Computational Hydraulics

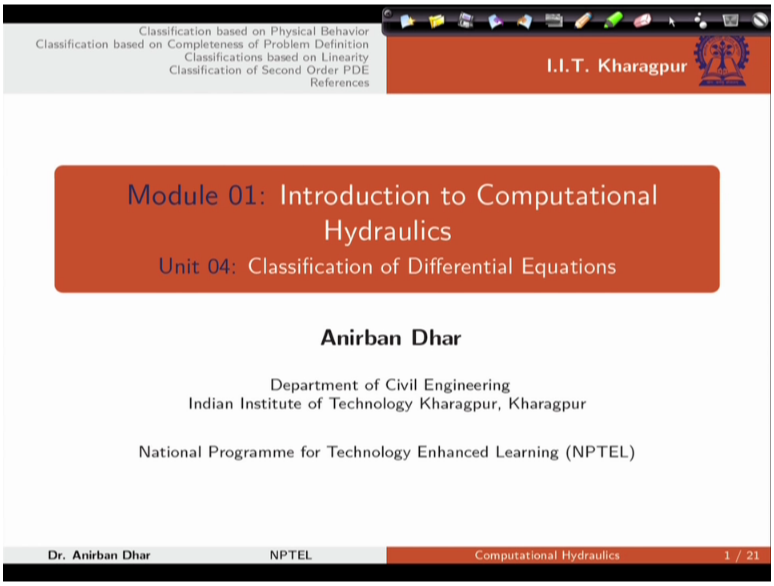
ProfessorAnirban Dhar
Department of Civil Engineering
Indian Institute of Technology Kharagpur

Lecture 4

Classification of Differential Equations

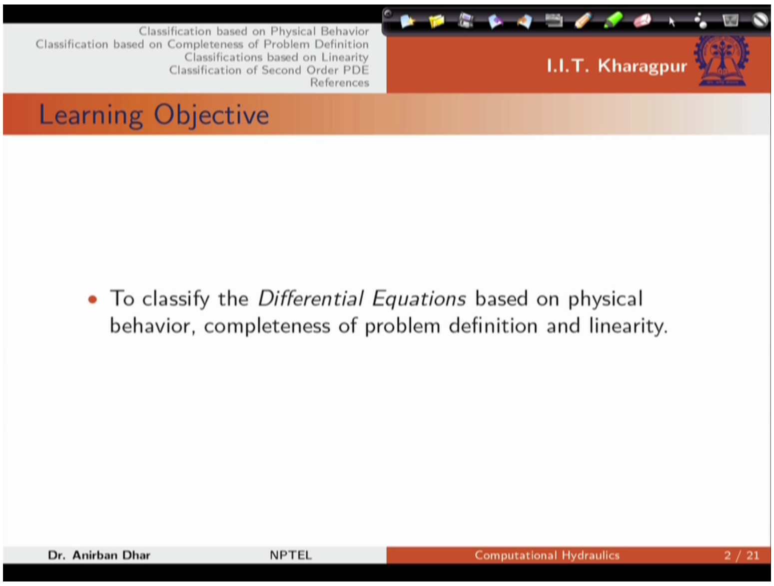
Welcome to this lecture number 4 and this is module number 1 and unit number 4. It is classification of differential equations.

(Refer Slide Time 00:26)



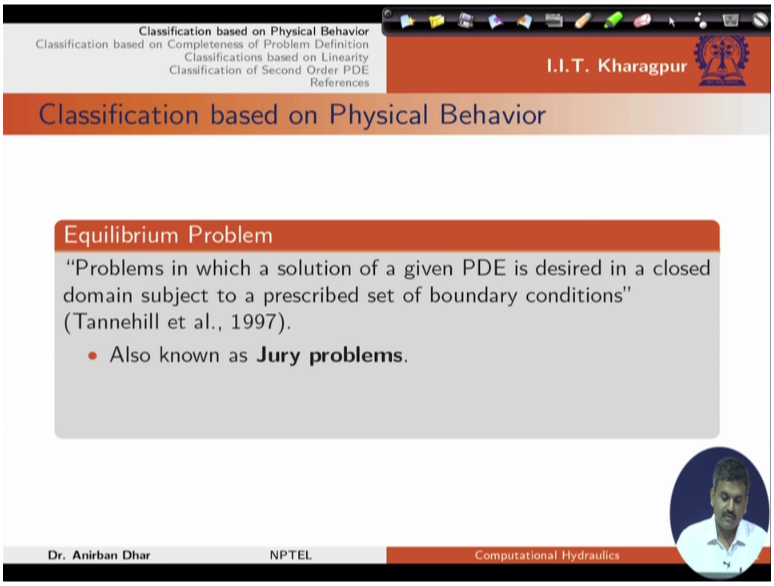
What is the learning objective of this particular unit? At the end of this unit student will be able to classify the differential equation based on physical behavior, completeness of problem definition and linearity.

(Refer Slide Time 00:54)



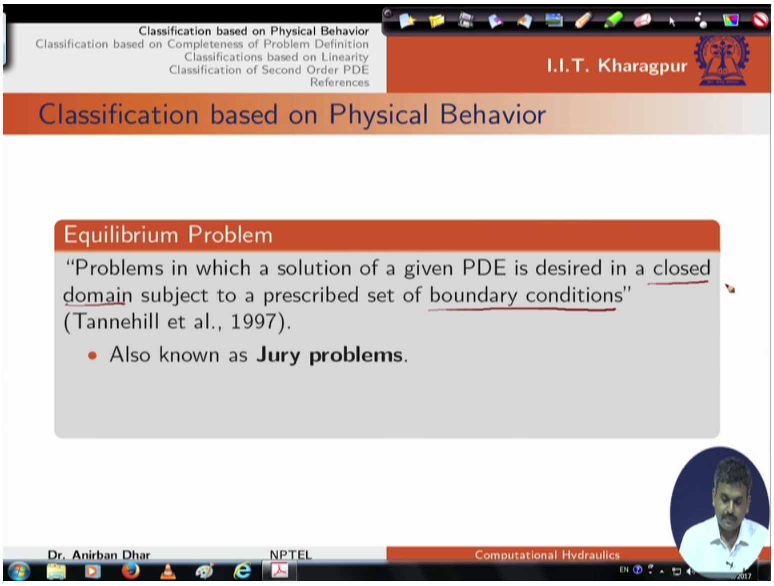
Classification based on physical behavior. First kind of problem that is called as equilibrium problem. As per Tannehill et al. problems in which solution (diff) of a given different partial differential equation is desired in a closed domain subject to prescribed set of boundary conditions.

(Refer Slide Time 01:31)



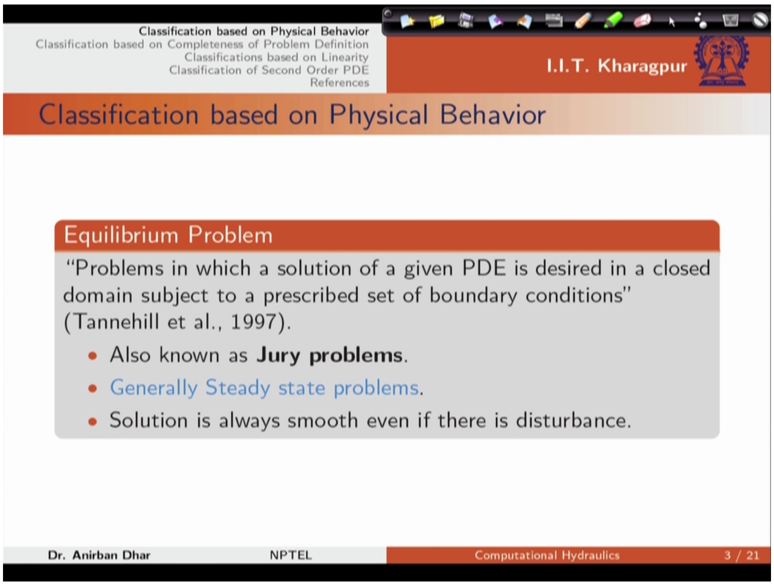
In this one this closed domain is important point and there should be only boundary condition. In our previous unit we have discussed about initial conditions and boundary condition. So for equilibriumtype of problem we need one governing equation and corresponding boundary condition. This kind of problem is also known as jury problems. So boundary conditions act as jury.

(Refer Slide Time 02:15)



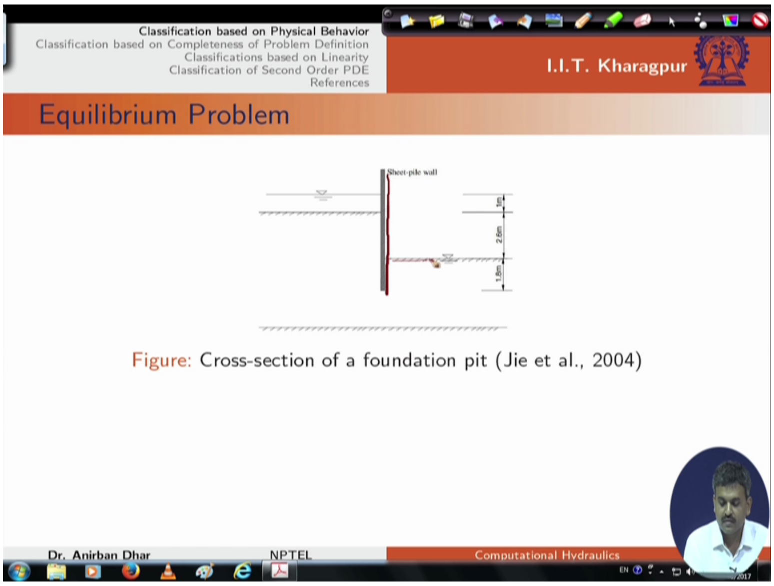
Generally these problems are steady state problems and solution is always smooth even if there is disturbance. So from all sides boundaries are defined. Mostly these problems are steady state problems.

(Refer Slide Time 02:45)



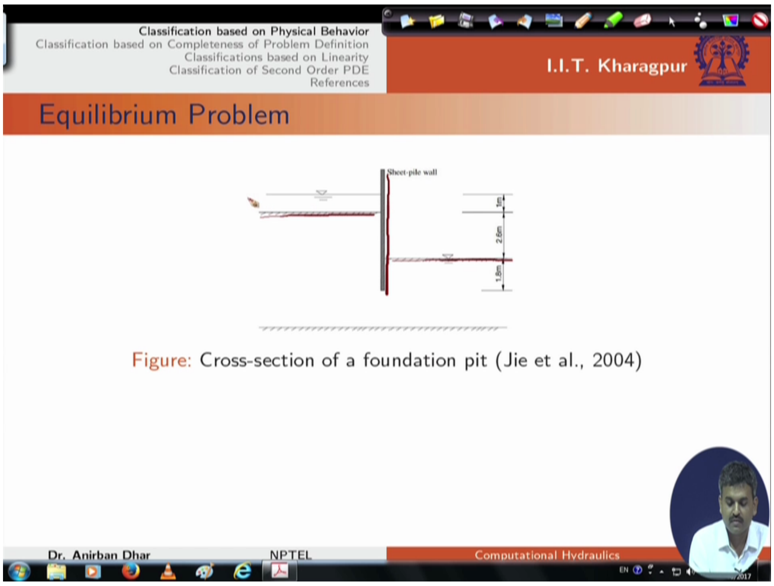
Let us consider one example where we have one sheet pile at the middle and this is our ground surface as well as water level.

(Refer Slide Time 03:13)



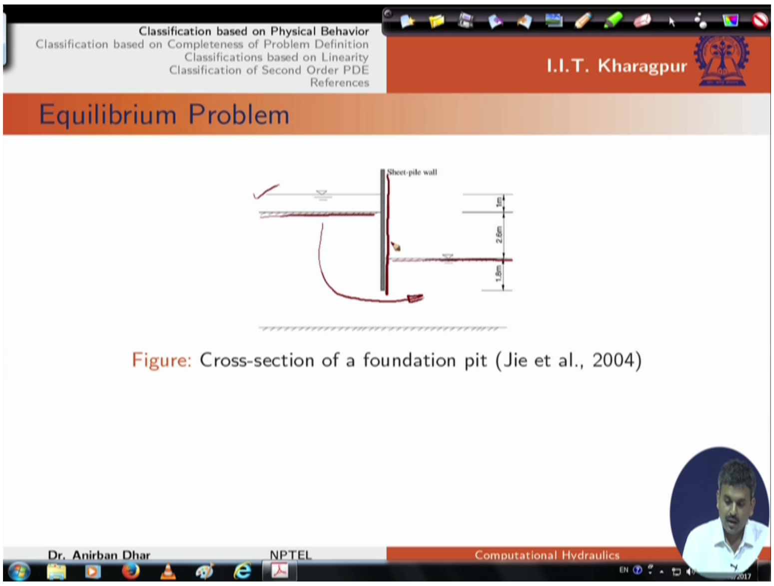
In this case this is your ground surface and this is our water level.

(Refer Slide Time 03:20)



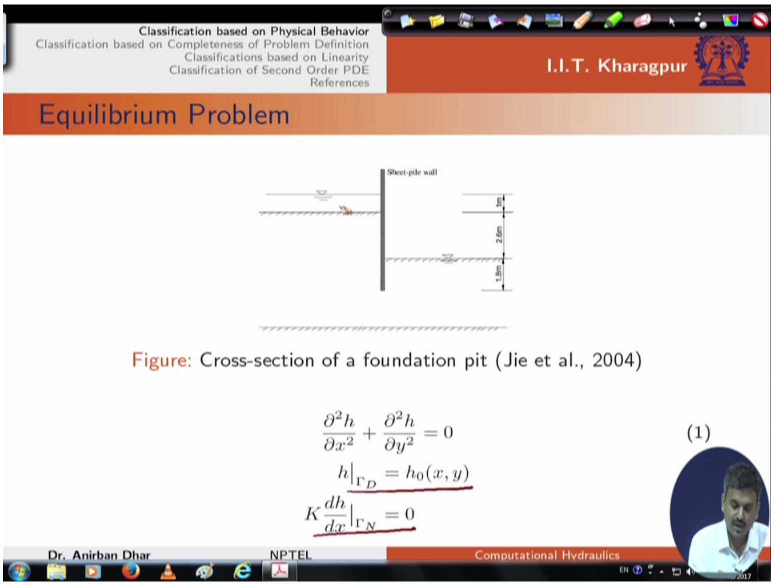
So obviously there will be flow from this side to left to right.

(Refer Slide Time 03:30)



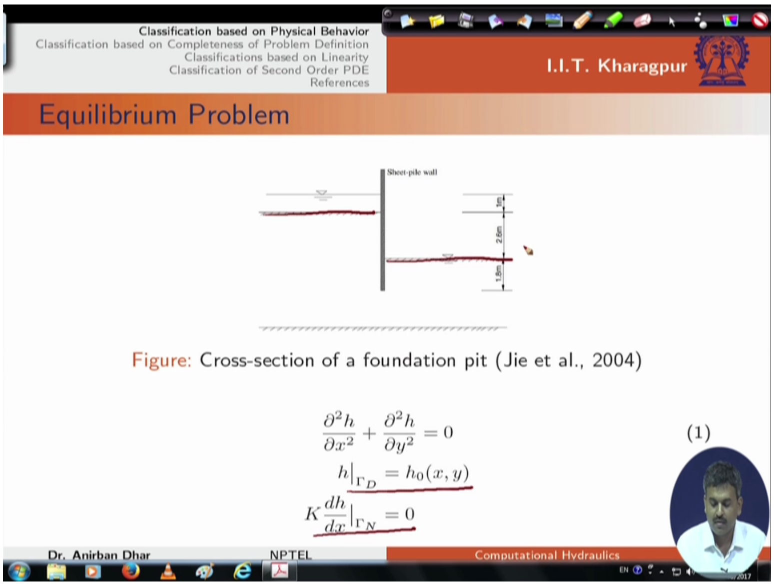
And this problem can be defined in terms of the governing equation that is del 2h del x2 plus del 2h dely2 that is actually Laplace equation. And in terms of Dirichlet boundary condition and zero Neumann condition.

(Refer Slide Time 03:58)



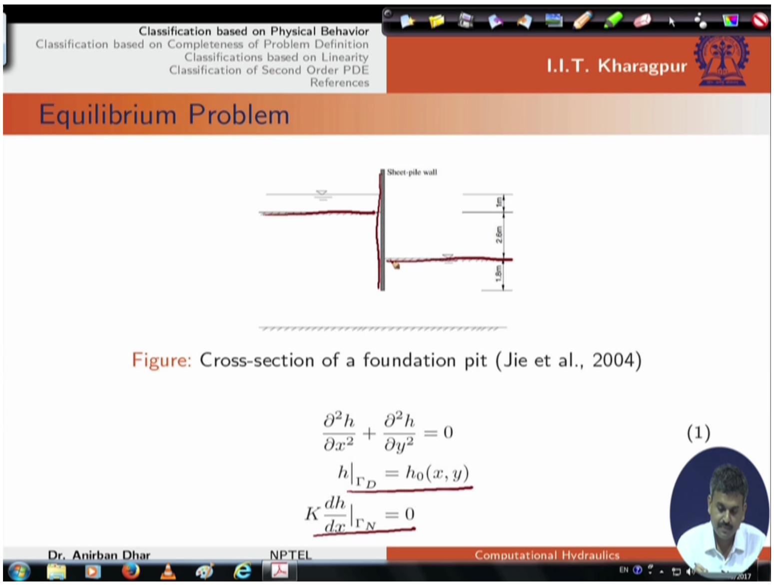
Dirichlet boundary condition is applicable for this boundary where our head is specified again it is applicable for this boundary where head is known.

(Refer Slide Time 04:07)



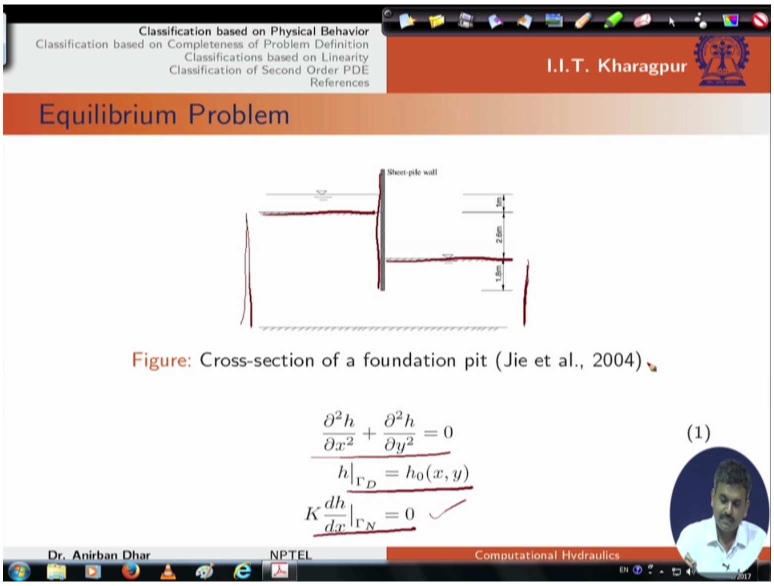
In the middle the sheet pile act as a no flow boundary. That's why zero Neumann boundary condition can be directly applied for this problem.

(Refer Slide Time 04:16)



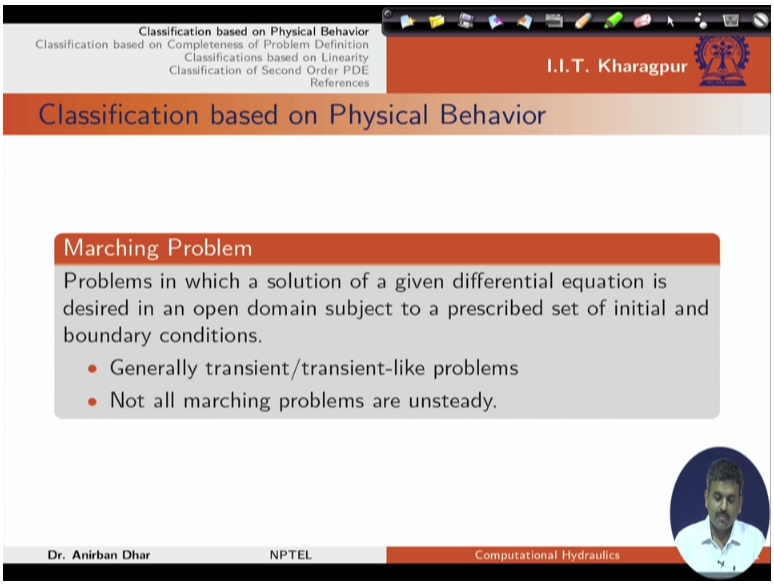
So this side that means left and right side also it can have specified head condition or if you have no flow boundaries then also things are defined. So we have one differential equations and boundary condition. So in this case the problem is space dependent. This is not time dependent that's why this is steady state problem.

(Refer Slide Time 05:10)



Second kind of problem is called as marching problem where the problems in which a solution of a given differential equation is desired in an open domain subject to a prescribed set of initial and boundary condition. Generally these problems are transient or transient like problems. And the important point is that not all marching problem are unsteady problem.

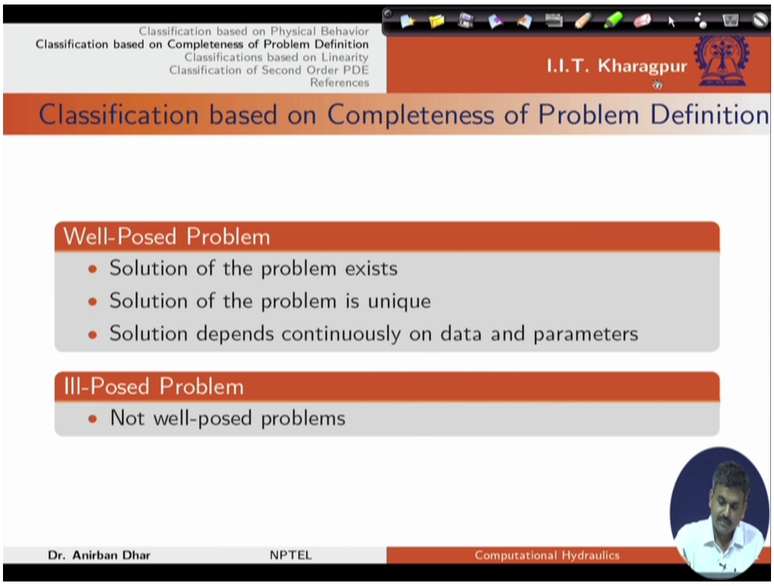
(Refer Slide Time 05:52)



So in this case we need one governing equation one initial condition and boundary condition for the hydraulic system. So initial condition means that your time domain one side your values are defined but the other side is open. That's why this is open domain problem. So mostly these are transient problem.

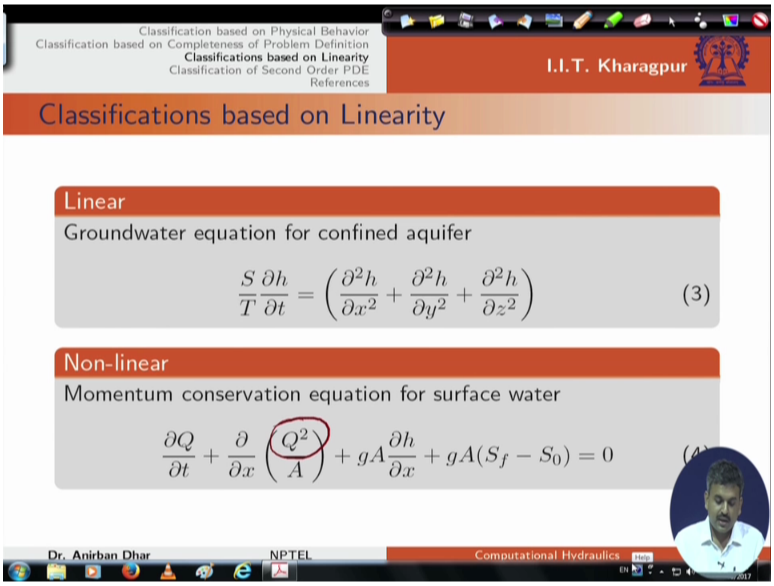
Classification based on completeness of problem definition. This problem is called as well posed problem where solution of the problem exists. Second condition is solution of the problem is unique and third condition is solution depends continuously on data and parameters. If we change the initial condition or parameter values obviously there should be change in the solution. Otherwise the problem is called as ill posed problem. That means if there is evaluation of any of these conditions.

(Refer Slide Time 07:34)



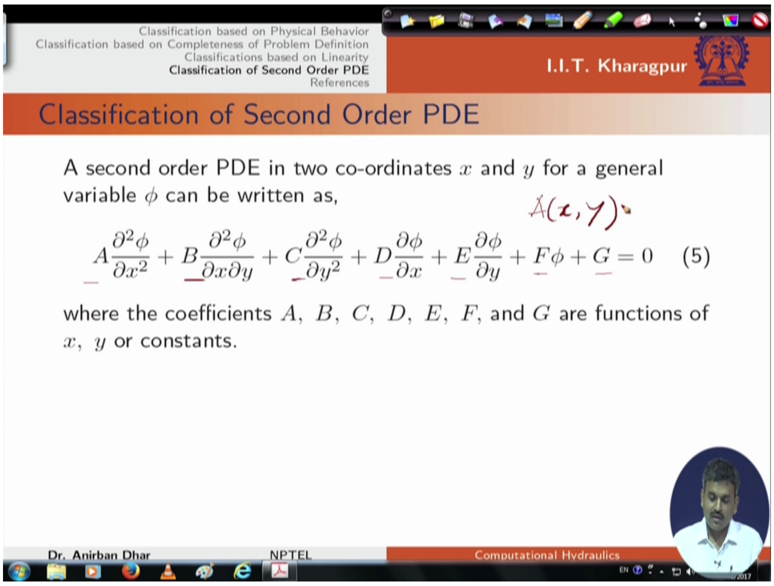
Then classification based on linearity. Classic example is this groundwater equation for confined aquifer. So obviously this equation is linear in nature because there is no multiplication of dependent variables. Then there can be nonlinear problems maybe this momentum conservation equation for surface water. In this case we can see that this q square term is there which is nonlinear in nature. So in this case we can say that equation is nonlinear.

(Refer Slide Time 08:32)



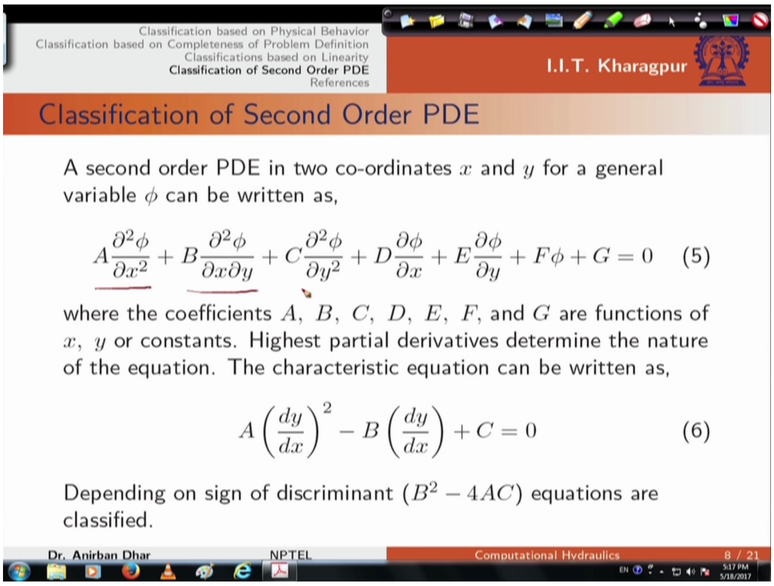
If we consider our hydraulic system most of the differential equation we can consider as linear second order partial differential equation. So if we have two independent variables and phi is the dependent variable then we can write this equation where a, b, c, d, e, f, g these are coefficients.And these are actually functions of x and y. Otherwise these values are constant.

(Refer Slide Time 09:33)



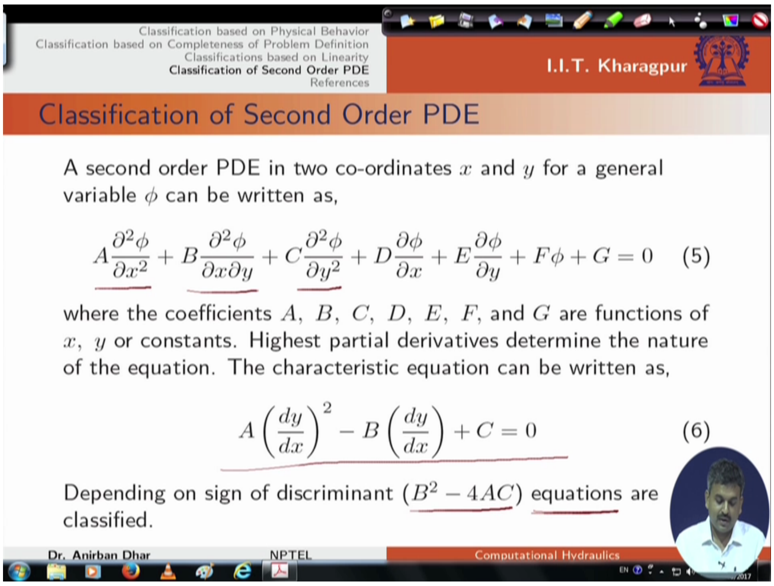
Highest partial derivatives determine the nature of the equation.So in this case highest partial derivatives are 1, 2, 3.

(Refer Slide Time 09:51)



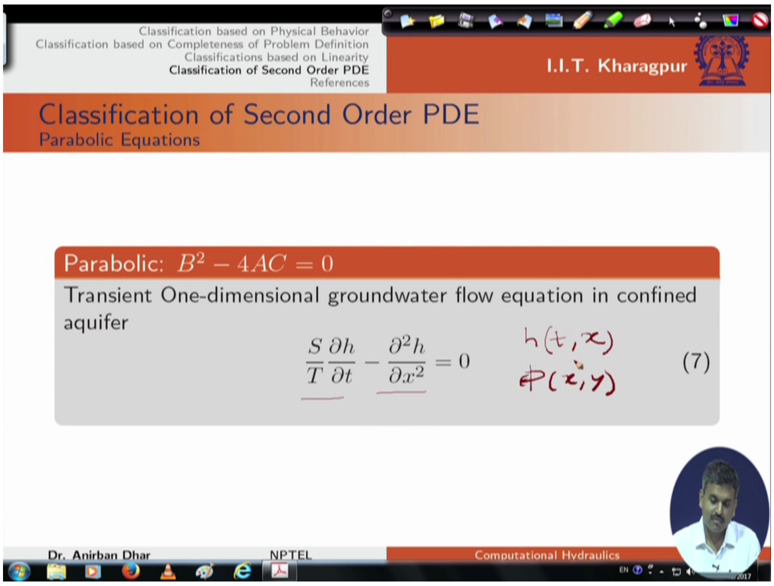
So from this equation we can get the characteristic equation which can be written in this form and depending on sign of discriminant that is B square minus 4AC equations can be classified.

(Refer Slide Time 10:18)



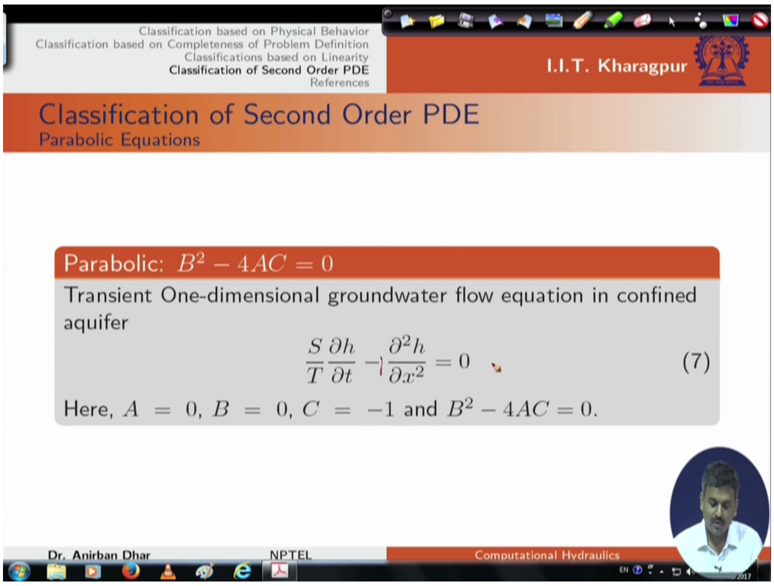
So classification of second order PDE first is parabolic, B square minus 4AC is equal to zero. So transient one dimensional groundwater flow equation in confined aquifer this is phi t del h by del t minus del 2 h by delx2. In this case h is the function of t and x only. So if we compare this situation with our linear or PDE equation that is x and y, so we can find outthe coefficient a, b, c and based on that we can comment on the nature of the partial differential equation.

(Refer Slide Time 11:27)



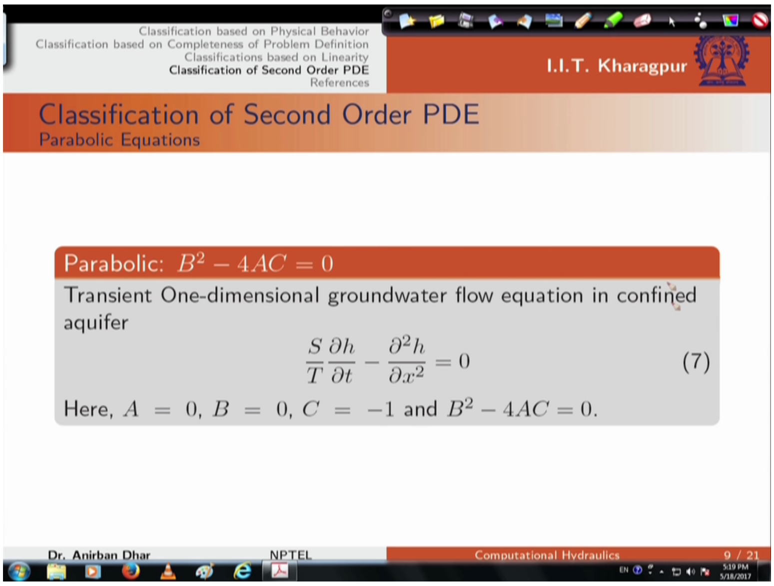
So in this case A is zero because the coefficient of del2h by delt2 is zero, B equal to zero because there is no cross term. C is minus one which is the coefficient of del 2h delx2. This is actually minus one.

(Refer Slide Time 11:57)



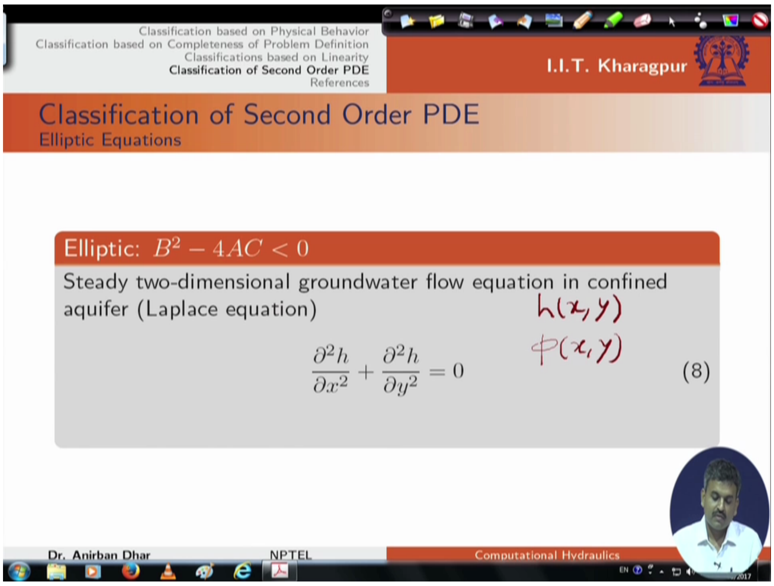
So obviously B square minus 4AC this term is zero. So we can say that this situation is parabolic in nature.

(Refer Slide Time 12:07)



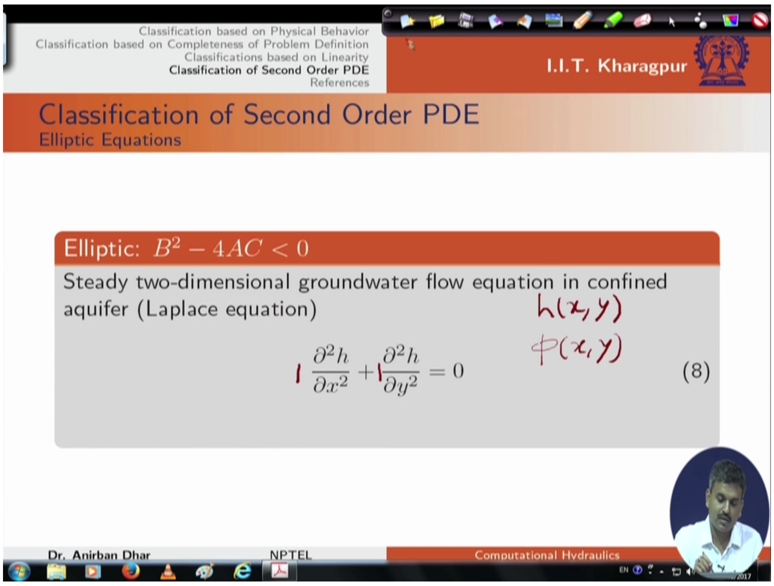
Then comes this elliptic equation where B square minus 4AC is less than zero. Steady two dimensional groundwater flow equation in confined aquifer. This is Laplace equation. Again if we compare this equation with our original phi xy, this is actually x, hxy.

(Refer Slide Time 12:41)



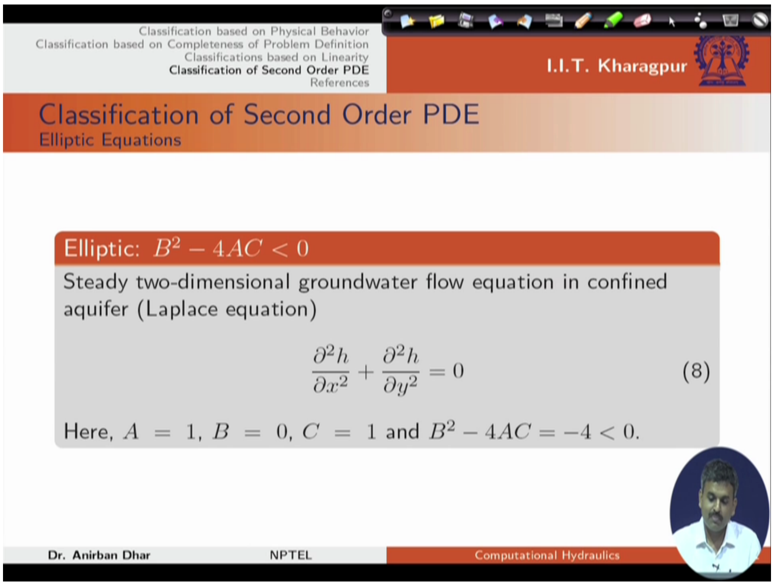
So, we have coefficient 1, 1 but for cross term the coefficient is zero.

(Refer Slide Time 12:47)



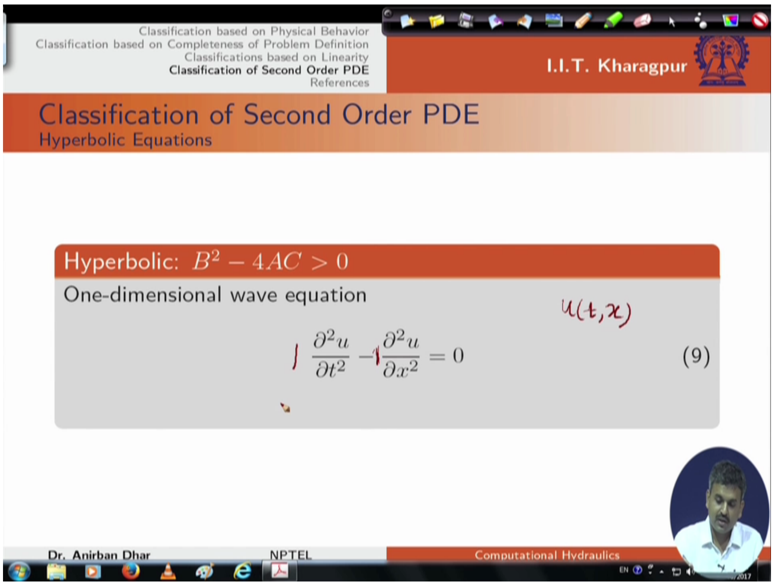
So A is one, B is zero, C is one. So B square minus 4AC is minus 4 which is less than zero. So we can see say that this equation is elliptic in nature.

(Refer Slide Time 13:04)



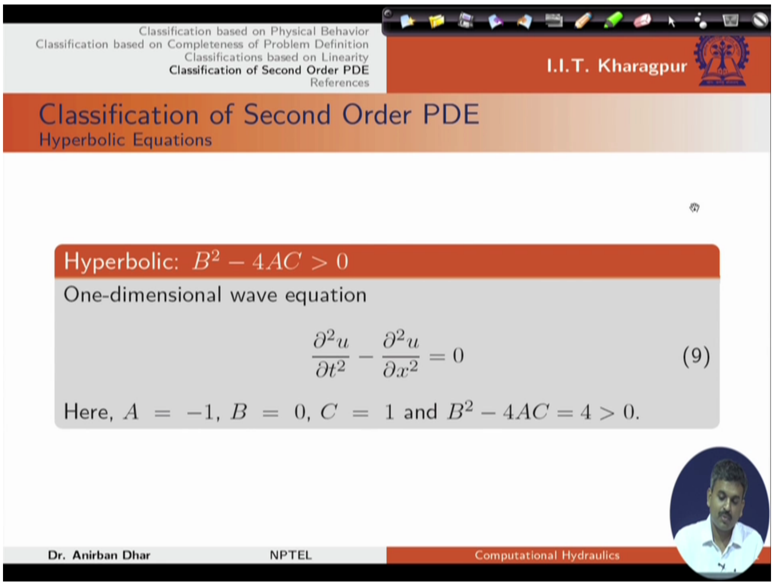
Then hyperbolic equation where B square minus 4AC is greater than zero. One dimensional wave equation. In this case we can see that the coefficient of u which is the function of p and x is one and in this case this is minus one there is no cross term available for this equation.

(Refer Slide Time 13:52)



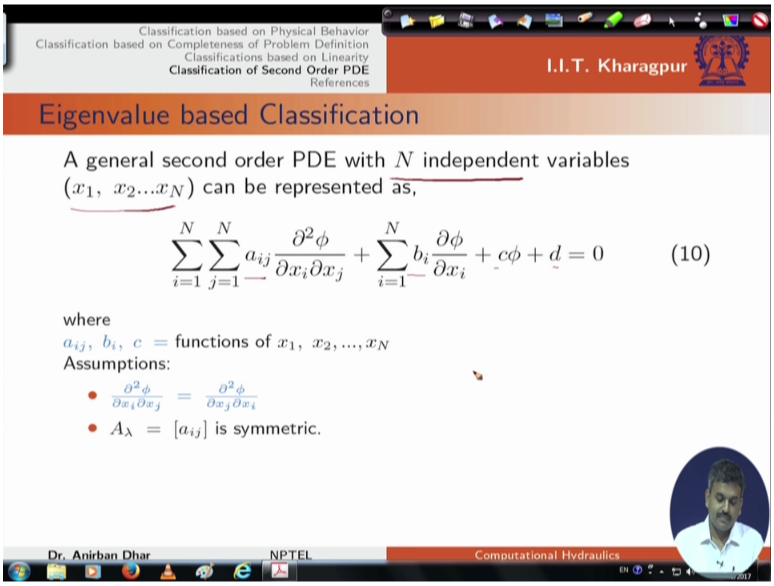
So obviously we have a equals to minus one which is coefficient of u del u by delx2, B is zero and C is one. So this is b square minus 4AC which is positive in nature. So we can say that this equation is hyperbolic equation.

(Refer Slide Time 14:20)



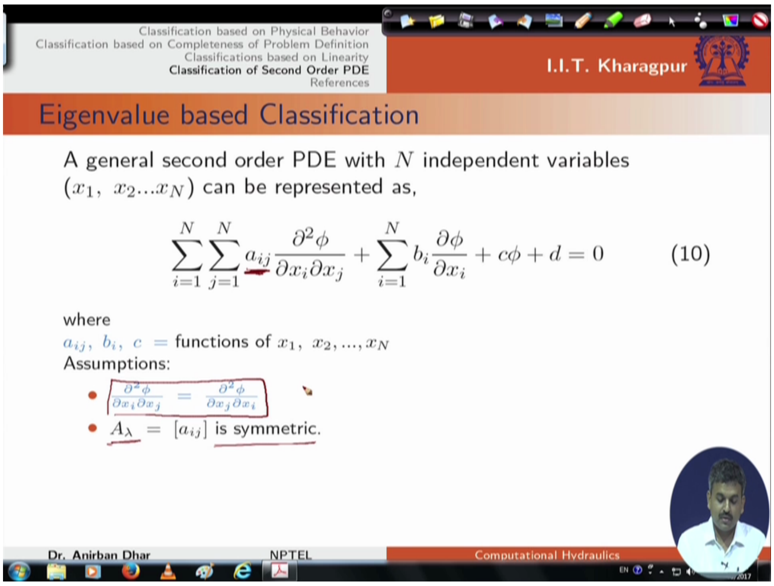
Now eigenvalue based classification. So, we can havemultiple independent variables for the differential equation. So let us consider a PDE where we have N independent variables. x1, x2 to xN and A ij and B ij, C, D, are coefficients. Again these are either functions of x1 to xN or these are constant values.

(Refer Slide Time 15:05)



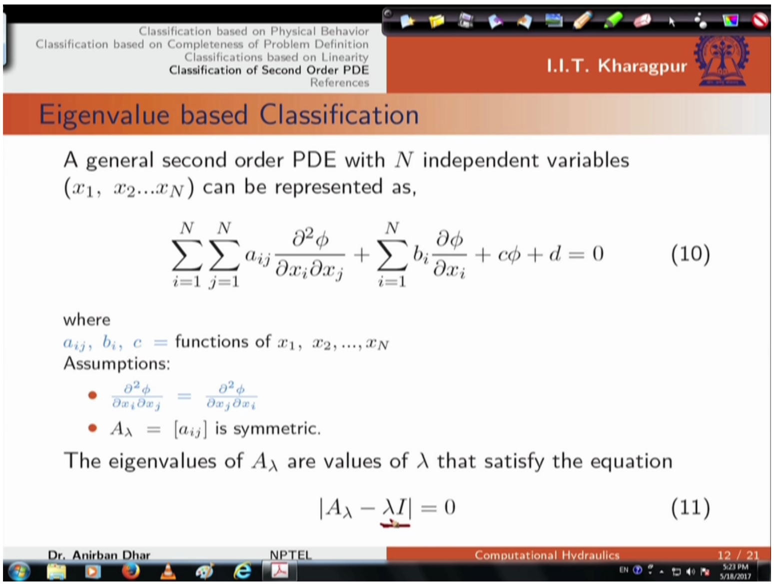
With the assumption that the cross term we have symmetry for this coefficient which is corresponding to the higher derivative terms we can construct a matrix A lambda and this is symmetric in nature. Symmetric because in this case we are considering that the coefficients of the cross term should be same.

(Refer Slide Time 15:48)



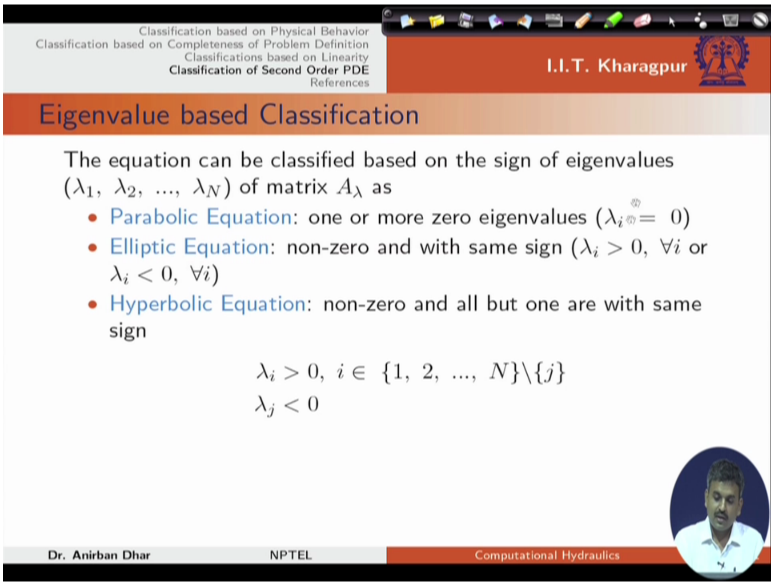
So with this assumption the eigenvalue of A lambda are the values of lambda that satisfy the equation a lambda minus lambda into I, which is identity matrix and determinant of that should be zero. If we can solve this equation we can find out the values or value of lambda depending on the order of equation.

(Refer Slide Time 16:15)



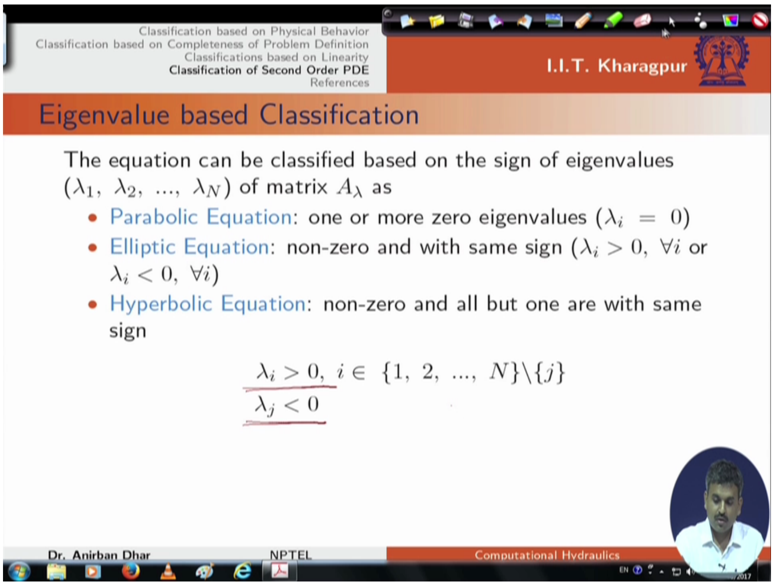
So equation can be classified based on sign of eigenvalue lambda 1 to lambda N of the matrix A lambda. For parabolic equations one or more zero eigenvalues. If we have elliptic equation non zero and with same sign. That means lambda I is greater than zero for all I or lambda I less than zero for all I. If we have hyperbolic equation then non zero and all but 1 are with same sign.

(Refer Slide Time 17:30)



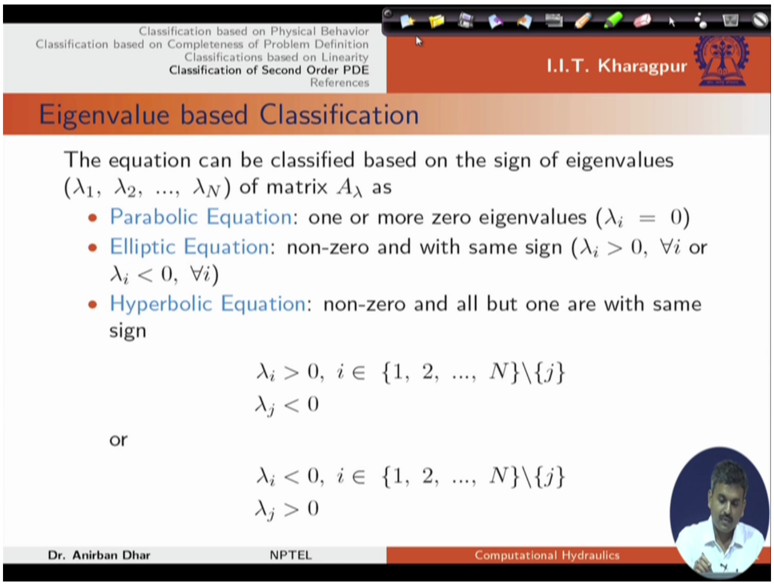
That means for a particular J eigenvalue this is listen zero, for others it will be greater than zero.

(Refer Slide Time 17:44)



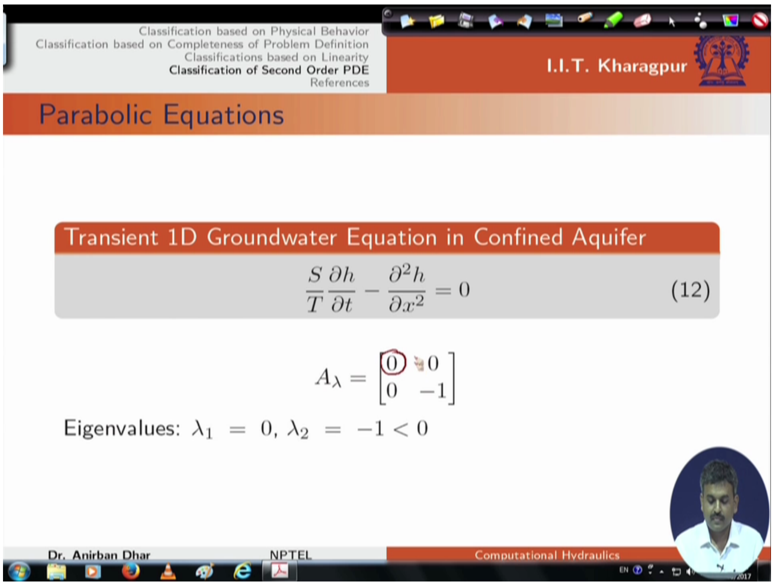
Or combination can be lambda I is less than zero and lambda J is greater than zero.

(Refer Slide Time 17:58)



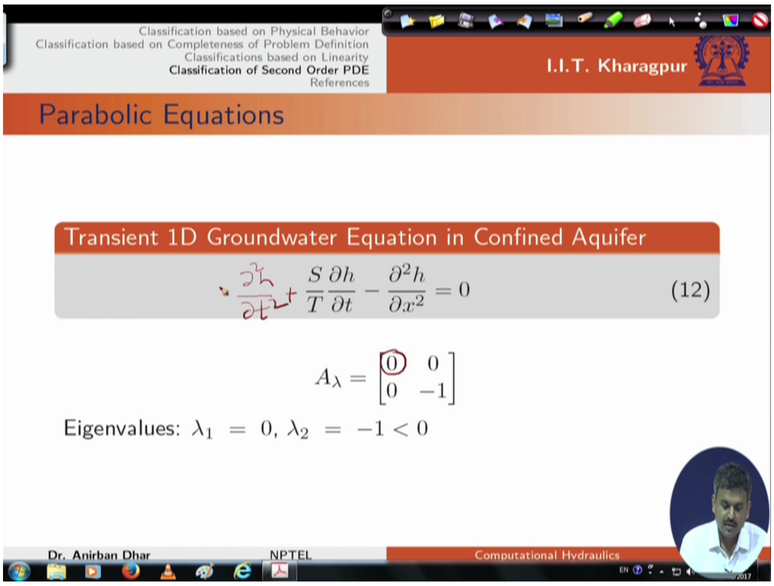
If we consider examples for parabolic equations we can take the same transient 1D groundwater equation in confined aquifer. In this case if we construct the coefficient matrix A lambda then we can see that this zero which is diagonal term A one one.

(Refer Slide Time 18:30)



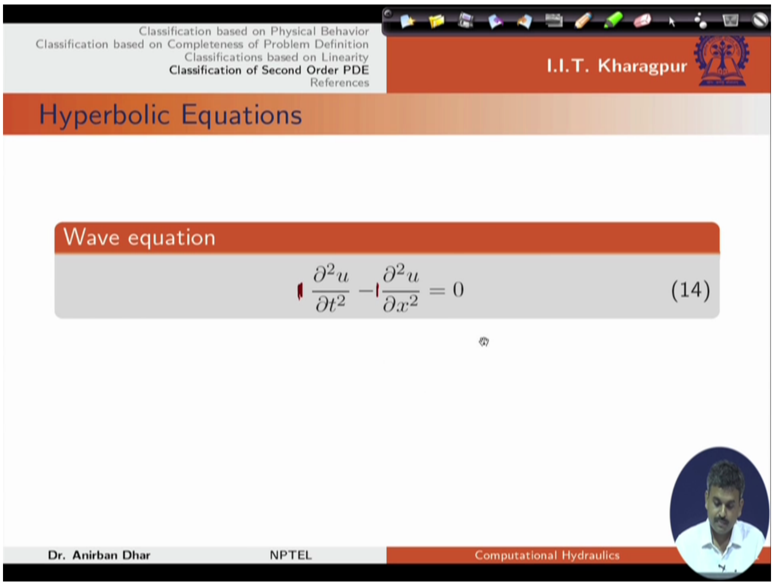
One one means there is no coefficient for del 2h by delt2. This is zero so obviously this is zero.

(Refer Slide Time 18:48)



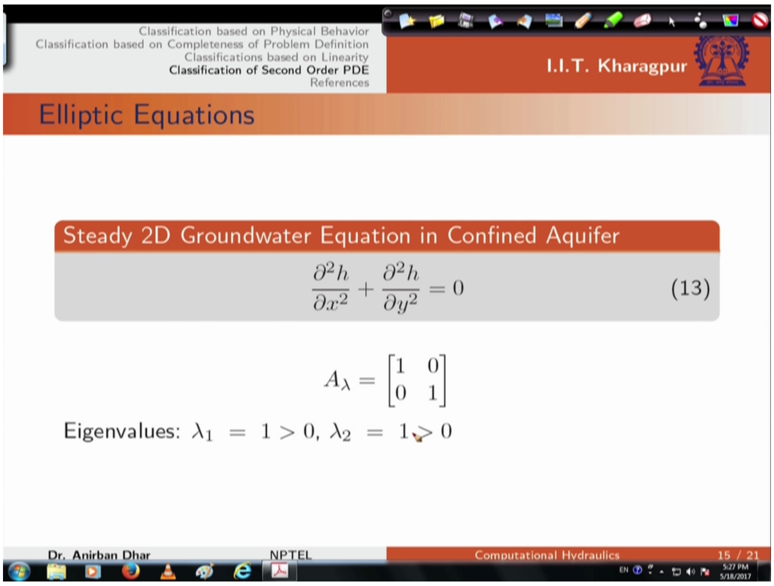
There is no cross term that's why zero values corresponding to cross term and minus one as coefficient of del 2h delx2. So obviously eigenvalues lambda 1 zero, Lambda 2 minus one which is less than zero. So one of the eigenvalue is zero. We can say that the equation is parabolic in nature.

(Refer Slide Time 19:25)



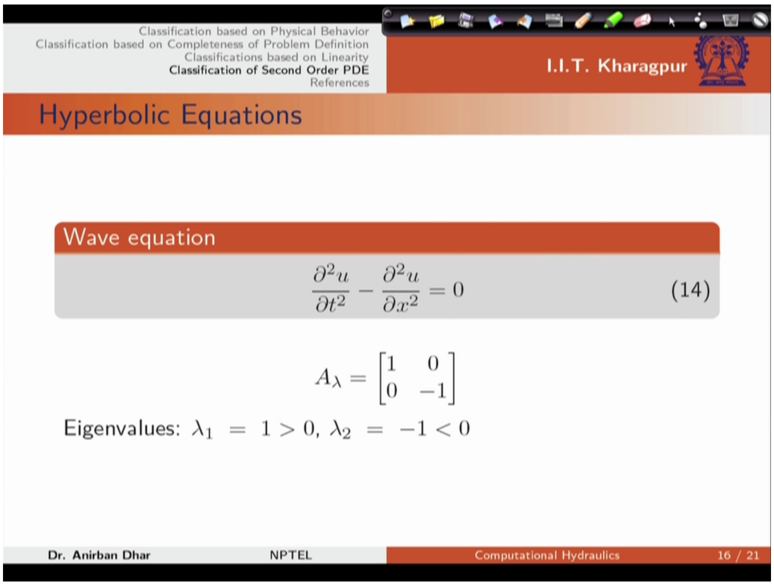
If we consider elliptic equation again with the Laplace equation the coefficient is one. In this case also this is one. So if we consider the coefficient matrix this is one one and cross terms are zero zero. So if we find out the eigenvalue lambda 1 is one lambda 2 is one both are greater than zero that means they have same sign and both are nonzero values.So we can call this as elliptic equation.

(Refer Slide Time 20:21)



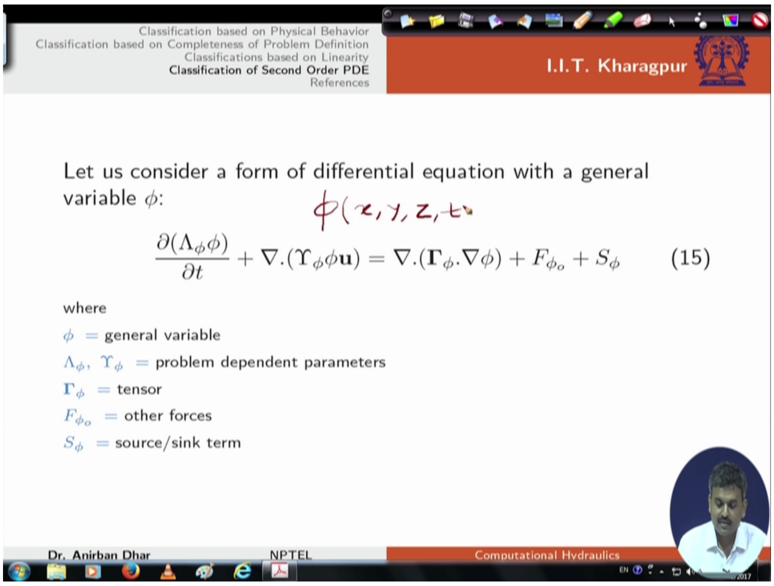
In case of hyperbolic equation that is wave equation we can see that the coefficient is one, here it is minus one and with this information if you proceed we can construct the coefficient matrix this is one minus one and cross terms are zero. So lambda 1 is one which is greater than zero, lambda 2 is minus one which is less than zero. That means the opposite sign is there and both are nonzero values. So obviously we can say that the equation is hyperbolic in nature.

(Refer Slide Time 21:18)



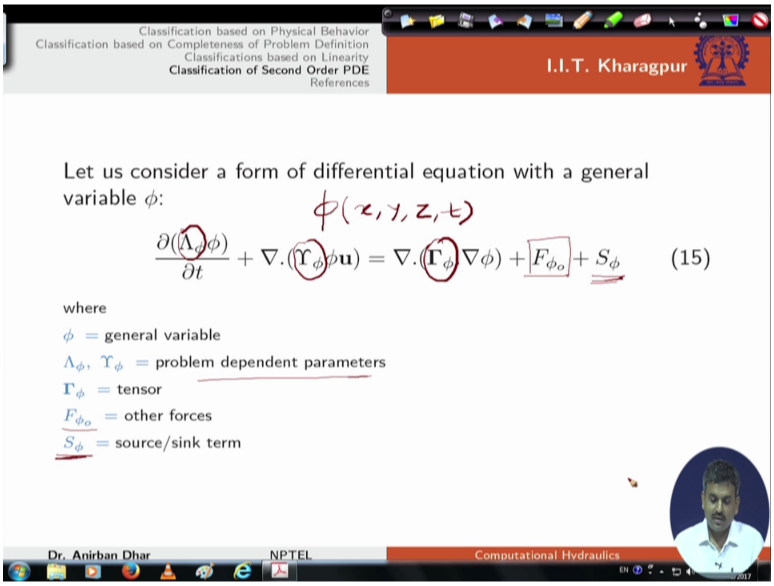
So let us consider form of differential equation with general variable phi. In this case phi in general form it is a function of x, y, z and t.

(Refer Slide Time 21:41)



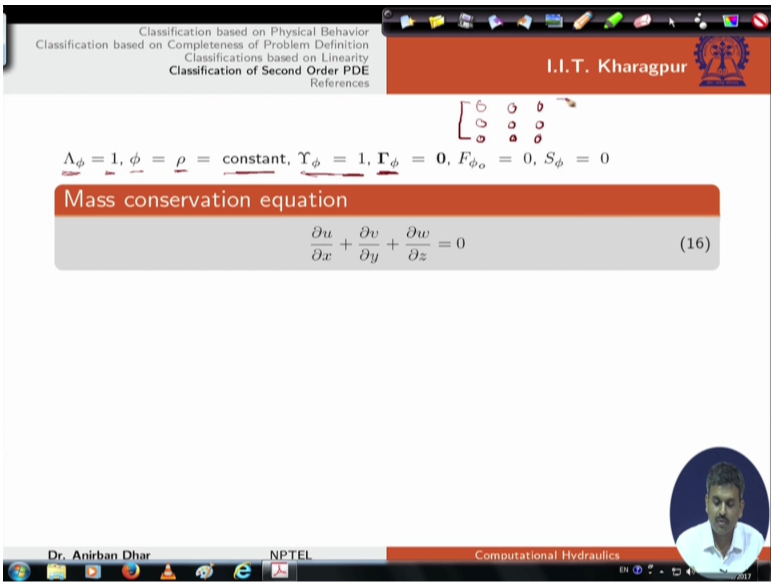
Gamma phi and upsilon phi these are problem dependent parameters and this lambda phi and upsilon phithese are problem dependent parameters. And gamma phi this is a tensor and this is equivalent to diffusion tensor and F phi zero or F phi others this term considers other forces and S phi is a source sink term. So we can use this general form of equation to represent all of our hydraulic system related problems.

(Refer Slide Time 22:35)



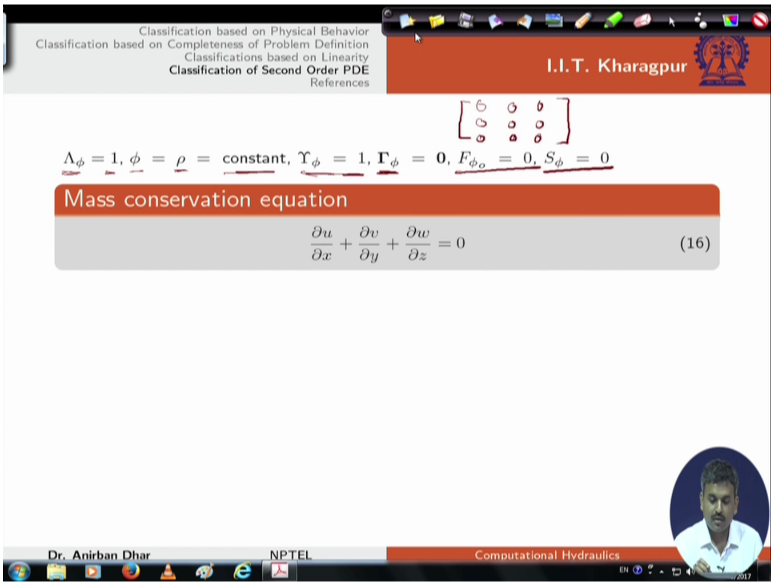
So example is mass conversation equation that is delu, delx, delv, dely, delw, delz. In this case we can see that lambda phi is 1 and phi is row and which is constant. So obviously the temporal term in the left hand side is not present. Upsilon again that is 1 and this dispersion coefficient tensor this is again having zero zero values that means all terms are zero in this case.

(Refer Slide Time 23:50)



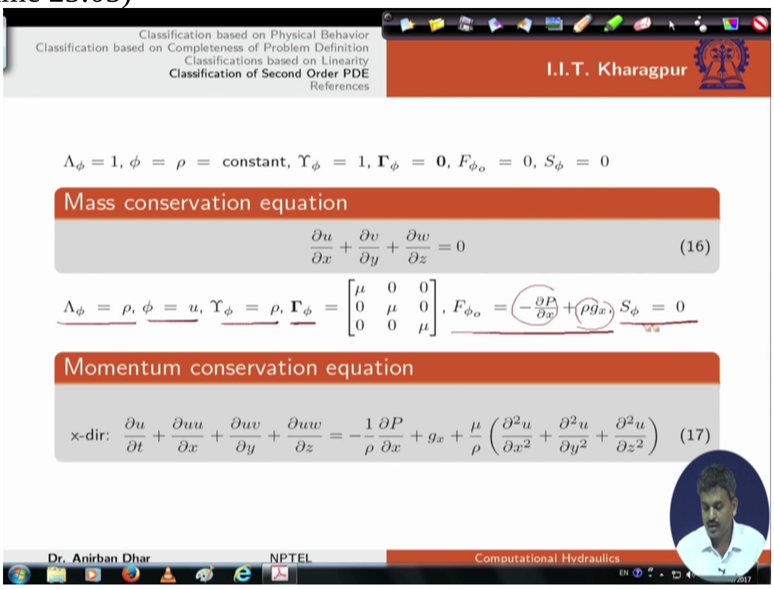
So there is no diffuse in term only advection related term in left hand side. So other force this is zero, S phi is zero. So if we use this parameters forces and source sink terms then we can get this mass conservation equation from our general equations.

(Refer Slide Time 24:20)



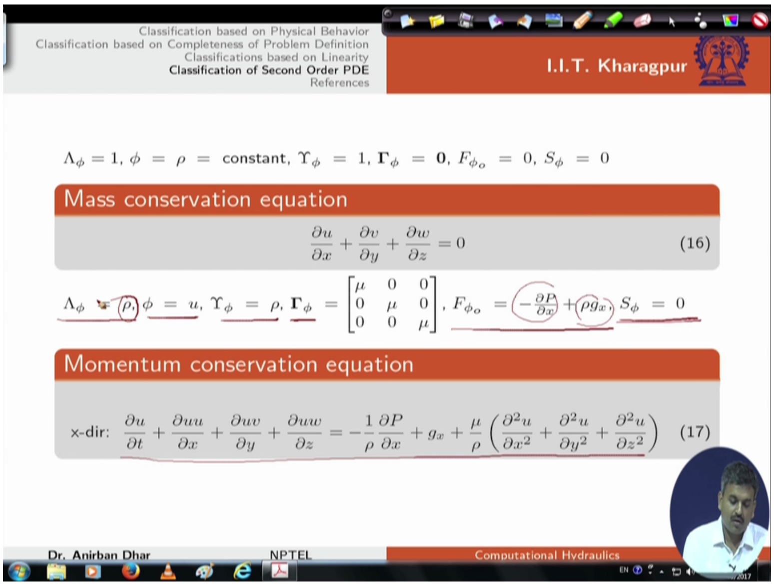
Again if we use this lambda equals to row,phi equals to u, which is velocity in x direction, upsilon equals to row and this (disper)division coefficient tensor as a with dynamic viscosity and body force including pressure and this gravity force S pie is zero.

(Refer Slide Time 25:05)



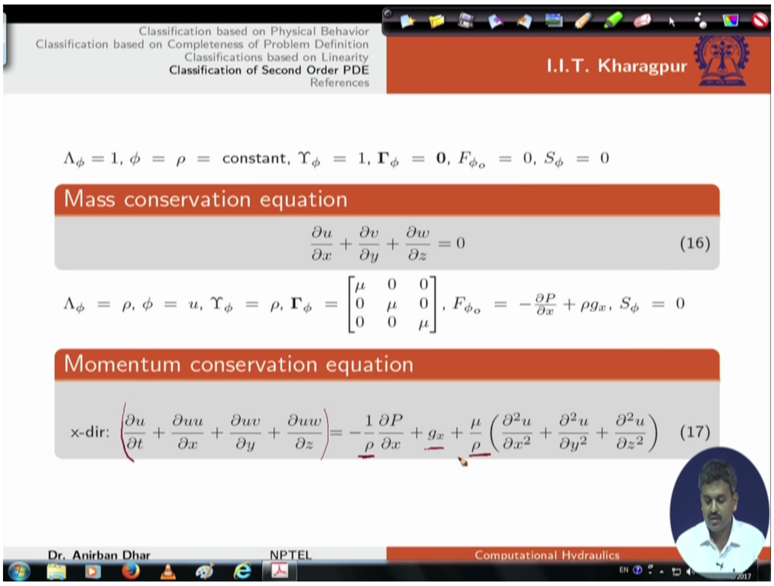
So for this moment conversation equation we can get this form for incomprehensible fluid flow because we have considered constant value of row upsilon.

(Refer Slide Time 25:23)



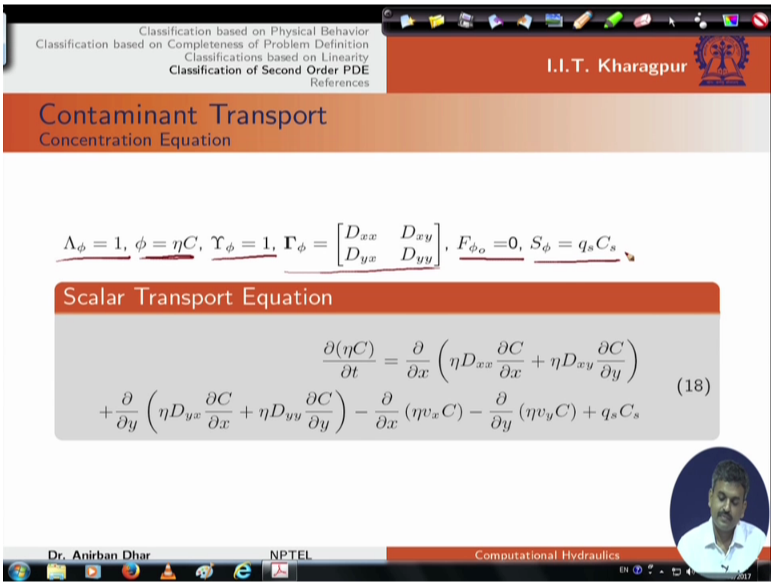
So obviously the row will be in multiplied form on this side and this side and that row can be transferred here so that uh we can get this incompressible fluid flow x direction momentum conversation equation.

(Refer Slide Time 25:56)



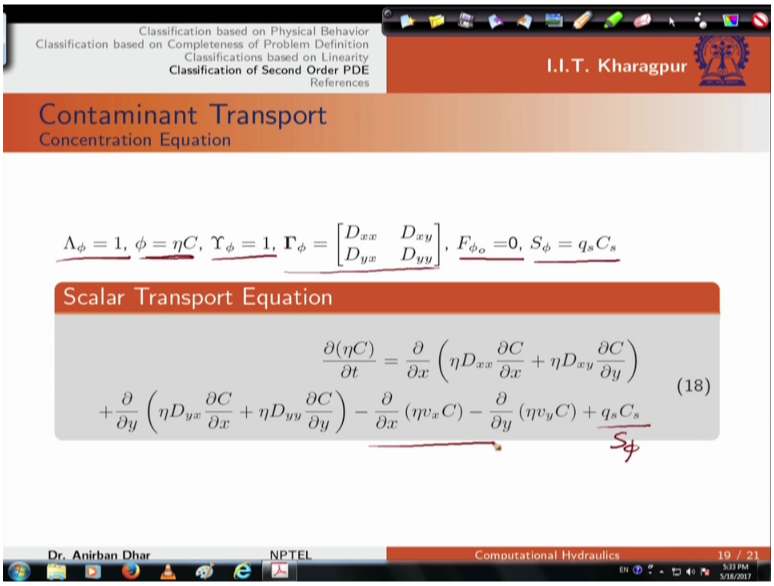
So with this information we can proceed and further for concentration which is scalar transport equation. Again lambda phi is one, phi is eta C, eta is porosity, C is concentration, upsilon phi is one. Again for two dimensional case this is deviation coefficient tensor other forces are zero and source sink term that is qs, cs.

(Refer Slide Time 26:39)



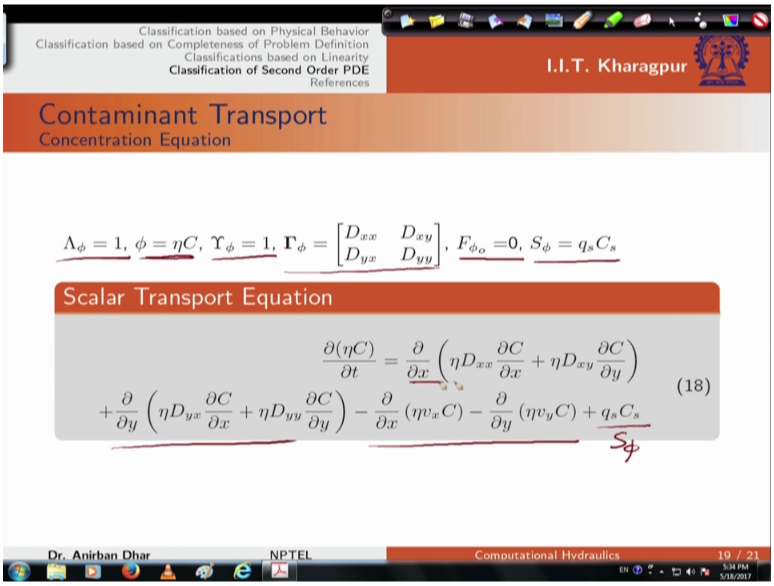
So, this is actually source sink term and this is advection term.

(Refer Slide Time 26:50)



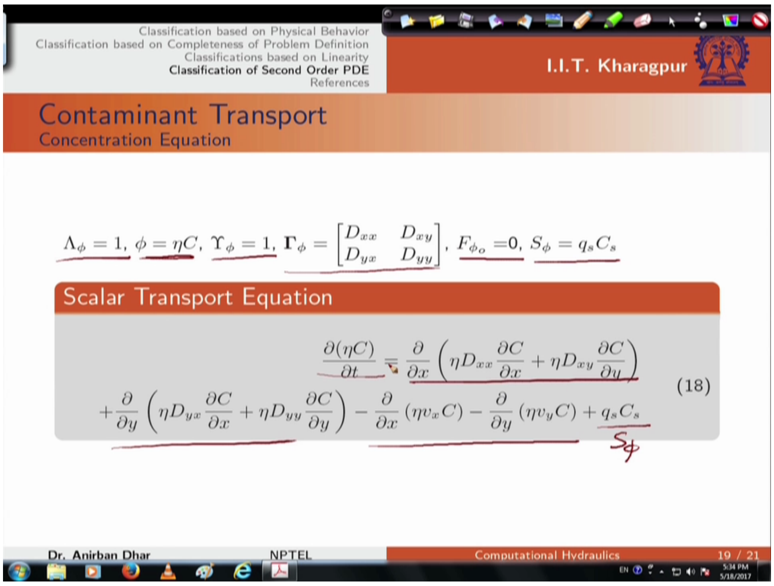
These two are related to diffusion term.

(Refer Slide Time 26:55)



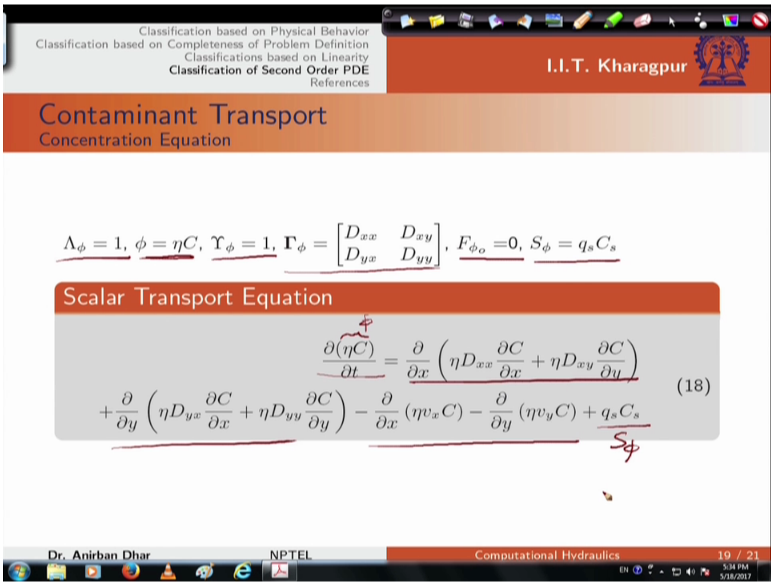
And this one is temporal term.

(Refer Slide Time 27:00)



So, obviously we can get the (con) concentration equation from our generalized equations.

(Refer Slide Time 27:15)



So we can utilize this generalized equation and we can get the discretization for that equation so that we can use that general discretization for all kinds of problems. So, this is the end of module number one. Next module we will talk about numerical methods. Thank you.