Computational Hydraulics
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Lecture 8
Ordinary Differential Equation: BVP

Welcome to this lecture number 8 of the course computational hydraulics. We are in module number 2 numerical methods and we will be covering unit number 4 which is ordinary differential equation and boundary value problem, BVP.

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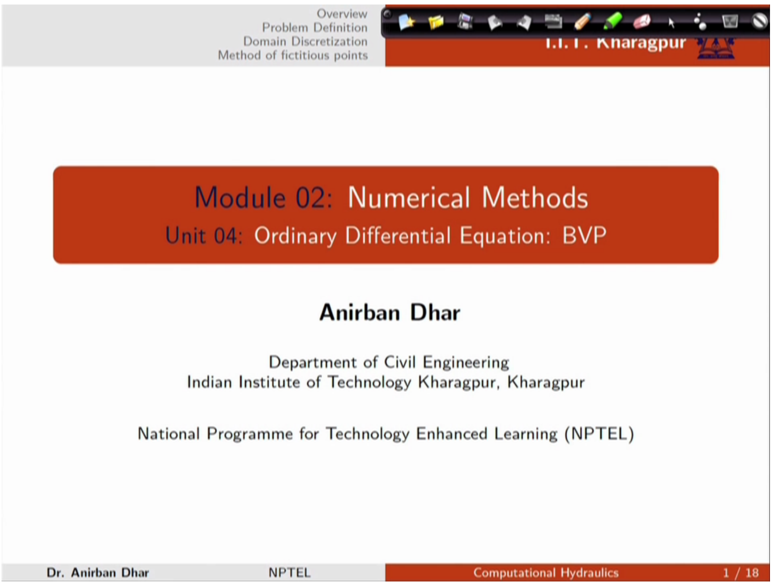

Our learning objectiveof this particular unit at the end of the unit, students will be able to discretize ordinary differential equation along with boundary conditions. Also students will be able to derive algebraic form using discretizedODE and BCs. That means ordinary differential equations and boundary condition.

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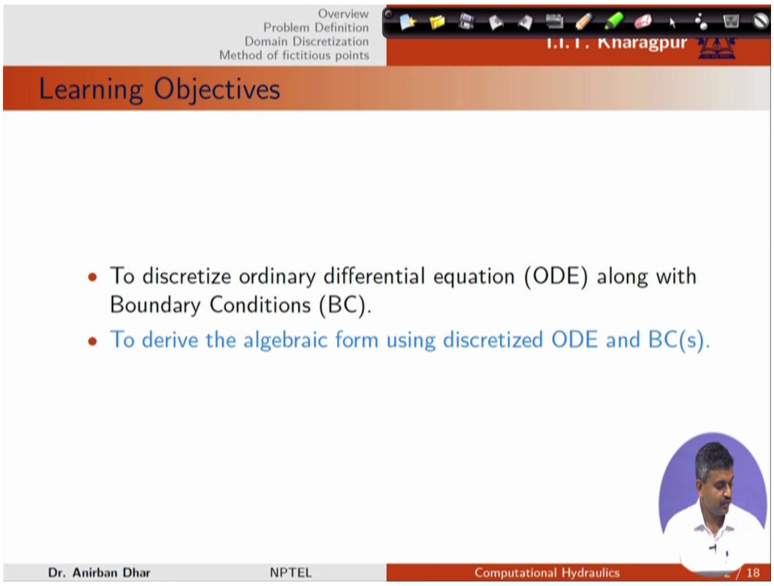

Ordinary differential equation with space discretizationwe call it as boundary value problem, time or time like discretization that is initial value problem. Now our lecture number 7 we have covered this initial value problem. Physical problem in one dimension can be mathematically conceptualized using ordinary differential equation along with boundary conditions. Ordinary differential equation can be solved by using finite difference approach. Accuracy of the solution depends on discretization of ordinary differential equation and boundary conditions.

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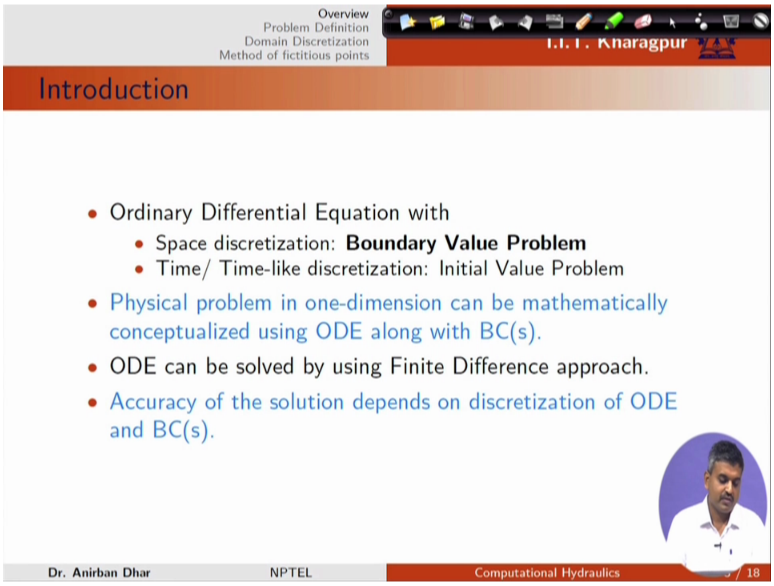

This is specific to boundary value problems. In case of initial value problem we have seen that accuracy of the solution depends only on the ordinary differential equation.

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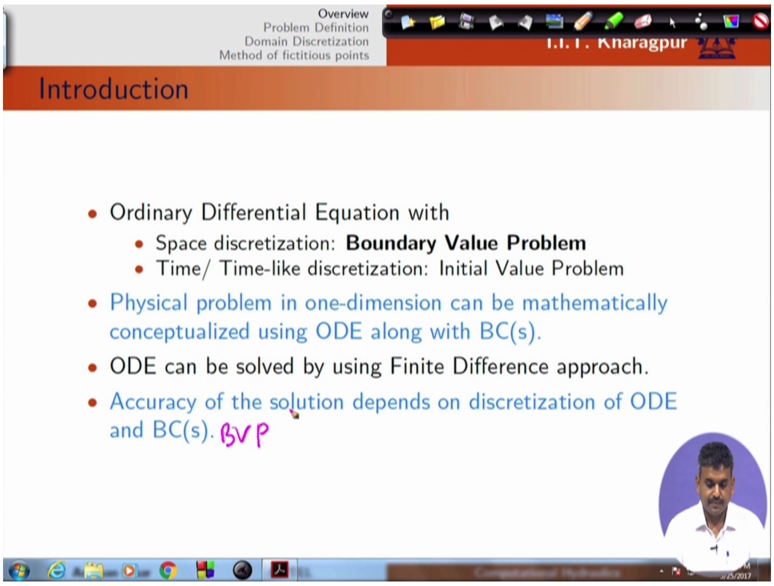

Let us consider a physical problem. We have one water body on the left side, we have confined unconfined aquifer system. Confined and unconfined aquifer system with leaky confining layer.

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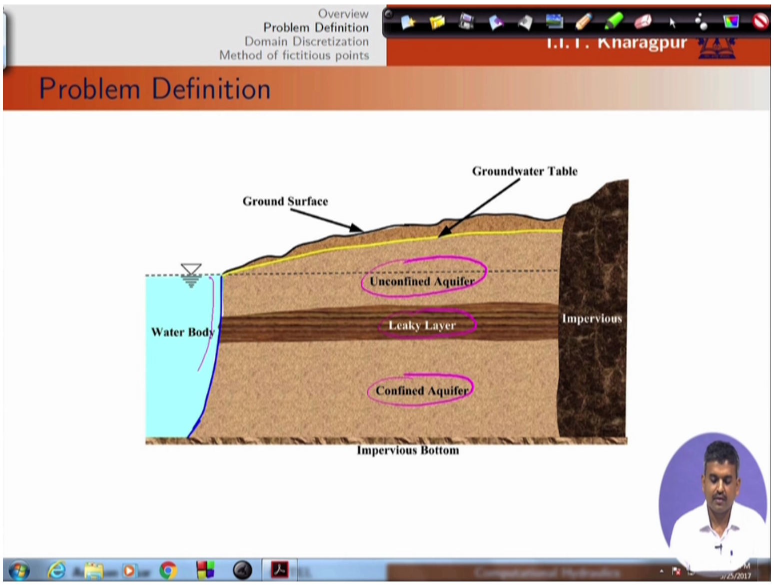

This is impervious unit. So in impervious unit is like mountain or any impervious rocks structure present there.

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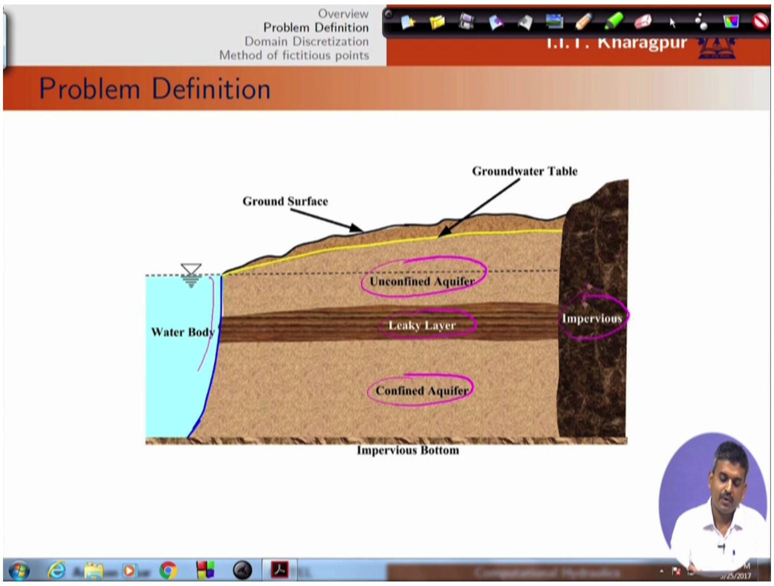

We have impervious water and this is our groundwater table with yellow line and on top we have ground service.

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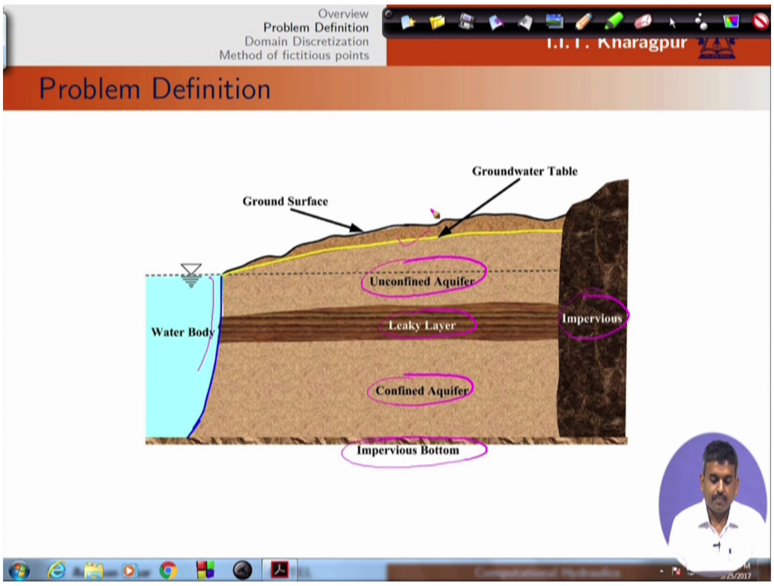

So problem is we need to find out what is the variation of groundwater table withthis direction. That means we are concerned about only one direction.

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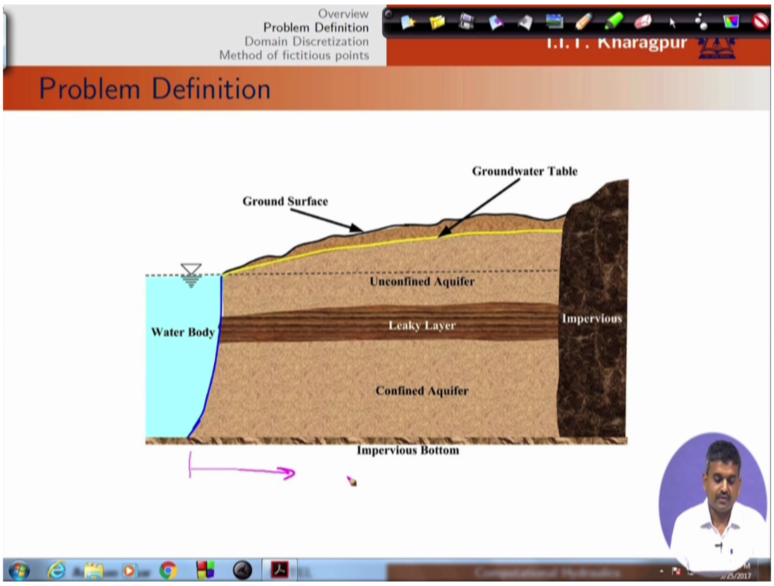

On the left hand side we have this water body which is specified condition or Dirichlet kind of condition, specified head.

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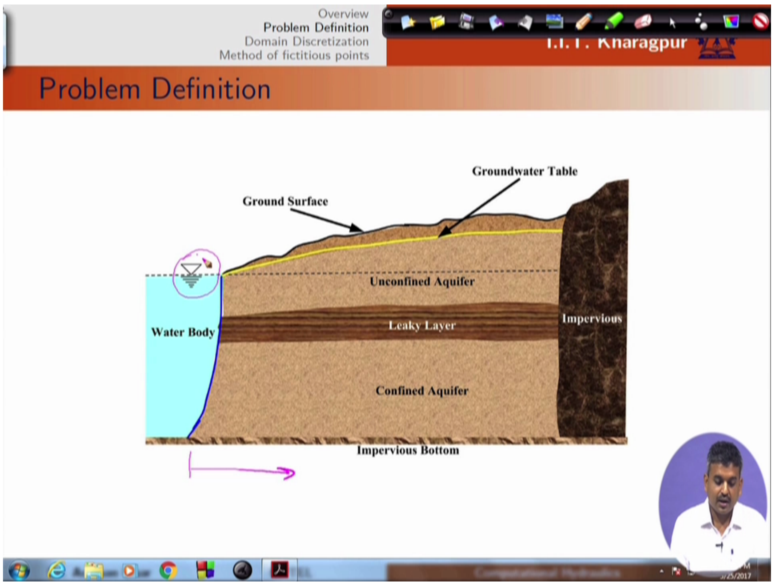

On the right hand side we have impervious condition which is zero flux or zero Neumann condition. So with this information we can proceed and we can mathematically conceptualize the problem using differential equations and boundary condition.

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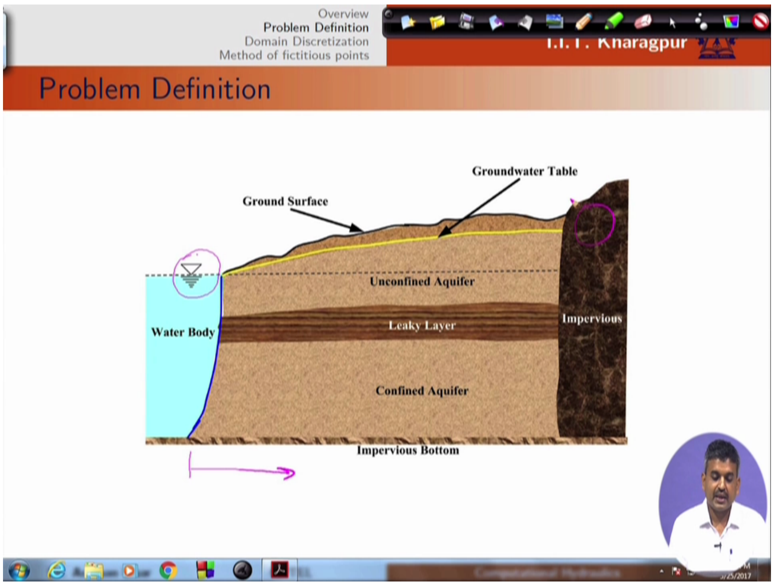

So mathematical conceptualization we can write this differential equation, where h is the head, which is the function of X only and T aquifer transmissivity, Cconfis a hydraulic conductivity divided by thickness of confining layer. This is related to confining unit and H wt is the overlying water table elevation. So this is our governing equation or GE.

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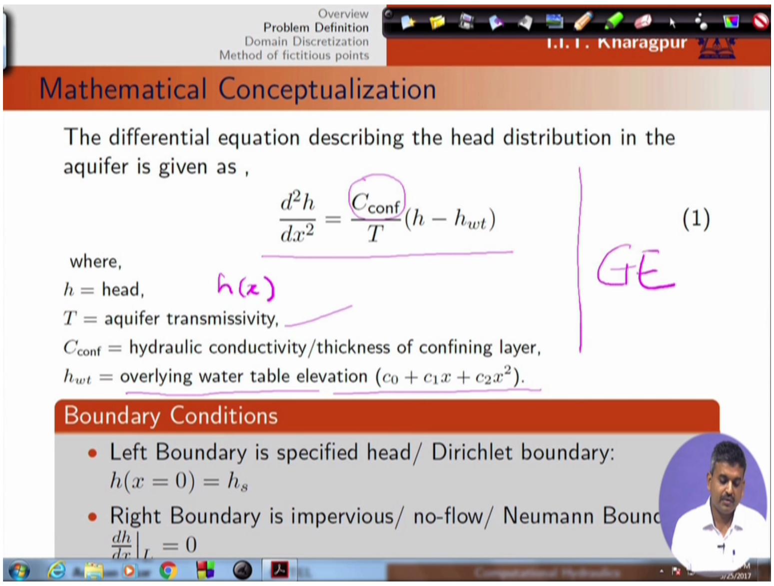

Now we need boundary conditions or BC. Left hand side as I have discussed, we have Dirichlet boundary condition. X is equal to zero, we have Hs or specified boundary. On the right hand side we have impervious boundary or no flow boundary or Neumann boundary which is dh by sx at L. L is the distance between the water body and the impervious unit on the right hand sidethis value is zero.

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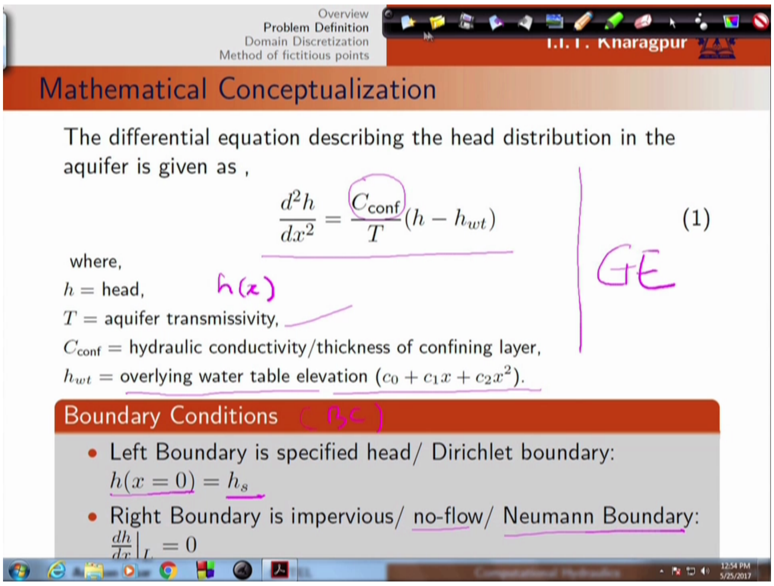

So we need to discretize the domain. So we have some information, with that information we can, we have conceptualized the problemin terms of differential equation. Now we need to discretize our physical domain with number of grids.

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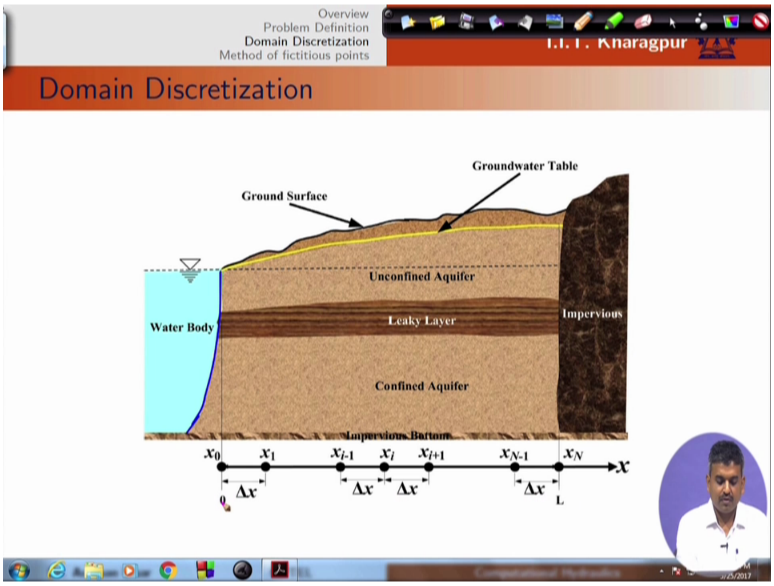

So let us say, we are concerned about only X direction. That's why we are considering X not to Xn as grid points and these are equally spaced and spacing size is delta x. Now we can discretize our governing equation. On the left hand side we have d2h by dx2. So with that if we discretize, this is our second order derivative and second order derivative has got this second order accuracy.

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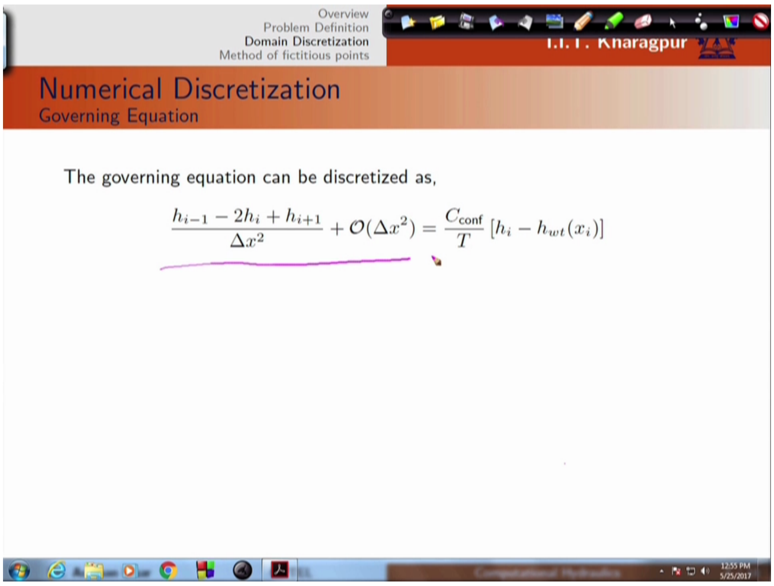

On the right hand side we have C conf by T and this h is basically evaluated at hI. This is exact value and hwt, this is a function of X. So this is again exact value. So overall accuracy of the governing equation issecond order.

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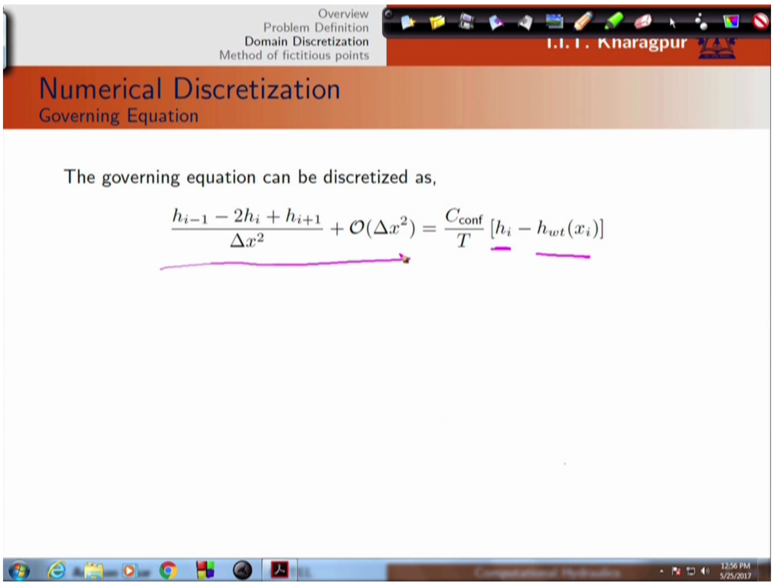

Now we can discretize this equation and we can (wri) rearrange it. So with h minus 1 term, hI, hI plus 1, we have rearrange it and this is our right hand side term.

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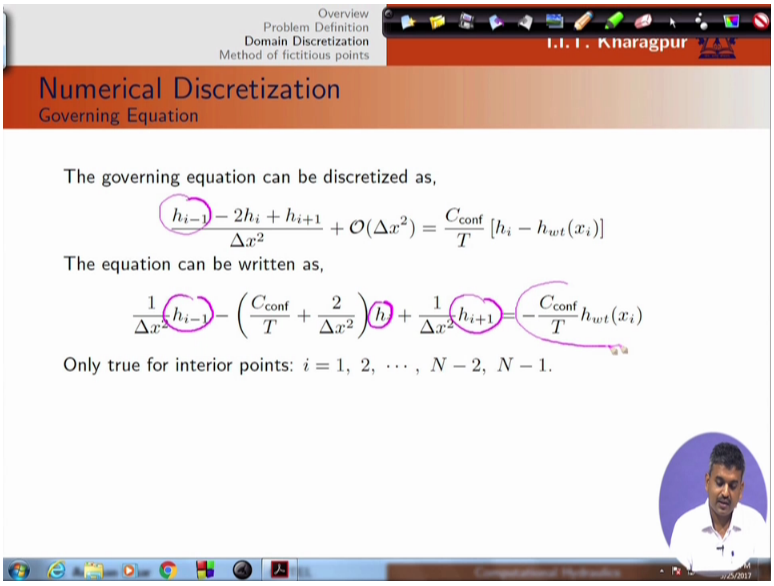

So basically these are coefficients.

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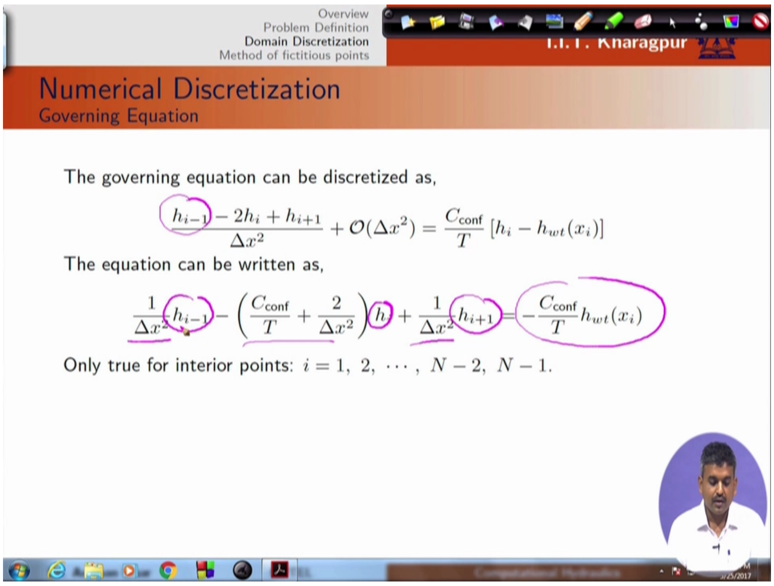

This governing equation is true for interior points. Interior point means that from point number 1 to N minus 1 excludingwe are excluding the points zero and N in this case.

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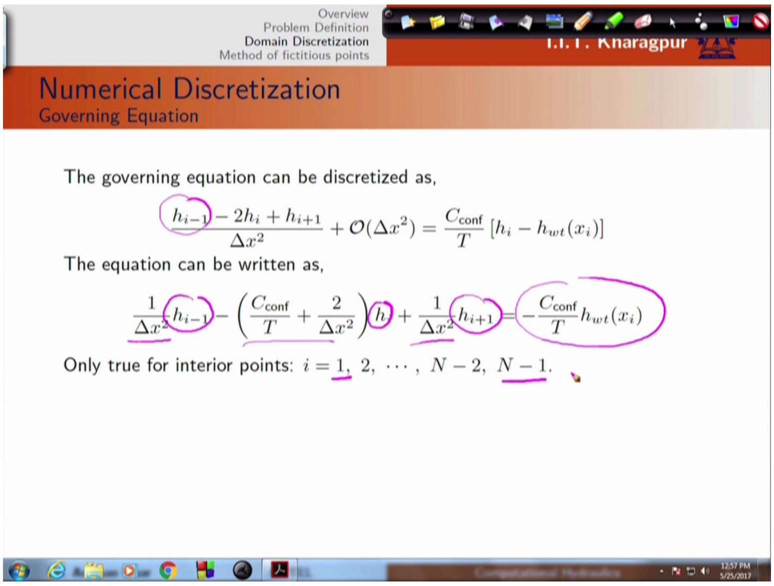

So equation can be further simplified.We can write it in a compact form with these coefficients where B is del x square, this dIis minus C conf to del x square, aI which is again 1 by del x square and rI which is on the right hand side as minus C conf T hwt xI.

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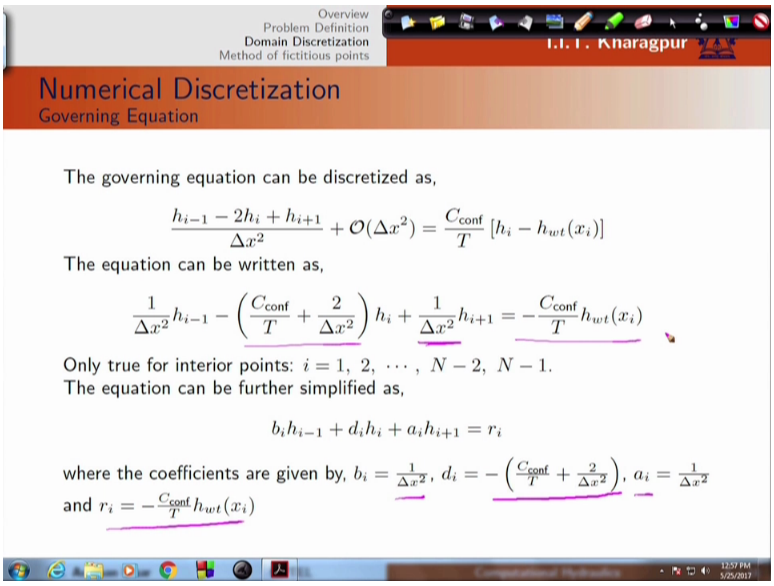

Now let us discretize the boundary condition. Left hand boundary is Dirichlet boundary. Dirichlet boundary is without any truncation error in general equation format is B not,because this is valid for ourfirst X not point. So if we use the general equation that means whatever we have used for our governing equation, B, D, A not and R not, then we can write B not is zero, D not is 1, A not is zero, R not is specified value hs.

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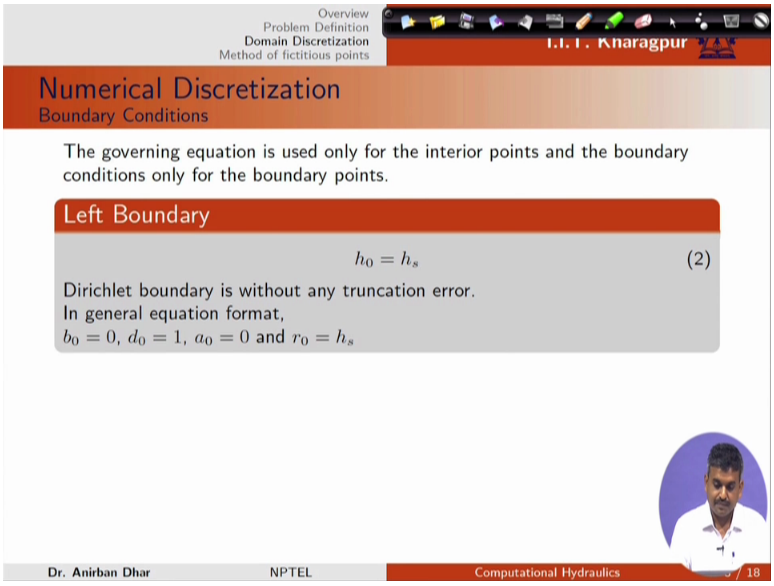

So if we consider right boundary then we can discretize it with backward difference. That's why hN minus hN minus 1 divided by del x plus first order accurate and this is first order accurate scheme. So order of accuracy is del X. In general (form) equation, this is bN is minus 1 by del X, dN is 1 by del X, aN is zero, rN is zero. So we have written (abl) algebraic equation for each of the interior points and boundary points. For interior points we are using governing equations and for boundary points we are using boundary condition in discretized form.

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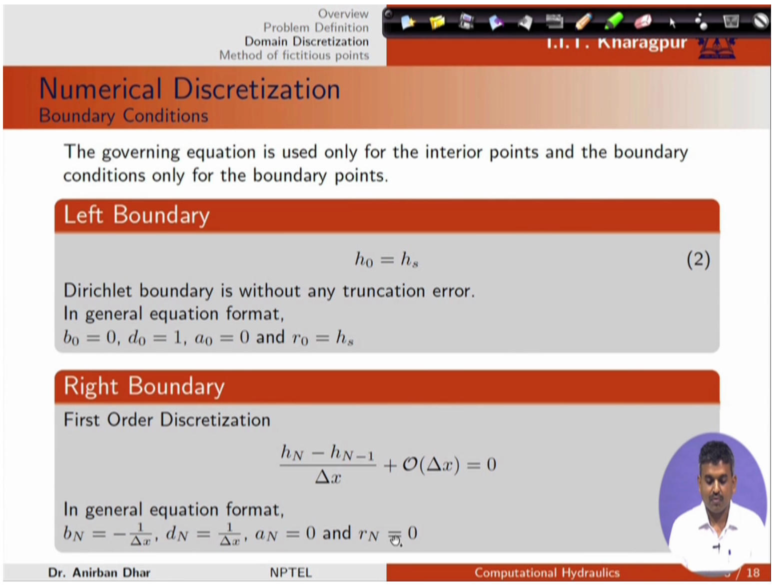

So if we see the algebraic form thenit is obvious that this portion is for interior points and last two points are actually boundary points. On the right hand side which is impermeable boundary. On the left hand side we have this D not A not although A not is zero, because this is specified boundary.

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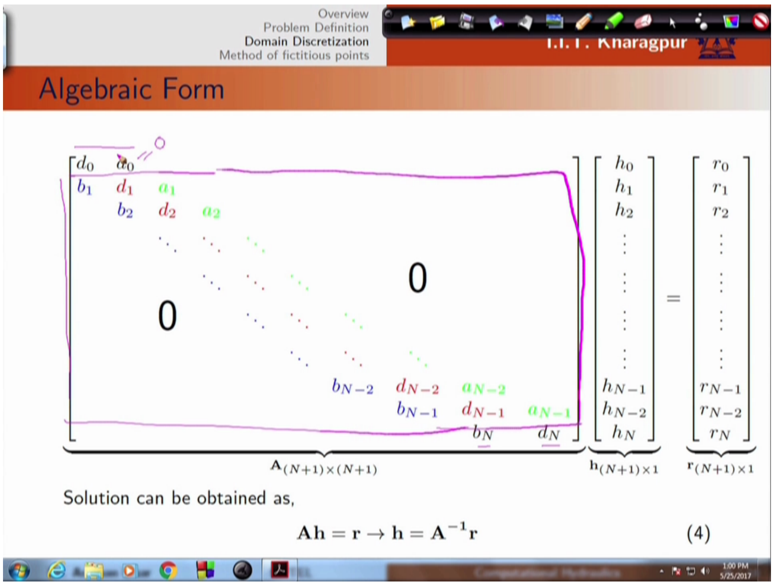

So D not is 1 only, A not is zero. So in general structure this coefficient matrix N plus 1 into N plus 1 and this H is N plus 1 into 1 and R is N plus 1 into 1.

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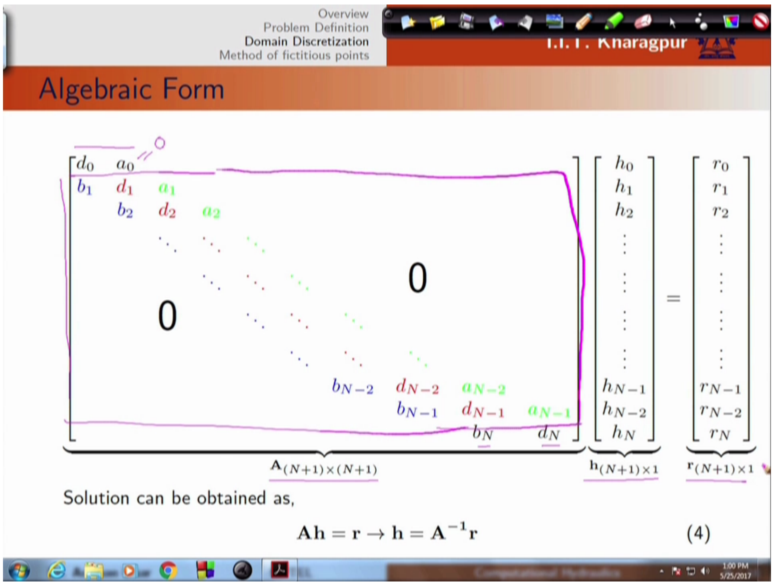

So solution can be obtained easily with this AH equals to R matrix R equation, H is A inverse R and in this case we can easily get the solution.

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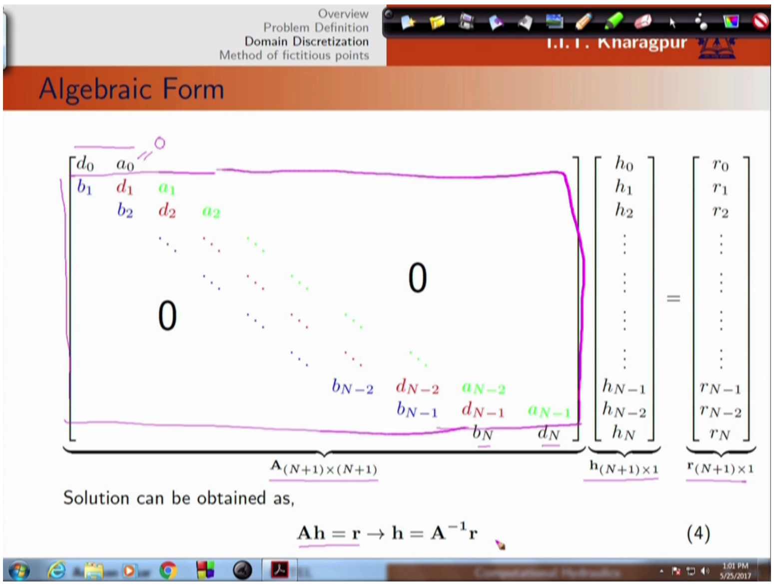

But the problem is this solution accuracy. We have discretized our governing equation with second order accuracy and our boundary condition specifically right hand boundary with first order accuracy. So overall accuracy of the solution will be of first order. So in tridiagonal form we haveboundary condition on top then we have interior notes and at the end again we have this boundary nodes.

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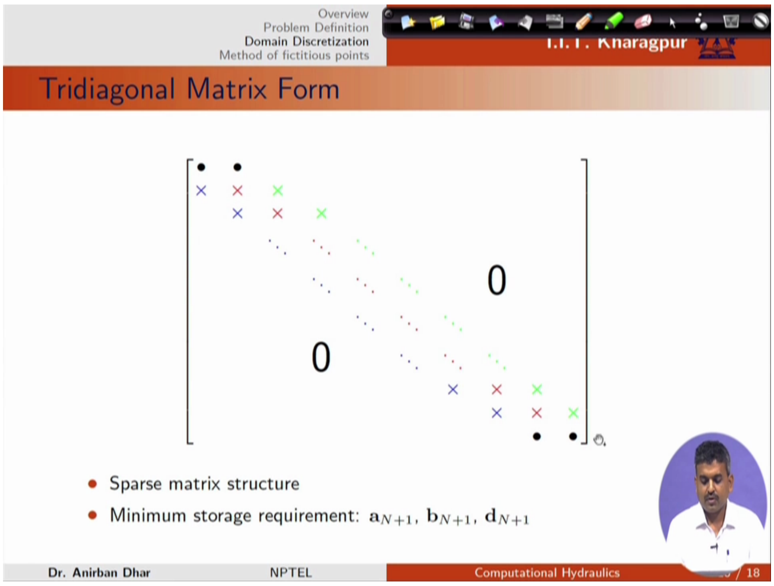

So it has got sparse matrix structure. We have zero values on both sides and it has got minimum storage requirement. If westore this diagonal values this is our B matrix, this is D matrix and D column vector and this one is A.

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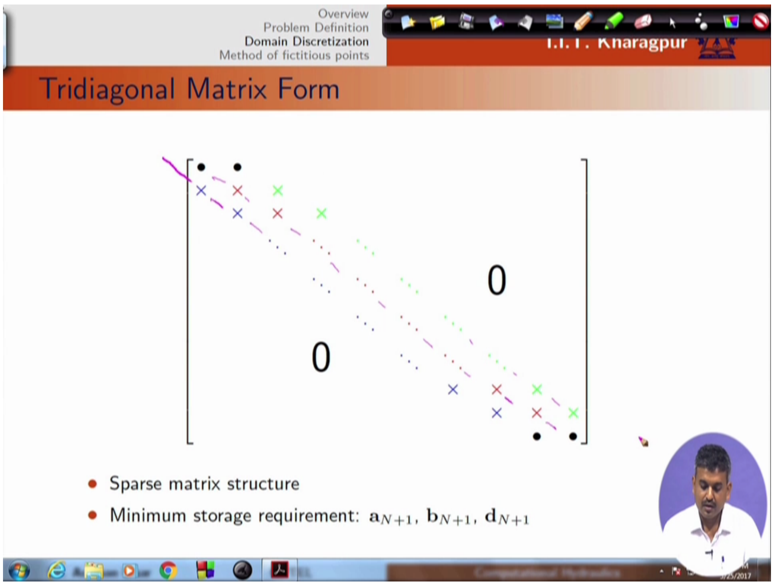

So in this case aN is always zero and in this case b0 is always zero. So considering these two external points ornonphysical points we canconstruct the matrix and we can store these values in individual column vectors.

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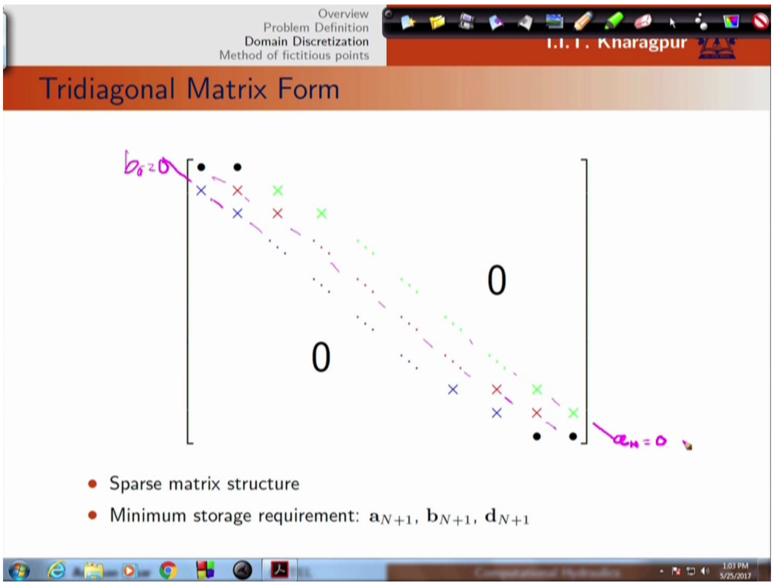

So if it store the full matrix, the storage requirement is N plus 1 into N plus 1, but if we store it in column vectors the storage requirement is 3 into N plus 1. So obviously we are gaining in the terms of storage. So obviously computational cost will be reduced in this case.

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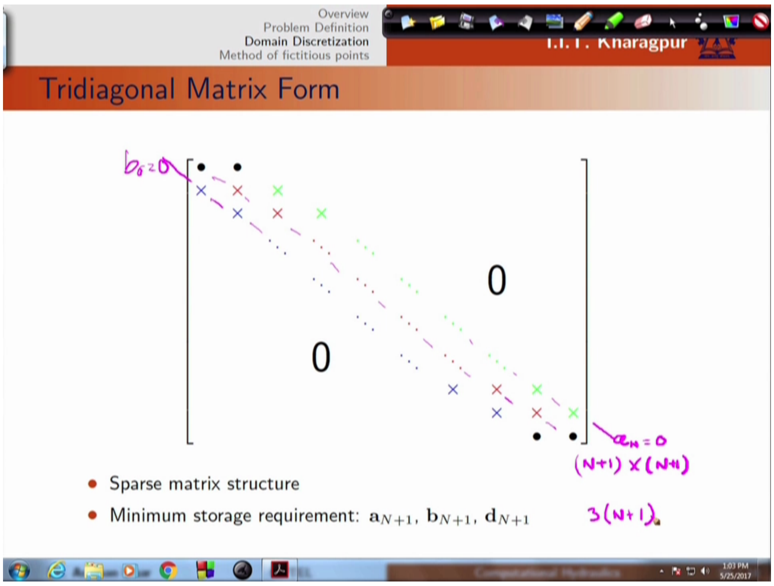

So accuracy of boundary condition. If we use the second order discretization for first order derivative, then we can write this for right hand boundary with N, N minus 1 and N minus 2 points and these are the coefficients.

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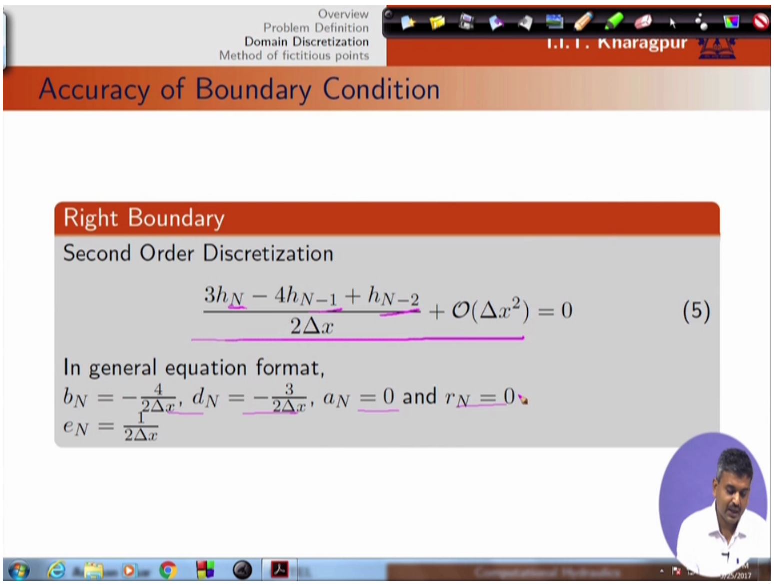

Now we are getting one extra coefficient here which is represented as eN 1 by 2 delta X.

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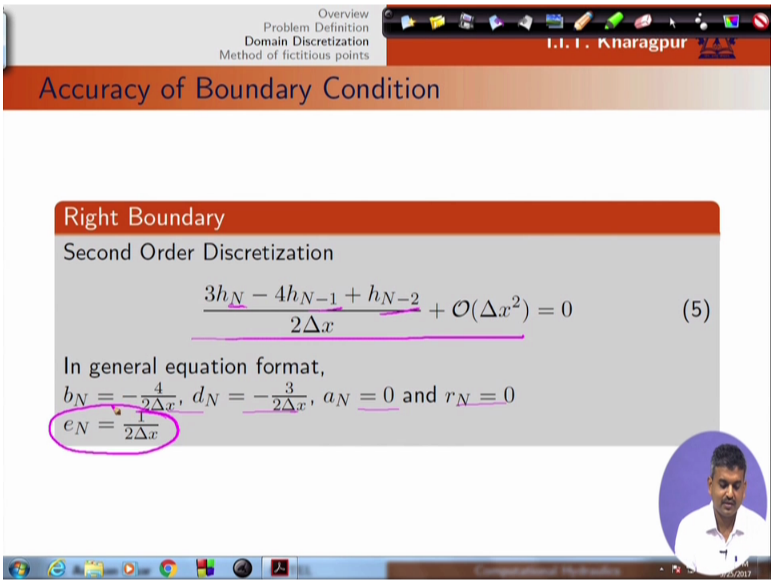

So how to incorporate this within Matrix structure. If we have algebraic form, so obviously if you want to store the full matrix this kind of arrangement is okay.

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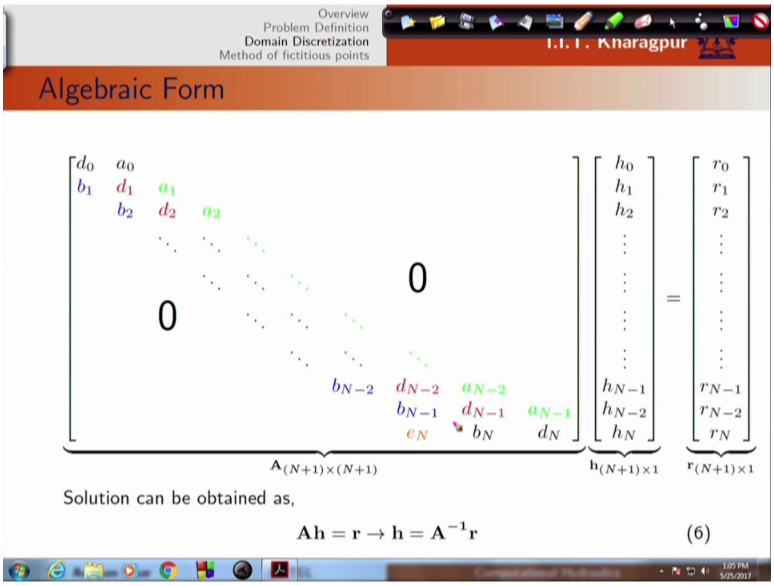

But problem is with this eN. EN is breaking the tridiagonal structure.

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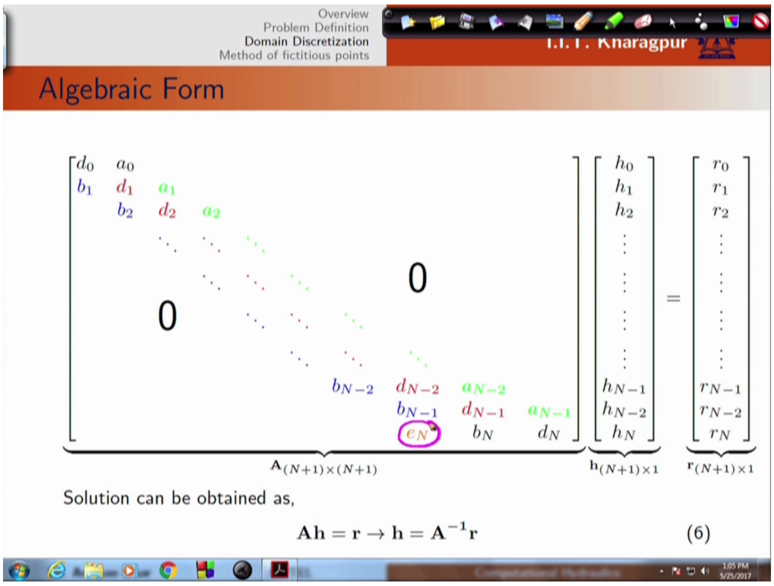

So this is increasing the storage requirement however we are getting second orders accurate solution. If you want to use direct conversion approach then we can store the full matrix and we can get the solution directly.

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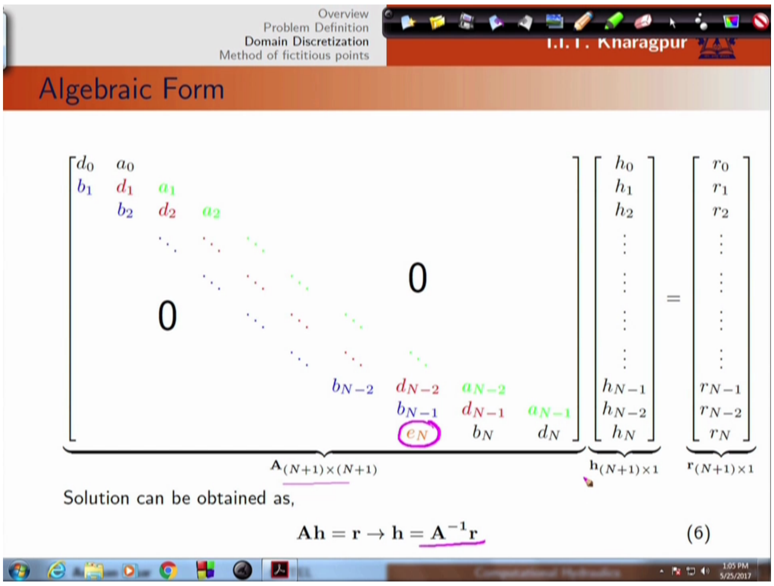

However in this case due to this extra point, tridiagonal structure is broken. Although this is second order scheme we need to preserve the matrix structure, with second order accuracy.

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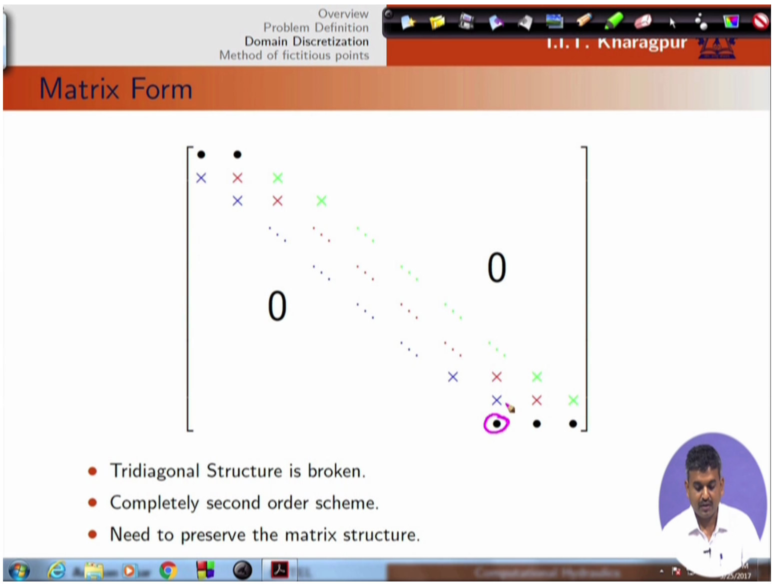

So in first algebraic form we have seen that the tridiagonal structure is intact. Second order accurate discretization of our right hand boundary condition, this is breaking the tridiagonal structure. Now we can use governing equation for interior.

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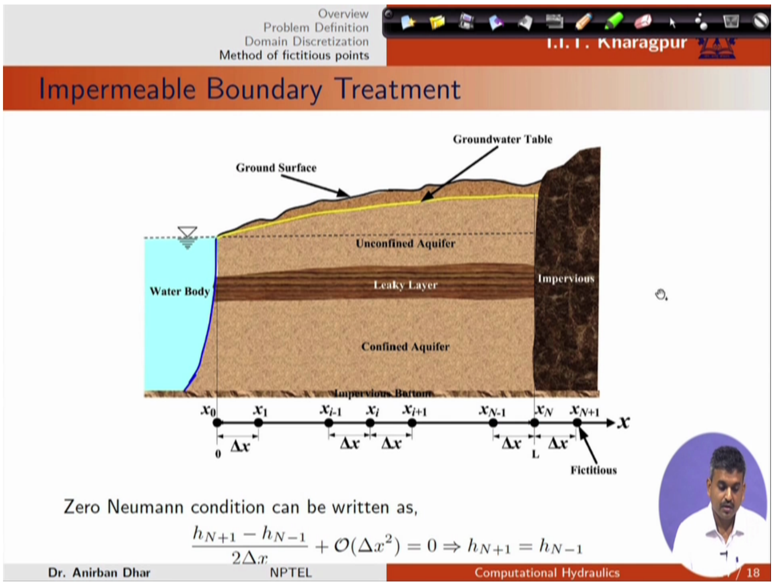

So impermeable boundary treatment. We can create one fictitious point, N plus 1.

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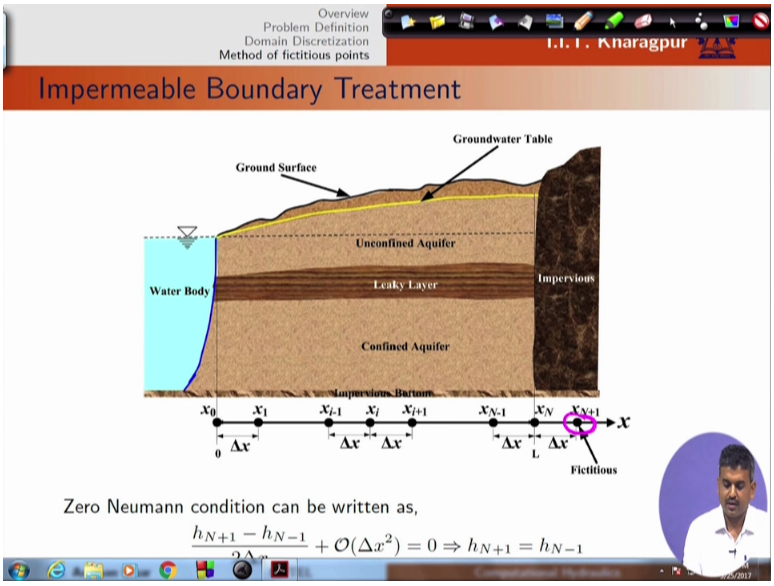

On the right hand side where we have this impermeable boundary.

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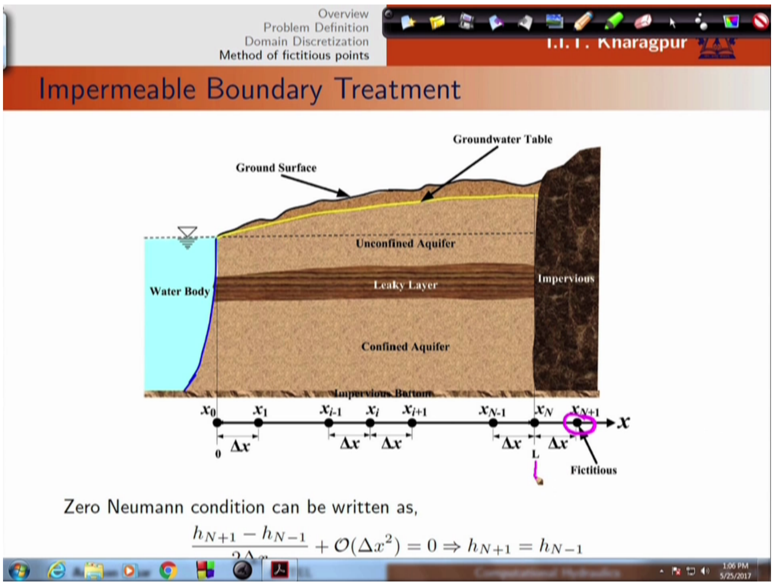

So zero Neumann boundary condition can be written as with our previous knowledge that is center difference hN plus 1 minus hN minus 1 divided by 2 delta X. This is second order accurate. This implies that we have this value hN plus 1 equals to hN minus 1.

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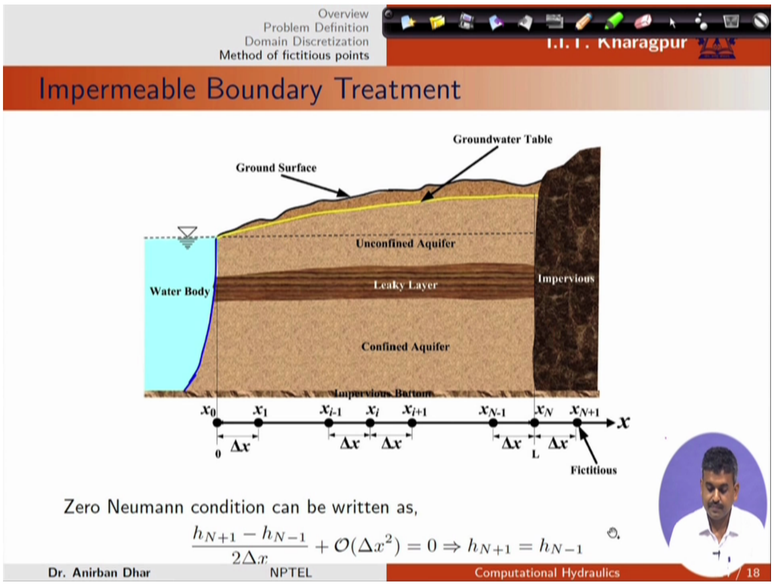

Now we can use this information in our governing equation. So if we discretized the governing equation at node I, then we can write bNhN minus 1. This dN, hN, aN hN plus 1 and rN.

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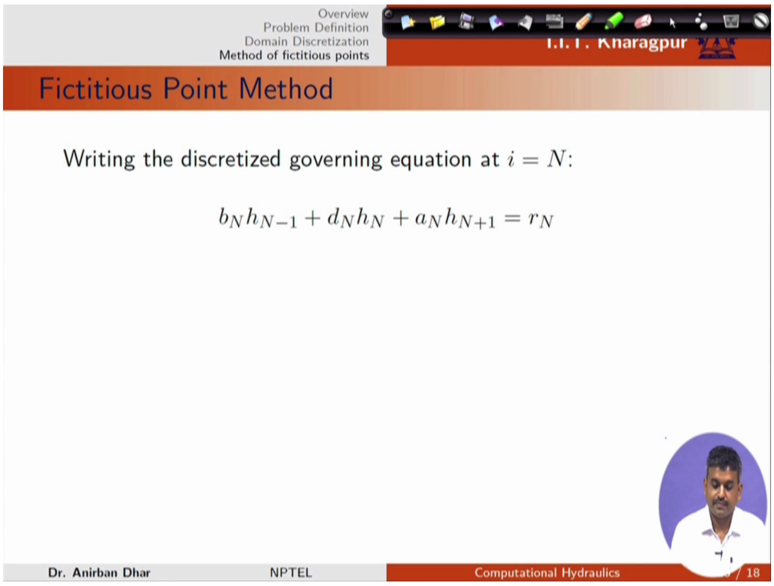

So for this one using the boundary condition, this can be written as. So in this case hN plus 1 and in this case we can write it as hN minus 1.

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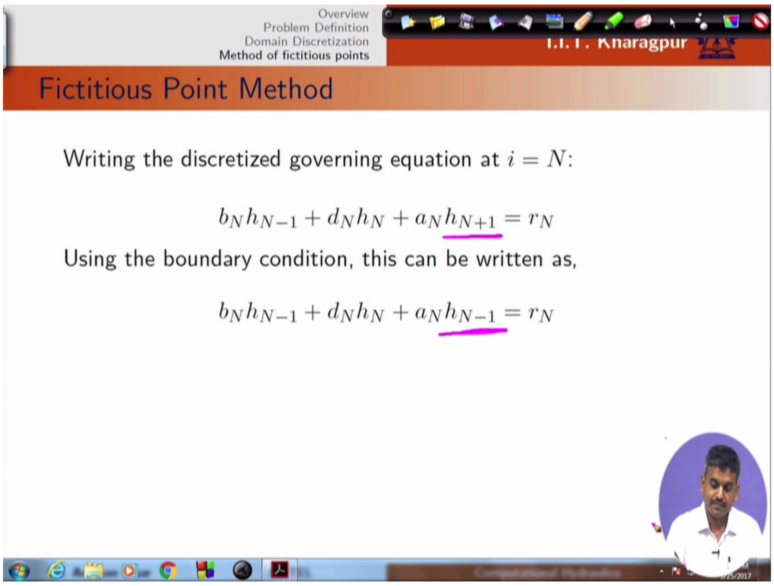

So this can be simplified as bN plus aN as coefficient of hN minus 1 and dN and these are the coefficients. DN, aN and rN for the corresponding point xN.

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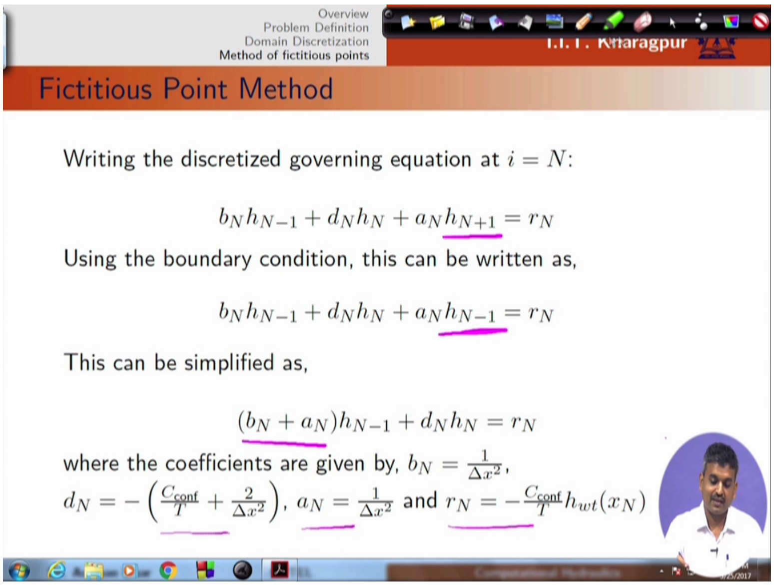

Again we are getting the algebraic form. We have second order accurate scheme because our governing equation is second order accurate. We are utilizing it for our boundary condition and in this case tridiagonal structure is also intact. So either we can use direct inversion or we can use special algorithm with tridiagonal matrix solution.

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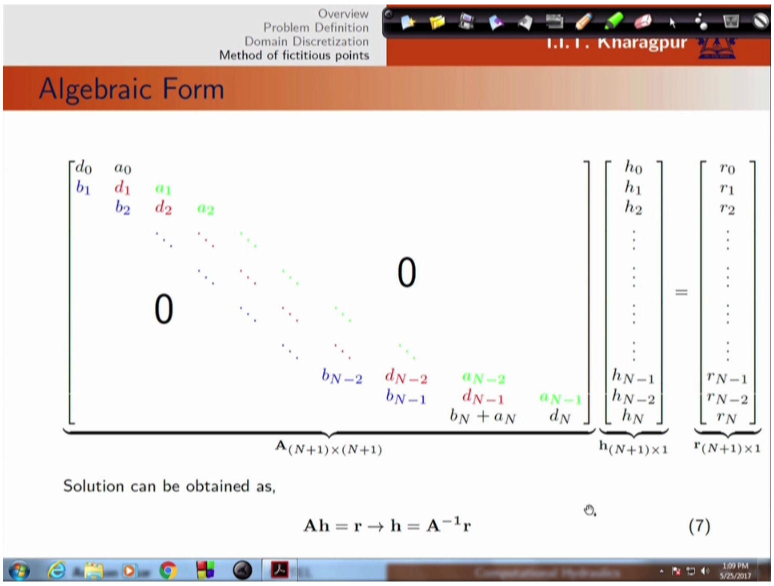

So matrix form again we have two points and our tridiagonal structure is there. So tridiagonal structure is preserved and completely second order scheme. So at the same time we are getting two important points.

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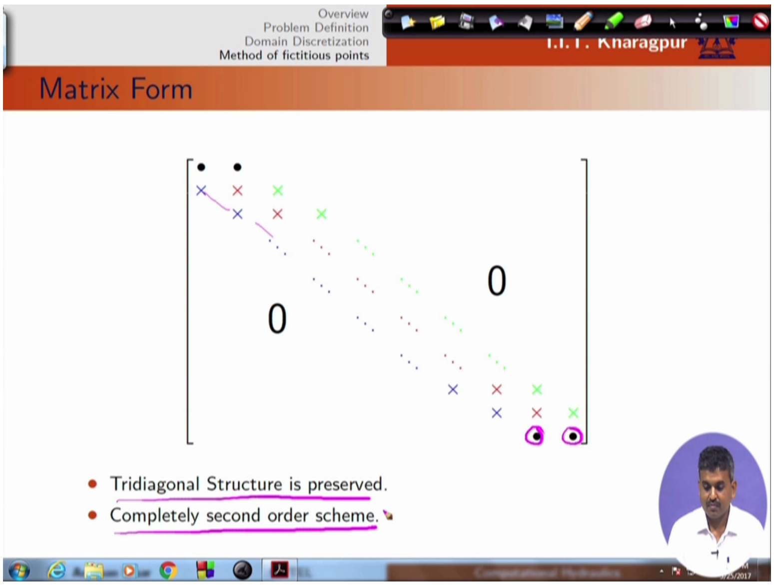

One is computational this will be better. At the same time the solution accuracy will be better. Thank you.