

Computational Hydraulics
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Indian Institute of Technology Kharagpur
Lecture 4
Classification of Differential Equations

Welcome to this lecture number 4 and this is module number 1 and unit number 4. It is classification of differential equations.

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The image shows a presentation slide with a white background and a dark blue header. The header contains the text: "Classification based on Physical Behavior", "Classification based on Completeness of Problem Definition", "Classifications based on Linearity", "Classification of Second Order PDE", and "References". To the right of the header is the I.I.T. Kharagpur logo. The main content of the slide is centered and includes: "Module 01: Introduction to Computational Hydraulics", "Unit 04: Classification of Differential Equations", "Anirban Dhar", "Department of Civil Engineering", "Indian Institute of Technology Kharagpur, Kharagpur", and "National Programme for Technology Enhanced Learning (NPTEL)". At the bottom, there is a footer with "Dr. Anirban Dhar", "NPTEL", "Computational Hydraulics", and "1 / 21".

What is the learning objective of this particular unit? At the end of this unit student will be able to classify the differential equation based on physical behavior, completeness of problem definition and linearity.

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Classification based on Physical Behavior
Classification based on Completeness of Problem Definition
Classifications based on Linearity
Classification of Second Order PDE
References

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Learning Objective

- To classify the *Differential Equations* based on physical behavior, completeness of problem definition and linearity.

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Classification based on physical behavior. First kind of problem that is called as equilibrium problem. As per Tannehill et al. problems in which solution (diff) of a given different partial differential equation is desired in a closed domain subject to prescribed set of boundary conditions.

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Classification based on Physical Behavior
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Classification based on Physical Behavior

Equilibrium Problem

"Problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions" (Tannehill et al., 1997).

- Also known as **Jury problems**.

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In this one this closed domain is important point and there should be only boundary condition. In our previous unit we have discussed about initial conditions and boundary condition. So for equilibriumtype of problem we need one governing equation and corresponding boundary condition. This kind of problem is also known as jury problems. So boundary conditions act as jury.

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Classification based on Physical Behavior
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Classification based on Physical Behavior

Equilibrium Problem

"Problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions"
(Tannehill et al., 1997).

- Also known as **Jury problems**.

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Generally these problems are steady state problems and solution is always smooth even if there is disturbance. So from all sides boundaries are defined. Mostly these problems are steady state problems.

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Classification based on Physical Behavior

Equilibrium Problem

"Problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions"
(Tannehill et al., 1997).

- Also known as **Jury problems**.
- Generally **Steady state problems**.
- Solution is always smooth even if there is disturbance.

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Let us consider one example where we have one sheet pile at the middle and this is our ground surface as well as water level.

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Equilibrium Problem

Sheet pile wall

Figure: Cross-section of a foundation pit (Jie et al., 2004)

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In this case this is your ground surface and this is our water level.

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Equilibrium Problem

Sheet pile wall

Figure: Cross-section of a foundation pit (Jie et al., 2004)

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So obviously there will be flow from this side to left to right.

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Equilibrium Problem

Figure: Cross-section of a foundation pit (Jie et al., 2004)

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And this problem can be defined in terms of the governing equation that is $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$ that is actually Laplace equation. And in terms of Dirichlet boundary condition and zero Neumann condition.

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Equilibrium Problem

Figure: Cross-section of a foundation pit (Jie et al., 2004)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (1)$$

$$h|_{\Gamma_D} = h_0(x, y)$$

$$K \frac{dh}{dx}|_{\Gamma_N} = 0$$

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Dirichlet boundary condition is applicable for this boundary where our head is specified again it is applicable for this boundary where head is known.

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Equilibrium Problem

Figure: Cross-section of a foundation pit (Jie et al., 2004)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (1)$$
$$h|_{\Gamma_D} = h_0(x, y)$$
$$K \frac{dh}{dx}|_{\Gamma_N} = 0$$

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In the middle the sheet pile act as a no flow boundary. That's why zero Neumann boundary condition can be directly applied for this problem.

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Equilibrium Problem

Figure: Cross-section of a foundation pit (Jie et al., 2004)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (1)$$
$$h|_{\Gamma_D} = h_0(x, y)$$
$$K \frac{dh}{dx}|_{\Gamma_N} = 0$$

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So this side that means left and right side also it can have specified head condition or if you have no flow boundaries then also things are defined. So we have one differential equations and boundary condition. So in this case the problem is space dependent. This is not time dependent that's why this is steady state problem.

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Equilibrium Problem

Figure: Cross-section of a foundation pit (Jie et al., 2004)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (1)$$
$$h|_{\Gamma_D} = h_0(x, y)$$
$$K \frac{dh}{dx}|_{\Gamma_N} = 0$$

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Second kind of problem is called as marching problem where the problems in which a solution of a given differential equation is desired in an open domain subject to a prescribed set of initial and boundary condition. Generally these problems are transient or transient like problems. And the important point is that not all marching problem are unsteady problem.

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Classification based on Physical Behavior

Marching Problem

Problems in which a solution of a given differential equation is desired in an open domain subject to a prescribed set of initial and boundary conditions.

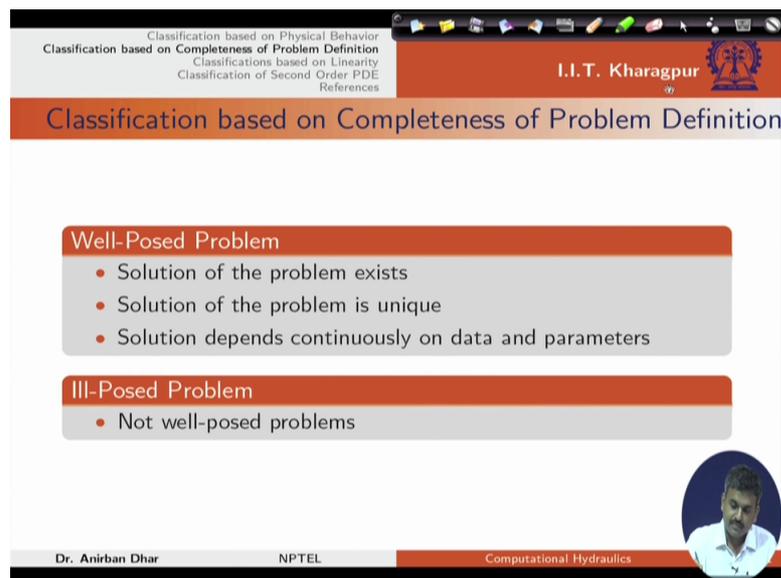
- Generally transient/transient-like problems
- Not all marching problems are unsteady.

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So in this case we need one governing equation one initial condition and boundary condition for the hydraulic system. So initial condition means that your time domain one side your values are defined but the other side is open. That's why this is open domain problem. So mostly these are transient problem.

Classification based on completeness of problem definition. This problem is called as well posed problem where solution of the problem exists. Second condition is solution of the problem is unique and third condition is solution depends continuously on data and parameters. If we change the initial condition or parameter values obviously there should be change in the solution. Otherwise the problem is called as ill posed problem. That means if there is evaluation of any of these conditions.

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The slide content is as follows:

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Classification based on Completeness of Problem Definition
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Classification based on Completeness of Problem Definition

- Well-Posed Problem**
 - Solution of the problem exists
 - Solution of the problem is unique
 - Solution depends continuously on data and parameters
- Ill-Posed Problem**
 - Not well-posed problems

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Then classification based on linearity. Classic example is this groundwater equation for confined aquifer. So obviously this equation is linear in nature because there is no multiplication of dependent variables. Then there can be nonlinear problems maybe this momentum conservation equation for surface water. In this case we can see that this q^2 term is there which is nonlinear in nature. So in this case we can say that equation is nonlinear.

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Classifications based on Linearity

Linear
Groundwater equation for confined aquifer

$$\frac{S}{T} \frac{\partial h}{\partial t} = \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) \quad (3)$$

Non-linear
Momentum conservation equation for surface water

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + gA \frac{\partial h}{\partial x} + gA(S_f - S_0) = 0 \quad (4)$$

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If we consider our hydraulic system most of the differential equation we can consider as linear second order partial differential equation. So if we have two independent variables and phi is the dependent variable then we can write this equation where a, b, c, d, e, f, g these are coefficients. And these are actually functions of x and y. Otherwise these values are constant.

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Classification of Second Order PDE

A second order PDE in two co-ordinates x and y for a general variable ϕ can be written as,

$$A \frac{\partial^2 \phi}{\partial x^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \frac{\partial \phi}{\partial x} + E \frac{\partial \phi}{\partial y} + F \phi + G = 0 \quad (5)$$

where the coefficients A , B , C , D , E , F , and G are functions of x , y or constants.

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Highest partial derivatives determine the nature of the equation. So in this case highest partial derivatives are 1, 2, 3.

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Classification of Second Order PDE

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where the coefficients A , B , C , D , E , F , and G are functions of x , y or constants. Highest partial derivatives determine the nature of the equation. The characteristic equation can be written as,

$$A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C = 0 \quad (6)$$

Depending on sign of discriminant ($B^2 - 4AC$) equations are classified.

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So from this equation we can get the characteristic equation which can be written in this form and depending on sign of discriminant that is B square minus 4AC equations can be classified.

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Classification of Second Order PDE

A second order PDE in two co-ordinates x and y for a general variable ϕ can be written as,

$$A \frac{\partial^2 \phi}{\partial x^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \frac{\partial \phi}{\partial x} + E \frac{\partial \phi}{\partial y} + F \phi + G = 0 \quad (5)$$

where the coefficients A , B , C , D , E , F , and G are functions of x , y or constants. Highest partial derivatives determine the nature of the equation. The characteristic equation can be written as,

$$A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C = 0 \quad (6)$$

Depending on sign of discriminant ($B^2 - 4AC$) equations are classified.

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So classification of second order PDE first is parabolic, B square minus 4AC is equal to zero. So transient one dimensional groundwater flow equation in confined aquifer this is $\phi = t \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2}$. In this case h is the function of t and x only. So if we compare this situation with our linear or PDE equation that is x and y , so we can find out the coefficient a , b , c and based on that we can comment on the nature of the partial differential equation.

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Classification of Second Order PDE

Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Transient One-dimensional groundwater flow equation in confined aquifer

$$\frac{S}{T} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \quad \begin{matrix} h(t, x) \\ \phi(x, y) \end{matrix} \quad (7)$$

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So in this case A is zero because the coefficient of $\frac{\partial^2 h}{\partial x^2}$ is zero, B equal to zero because there is no cross term. C is minus one which is the coefficient of $\frac{\partial h}{\partial t}$. This is actually minus one.

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Classification of Second Order PDE

Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Transient One-dimensional groundwater flow equation in confined aquifer

$$\frac{S}{T} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \quad (7)$$

Here, $A = 0, B = 0, C = -1$ and $B^2 - 4AC = 0$.

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So obviously $B^2 - 4AC$ this term is zero. So we can say that this situation is parabolic in nature.

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Parabolic Equations

Parabolic: $B^2 - 4AC = 0$

Transient One-dimensional groundwater flow equation in confined aquifer

$$\frac{S}{T} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \quad (7)$$

Here, $A = 0$, $B = 0$, $C = -1$ and $B^2 - 4AC = 0$.

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Then comes this elliptic equation where $B^2 - 4AC$ is less than zero. Steady two dimensional groundwater flow equation in confined aquifer. This is Laplace equation. Again if we compare this equation with our original $\phi(x,y)$, this is actually $x, h(x,y)$.

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Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Steady two-dimensional groundwater flow equation in confined aquifer (Laplace equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (8)$$

$h(x,y)$
 $\phi(x,y)$



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So, we have coefficient 1, 1 but for cross term the coefficient is zero.

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Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Steady two-dimensional groundwater flow equation in confined aquifer (Laplace equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (8)$$

$h(x,y)$
 $\phi(x,y)$

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So A is one, B is zero, C is one. So B square minus 4AC is minus 4 which is less than zero. So we can see say that this equation is elliptic in nature.

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Elliptic Equations

Elliptic: $B^2 - 4AC < 0$

Steady two-dimensional groundwater flow equation in confined aquifer (Laplace equation)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (8)$$

Here, $A = 1, B = 0, C = 1$ and $B^2 - 4AC = -4 < 0$.

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Then hyperbolic equation where B square minus 4AC is greater than zero. One dimensional wave equation. In this case we can see that the coefficient of u which is the function of p and x is one and in this case this is minus one there is no cross term available for this equation.

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Hyperbolic Equations

Hyperbolic: $B^2 - 4AC > 0$

One-dimensional wave equation $u(t, x)$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (9)$$

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So obviously we have a equals to minus one which is coefficient of u del u by del x^2 , B is zero and C is one. So this is b square minus $4AC$ which is positive in nature. So we can say that this equation is hyperbolic equation.

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Hyperbolic Equations

Hyperbolic: $B^2 - 4AC > 0$

One-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (9)$$

Here, $A = -1$, $B = 0$, $C = 1$ and $B^2 - 4AC = 4 > 0$.

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Now eigenvalue based classification. So, we can have multiple independent variables for the differential equation. So let us consider a PDE where we have N independent variables. x_1, x_2 to x_N and A_{ij} and B_{ij} , C , D , are coefficients. Again these are either functions of x_1 to x_N or these are constant values.

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Eigenvalue based Classification

A general second order PDE with N independent variables (x_1, x_2, \dots, x_N) can be represented as,

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial \phi}{\partial x_i} + c\phi + d = 0 \quad (10)$$

where
 $a_{ij}, b_i, c =$ functions of x_1, x_2, \dots, x_N

Assumptions:

- $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$
- $A_\lambda = [a_{ij}]$ is symmetric.

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With the assumption that the cross term we have symmetry for this coefficient which is corresponding to the higher derivative terms we can construct a matrix A_λ and this is symmetric in nature. Symmetric because in this case we are considering that the coefficients of the cross term should be same.

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Eigenvalue based Classification

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$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial \phi}{\partial x_i} + c\phi + d = 0 \quad (10)$$

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 $a_{ij}, b_i, c =$ functions of x_1, x_2, \dots, x_N

Assumptions:

- $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$
- $A_\lambda = [a_{ij}]$ is symmetric.

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So with this assumption the eigenvalue of A_λ are the values of λ that satisfy the equation $A_\lambda - \lambda I$, which is identity matrix and determinant of that should be zero. If we can solve this equation we can find out the values or value of λ depending on the order of equation.

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Eigenvalue based Classification

A general second order PDE with N independent variables (x_1, x_2, \dots, x_N) can be represented as,

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial \phi}{\partial x_i} + c\phi + d = 0 \quad (10)$$

where
 $a_{ij}, b_i, c =$ functions of x_1, x_2, \dots, x_N

Assumptions:

- $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$
- $A_\lambda = [a_{ij}]$ is symmetric.

The eigenvalues of A_λ are values of λ that satisfy the equation

$$|A_\lambda - \lambda I| = 0 \quad (11)$$

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So equation can be classified based on sign of eigenvalue lambda 1 to lambda N of the matrix A lambda. For parabolic equations one or more zero eigenvalues. If we have elliptic equation non zero and with same sign. That means lambda I is greater than zero for all I or lambda I less than zero for all I. If we have hyperbolic equation then non zero and all but 1 are with same sign.

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Eigenvalue based Classification

The equation can be classified based on the sign of eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_N)$ of matrix A_λ as

- Parabolic Equation:** one or more zero eigenvalues ($\lambda_i = 0$)
- Elliptic Equation:** non-zero and with same sign ($\lambda_i > 0, \forall i$ or $\lambda_i < 0, \forall i$)
- Hyperbolic Equation:** non-zero and all but one are with same sign

$$\lambda_i > 0, i \in \{1, 2, \dots, N\} \setminus \{j\}$$

$$\lambda_j < 0$$

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That means for a particular J eigenvalue this is listen zero, for others it will be greater than zero.

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The equation can be classified based on the sign of eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_N)$ of matrix A_λ as

- **Parabolic Equation:** one or more zero eigenvalues ($\lambda_i = 0$)
- **Elliptic Equation:** non-zero and with same sign ($\lambda_i > 0, \forall i$ or $\lambda_i < 0, \forall i$)
- **Hyperbolic Equation:** non-zero and all but one are with same sign

$$\frac{\lambda_i > 0, i \in \{1, 2, \dots, N\} \setminus \{j\}}{\lambda_j < 0}$$

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Or combination can be lambda I is less than zero and lambda J is greater than zero.

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Eigenvalue based Classification

The equation can be classified based on the sign of eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_N)$ of matrix A_λ as

- **Parabolic Equation:** one or more zero eigenvalues ($\lambda_i = 0$)
- **Elliptic Equation:** non-zero and with same sign ($\lambda_i > 0, \forall i$ or $\lambda_i < 0, \forall i$)
- **Hyperbolic Equation:** non-zero and all but one are with same sign

$$\lambda_i > 0, i \in \{1, 2, \dots, N\} \setminus \{j\}$$
$$\lambda_j < 0$$

or

$$\lambda_i < 0, i \in \{1, 2, \dots, N\} \setminus \{j\}$$
$$\lambda_j > 0$$

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If we consider examples for parabolic equations we can take the same transient 1D groundwater equation in confined aquifer. In this case if we construct the coefficient matrix A lambda then we can see that this zero which is diagonal term A one one.

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Parabolic Equations

Transient 1D Groundwater Equation in Confined Aquifer

$$\frac{S}{T} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \quad (12)$$
$$A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0, \lambda_2 = -1 < 0$

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One one means there is no coefficient for del 2h by delt2. This is zero so obviously this is zero.

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Parabolic Equations

Transient 1D Groundwater Equation in Confined Aquifer

$$0 \frac{\partial h}{\partial t} + \frac{S}{T} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = 0 \quad (12)$$
$$A_\lambda = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0, \lambda_2 = -1 < 0$

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There is no cross term that's why zero values corresponding to cross term and minus one as coefficient of del 2h delx2. So obviously eigenvalues lambda 1 zero, Lambda 2 minus one which is less than zero. So one of the eigenvalue is zero. We can say that the equation is parabolic in nature.

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Hyperbolic Equations

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (14)$$

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If we consider elliptic equation again with the Laplace equation the coefficient is one. In this case also this is one. So if we consider the coefficient matrix this is one one and cross terms are zero zero. So if we find out the eigenvalue lambda 1 is one lambda 2 is one both are greater than zero that means they have same sign and both are nonzero values. So we can call this as elliptic equation.

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Elliptic Equations

Steady 2D Groundwater Equation in Confined Aquifer

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (13)$$
$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = 1 > 0$

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In case of hyperbolic equation that is wave equation we can see that the coefficient is one, here it is minus one and with this information if you proceed we can construct the coefficient matrix this is one minus one and cross terms are zero. So lambda 1 is one which is greater than zero, lambda 2 is minus one which is less than zero. That means the opposite sign is

there and both are nonzero values. So obviously we can say that the equation is hyperbolic in nature.

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Hyperbolic Equations

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (14)$$

$$A_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = -1 < 0$

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So let us consider form of differential equation with general variable phi. In this case phi in general form it is a function of x, y, z and t.

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Let us consider a form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \cdot \nabla \phi) + F_{\phi_o} + S_\phi \quad (15)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_o} = other forces
- S_ϕ = source/sink term

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Gamma phi and upsilon phi these are problem dependent parameters and this lambda phi and upsilon phi these are problem dependent parameters. And gamma phi this is a tensor and this is equivalent to diffusion tensor and F phi zero or F phi others this term considers other forces

and S_ϕ is a source/sink term. So we can use this general form of equation to represent all of our hydraulic system related problems.

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Let us consider a form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \nabla \phi) + F_{\phi_\alpha} + S_\phi \quad (15)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_α} = other forces
- S_ϕ = source/sink term

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So example is mass conservation equation that is $\text{del}_t, \text{del}_x, \text{del}_y, \text{del}_z$. In this case we can see that λ_ϕ is 1 and ϕ is ρ and which is constant. So obviously the temporal term in the left hand side is not present. Υ_ϕ again that is 1 and this dispersion coefficient tensor this is again having zero zero values that means all terms are zero in this case.

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$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Lambda_\phi = 1, \phi = \rho = \text{constant}, \Upsilon_\phi = 1, \Gamma_\phi = 0, F_{\phi_\alpha} = 0, S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (16)$$

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So there is no diffuse in term only advection related term in left hand side. So other force this is zero, S phi is zero. So if we use this parameters forces and source sink terms then we can get this mass conservation equation from our general equations.

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$\Lambda_\phi = 1, \phi = \rho = \text{constant}, \Upsilon_\phi = 1, \Gamma_\phi = 0, F_{\phi_o} = 0, S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (16)$$

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Again if we use this lambda equals to row, phi equals to u, which is velocity in x direction, epsilon equals to row and this (disper)division coefficient tensor as a with dynamic viscosity and body force including pressure and this gravity force S pie is zero.

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$\Lambda_\phi = 1, \phi = \rho = \text{constant}, \Upsilon_\phi = 1, \Gamma_\phi = 0, F_{\phi_o} = 0, S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (16)$$

$\Lambda_\phi = \rho, \phi = u, \Upsilon_\phi = \rho, \Gamma_\phi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, F_{\phi_o} = \left(-\frac{\partial P}{\partial x}\right) + (\rho g_x), S_\phi = 0$

Momentum conservation equation

x-dir: $\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (17)$

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So for this moment conversation equation we can get this form for incompressible fluid flow because we have considered constant value of row epsilon.

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$\Lambda_\phi = 1, \phi = \rho = \text{constant}, \Upsilon_\phi = 1, \Gamma_\phi = \mathbf{0}, F_{\phi_o} = 0, S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (16)$$

$\Lambda_\phi = \rho, \phi = u, \Upsilon_\phi = \rho, \Gamma_\phi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, F_{\phi_o} = \left(-\frac{\partial P}{\partial x}\right) + (\rho g_x), S_\phi = 0$

Momentum conservation equation

x-dir: $\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (17)$

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So obviously the row will be in multiplied form on this side and this side and that row can be transferred here so that uh we can get this incompressible fluid flow x direction momentum conservation equation.

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$\Lambda_\phi = 1, \phi = \rho = \text{constant}, \Upsilon_\phi = 1, \Gamma_\phi = \mathbf{0}, F_{\phi_o} = 0, S_\phi = 0$

Mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (16)$$

$\Lambda_\phi = \rho, \phi = u, \Upsilon_\phi = \rho, \Gamma_\phi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}, F_{\phi_o} = -\frac{\partial P}{\partial x} + \rho g_x, S_\phi = 0$

Momentum conservation equation

x-dir: $\left(\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \right) = -\frac{1}{\rho} \frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (17)$

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So with this information we can proceed and further for concentration which is scalar transport equation. Again lambda phi is one, phi is eta C, eta is porosity, C is concentration, epsilon phi is one. Again for two dimensional case this is deviation coefficient tensor other forces are zero and source sink term that is q_s, c_s.

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Contaminant Transport Concentration Equation

$$\Lambda_\phi = 1, \phi = \eta C, \Upsilon_\phi = 1, \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}, F_{\phi_o} = 0, S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x} \left(\eta D_{xx} \frac{\partial C}{\partial x} + \eta D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left(\eta D_{yx} \frac{\partial C}{\partial x} + \eta D_{yy} \frac{\partial C}{\partial y} \right) - \frac{\partial}{\partial x} (\eta v_x C) - \frac{\partial}{\partial y} (\eta v_y C) + q_s C_s \quad (18)$$

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So, this is actually source sink term and this is advection term.

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Contaminant Transport Concentration Equation

$$\Lambda_\phi = 1, \phi = \eta C, \Upsilon_\phi = 1, \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}, F_{\phi_o} = 0, S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x} \left(\eta D_{xx} \frac{\partial C}{\partial x} + \eta D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left(\eta D_{yx} \frac{\partial C}{\partial x} + \eta D_{yy} \frac{\partial C}{\partial y} \right) - \frac{\partial}{\partial x} (\eta v_x C) - \frac{\partial}{\partial y} (\eta v_y C) + q_s C_s \quad (18)$$

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These two are related to diffusion term.

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Contaminant Transport Concentration Equation

$$\Lambda_\phi = 1, \phi = \eta C, \Upsilon_\phi = 1, \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}, F_{\phi_o} = 0, S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x} \left(\eta D_{xx} \frac{\partial C}{\partial x} + \eta D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left(\eta D_{yx} \frac{\partial C}{\partial x} + \eta D_{yy} \frac{\partial C}{\partial y} \right) - \frac{\partial}{\partial x} (\eta v_x C) - \frac{\partial}{\partial y} (\eta v_y C) + \frac{q_s C_s}{S_\phi} \quad (18)$$

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And this one is temporal term.

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Contaminant Transport Concentration Equation

$$\Lambda_\phi = 1, \phi = \eta C, \Upsilon_\phi = 1, \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}, F_{\phi_o} = 0, S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x} \left(\eta D_{xx} \frac{\partial C}{\partial x} + \eta D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left(\eta D_{yx} \frac{\partial C}{\partial x} + \eta D_{yy} \frac{\partial C}{\partial y} \right) - \frac{\partial}{\partial x} (\eta v_x C) - \frac{\partial}{\partial y} (\eta v_y C) + \frac{q_s C_s}{S_\phi} \quad (18)$$

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So, obviously we can get the (con) concentration equation from our generalized equations.

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Contaminant Transport

Concentration Equation

$$\Lambda_\phi = 1, \phi = \eta C, \Upsilon_\phi = 1, \Gamma_\phi = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix}, F_{\phi\alpha} = 0, S_\phi = q_s C_s$$

Scalar Transport Equation

$$\frac{\partial(\eta C)}{\partial t} = \frac{\partial}{\partial x} \left(\eta D_{xx} \frac{\partial C}{\partial x} + \eta D_{xy} \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial y} \left(\eta D_{yx} \frac{\partial C}{\partial x} + \eta D_{yy} \frac{\partial C}{\partial y} \right) - \frac{\partial}{\partial x} (\eta v_x C) - \frac{\partial}{\partial y} (\eta v_y C) + \frac{q_s C_s}{S_\phi} \quad (18)$$

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So we can utilize this generalized equation and we can get the discretization for that equation so that we can use that general discretization for all kinds of problems. So, this is the end of module number one. Next module we will talk about numerical methods. Thank you.