

Computational Hydraulics
Professor Anirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur
Lecture 7
Ordinary Differential Equation: IVP

Start? Welcome to this lecture number 7 of the course computational hydraulics. We are in the module 2 numerical methods. And in this specific lecture we will be covering unit three, ordinary differential equation and specifically initial value problem or IVP.

(Refer Slide Time 00:30)

The slide features a navigation menu at the top left with the following items: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The top right corner displays the I.I.T. Kharagpur logo. The main content area has a red header box containing the text: "Module 02: Numerical Methods" and "Unit 03: Ordinary Differential Equation: IVP". Below this, the author's name "Anirban Dhar" is listed, followed by his affiliation: "Department of Civil Engineering, Indian Institute of Technology Kharagpur, Kharagpur". The slide also mentions the "National Programme for Technology Enhanced Learning (NPTEL)". The footer contains the text: "Dr. Anirban Dhar", "NPTEL", "Computational Hydraulics", and "1 / 19".

So what is the learning objective for this particular unit? At the end of this unit (stu) students will be able to discretize first order differential equation along with boundary condition initial condition because in initial value problem we have only ordinary differential equation and initial conditions.

(Refer Slide Time 1:20)

The slide is titled "Learning Objective" and contains a single bullet point: "To discretize first-order ordinary differential equation (ODE) along with Initial Condition (IC)." The slide includes a navigation menu at the top with options: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The I.I.T. Kharagpur logo is visible in the top right. A small circular portrait of the speaker is in the bottom right. The footer contains "Dr. Anirban Dhar", "NPTEL", "Computational Hydraulics", and the slide number "19".

Ordinary differential equation with initial condition can be solved as initial value problem with time or time like discretization. This point is important because always the first order equation may not be time dependent maybe it is space dependent. And we need to solve that ODE with initial condition to get the solution of initial value problem.

(Refer Slide Time 02:04)

The slide is titled "Introduction" and contains two bullet points: "Ordinary Differential Equation with initial condition can be solved as Initial Value Problem with time/ time-like discretization." and "ODE can be solved by using Finite Difference approach." The slide includes a navigation menu at the top with options: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The I.I.T. Kharagpur logo is visible in the top right. A small circular portrait of the speaker is in the bottom right. The footer contains "Dr. Anirban Dhar", "NPTEL", "Computational Hydraulics", and the slide number "19".

ODE can be solved by using finite difference approach. Whatever we have covered in lecture 6. And we can use different discretization to get the solution of ordinary differential equation. Then comes the accuracy. Accuracy of the solution depends only on discretization of ODE. In case of initial value problem, initial condition is like Dirichlet kind of condition which is specified condition.

(Refer Slide Time 02:50)

The slide is titled "Introduction" and contains the following content:

- Ordinary Differential Equation with initial condition can be solved as Initial Value Problem with time/ time-like discretization.
- ODE can be solved by using Finite Difference approach.
- Accuracy of the solution depends only on discretization of ODE.

At the bottom of the slide, there is a small circular portrait of a man and the text: "Dr. Anirban Dhar NPTEL Computational Hydraulics 19".

So there is no error associated with that initial condition. So the error comes only from the ordinary differential equation. That's why discretization of ordinary differential equation is important. In general we can write that ordinary differential equation with dependent variable phi in this format where $d\phi$ by dt is the derivative term. And $\psi(t, \phi)$ is the function which is representing the derivative.

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The slide is titled "General Structure of IVP" and contains the following text and equation:

In general, first order ODE with dependent variable ϕ can be written as

$$\frac{d\phi}{dt} = \Psi(t, \phi)$$

The slide also includes a navigation bar at the top with "Euler Method", "Modified Euler Method", "Runge-Kutta Methods", and "References", and the text "I.I.T. Kharagpur". At the bottom, there is a Windows taskbar showing the time as 12:26 PM on 5/26/2017.

Now with this we need initial condition. This phi at t_0 which is at the starting or initial location or initial time level value. ϕ_0 is the corresponding value. Ψ is a general function. So this is the basic definition of ordinary differential equation along with our initial condition for initial value problem. In a particular problem we can have multiple dependent variable. So

we will have multiple ordinary differential equation and there should be initial condition for each of those differential equations.

(Refer Slide Time 05:00)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

I.I.T. Kharagpur

General Structure of IVP

In general, first order ODE with dependent variable ϕ can be written as

$$\frac{d\phi}{dt} = \Psi(t, \phi)$$

subject to the initial condition

$$\phi(t_0) = \phi_0$$

where
 $\Psi()$ = a general function

12:27 PM
5/26/2017

Now we need to discretize it numerically. So from governing equation if we integrate it from t_n and t_n plus 1 is representing a general time interval. So we can use mean value theorem to evaluate the right hand side of the above equation.

(Refer Slide Time 05:40)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Numerical Discretization (Sengupta, 2013)

Integrating both sides of ODE from t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} = \int_{t_n}^{t_{n+1}} \Psi(t, \phi)$$

Using Mean Value Theorem to evaluate the RHS of the above equation,

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$$

Dr. Anirban Dhar NPTEL Computational Hydraulics 19

In this case we can see that this ϕ^{n+1} , which is future time level value, ϕ^n which is present time level value, plus Δt , this is time increment and Ψ is the function. This Ψ is the function of t_n plus $\theta \Delta t$. What is this θ ? θ is again a value which is in

between zero to one. That means we can use any value theta to theta within this zero to one. If we use (ze) zero, we can get some method. If we use 1 we can get some different method but with intermediate values we will get intermediate methods.

(Refer Slide Time 07:04)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

I.I.T. Kharagpur

Numerical Discretization (Sengupta, 2013)

Integrating both sides of ODE from t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} = \int_{t_n}^{t_{n+1}} \Psi(t, \phi)$$

Using Mean Value Theorem to evaluate the RHS of the above equation,

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$$

where $0 < \theta < 1$.

Different values of θ and evaluation of $\Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$ yields different numerical methods.

And in this case phi is also function of t_n plus theta into del t. In this case we can also include this lower and upper limit for our problems. So different values of theta we need to evaluate this psi function.

(Refer Slide Time 07:33)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Numerical Discretization (Sengupta, 2013)

Integrating both sides of ODE from t_n to t_{n+1}

$$\int_{t_n}^{t_{n+1}} \frac{d\phi}{dt} = \int_{t_n}^{t_{n+1}} \Psi(t, \phi)$$

Using Mean Value Theorem to evaluate the RHS of the above equation,

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$$

where $0 < \theta < 1$.

Different values of θ and evaluation of $\Psi(t_n + \theta \Delta t, \phi(t_n + \theta \Delta t))$ yields different numerical methods.

Now let us see the truncation error. So ϕ at t_{n+1} can be expanded as $\phi(t_n) + \Delta t \phi'(t_n) + \dots$. This is first order derivative. If we expand it further this is p th order term and this is $p+1$ order term.

(Refer Slide Time 08:07)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Truncation Error Analysis

The function ϕ at t_{n+1} can be expanded as

$$\phi(t_{n+1}) = \phi(t_n) + \underbrace{\Delta t \phi'(t_n) + \dots + \frac{\Delta t^p}{p!} \phi^{(p)}(t_n)}_{\Delta t \Psi(t_n, \phi(t_n), \Delta t)} + \frac{\Delta t^{(p+1)}}{(p+1)!} \phi^{(p+1)}(t_n + \theta \Delta t)$$

where $0 < \theta < 1$.

In this case again θ is in between zero to 1. We can include the zero and 1 value for our problem. The equation can be written as $\phi(t_n) + \Delta t \psi(t_n, \phi(t_n), \Delta t) + \text{Truncation Error}$. This is essentially $\phi(t_n) + \Delta t \psi(t_n, \phi(t_n), \Delta t) + \text{Truncation Error}$. So this function is there plus there will be some truncation error associated with it. So truncation error. We can use this concept for our initial value problems.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Truncation Error Analysis

The function ϕ at t_{n+1} can be expanded as

$$\phi(t_{n+1}) = \phi(t_n) + \underbrace{\Delta t \phi'(t_n) + \dots + \frac{\Delta t^p}{p!} \phi^{(p)}(t_n)}_{\Delta t \Psi(t_n, \phi(t_n), \Delta t)} + \frac{\Delta t^{(p+1)}}{(p+1)!} \phi^{(p+1)}(t_n + \theta \Delta t)$$

where $0 < \theta < 1$.

Thus the equation can be written as,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \Psi(t_n, \phi(t_n), \Delta t) + \underbrace{\frac{\Delta t^{(p+1)}}{(p+1)!} \phi^{(p+1)}(t_n + \theta \Delta t)}_{\text{Truncation Error}}$$

$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n, \phi(t_n), \Delta t) + TE$

If you have theta equals to zero then we call it as Euler method. This phi n plus 1 phi n, delta t and this is evaluated at tn phi n. That means at the starting point we are evaluating the function.

(Refer Slide Time 10:03)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler Method

For $\theta = 0$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \psi(t_n, \phi^n)$$

Now if we use p equals to 1 and theta is equal to zero in our Taylor series expansion, obviously we will get this thing only first order term, because this phi tn is essentially psi tn phi n. So this leading error term is having delta t square value.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler Method

For $\theta = 0$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \psi(t_n, \phi^n)$$

For $p = 1$ and $\theta = 0$,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_n) + \frac{\Delta t^2}{2!} \phi''(t_n)$$

Leading Error

$\phi'(t_n) = \psi(t_n, \phi^n)$

So we can say that the order of accuracy for this step is delta t square. But the problem is when we are considering the Euler's method we are getting delta t as order of accuracy. Why?

Because if we consider this thing we are to get the derivative we need to divide it by delta t. So truncation error in this case divided by delta t this will give you the actual order of accuracy of the method. So in this case we are getting first order accuracy.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler Method

For $\theta = 0$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_n, \phi^n)$$

For $p = 1$ and $\theta = 0$,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_n) + \underbrace{\frac{\Delta t^2}{2!} \phi''(t_n)}_{\text{Leading Error}}$$

Or,

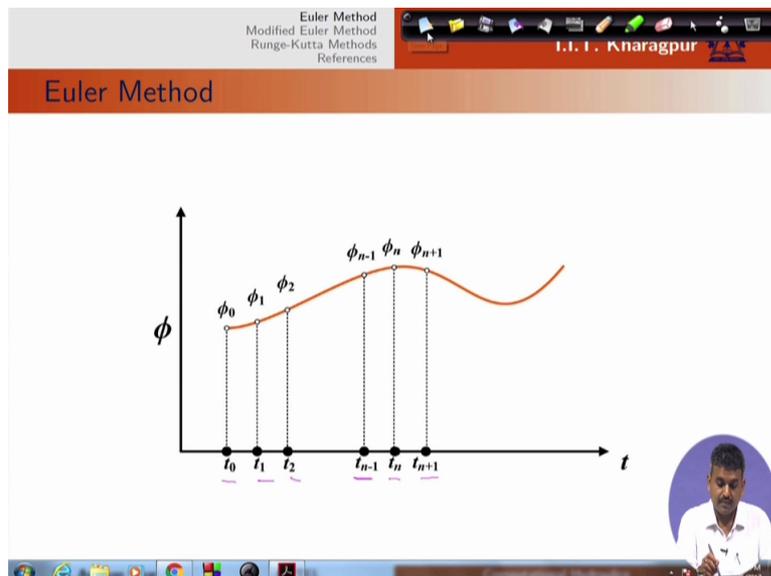
$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_n) + \mathcal{O}(\Delta t^2)$$

Order of Euler's method: $\mathcal{O}(\Delta t)$

$\frac{TE}{\Delta t}$

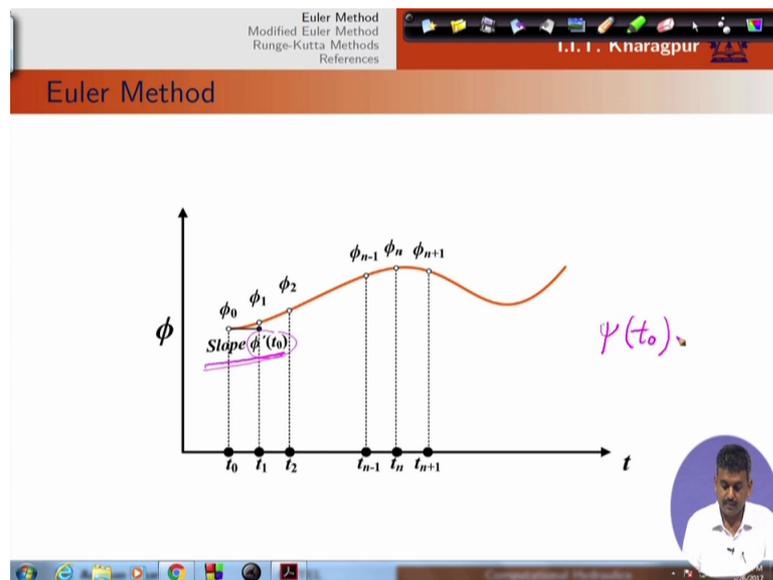
Next if we geometrically see this thing then we have one continuous function and $t_0, t_1, t_2, \dots, t_n, t_{n+1}$ these are nodal points in the t axis.

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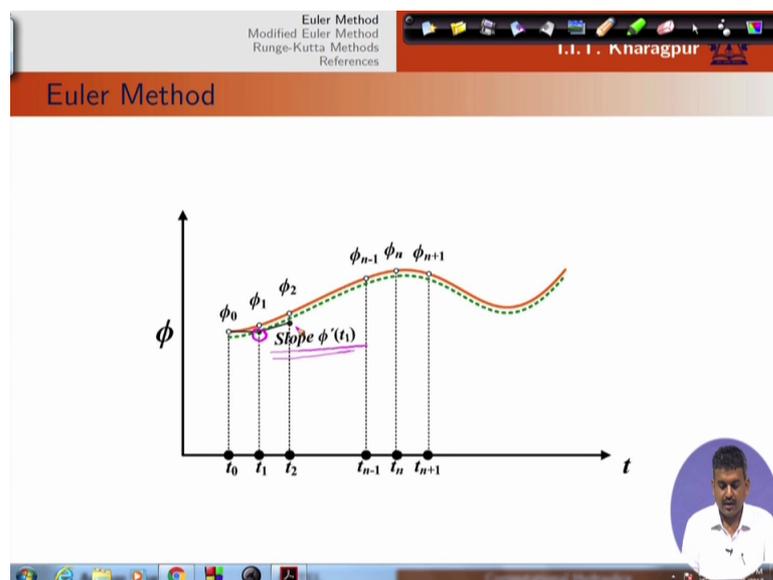
So with this if we calculate the slope at t_0 point that is basically calculating the derivative. This is essentially $\psi(t_0)$ calculation.

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Now with this $\psi(t_0)$ value we can take first step to get this black dot points. White dot points are given actual values and black dot points in this case is giving the value that we are getting from Euler's method. So in this case if we draw another slope at this black dot point, so obviously virtually we are shifting the curve here and we are taking the derivative at t_1 point which is the slope at this particular point. It is important to note that obviously we are getting some amount of deviation from the actual curve. So error is there.

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Next is backward Euler method. In backward Euler method we are calculating the derivative function at future time level. In our Euler's method we have used t_n and ψ_n . In this case we are utilizing future time level value.

(Refer Slide Time 14:59)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Backward Euler Method

For $\theta = 1$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_{n+1}, \phi^{n+1})$$

With this for phi if we write the expansion then t_n is $t_{n+1} - \Delta t$ into $\phi'(t_{n+1})$ plus 1 and this is our leading error. So that's basically calculating the $\psi(t_{n+1})$ and ϕ^{n+1} plus 1.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Backward Euler Method

For $\theta = 1$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_{n+1}, \phi^{n+1})$$

The function ϕ at t_n can be expanded as

$$\phi(t_n) = \phi(t_{n+1}) - \Delta t \phi'(t_{n+1}) + \underbrace{\frac{\Delta t^2}{2!} \phi''(t_{n+1})}_{\text{Leading Error}}$$

$\Psi(t_{n+1}, \phi^{n+1})$

So we can say that we have again this is second order accuracy but actually we are getting first order accurate backward Euler method. So with this if we use any intermediate value so what will be the situation? We need to examine that.

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The slide is titled "Backward Euler Method" and is part of a presentation on numerical methods. The navigation menu at the top includes "Euler Method", "Modified Euler Method", "Runge-Kutta Methods", and "References". The slide content is as follows:

For $\theta = 1$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi(t_{n+1}, \phi^{n+1})$$

The function ϕ at t_n can be expanded as

$$\phi(t_n) = \phi(t_{n+1}) - \Delta t \phi'(t_{n+1}) + \underbrace{\frac{\Delta t^2}{2!} \phi''(t_{n+1})}_{\text{Leading Error}}$$

Or,

$$\phi(t_{n+1}) = \phi(t_n) + \Delta t \phi'(t_{n+1}) + \mathcal{O}(\Delta t^2)$$

Order of Backward Euler method: $\mathcal{O}(\Delta t)$

A small video inset in the bottom right corner shows a man in a white shirt speaking.

So in this case we are using theta equals to 1 by 2. You can see that in this case we are getting this n plus del t by 2 and phi also evaluated tn plus del t by 2.

(Refer Slide Time 17:02)

The slide is titled "Modified Euler Method" and is part of a presentation on numerical methods. The navigation menu at the top includes "Euler Method", "Modified Euler Method", "Runge-Kutta Methods", and "References". The slide content is as follows:

For $\theta = \frac{1}{2}$, we can write

$$\phi^{n+1} = \phi^n + \Delta t \Psi \left[t_n + \frac{\Delta t}{2}, \phi \left(t_n + \frac{\Delta t}{2} \right) \right]$$

A small video inset in the bottom right corner shows a man in a white shirt speaking.

Important thing to note that t_n plus half by delta t this is not a nodal point. This is somewhat intermediate point between t_n plus 1. This is intermediate point between t_n and t_n plus 1. So we need to figure out how to get the approximation for this term.

(Refer Slide Time 17:52)

The screenshot shows a presentation slide with a title bar containing 'Euler Method', 'Modified Euler Method', 'Runge-Kutta Methods', and 'References'. The main title is 'Modified Euler Method'. The slide content includes the text 'For $\theta = \frac{1}{2}$, we can write' followed by the equation
$$\phi^{n+1} = \phi^n + \Delta t \Psi \left[t_n + \frac{\Delta t}{2}, \phi \left(t_n + \frac{\Delta t}{2} \right) \right]$$
 with handwritten annotations in pink. Above the equation, 't_n' and 't_{n+1}' are written with a multiplication sign between them. The entire term $\left[t_n + \frac{\Delta t}{2}, \phi \left(t_n + \frac{\Delta t}{2} \right) \right]$ is underlined. A small video inset of the presenter is visible in the bottom right corner.

So first approach we can utilize with Euler method. With Euler's method we can take this phi steps. So with Euler's approximation we are basically multiplying the time interval into this is evaluated at present time step. That is t_n phi n.

(Refer Slide Time 18:27)

The screenshot shows a presentation slide with a title bar containing 'Euler Method', 'Modified Euler Method', 'Runge-Kutta Methods', and 'References'. The main title is 'Modified Euler Method' and the subtitle is 'First Approach'. The slide content includes the text 'If we evaluate $\phi(t_n + \frac{\Delta t}{2})$ by the Euler method, i.e.,' followed by the equation
$$\phi \left(t_n + \frac{\Delta t}{2} \right) = \phi(t_n) + \frac{\Delta t}{2} \Psi(t_n, \phi^n)$$
 with handwritten annotations in pink. The term $\phi \left(t_n + \frac{\Delta t}{2} \right)$ is underlined, and the term $\frac{\Delta t}{2} \Psi(t_n, \phi^n)$ is circled. A small video inset of the presenter is visible in the bottom right corner.

So with this information if we write this, this is specifically t_n plus $\frac{\Delta t}{2}$ plus $\phi(t_n)$ plus $\frac{\Delta t}{2}$ into Ψ . Whatever we are getting from Euler's method.

(Refer Slide Time 19:06)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Modified Euler Method

First Approach

If we evaluate $\phi(t_n + \frac{\Delta t}{2})$ by the Euler method, i.e.,

$$\phi\left(t_n + \frac{\Delta t}{2}\right) = \phi(t_n) + \frac{\Delta t}{2} \Psi(t_n, \phi^n)$$

In the next step,

$$\phi^{n+1} = \phi^n + \Delta t \Psi\left[t_n + \frac{\Delta t}{2}, \phi(t_n) + \frac{\Delta t}{2} \Psi(t_n, \phi^n)\right]$$

Euler

So with this information we can get the phi future value that is phi n plus 1. In simplified form we can write this term as K2, where K2 is dependent on K1. K1 is the increment that we are getting from Euler time step.

(Refer Slide Time 19:47)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Modified Euler Method

First Approach

If we evaluate $\phi(t_n + \frac{\Delta t}{2})$ by the Euler method, i.e.,

$$\phi\left(t_n + \frac{\Delta t}{2}\right) = \phi(t_n) + \frac{\Delta t}{2} \Psi(t_n, \phi^n)$$

In the next step,

$$\phi^{n+1} = \phi^n + \Delta t \Psi\left[t_n + \frac{\Delta t}{2}, \phi(t_n) + \frac{\Delta t}{2} \Psi(t_n, \phi^n)\right]$$

In simplified form

$$\phi^{n+1} = \phi^n + K_2 + \mathcal{O}(\Delta t^3)$$

with $K_2 = \Delta t \Psi\left(t_n + \frac{\Delta t}{2}, \phi^n + \frac{1}{2} K_1\right)$ and $K_1 = (\Delta t) \Psi^n$.

So in this method we are getting third order accuracy. However the resulting accuracy for this method will be second order.

(Refer Slide Time 20:00)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Modified Euler Method

First Approach

If we evaluate $\phi(t_n + \frac{\Delta t}{2})$ by the Euler method, .i.e.,

$$\phi\left(t_n + \frac{\Delta t}{2}\right) = \phi(t_n) + \frac{\Delta t}{2}\Psi(t_n, \phi^n)$$

In the next step,

$$\phi^{n+1} = \phi^n + \Delta t\Psi\left[t_n + \frac{\Delta t}{2}, \phi(t_n) + \frac{\Delta t}{2}\Psi(t_n, \phi^n)\right]$$

In simplified form

$$\phi^{n+1} = \phi^n + K_2 + \mathcal{O}(\Delta t^3)$$

with $K_2 = \Delta t\Psi(t_n + \frac{\Delta t}{2}, \phi^n + \frac{1}{2}K_1)$ and $K_1 = \Delta t\Psi^n$.

In second approach we can take average of present time level and future time level. But these are node point values. This is t_n this t_n plus 1.

(Refer Slide Time 20:26)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler-Cauchy method

Second Approach

Using averaging approach,

$$\phi'\left(t_n + \frac{\Delta t}{2}\right) = \frac{1}{2} [\phi'(t_n) + \phi'(t_n + \Delta t)]$$

With this we can use Euler's approximation. So Euler's approximation for this term, second term and in this case we can write this with Euler approximation.

(Refer Slide Time 20:51)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

I.I.T. Kharagpur

Euler-Cauchy method

Second Approach

Using averaging approach,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\phi'(t_n) + \phi'(t_{n+1})]$$

With Euler approximation,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)]$$

With this we can write the final equation in terms of derivative values psi. This is evaluated at Euler's step.

(Refer Slide Time 21:09)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler-Cauchy method

Second Approach

Using averaging approach,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\phi'(t_n) + \phi'(t_n + \Delta t)]$$

With Euler approximation,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)]$$

The equation can be written as,

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2} [\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)]$$

Finally if we write it in a compact form we can write it as K_1 and this is half K_1 plus K_2 plus order of accuracy is Δt^3 . K_2 is calculated as Δt into $\psi(t_n, \phi^n) + K_1$ and K_1 is $\Delta t \psi^n$. This is again third order accurate and resulting thing will be second order accurate.

(Refer Slide Time 22:05)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Euler-Cauchy method

Second Approach

Using averaging approach,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\phi'(t_n) + \phi'(t_n + \Delta t)]$$

With Euler approximation,

$$\phi' \left(t_n + \frac{\Delta t}{2} \right) = \frac{1}{2} [\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)]$$

The equation can be written as,

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2} [\Psi(t_n, \phi^n) + \Psi(t_{n+1}, \phi^n + \Delta t \Psi^n)]$$

In simplified form,

$$\phi^{n+1} = \phi^n + \frac{1}{2} [K_1 + K_2] + \mathcal{O}(\Delta t^3)$$

with $K_2 = \Delta t \Psi(t_{n+1}, \phi^n + K_1)$ and $K_1 = \Delta t \Psi^n$.

Then comes this Runge Kutta method. Runge Kutta method we can use this different values of increments and this increments are explicit in nature. Explicit means that on the right hand side in the function psi only known value are there. In this case K2 in the left hand side, K1 in the right left hand right hand side. So whatever value is calculated from K1 that will be transferred to this psi calculation for K2.

(Refer Slide Time 22:55)

Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Runge-Kutta Methods

Individual m increments are defined as,

$$\begin{cases} K_1 = \Delta t \Psi(t_n, \phi^n) \\ K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\ K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\ \vdots \\ K_m = \Delta t \Psi(t_n + c_m \Delta t, \phi^n + a_{m1} K_1 + a_{m2} K_2 + \dots + a_{m,m-1} K_{m-1}) \end{cases}$$

Again this calculated K2 will be transferred here K1 and K2. Similarly for Km we can get K1, K2 to Km minus 1. So this method is explicit in nature. We can directly get the value of Km without any matrix solution or any other method.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

I. I. I. Kharagpur

Runge-Kutta Methods

Individual m increments are defined as,

$$\begin{cases} K_1 = \Delta t \Psi(t_n, \phi^n) \\ K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\ K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\ \vdots \\ K_m = \Delta t \Psi(t_n + c_m \Delta t, \phi^n + a_{m1} K_1 + a_{m2} K_2 + \dots + a_{m,m-1} K_{m-1}) \end{cases}$$

Further this Runge Kutta method is defined in terms of this weighted assembly of increments. This is $\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + \dots + W_m K_m$. And this parameters c_j , a_{ij} , w_j can be obtained by matching the corresponding expansions with Taylor series.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Runge-Kutta Methods

Individual m increments are defined as,

$$\begin{cases} K_1 = \Delta t \Psi(t_n, \phi^n) \\ K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1) \\ K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2) \\ \vdots \\ K_m = \Delta t \Psi(t_n + c_m \Delta t, \phi^n + a_{m1} K_1 + a_{m2} K_2 + \dots + a_{m,m-1} K_{m-1}) \end{cases}$$

The Runge-Kutta method is defined as weighted assembly of increments by,

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + \dots + W_m K_m$$

The parameters c_j , a_{ij} and W_j can be obtained by matching the corresponding expansions with Taylor Series.

So we can expand these increments around these points t_n and ϕ^n and we can get different values of c_j , a_j , w_j .

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Runge-Kutta Methods

Individual m increments are defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$\vdots$$

$$K_m = \Delta t \Psi(t_n + c_m \Delta t, \phi^n + a_{m1} K_1 + a_{m2} K_2 + \dots + a_{m,m-1} K_{m-1})$$

The Runge-Kutta method is defined as weighted assembly of increments by,

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + \dots + W_m K_m$$

The parameters (c_i, a_{ij}) and (W_i) can be obtained by matching the corresponding expansions with Taylor Series.

If we consider second order Runge Kutta then we have only two terms or two increments, K_1 and K_2 . We need to find out what is K_1 , c_2 and a_{21} and W_1, W_2 . We can get it from Taylor series expansion.

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Euler Method
Modified Euler Method
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References

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Second Order RK Method

In general terms, second order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2$$

And if we finally write the solution this is K_1 and K_2 is evaluated at two third delta t and t_n plus two third delta t and $\phi^n + \frac{2}{3} K_1$. And overall accuracy is Δt^3 . But finally this is second order accurate method.

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Euler Method
Modified Euler Method
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References

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Second Order RK Method

In general terms, second order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

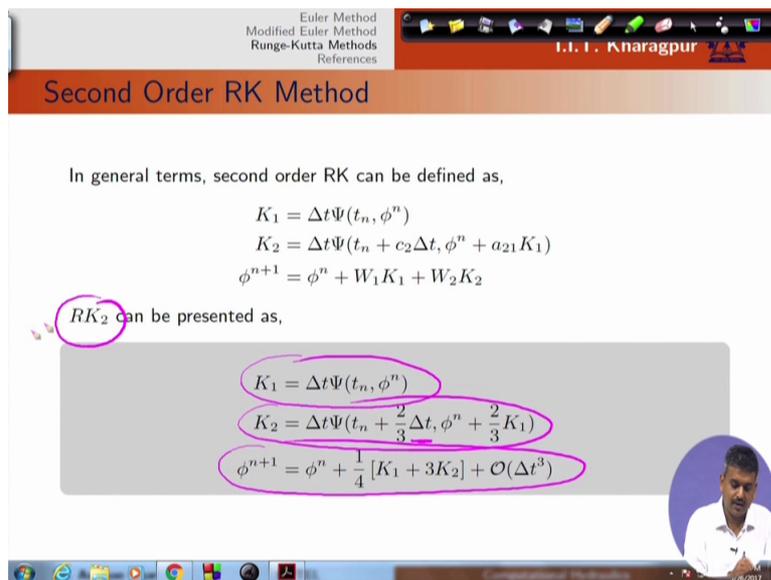
$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2$$

RK_2 can be presented as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + \frac{2}{3} \Delta t, \phi^n + \frac{2}{3} K_1)$$

$$\phi^{n+1} = \phi^n + \frac{1}{4} [K_1 + 3K_2] + \mathcal{O}(\Delta t^3)$$


In third order RK we need to consider three terms one, two and three.

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Euler Method
Modified Euler Method
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References

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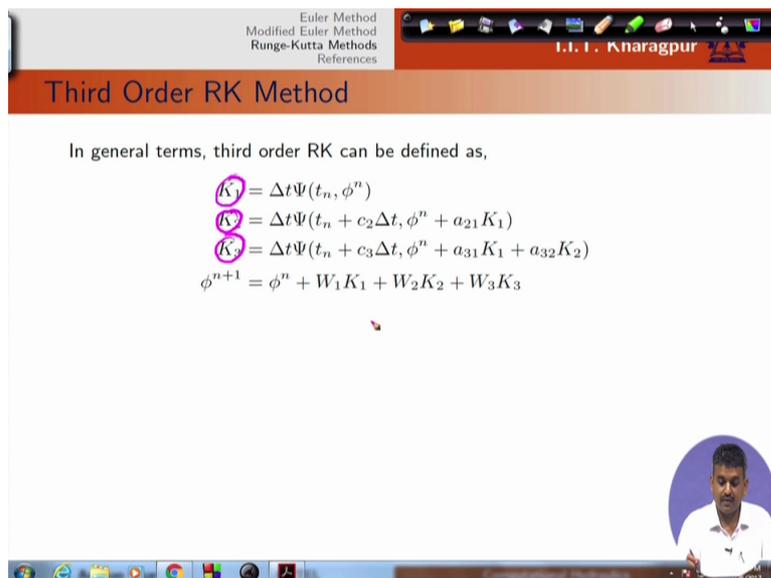
Third Order RK Method

In general terms, third order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3$$


Similarly we can get the values for C2, C3, C2, C C3, A21, A32, W1, W2, W3 from Taylor series expansion.

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Euler Method
Modified Euler Method
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References

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Third Order RK Method

In general terms, third order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3$$

And if we write the final expression, we can write it as RK 3 and interestingly in this case is evaluated as half time step for the intermediate K2 and for others it at initial and this is that final time step.

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Euler Method
Modified Euler Method
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References

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Third Order RK Method

In general terms, third order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3$$

RR_3 can be presented as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1)$$

$$K_3 = \Delta t \Psi(t_n + \Delta t, \phi^n - K_1 + 2K_2)$$

$$\phi^{n+1} = \phi^n + \frac{1}{6} K_1 + \frac{4}{6} K_2 + \frac{1}{6} K_3 + \mathcal{O}(\Delta t^4)$$

And these are coefficients, minus 1, 2 and finally we can write it else 4th order accurate but finally accuracy will be of third order.

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Euler Method
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Third Order RK Method

In general terms, third order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3$$

RK_3 can be presented as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1)$$

$$K_3 = \Delta t \Psi(t_n + \Delta t, \phi^n + K_1 + 2K_2)$$

$$\phi^{n+1} = \phi^n + \frac{1}{6} K_1 + \frac{4}{6} K_2 + \frac{1}{6} K_3 + \mathcal{O}(\Delta t^4)$$

Similarly for 4th order RK we can use this approach and we can get 5th order accurate scheme but finally accuracy will be of order 4.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Fourth Order RK Method

In general terms, fourth order RK can be defined as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$K_4 = \Delta t \Psi(t_n + c_4 \Delta t, \phi^n + a_{41} K_1 + a_{42} K_2 + a_{43} K_3)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4$$

RK_4 can be presented as,

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1)$$

$$K_3 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_2)$$

$$K_4 = \Delta t \Psi(t_n + \Delta t, \phi^n + K_3)$$

$$\phi^{n+1} = \phi^n + \frac{1}{6} K_1 + \frac{1}{3} K_2 + \frac{1}{3} K_3 + \frac{1}{6} K_4 + \mathcal{O}(\Delta t^5)$$

And interesting in this case this K_2 and K_3 are calculated at halftime step or at the middle of the two nodal points.

(Refer Slide Time 27:35)

The slide is titled "Fourth Order RK Method" and is part of a presentation by I.I.T. Kharagpur. It lists navigation options: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The main content states: "In general terms, fourth order RK can be defined as," followed by the following equations:

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + c_2 \Delta t, \phi^n + a_{21} K_1)$$

$$K_3 = \Delta t \Psi(t_n + c_3 \Delta t, \phi^n + a_{31} K_1 + a_{32} K_2)$$

$$K_4 = \Delta t \Psi(t_n + c_4 \Delta t, \phi^n + a_{41} K_1 + a_{42} K_2 + a_{43} K_3)$$

$$\phi^{n+1} = \phi^n + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4$$

Below this, it says "RK4 can be presented as," and shows a boxed set of equations with some terms circled in pink:

$$K_1 = \Delta t \Psi(t_n, \phi^n)$$

$$K_2 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_1)$$

$$K_3 = \Delta t \Psi(t_n + \frac{1}{2} \Delta t, \phi^n + \frac{1}{2} K_2)$$

$$K_4 = \Delta t \Psi(t_n + \Delta t, \phi^n + K_3)$$

$$\phi^{n+1} = \phi^n + \frac{1}{6} K_1 + \frac{1}{3} K_2 + \frac{1}{3} K_3 + \frac{1}{6} K_4 + \mathcal{O}(\Delta t^5)$$

So if we consider one practical example of this. This is $\frac{d^2y}{dx^2} = S_0 - S_f$, $1 - Fr^2$. This is well known equation for describing the gradually varied flow in open channels.

(Refer Slide Time 28:06)

The slide is titled "Gradually Varied Flow in Open Channel" and is part of a presentation by I.I.T. Kharagpur. It lists navigation options: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The main content is under the heading "Initial Value Problem" and shows the "Governing Equation:"

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \quad (1)$$

This is ordinary differential equation first order and with initial condition. That means depth is specified at some special locations. We can see that on the right hand side we have this $S_0 - S_f$, $1 - Fr^2$ term.

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Euler Method
Modified Euler Method
Runge-Kutta Methods
References

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Gradually Varied Flow in Open Channel

Ordinary Differential Equation

Initial Value Problem

Governing Equation: $\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2}$ (1)

Initial Condition: $y|_{x=0} = y_0$ (2)

$$\Psi(x, y) = \frac{S_0 - S_f}{1 - Fr^2}$$



This is function psi which is function of xy, s_0 minus s_f , 1 minus fr square. S_0 is bed slope, s_f is energy slope, one minus fraud number square.

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Euler Method
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Gradually Varied Flow in Open Channel

Ordinary Differential Equation

Initial Value Problem

Governing Equation: $\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2}$ (1)

Initial Condition: $y|_{x=0} = y_0$ (2)

$$\Psi(x, y) = \frac{S_0 - S_f}{1 - Fr^2}$$

$\frac{S_0 - S_f}{1 - Fr^2}$

So essentially in this case we get psi as function of y only, which is flow depth in the channel.

(Refer Slide Time 29:09)

The slide is titled "Gradually Varied Flow in Open Channel" and "Ordinary Differential Equation". It lists navigation options: Euler Method, Modified Euler Method, Runge-Kutta Methods, and References. The slide content is as follows:

Initial Value Problem

Governing Equation:
$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \quad (1)$$

Initial Condition:
$$y|_{x=0} = y_0 \quad (2)$$

$$\Psi(x, y) = \frac{S_0 - S_f}{1 - Fr^2}$$

Handwritten in pink:
$$\Psi(y) = \frac{S_0 - S_f}{1 - Fr^2}$$

So this is a typical example of initial value problem. In the next lecture we will be discussing boundary value problem for ordinary differential equation. Thank you.