Computational Hydraulics
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Lecture 7
Ordinary Differential Equation: IVP

Start? Welcome to this lecture number 7 of the course computational hydraulics. We are in the module 2 numerical methods. And in this specific lecture we will be covering unit three, ordinary differential equation and specifically initial value problem or IVP.

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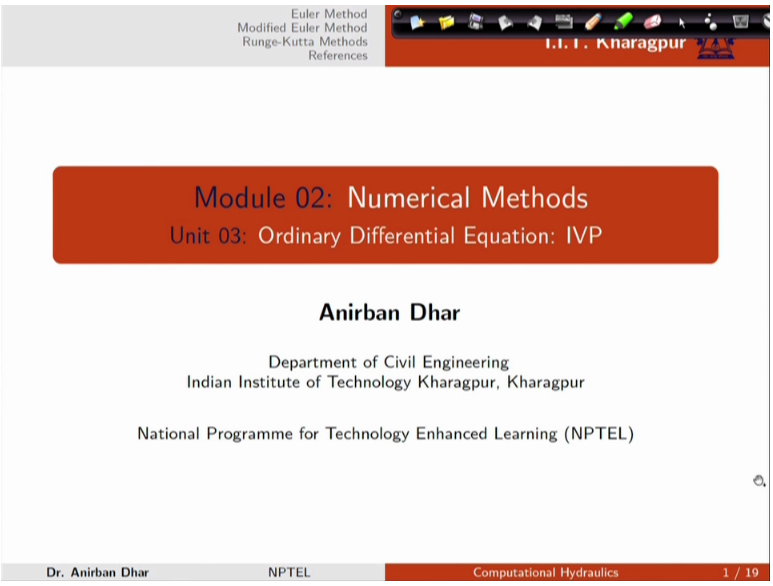

So what is the learning objective for this particular unit? At the end of this unit (stu) students will be able to discretize first order differential equation along with boundary condition initial condition because in initial value problem we have only ordinary differential equation and initial conditions.

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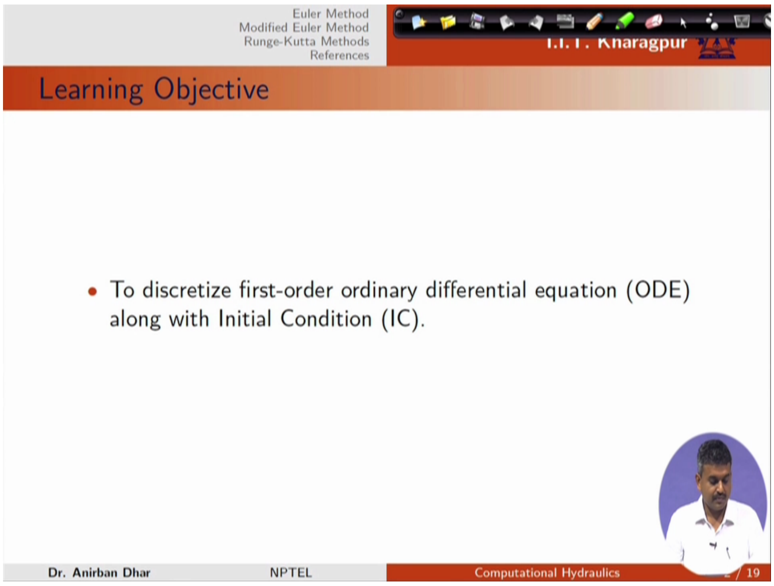

Ordinary differential equation with initial condition can be solved as initial value problem with time or time like discretization. This point is important because always the first order equation may not be time dependent maybe it is space dependent. And we need to solve that ODE with initial condition to get the solution of initial value problem.

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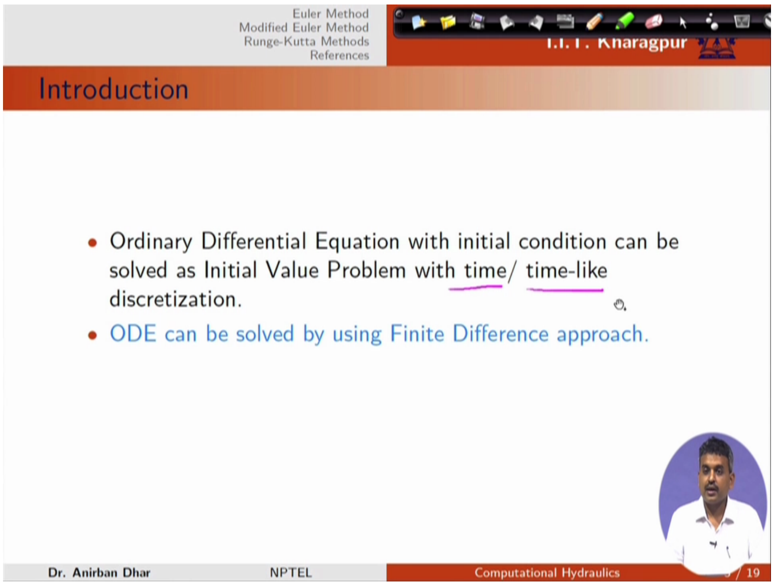

ODE can be solved by using finite difference approach. Whatever we have covered in lecture 6. And we can use different discretization to get the solution of ordinary differential equation.Then comes the accuracy. Accuracy of the solution depends only on discretization of ODE. In case of initial value problem, initial condition is like Dirichlet kind of condition which is specified condition.

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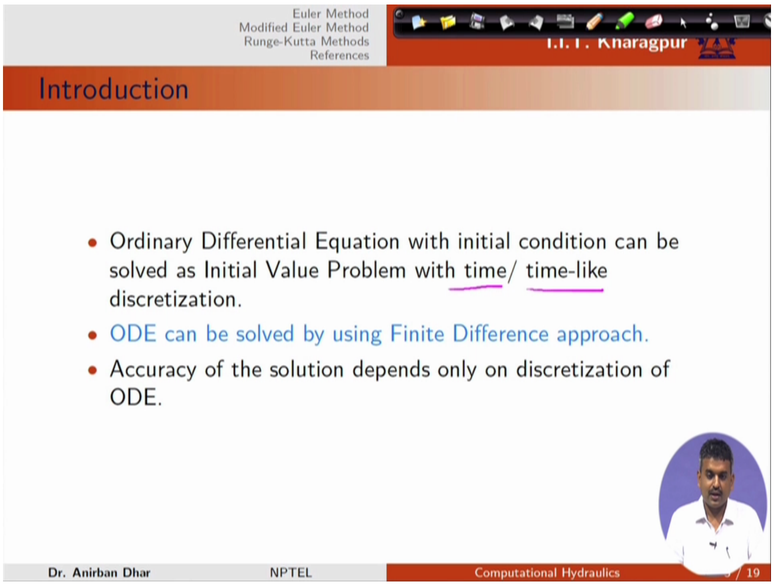

So there is no error associated with that initial condition. So the error comes only from the ordinary differential equation. That's why discretization of ordinary differential equation is important. In general we canwrite that ordinary differential equation with dependent variable phi in this format where dphi by dt is the derivative term. And psi t phi is the function which is representing the derivative.

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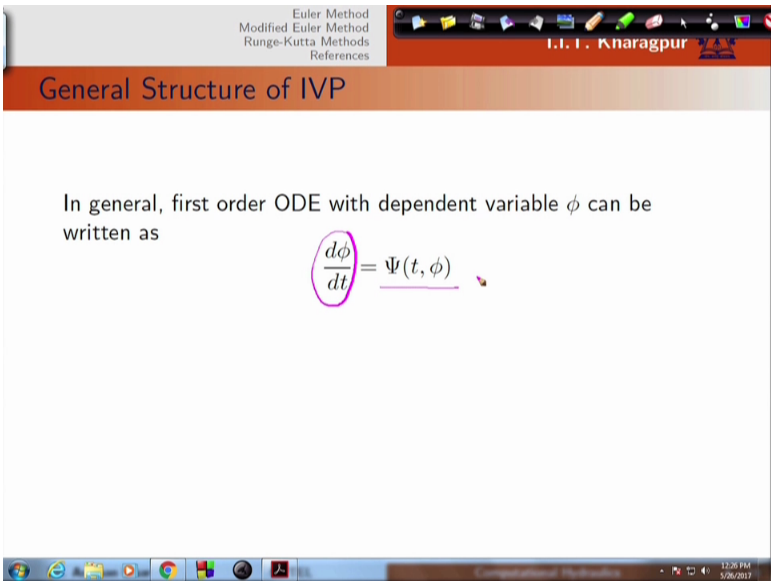

Now with this we need initial condition. This phi at t0 which is at the starting or initial location or initial time level value. Phi0 is the corresponding value. Psi isa general function. So this is the basic definition of ordinary differential equation along with our initial condition for initial value problem.In a particular problem we can have multiple dependent variable. So we will have multiple ordinary differential equation and there should be initial condition for each of those differential equations.

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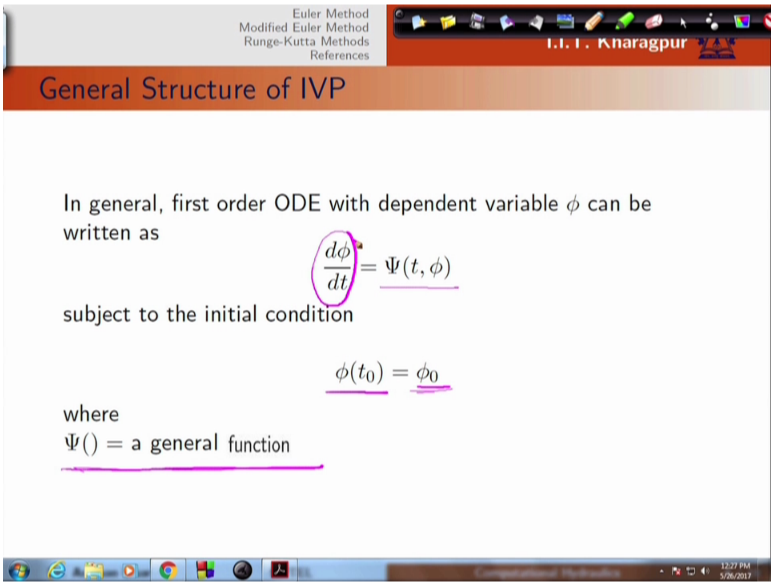

Now we need to discretize it numerically. So from governing equation if we integrate it from tn and tn plus 1 is representing a general time interval. So we can use mean value theorem to evaluate the right hand side of the above equation.

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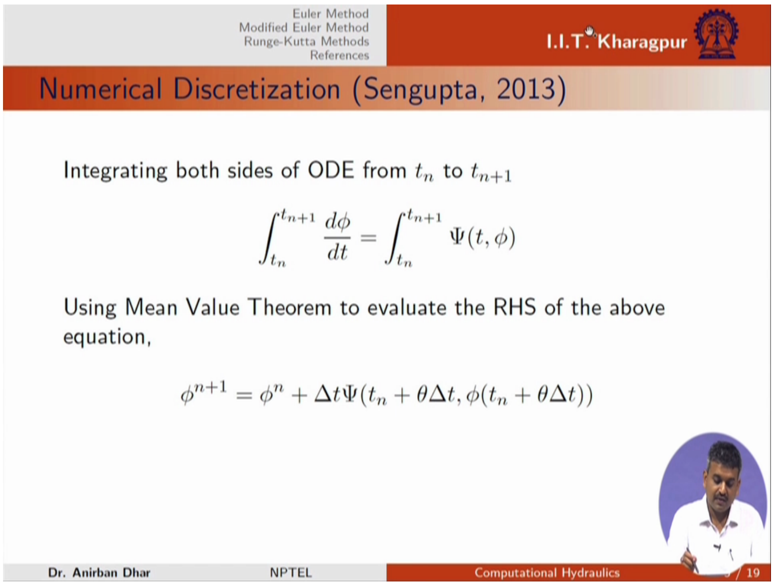

In this case we can see that this phi n plus 1, which is future time level value,phi n which is present time level value, plus del t, this is time increment and psi is the function. This psi is the function of tn plus theta into del t .What is this theta? Theta is again a value which is in between zero to one. That means we can use any value theta to theta within this zero to one. If we use (ze) zero, we can get some method. If we use 1 we can get some different method but with intermediate values we will get intermediate methods.

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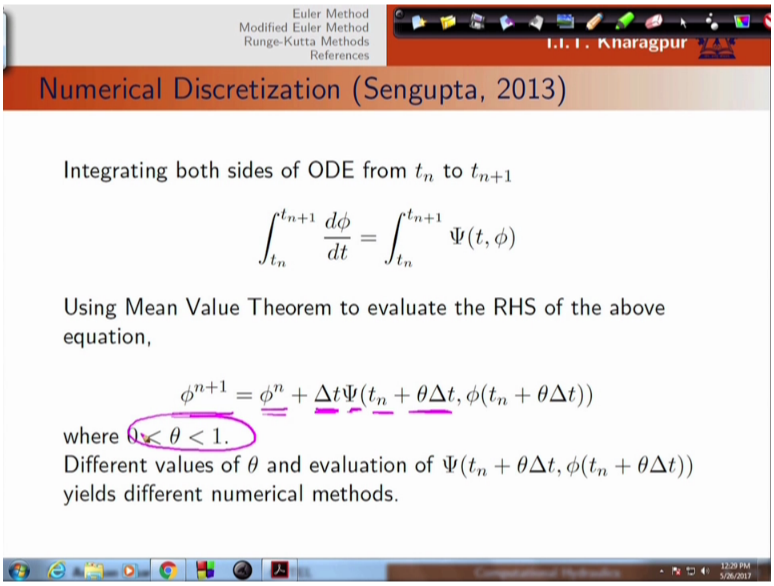

And in this case phi is also function of tn plus theta into del t. In this case we can also include this lower and upper limit for our problems. So different values of theta we need to evaluate this psi function.

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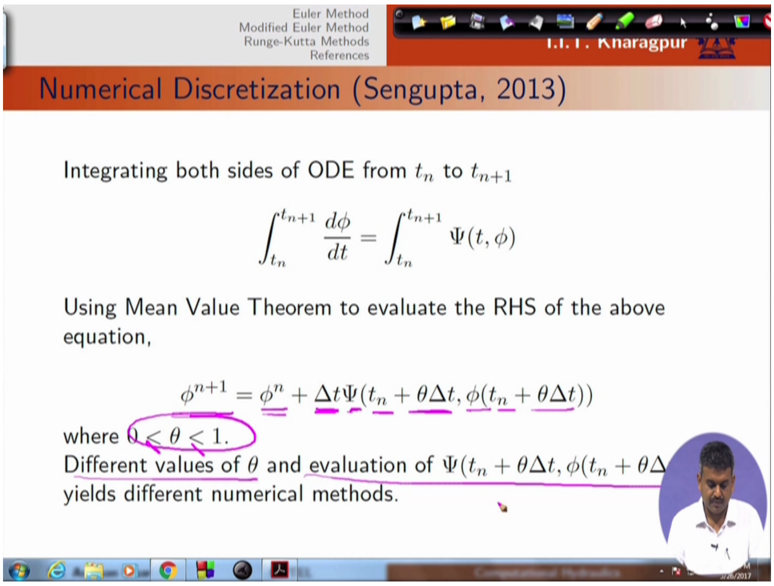

Now let us see the truncation error. So phi at tn plus 1 can be expanded as phi tn plus 1.Phi t, this is first order derivative. If we expand it further this is pth order term and this is p plus 1 order term.

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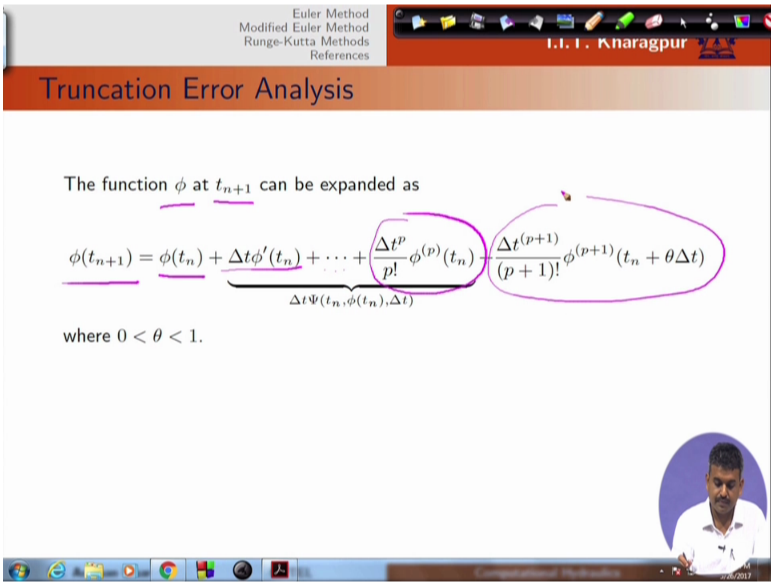

In this case again theta is in between zero to 1. We can include the zero and 1 value for our problem. The equation can be written as phi tn plus 1phi t. This is essentially phi n plus 1 and phi n plus del t into psi tn plus phi this is comma phi tn and delta t. So this function is there plus there will be some truncation error associated with it. So truncation error.We can use this concept for our initial value problems.

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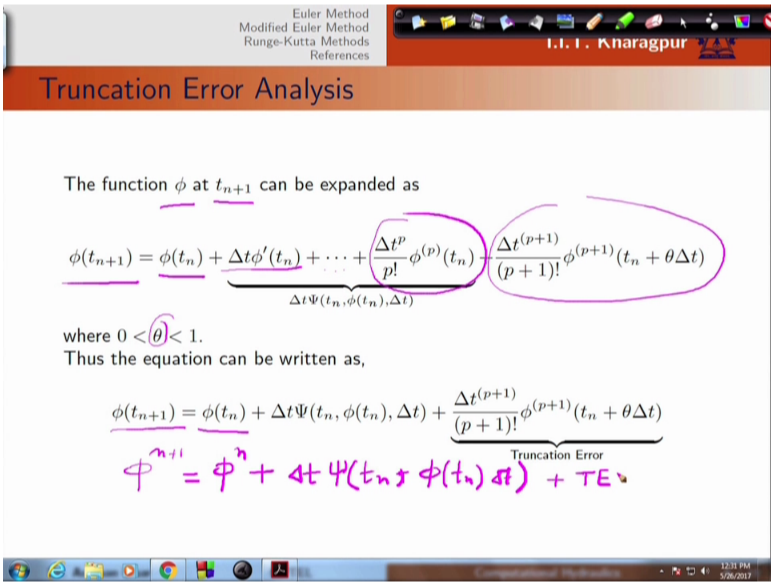

If you have theta equals to zero then we call it as Euler method. This phi n plus 1 phi n, delta t and this is evaluated at tn phi n. That means at the starting point we are evaluating the function.

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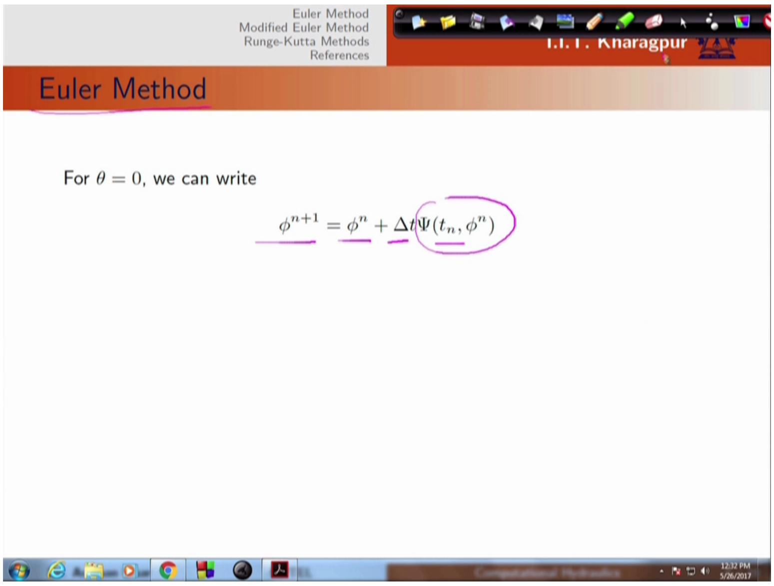

Now if we use p equals to 1 and theta is equal to zero in our Taylor series expansion, obviously we will get this thing only first order term, because this phi tn is essentially psi tn phi n.So this leading error term is having delta t square value.

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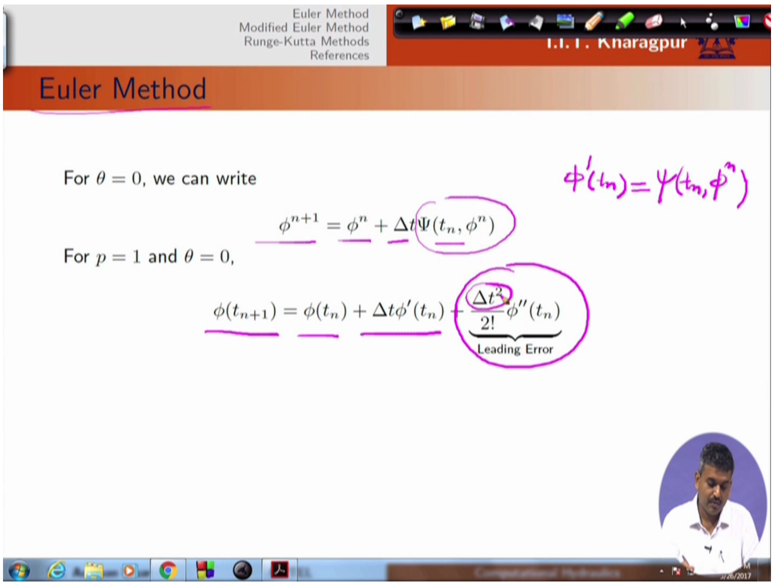

So we can say that the order of accuracy for this step is delta t square. But the problem is when we are considering the Euler's method we are getting delta t as order of accuracy. Why?

Because if we consider this thing we are to get the derivative we need to divide it by delta t. So truncation error in this case divided by delta t this will give you the actual order of accuracy of the method.So in this case we are getting first order accuracy.

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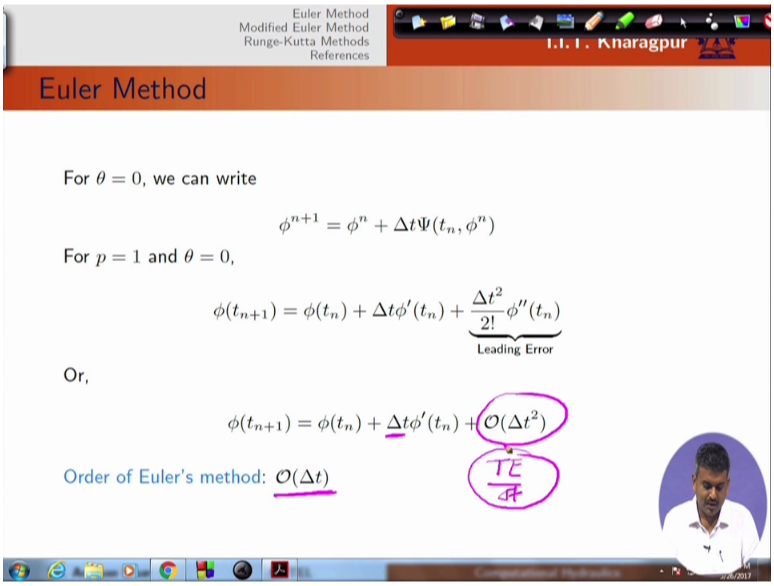

Next if we geometrically see this thing then we have one continuous function and t0, t1, t2, tn, tn plus 1 these are nodal points in the t axis.

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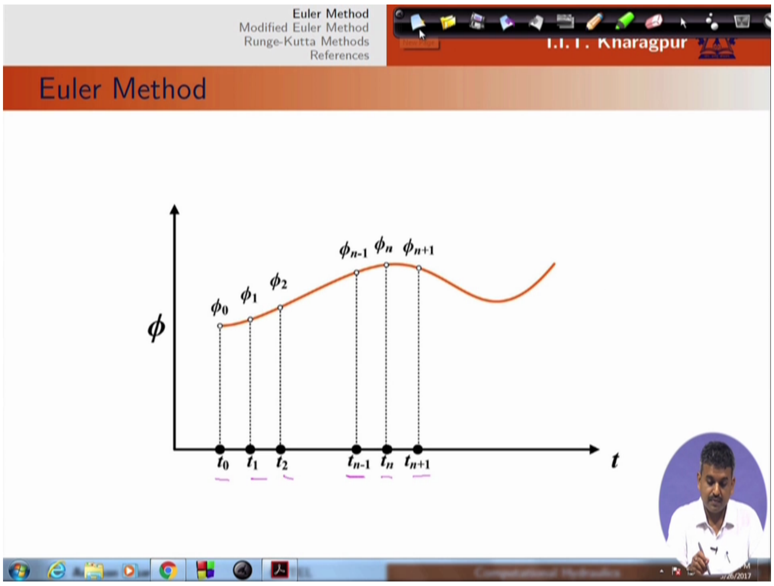

So with this if we calculate the slope at t0 point that is basically calculating the derivative. This is essentially psi t0 calculation.

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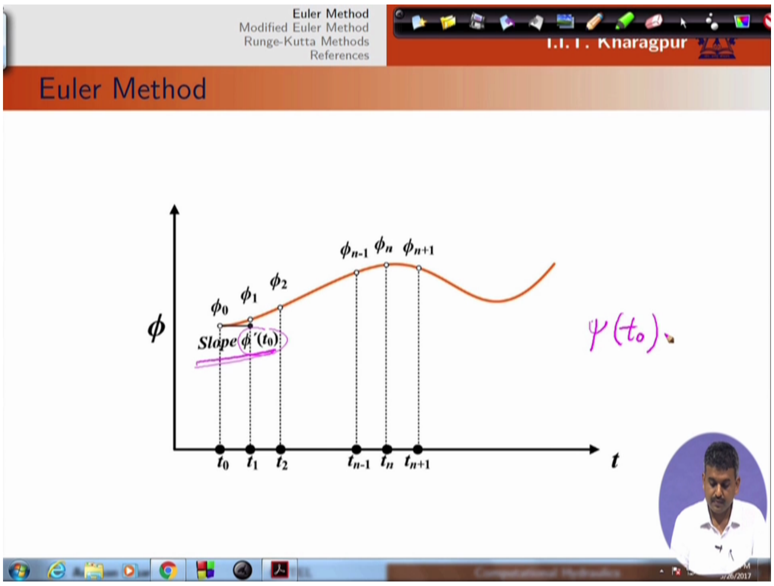

Now with this psi t0 value we can take first step to get this black dot points. White dot points are given actual values and black dot points in this case is giving the value that we are getting from Euler's method. So in this case if wedraw another slope at this black dot point, so obviously virtually we are shifting the curve here and we are taking the derivative at t 1 point which is the slope at this particular point.It is important to note that obviously we are getting some amount of deviation from the actual curve. So error is there.

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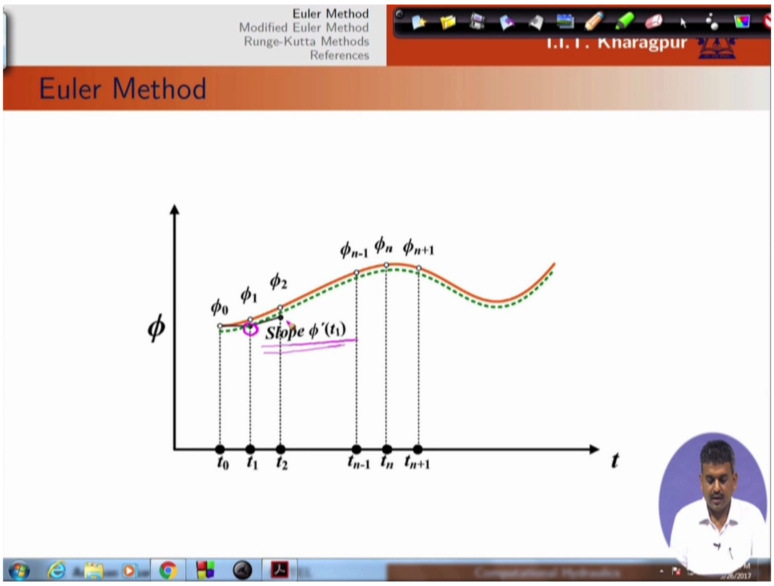

Next is backward Euler method. In backward Euler method we are calculating the derivative function at future time level. In our Euler's method we have used tn and phi n. In this case we are utilizing future time level value.

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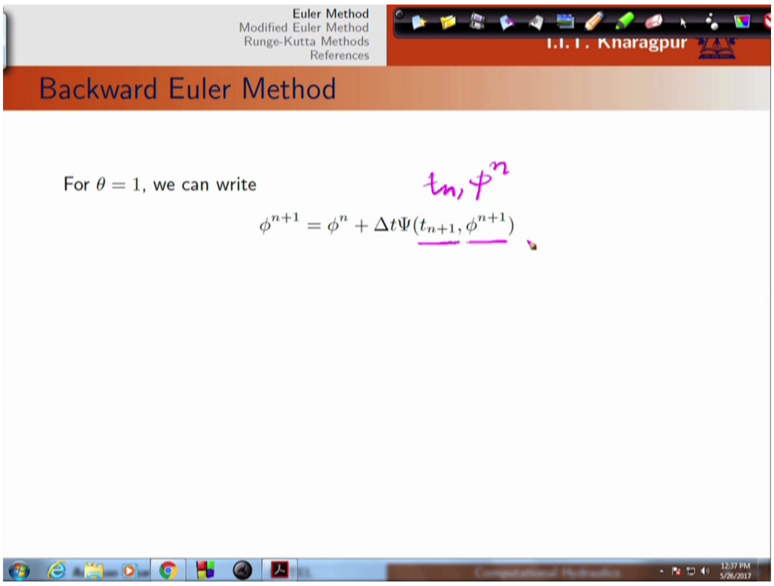

With this for phi if we write the expansion then tn is tn plus 1 minus del t into phi prime tn plus 1 and this is our leading error.So that's basically calculating the psi tn plus 1 and phi n plus 1.

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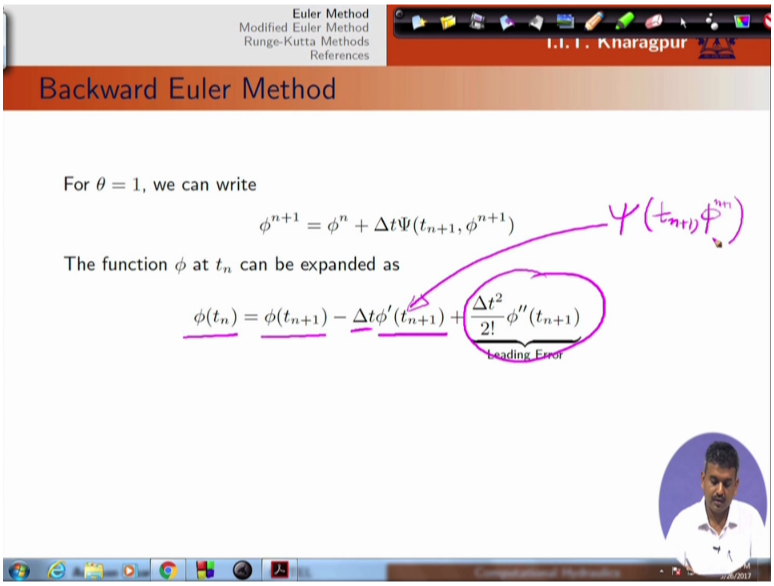

So we can say that we have again this is second order accuracy but actually we are getting first order accurate backward Euler method. So with this if we use any intermediate value so what will be the situation? We need to examine that.

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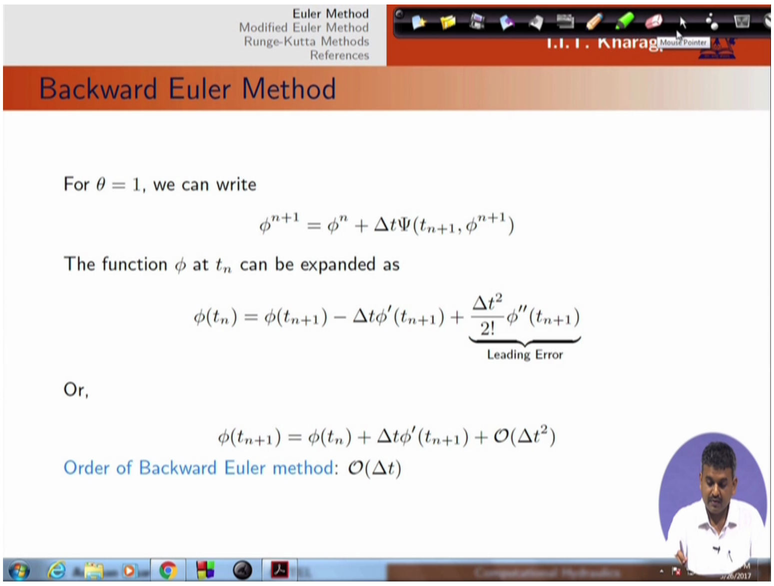

So in this case we are using theta equals to 1 by 2. You can see that in this case we are getting this n plus del t by 2 and phi also evaluated tn plus del t by 2.

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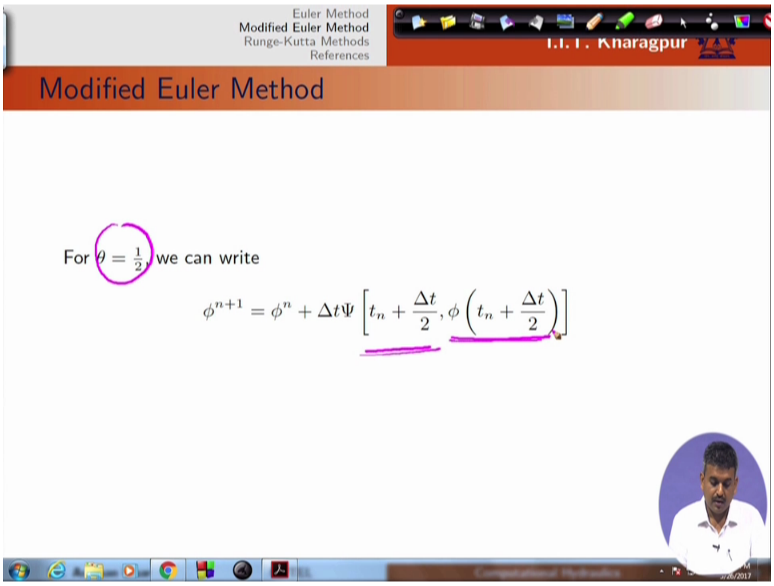

Important thing to note that tn plus half by delta t this is not a nodal point. This is somewhat intermediate point between tn plus 1. This is intermediate point between tn and tn plus 1. So we need to figure out how to get the approximation for this term.

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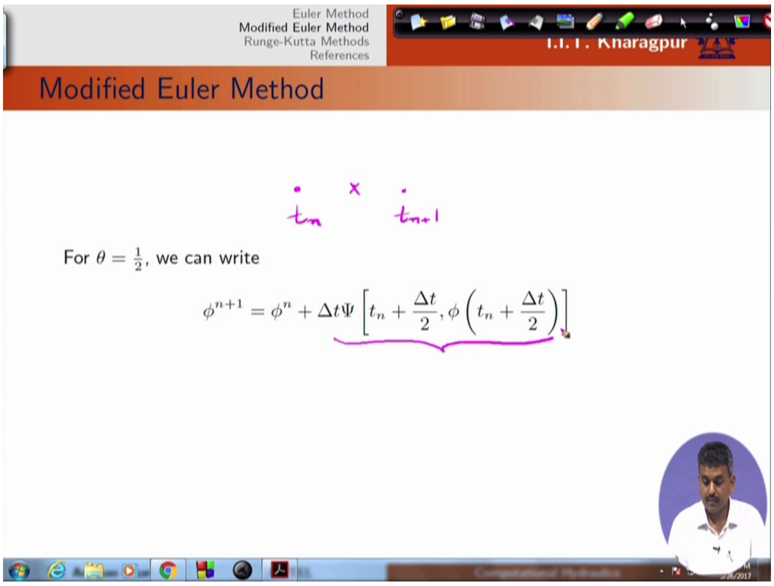

So first approach we can utilize with Euler method. With Euler's method we can take this phi steps. So with Euler’s approximation we are basicallymultiplying the time interval into this is evaluated at present time step.That is tn phi n.

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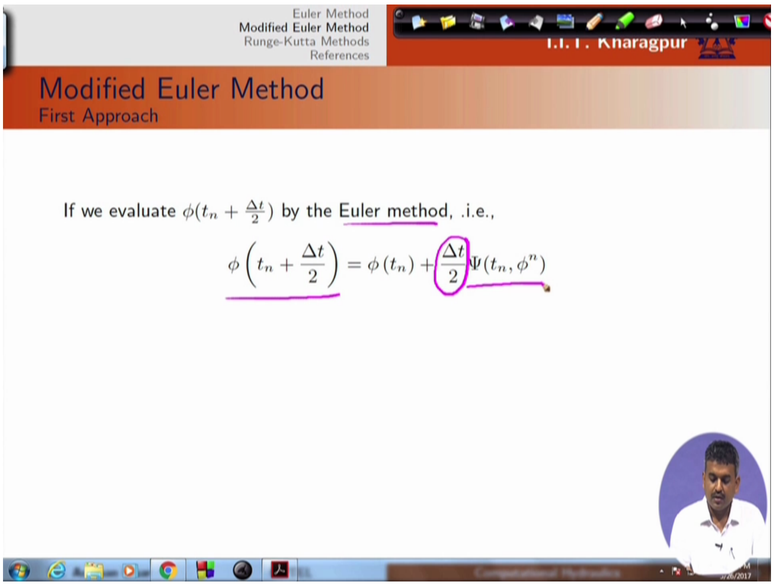

So with this information if we write this, this is specifically tn plus delta t by 2 plus phi tn plus delta t by 2 into psi. Whatever we are getting from Euler's method.

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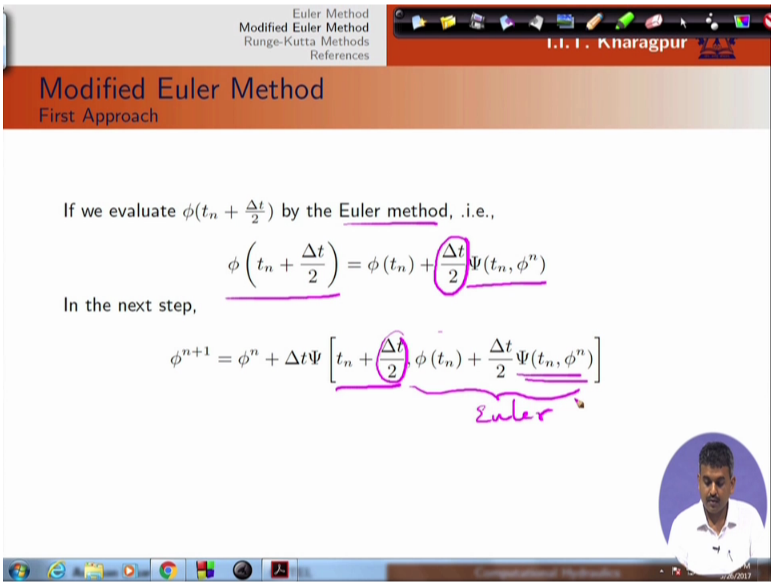

So with this information we can get the phi future value that is phi n plus 1. In simplified form we can write this term as K2, where K2 is dependent on K1. K1 is the increment that we are getting from Euler time step.

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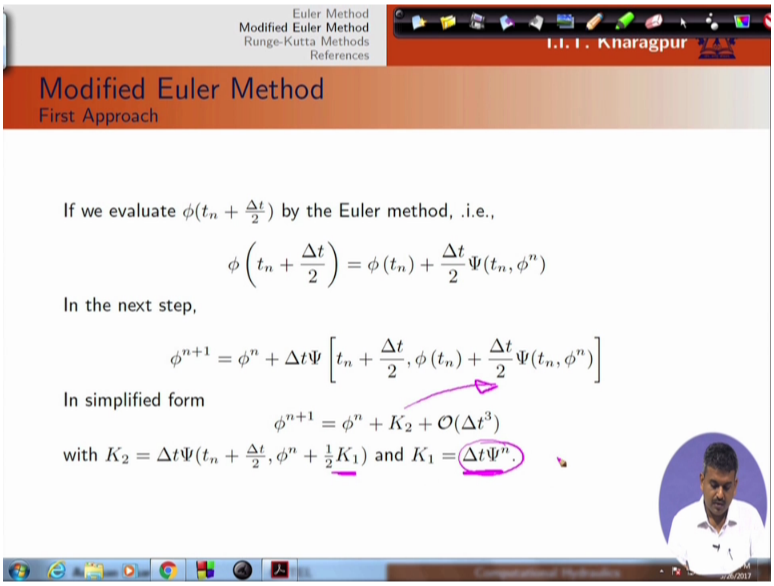

So in this method we are getting third order accuracy. However the resulting accuracy for this method will be second order.

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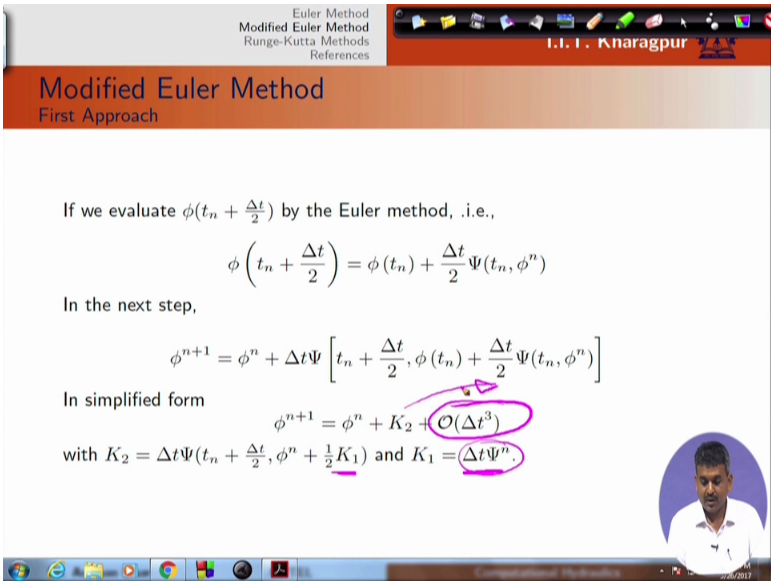

In second approach we can take average of present time level and future time level. But these are node point values. This is tn this tn plus 1.

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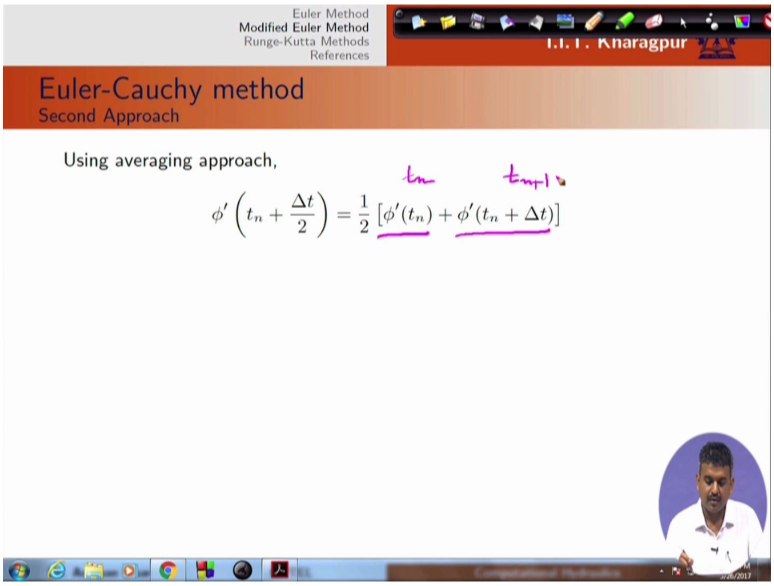

With this we can use Euler's approximation. So Euler’s approximation for this term, second term and in this case we can write this with Euler approximation.

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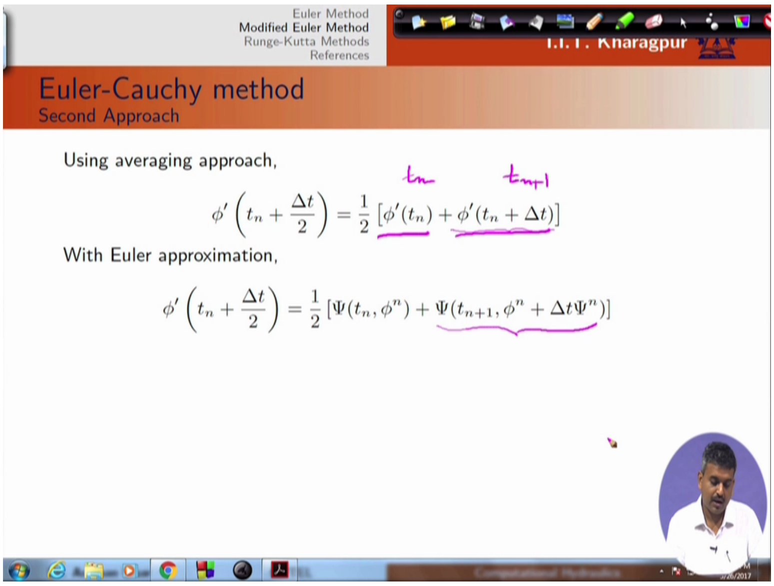

With this we can write the final equation in terms of derivative values psi. This is evaluated at Euler's step.

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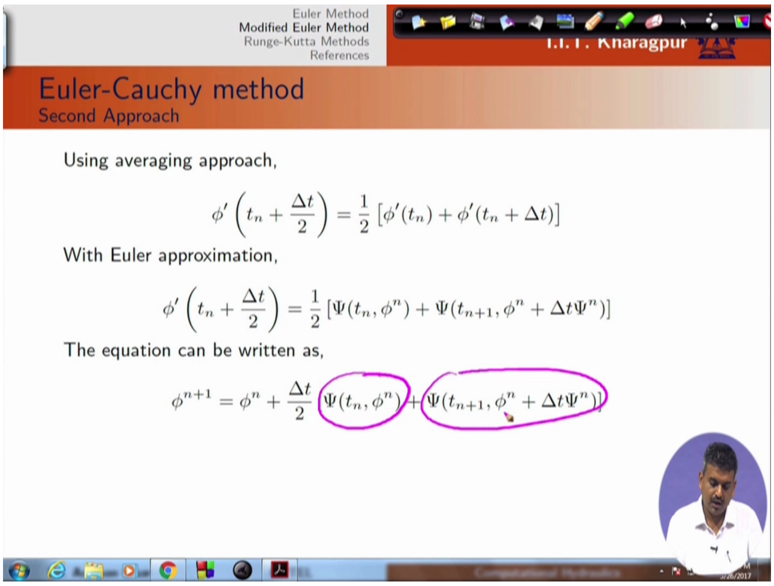

Finally if we write it in a compact form we can write it as K1 and this is half K1 plus K2 plus order of accuracy is delta tq. K2 is calculated as delta t into psi tn plus 1phi n plus K1 and K1 is delta t psi n. This is again third order accurate and resulting thing will be second order accurate.

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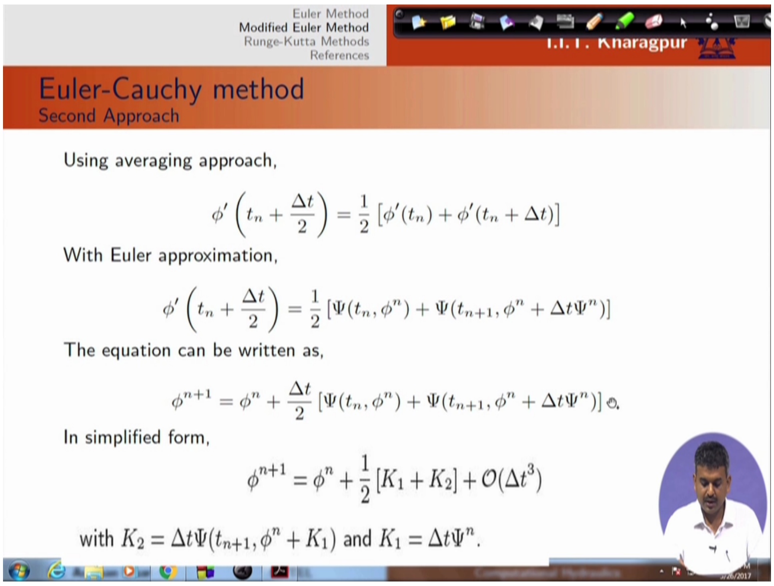

Then comes this Runge Kutta method. Runge Kutta method we can use this different values of increments and this increments are explicit in nature. Explicit means that on the right hand side in the function psi only known value are there. In this case K2 in the left hand side, K1 in the right left hand right hand side. So whatever value is calculated from K1 that will be transferred to this psi calculation for K2.

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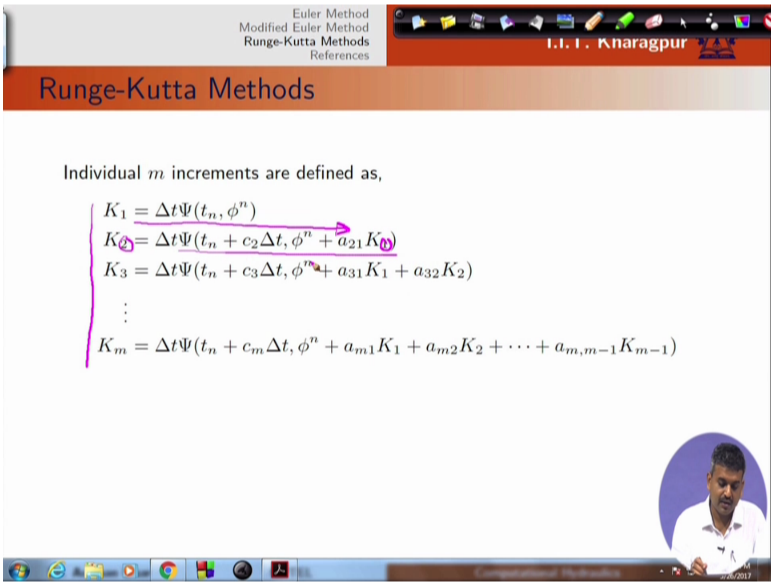

Again this calculated K2 will be transferred here K1 and K2. Similarly for Km we can get K1,K2 toKm minus 1. So this method is explicit in nature.We can directly get the value of Km without any matrix solution or any other method.

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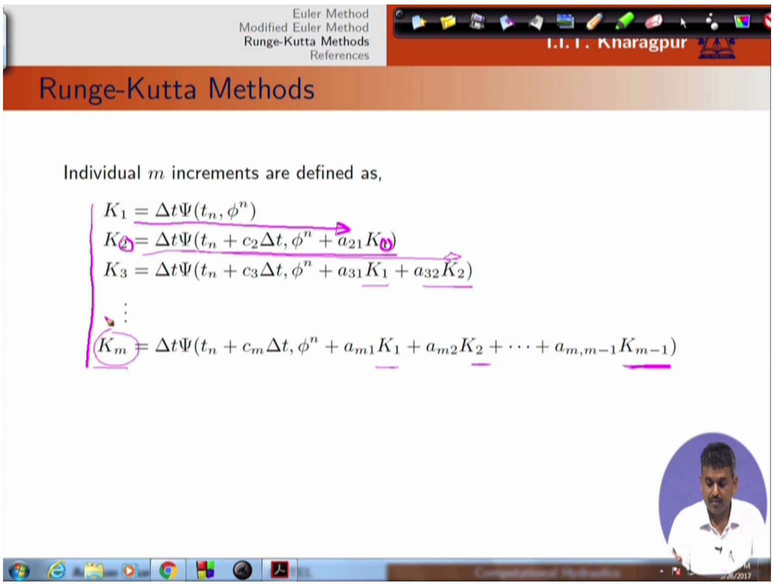

Further this Runge Kutta method is defined in terms of this weighted assembly of increments. This is phi n plus 1 equals to phi n. This W1 K1, W2 K2, Wm Km. And this parameters cj, aij, wj can be obtained by matching the corresponding expansions with Taylor series.

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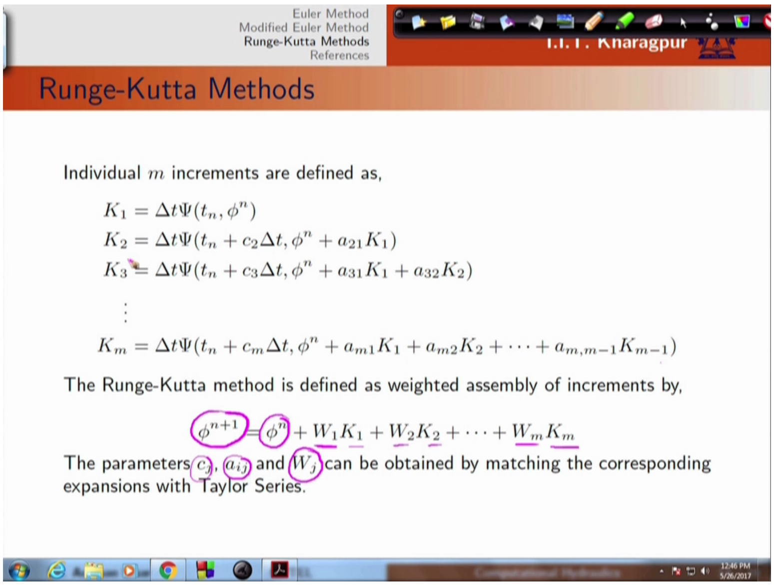

So we can expand these increments around these points tn and phi n and we can get different values of cj, aj, wj.

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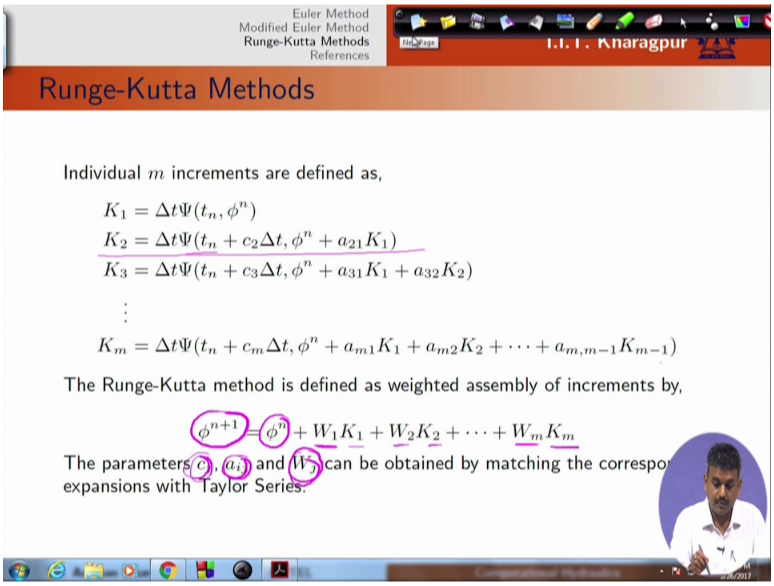

If we consider second order Runge Kutta then we have only two terms or two increments, K1 and K2. We need to find out what is K1 C2 and A21 and W1, W2. We can get it from Taylor series expansion.

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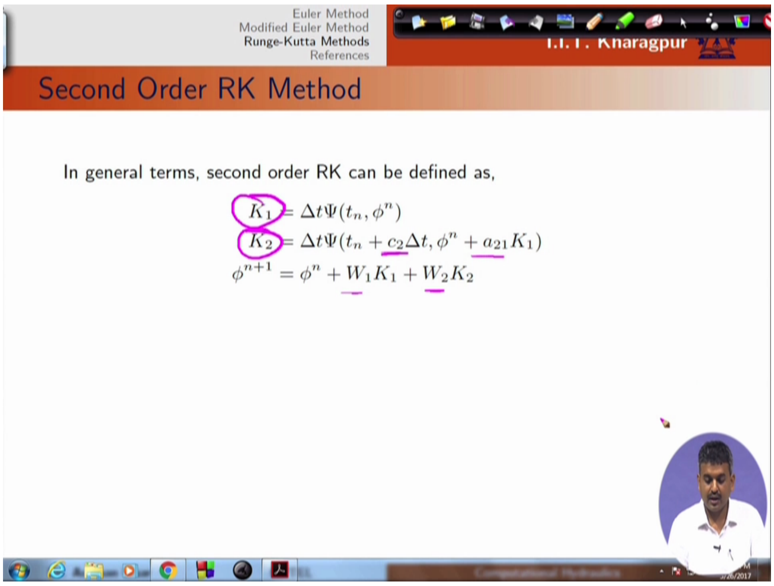

And if we finally write the solution this is K1 and K2 is evaluated at two third delta t and tn plus two third delta t and phi n two third K1. And overall accuracy is del tq. But finally this is second order accurate method.

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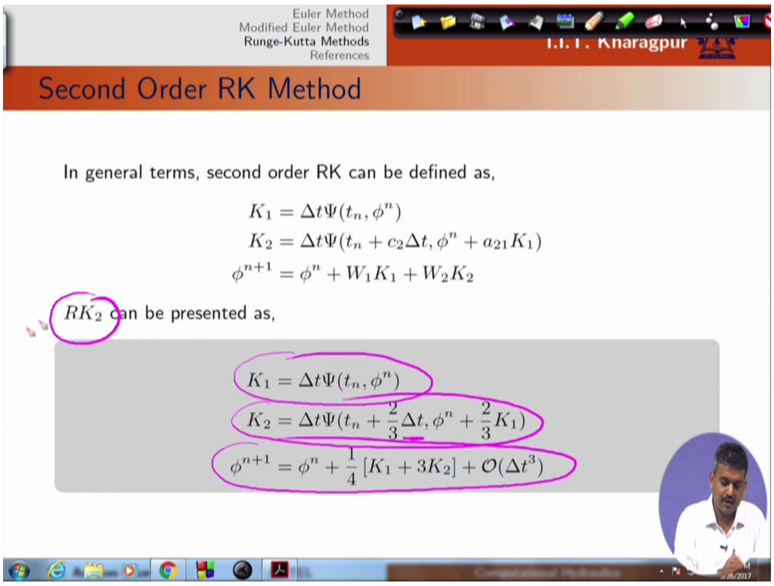

In third order RK we need to consider three terms one, two and three.

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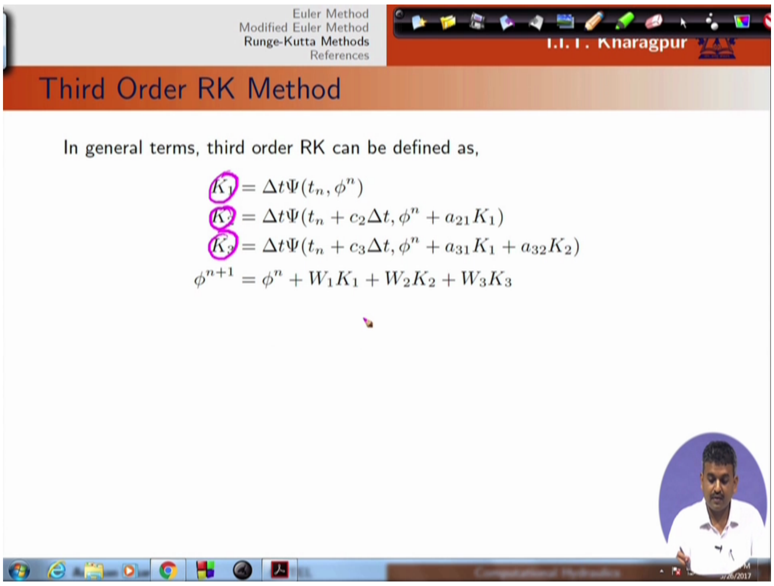

Similarly we can get the values for C2, C3, C2, C C3, A21, A32, W1, W2, W3 from Taylor series expansion.

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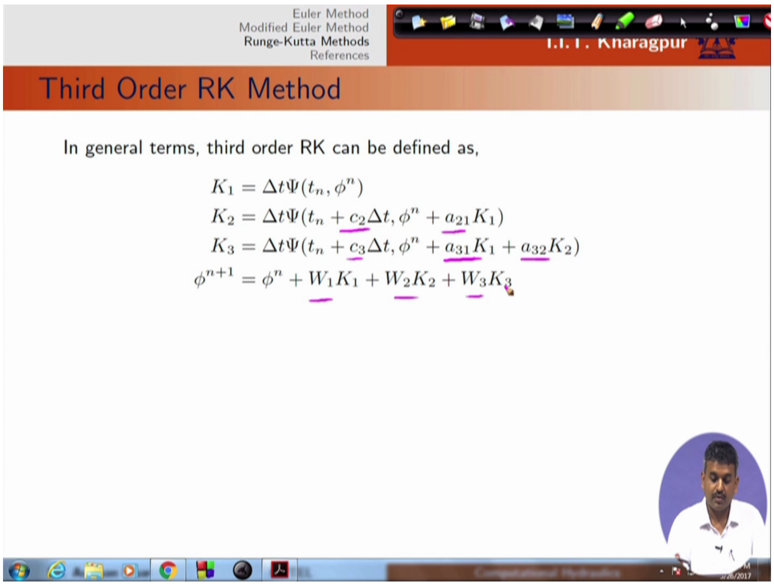

And if we write the final expression, we can write it as RK 3 and interestingly in this case is evaluated as half time step for the intermediate K2 and for others it at initial and this is that final time step.

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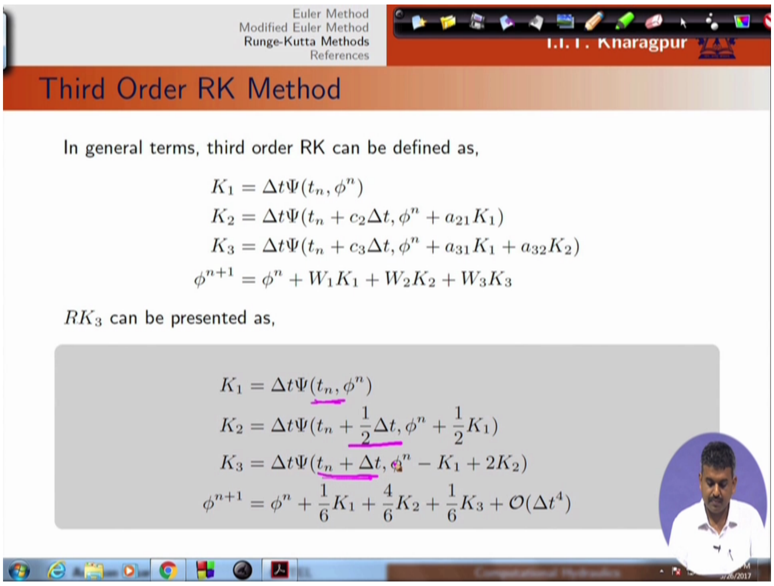

And these are coefficients, minus 1, 2 and finally we can write it else 4th order accurate but finally accuracy will be of third order.

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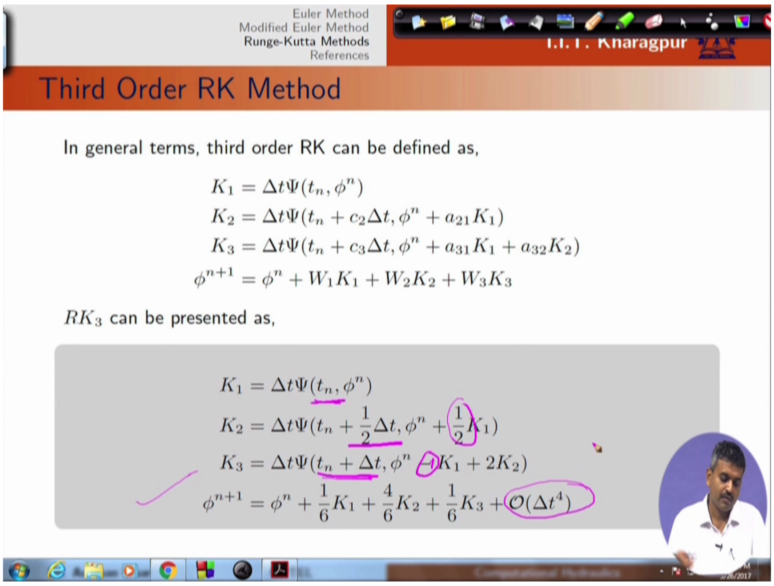

Similarly for 4th order RK we can use this approach and we can get 5th order accurate scheme but finally accuracy will be of order 4.

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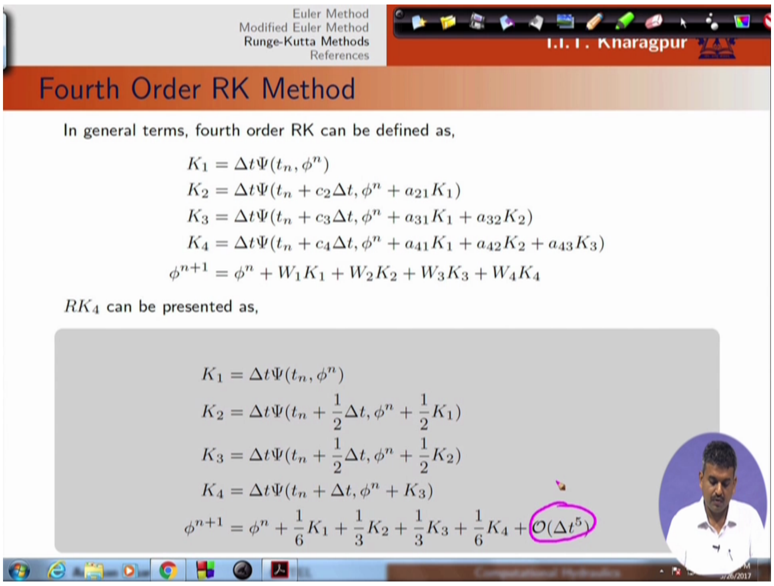

And interesting in this case this K2 and K3 are calculated at halftime step or at the middle of the two nodal points.

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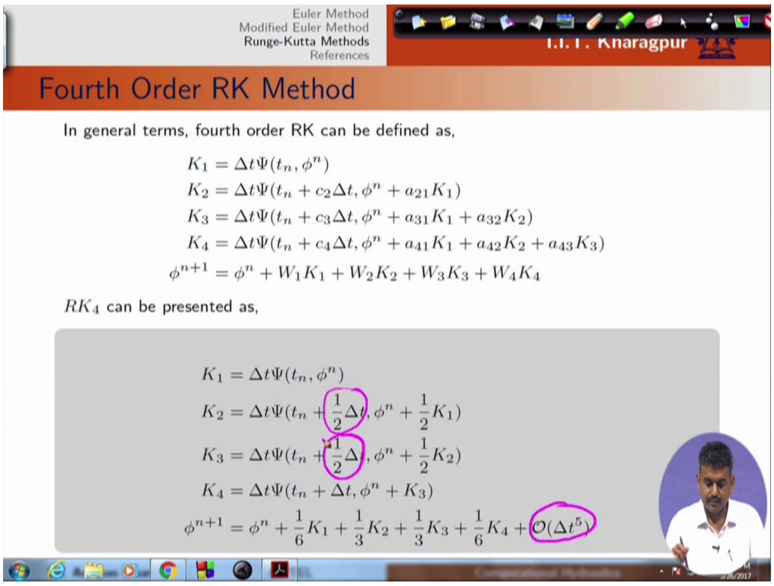

So if we consider one practical example of this. This is del2y by del del y by del x, s0 minus sf, 1 minus fr square. This is well known equation for describing the gradually varied flow in open channels.

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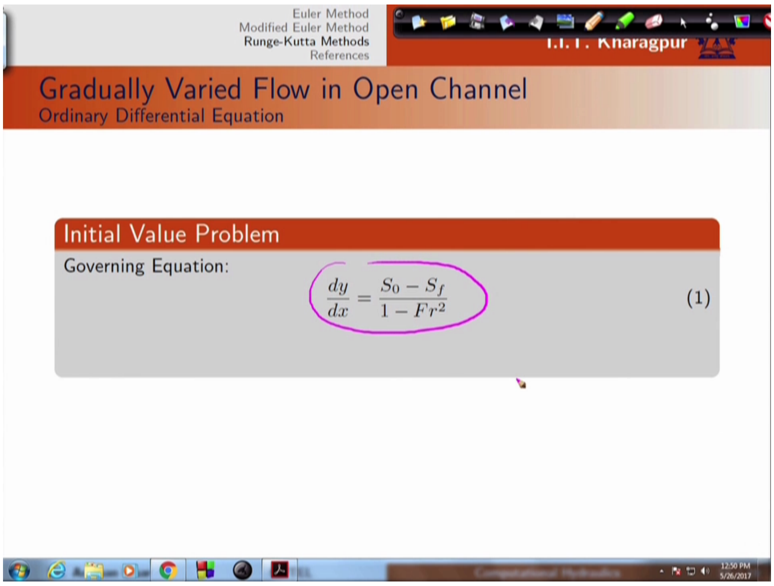

This is ordinary differential equation first order and with initial condition. That means depth is specified at some special locations. We can see thaton the right hand side we have this s0 minus sf, 1 minus fr square term.

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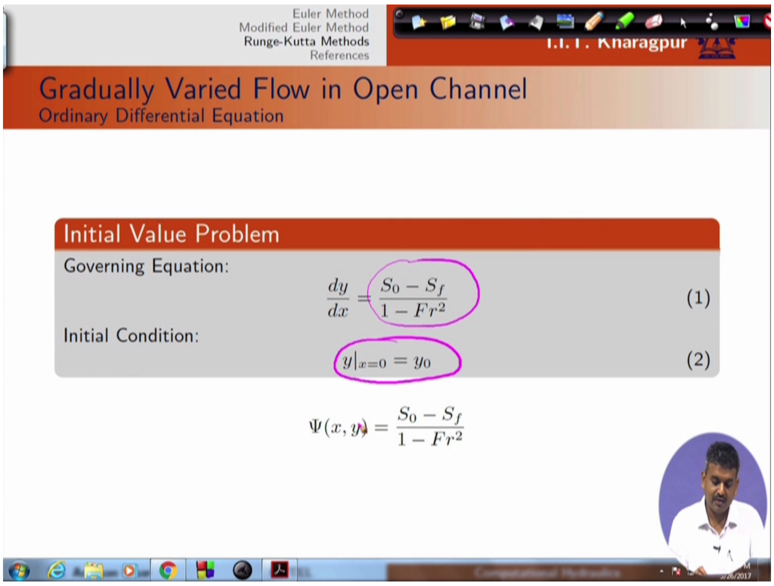

This is function psi which is function of xy, s0 minus sf, 1 minus fr square. S0 is bed slope, sf is energy slope, one minus fraud number square.

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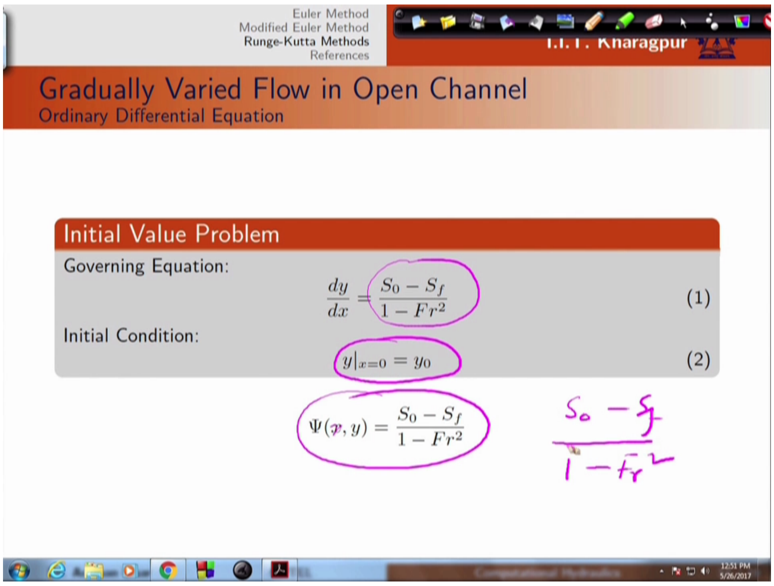

So essentially in this case we get psi as function of y only, which is flow depth in the channel.

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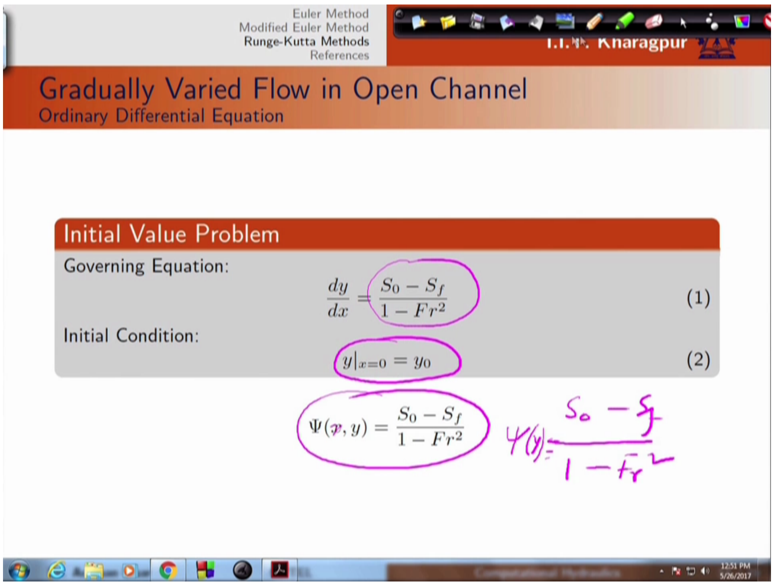

So this is a typical example of initial value problem. In the next lecture we will be discussing boundary value problem for ordinary differential equation. Thank you.