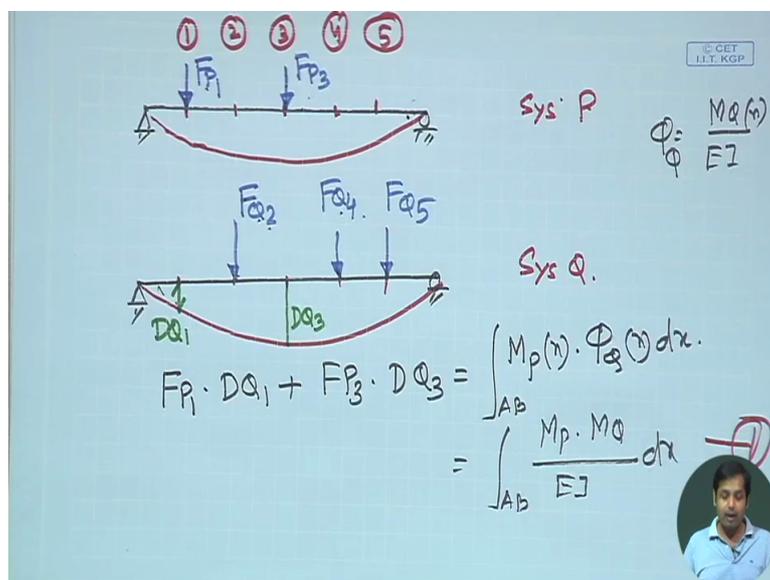
If we take this is the external force and this is the conjugate displacement. So this will be the external work done. And what will be the internal work done? Internal work done will be the internal forces in system P which is  $M_{px}$  and the displacement in system Q. Internal displacement means curvature on system Q. Curvature of system Q means  $\phi$  of Q x into dx, okay.

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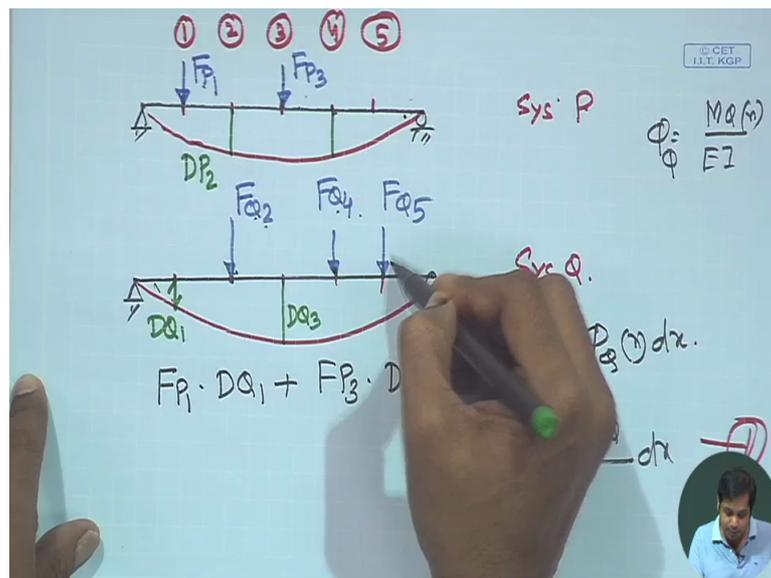
Now this becomes what? We know that phi of Q will be M in system Q of x divided by EI, right? So this is over AB. This becomes MP into MQ divided by EI into dx. Say this is equation number one, okay.

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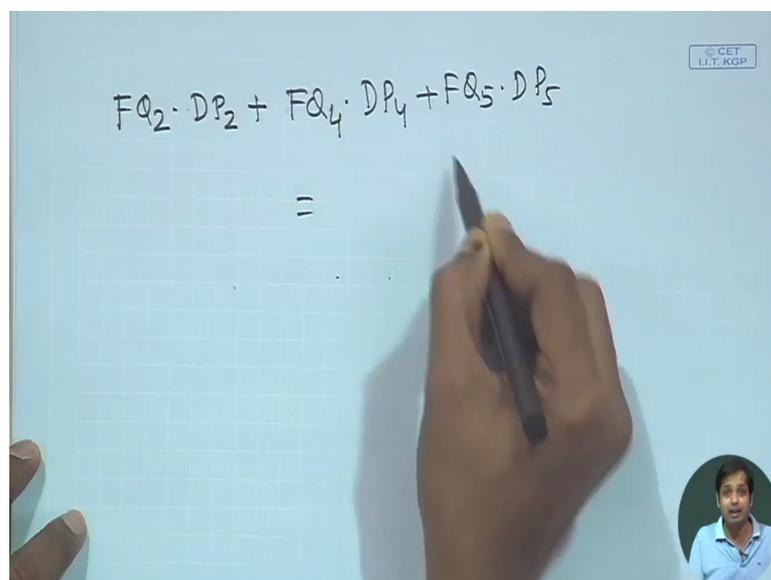
Now let us do the opposite. Opposite means that if we take the force in system Q (con) which is conjugate to the displacement in system P then what will happen? Happen the external work done will be FQ2 into displacement. This displacement is DP2. Force FQ2 into this displacement DP4. Force this displacement and this displacement will be DP5. So displacement in system 5.

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So what we have then? We will quickly have this. That force in system 2 into corresponding displacement in system P. Force in system Q, corresponding displacement in system 4 plus force in system 5 corresponding displacement in 5. This will be the external work done.

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If we take the system Q as force field and displacement in system P is the displacement field. So this force field in system Q (ca) and displacement field in system P are conjugate. Then this will be the internal work done. In this case what internal work done will be?  $M_Q$ , the force in system Q, internal force in system Q into displacement in system P, okay. And this is

equal to what is the displacement? This is  $MQ_x$  (displa) curvature in system P will be MP of  
x by EI dx is equal to  $MP_x MQ_x$  by EI dx integration.

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$$F_{Q2} \cdot DP_2 + F_{Q4} \cdot DP_4 + F_{Q5} \cdot DP_5$$

$$= \int M_Q(x) \cdot \Phi_P(x) dx$$

$$= \int M_Q(x) \frac{M_P(x)}{EI} dx$$

$$= \int \frac{M_P(x) M_Q(x)}{EI} dx$$

You see we also obtained the similar expression, (ex) this is 2, okay. Then what it says? This says that these two are same, right? This and this are same.

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$$F_{Q2} \cdot DP_2 + F_{Q4} \cdot DP_4 + F_{Q5} \cdot DP_5$$

$$F_{P1} \cdot DQ_1 + F_{P3} \cdot DP_3 = \int_{AB} M_P(x) \cdot \Phi_Q(x) dx$$

$$= \int_{AB} \frac{M_P \cdot M_Q}{EI} dx$$

What is this? This is force in system Q and displacement in system P and their work done should equal to force in system P and displacement in system Q, okay. So if you take force field in system Q and (piu) conjugate displacement in system P whatever work done we have and if we take the opposite means the force field in system P and displacement in the system Q they are conjugate the work done what you will get that work done will be same.

That is the reciprocal theorem which is a general version of reciprocal theorem. Just now we have (ex) experienced through one example, right? So the reciprocal theorem is this. In a linear elastic structure the total external virtual work associated with force  $F_P$ . Two systems if you take, this is system P and this is system Q.

The forces acting, two forces acting in system P and one force is acting at system Q. We have 3 points, point 1, point 2 and point 3 in system 1. In system P have force at point 1 and point 3 and in system 2 we have force at point 2.

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**Reciprocal Theorem: E Betti 1872**

In a linear elastic structure the total external virtual work associated with forces  $\{F_P\}$  in system P and the conjugate displacements  $\{D_Q\}$  in system Q is equal to that associated with forces  $\{F_Q\}$  in system Q and the conjugate displacements  $\{D_P\}$  in system P

System P: Forces  $F_{P1}$ ,  $F_{P3}$ ; Displacements  $D_{Q1}$ ,  $D_{Q3}$

System Q: Force  $F_{Q2}$ ; Displacement  $D_{P2}$

$$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$$

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Now displacement at point 2 in system P is  $D_{P2}$  and corresponding displacement at point 1 and point 3 in system 3 is  $D_{Q1}$  and  $D_{Q3}$ . And what this theorem says that this multiplied by this plus this multiplied by this means work done by this force and corresponding conjugate displacement will be equal to the work done by this force and corresponding conjugate displacement.

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**Reciprocal Theorem: E Betti 1872**

In a linear elastic structure the total external virtual work associated with forces  $\{F_P\}$  in system P and the conjugate displacements  $\{D_Q\}$  in system Q is equal to that associated with forces  $\{F_Q\}$  in system Q and the conjugate displacements  $\{D_P\}$  in system P

System P: Forces  $F_{P1}$ ,  $F_{P3}$ ; Displacements  $D_{P2}$

System Q: Forces  $F_{Q2}$ ; Displacements  $D_{Q1}$ ,  $D_{Q3}$

$$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$$

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And this is the Bettis Reciprocal Theorem which is the general version of the reciprocal theorem of that proposal. Maxwell if we look at the year it was proposed, it was proposed in 1872.

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**Reciprocal Theorem: E Betti 1872**

In a linear elastic structure the total external virtual work associated with forces  $\{F_P\}$  in system P and the conjugate displacements  $\{D_Q\}$  in system Q is equal to that associated with forces  $\{F_Q\}$  in system Q and the conjugate displacements  $\{D_P\}$  in system P

System P: Forces  $F_{P1}$ ,  $F_{P3}$ ; Displacements  $D_{P2}$

System Q: Forces  $F_{Q2}$ ; Displacements  $D_{Q1}$ ,  $D_{Q3}$

$$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$$

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Now let us see what is the Maxwell version? Maxwell version is the very special case of Betti reciprocal theorem. What you see? This is proposing much earlier than the Bettis generalization. It was proposed in 1864 for just one force and one displacement case and then later it was generalized by Betti in 72 and that generalized version we have just now discussed. Now this is the general version of the Bettis reciprocal theorem.

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Reciprocal Theorem: J C Maxwell 1864

$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$

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Now from that if you just remove the third point, okay. Now we have two forces and two corresponding displacements. Now make this force and this force same, okay.

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Reciprocal Theorem: J C Maxwell 1864

$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$

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Now this point is point number 1, this point is point number 2. Similarly this point is point number 1 and this point is point number 2.

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Reciprocal Theorem: J C Maxwell 1864

System P

System Q

$$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$$

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Then since  $D_{Q3}$  is gone so this becomes zero. So we have two forces and two displacements. Now in first case force is acting at point number 1 and then these are the displacement at 1, displacement at 2.

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Reciprocal Theorem: J C Maxwell 1864

System P

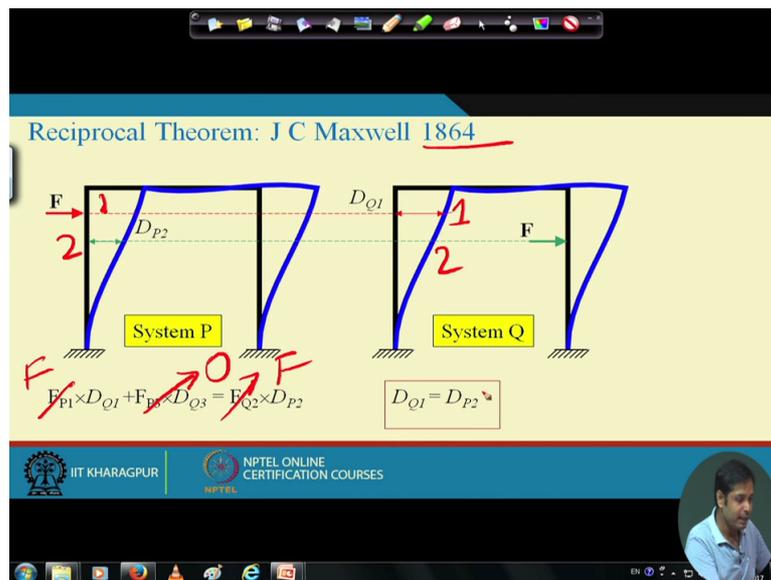
System Q

$$F_{P1} \times D_{Q1} + F_{P3} \times D_{Q3} = F_{Q2} \times D_{P2}$$

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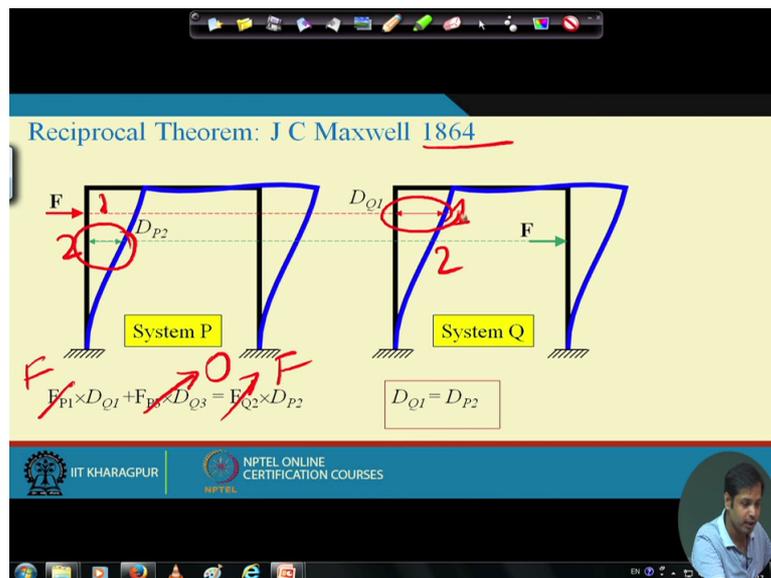
Now what we have is this becomes  $F$  and this also becomes  $F$  because now the forces are same. This  $F$  and this  $F$  gets cancelled out. And then we have  $D_{Q1}$  is equal to  $D_{P2}$ .

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Means this displacement is equal to this displacement.

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Now what is this displacement? This displacement is the force acting at 1 and measure the displacement at 2, here force acting at 2 and measure the displacement at 1. This is the Maxwell reciprocal theorem and we observed through a beam problem just now, right? This is the beam problem we demonstrated that theorem, right? This is the same theorem obtained from general version of the reciprocal theorem.

So Maxwell reciprocal theorem is a general version of the Betti reciprocal theorem. Now as I said I started this lecture by saying that this reciprocal theorem virtual work principle gives very interesting insight into the structural behaviour and that insight is the reciprocal theorem. But what is the physical interpretation of the reciprocal theorem, I leave it to you to think about it.

We will again come to this reciprocal theorem and discuss what is the physical interpretation of this theorem and how this (reci) structural behaviour, this property of the structure makes our life easier. We will discuss this when we learn displacement based on our force field method for solving or in general the direct stiffness method of structural analysis towards the end of this course. For the time being I leave it to you.

Please think in what way we can use this information in our structural analysis, okay. I stop here today. What we will do as I said this is a last class of this week.

Next week what we will do is we will start how to find out deflections of beams and frames, we will learn various methods and one method will be the virtual work principle unit load method. There we will have some demonstration of the unit load method for beams and frames, okay. See you in the next week. Thank you.