Computational Hydraulics
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Lecture 38
Gradually Varied Flow: Implicit Approach

Welcome to this lecture class of the course computational hydraulics. Where are in model 4, surface water hydraulics. And this is unit number 2, gradually varied flow, implicit approach.

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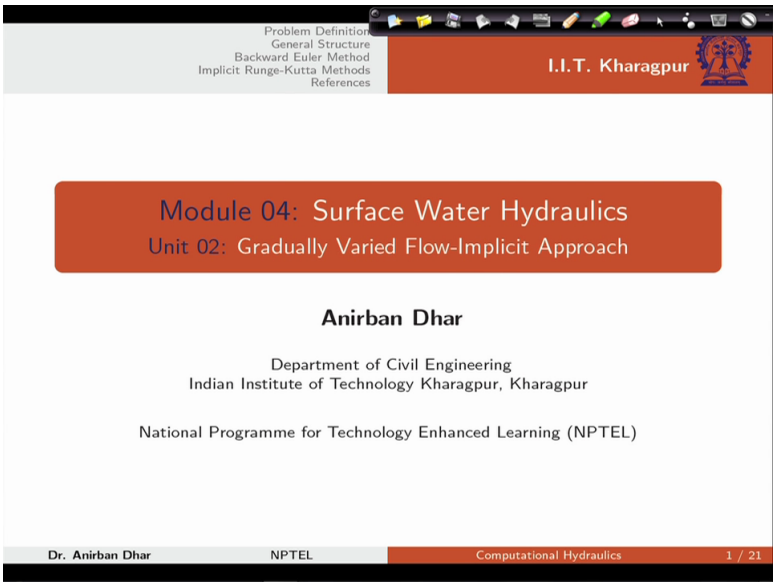

In our previous lecture class we have discussed about gradually varied flow using explicit approach. Now learning objective of this particular lecture class. At the end of this lecture class students will be able to solve gradually varied flow problem for open channels using implicit method.

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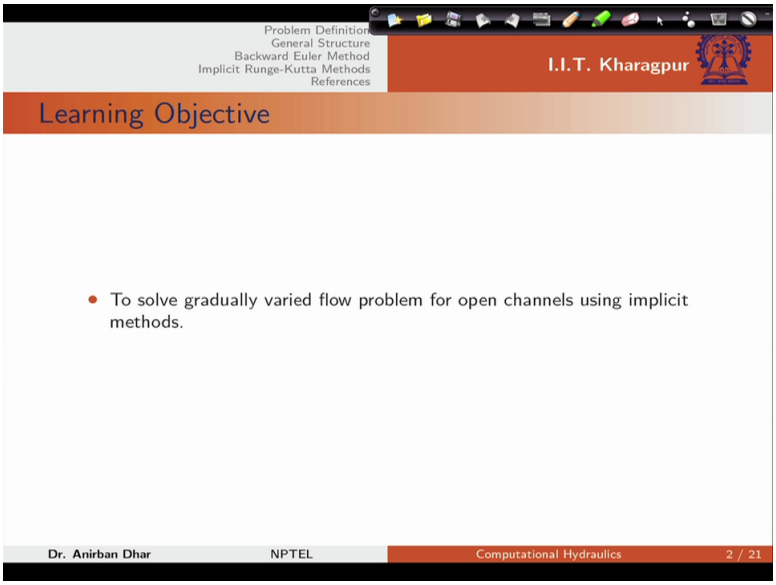

Now this gradually varied flow we are considering for second channels. Problem definition to solution, this is our general framework. In our previous lecture class we have got solution which is y as a function of x from the governing equation dy by dx equals to S not minus SF 1 minus Fr square. And this is initial value problem.

So from our definition we do not need boundary condition in this case or initial condition, one sided condition is there. So this is our first order ODE differential equation.

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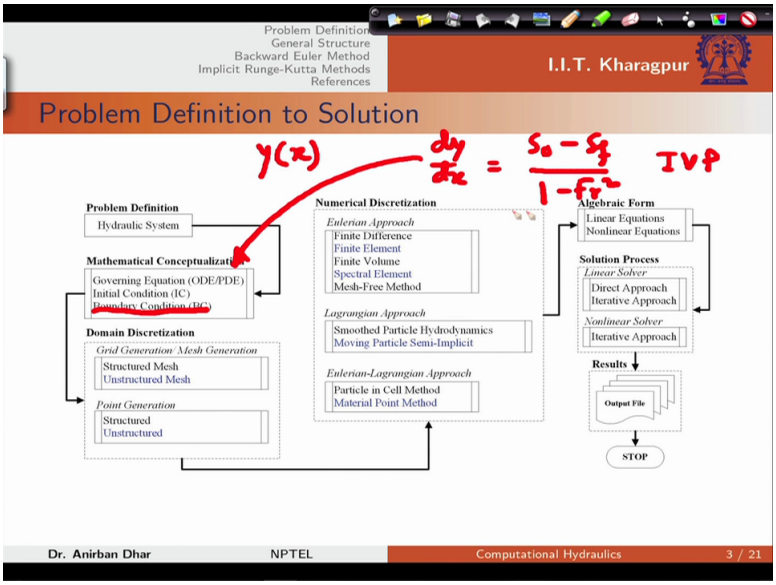

Now to solve this with the structure of explicit method we can directly transfer all the known quantities on the right hand side and only unknown quantities on the left hand side. So directly we can get the solution using explicit approach. However if we utilise the implicit approach for the solution of GVF or gradually varied flow, we need to solve it either using this linear equations or nonlinear equations as per our requirement.

But in this case the term SF which is a function of y and Fr which is again function of y, these are non linear in nature. Obviously we will be getting non linear equations. So we need the solution approach for those governing equations or discretized governing equations.

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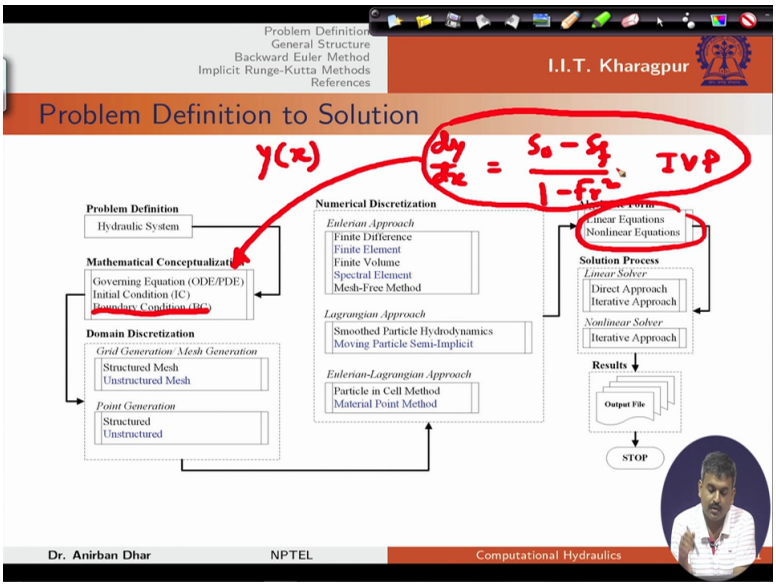

Now let us consider our basic problem statement. So problem definition is governing equation for gradually varied flow in prismatic channels can be written as dy by dx equals to psi xy. With psi xy equals to S not minus SF 1 minus Fr square. And in this case n is Mannings roughness, Q is discharge. Obviously for a channel maybe it is a specified discharge.

R to the power 4 by 3, R is hydraulic radius, A is area, T is top width, Q is again discharge, g is in this case this is acceleration due to gravity, A is area and we have this initial condition on one side. So these are the expressions that we can directly utilise for our problem.

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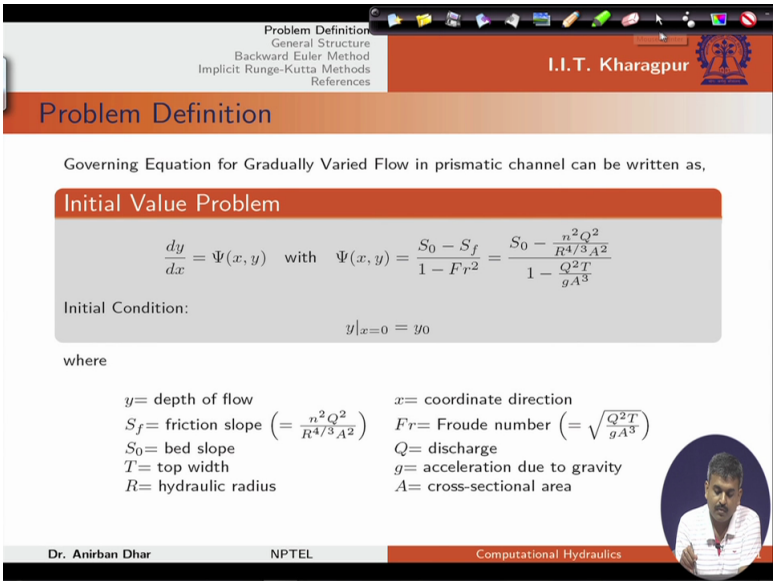

Now what is given and what is required for our problem solution? Given, let us say that we have a given channel cross section type. Channel cross section type is rectangular. Then y not which is the initial value at x equals to zero, this is point 8 metres. Then B which is the bottom width of rectangular channel, this is 15 metres. G is 9 point 81 metres per second square. Then as not is 0008. N is point not 15. Lx, length of the domain or length of the problem which we need to consider.

Lx is 200 metres and Q is 20 metres cube per second. Now in this case required is that we need to identify the type of GVF profile from the problem and plot of the GVF profile.

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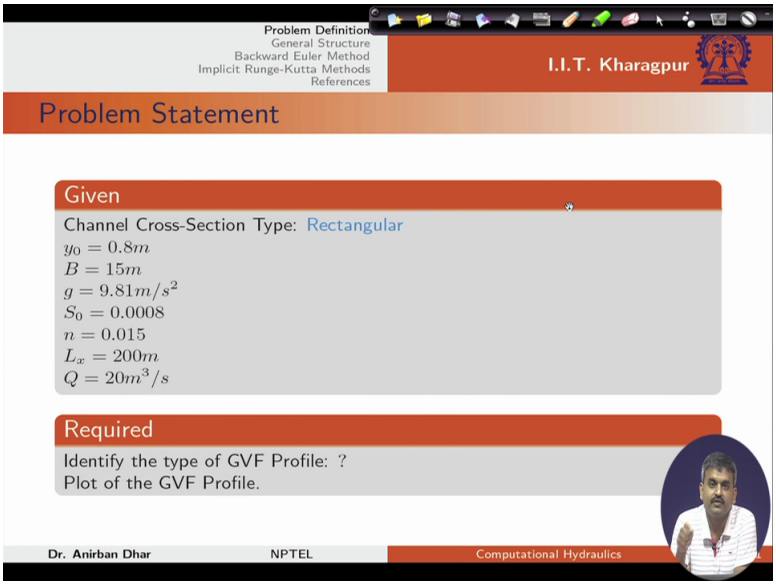

So this is for rectangular cross section. In our previous lecture class we have seen that we can have two different slopes for a trapezoidal cross section. So this is general slope feature. If this m equals to zero m1 and m2 then we can get rectangular cross section in that case.

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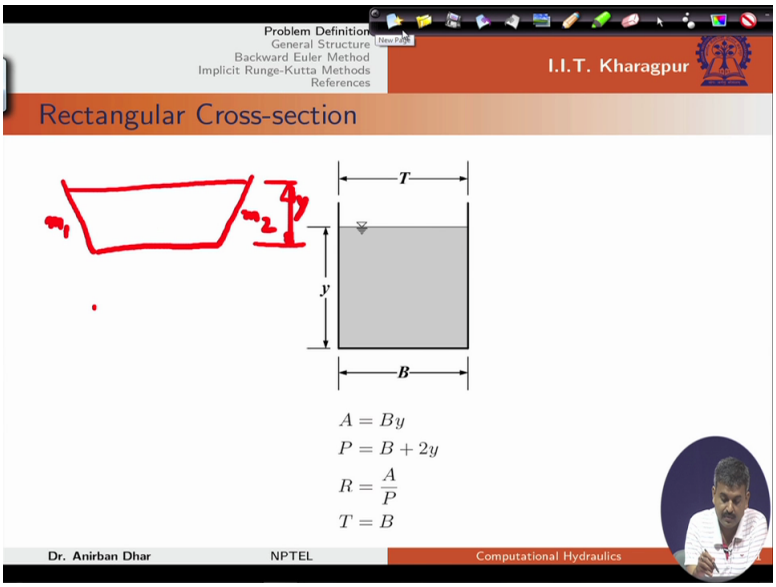

So in this case we have top width equals to bottom width or channel width and y. So this area is B into y, P is B plus 2y considering depth on both sides. So this is the weighted perimeter for the channel.

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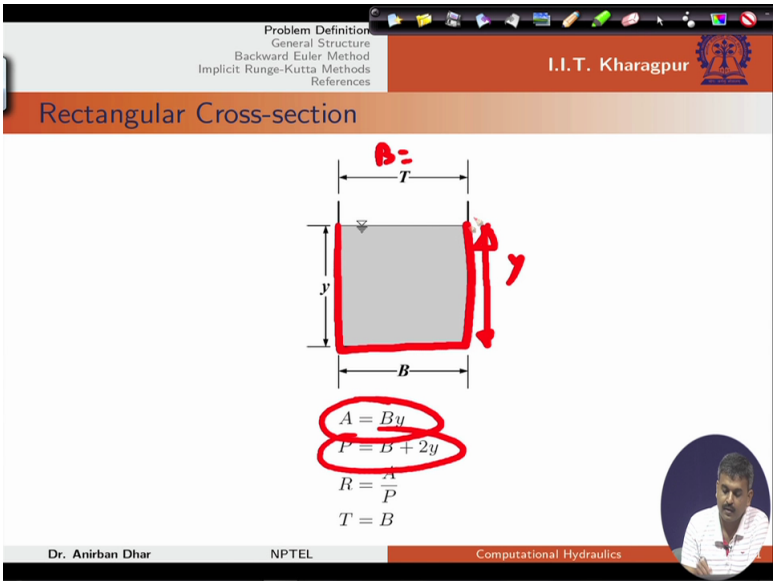

Now R which is hydraulic radius is A by P. This is By by B plus 2y. Now in this case R is again function of y. For given B your flow depth can vary. So in this case R is function of y.

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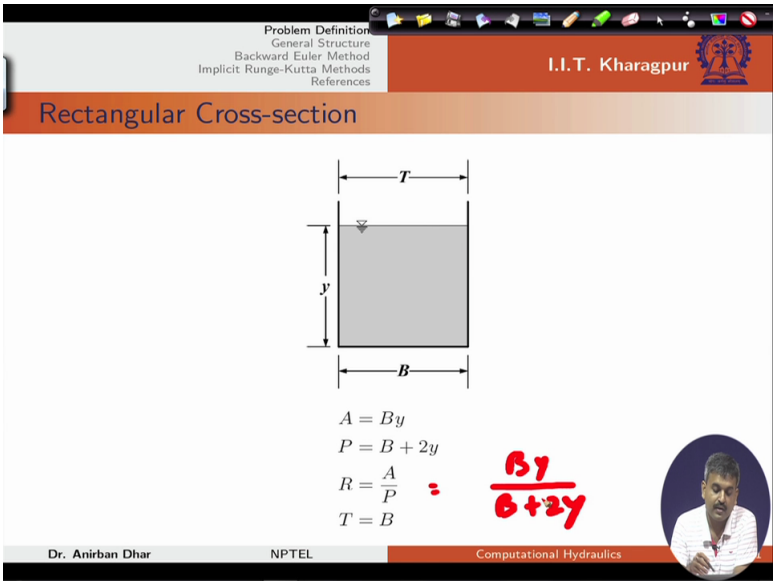

So in this case we can find out the critical depth to define the GVF or gradually varied flow profiles. We need the level for NDL which is normal depth line and critical depth line. Critical depth line is corresponding to this Fr equals to 1 and normal depth line we can find out from Mannings equation.

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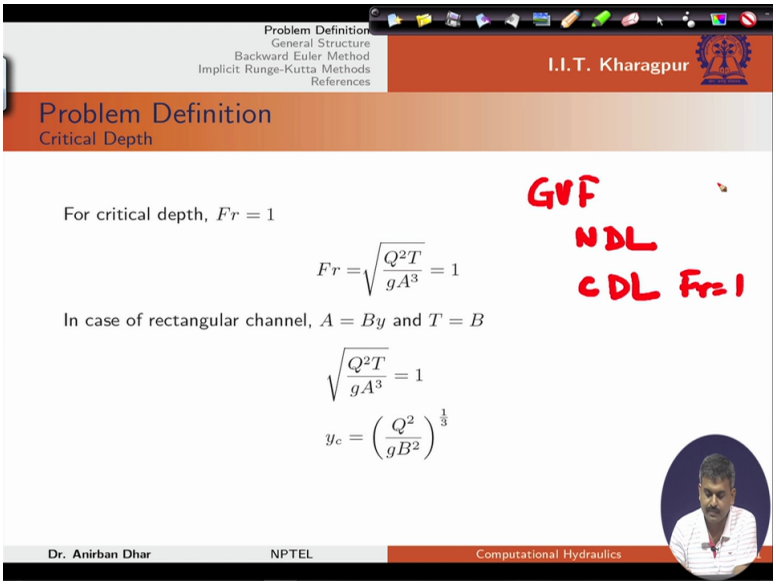

So for the first case if Fr equals to 1 then Q square T by gA cube this is 1 and from there we can get this QC which is Q square gB cube to the power 1 3rd.

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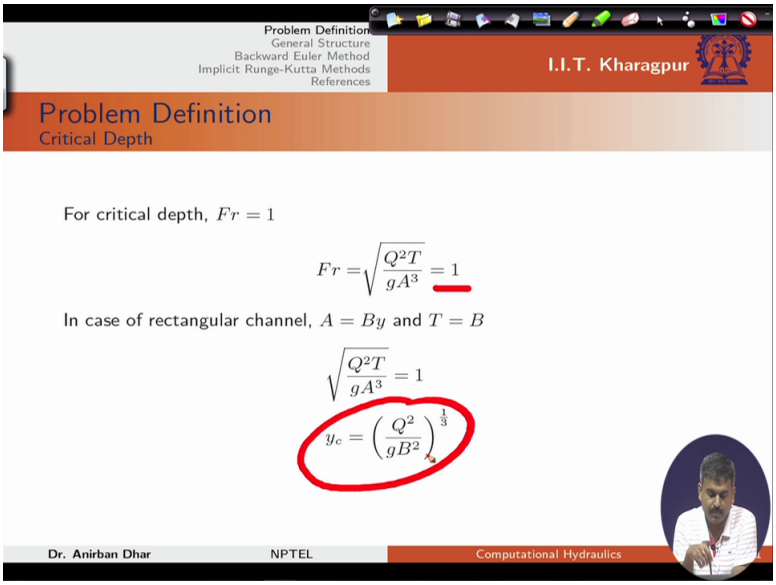

Now for rectangular channels if we need this normal depth we can use Mannings equation. In this case please remember that we are using S not. And rectangular channel case if we represent yn as normal depth so this is area and this is perimeter.

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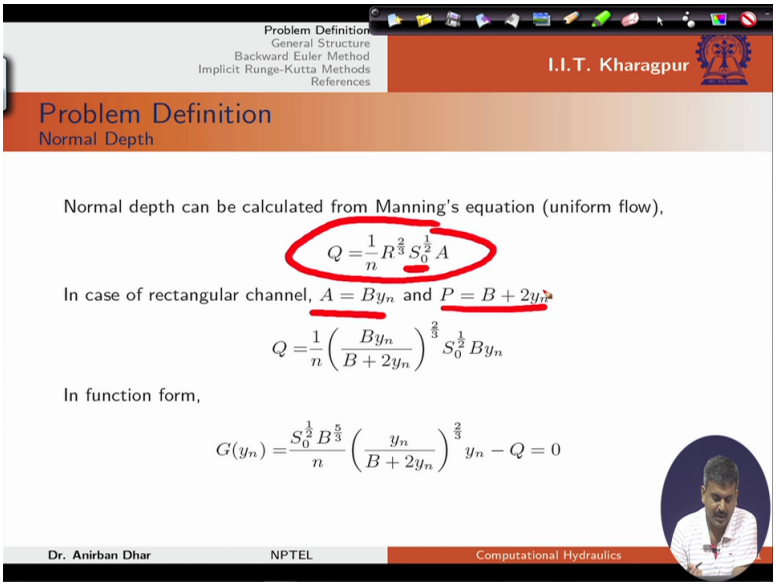

Now for a given discharge slope and n value we can find out unit value of yn. So that is the normal depth in this case.

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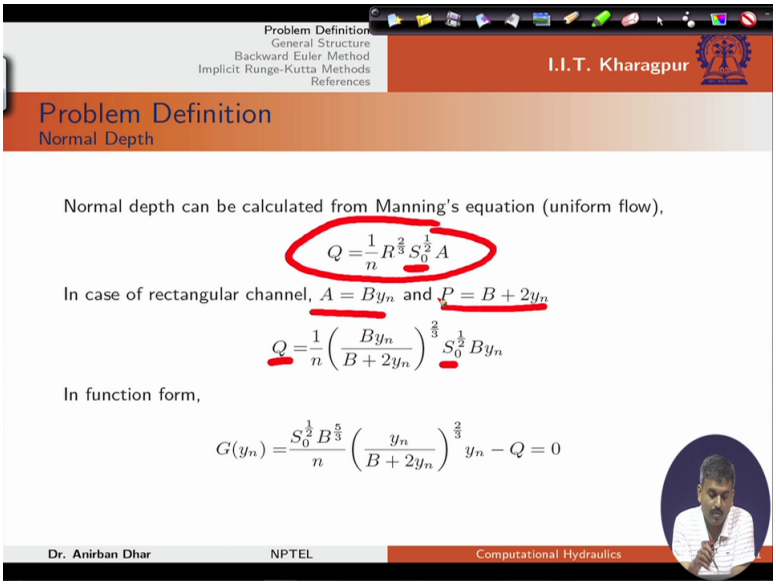

So we can write this as general function G yn which is nothing but this quantity minus the quantity on left hand side and we can equate it with zero.

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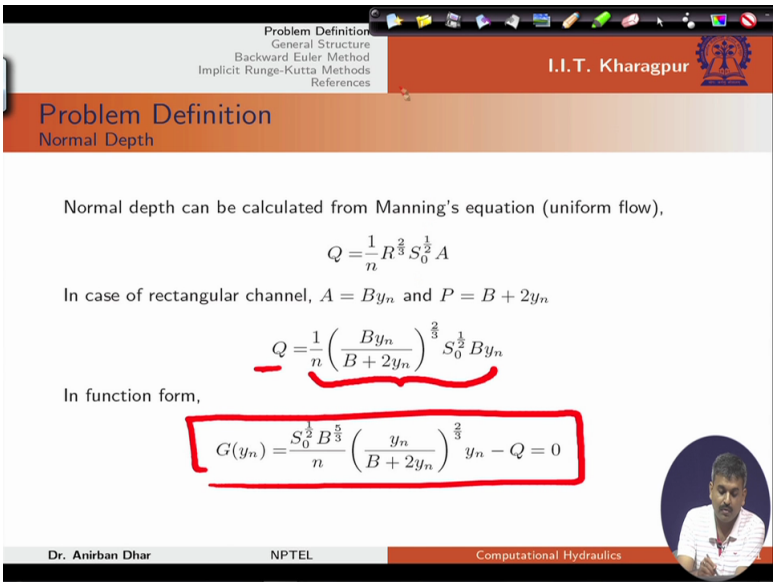

Now this is one non linear equation. Now if you want to solve this problem we need to use our Newton Raphson technique and from our Newton Raphson technique we can get the solution by iterative approach. This yn P, P is iteration count in this case. So yn P minus 1, this is coming from previous iteration. So we can start with yn zero and maybe we can start from yc value because we have already got the yc.

If we have this yc value or CDL location then either this NDL on the upper side or the lower side. So it is safe if we start from this yc which is the guess value for the normal depth iteration process.

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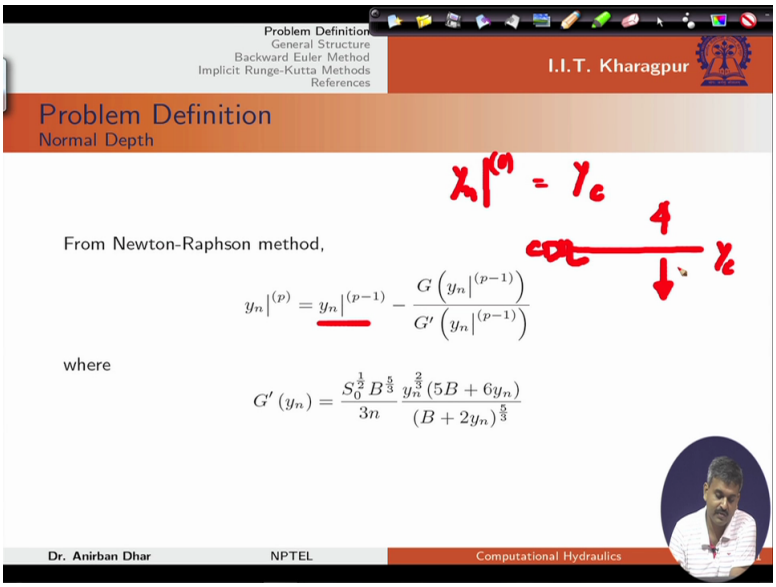

Now in this case we need to find out what is this G prime? G prime is nothing but this is dG divided by dYn. Now this quantity we need on this denominator and this should be calculated based on the previous iteration value and this is our original function value which is capital G function. Now from this iteration process we can get this y.

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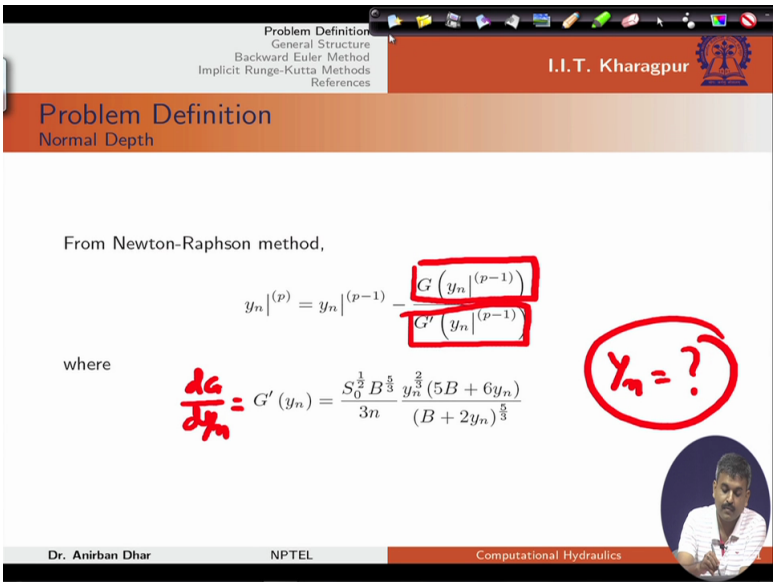

Now if we see this implicit Runge Kutta methods, in our previous (lec) lecture class we have seen that this counter which is i in case of explicit method this was up to i minus 1. So this was up to i minus 1.

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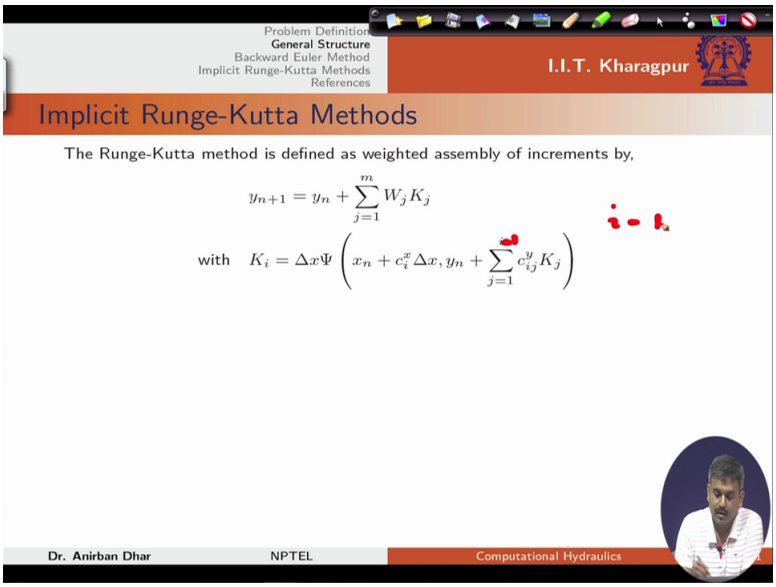

Now in this case we are going up to i. That means whatever value of this increment Ki is there on the left hand side, we are also incorporating that Ki value during calculation of psi. So in a way this increment equation is implicit in nature and we have different weights Wj, Kj. J is running from 1 to m, m is the number of weights or number of points or number of increments that are required for calculation of this weight.

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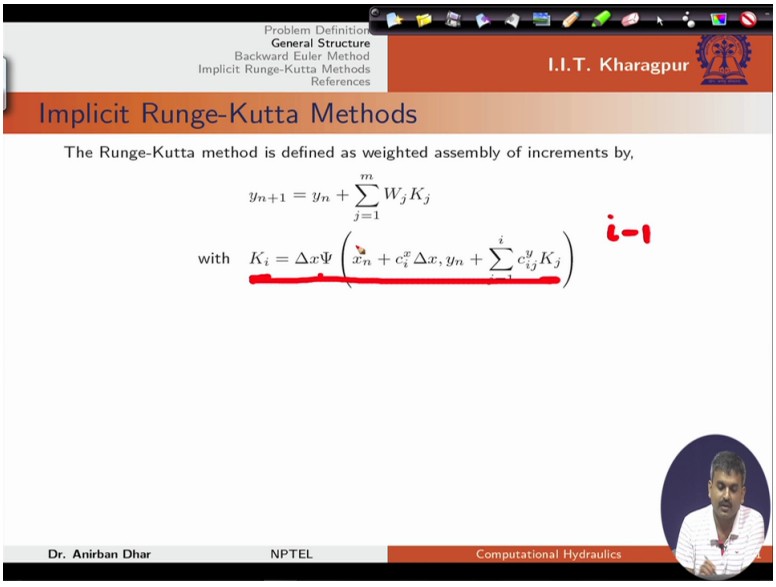

Now in this case we have Ci x or increment for x direction and increment for our y. So Cx this i is increment corresponding to ith increment parameter. And this C ij y, this is general increment parameter for ym.

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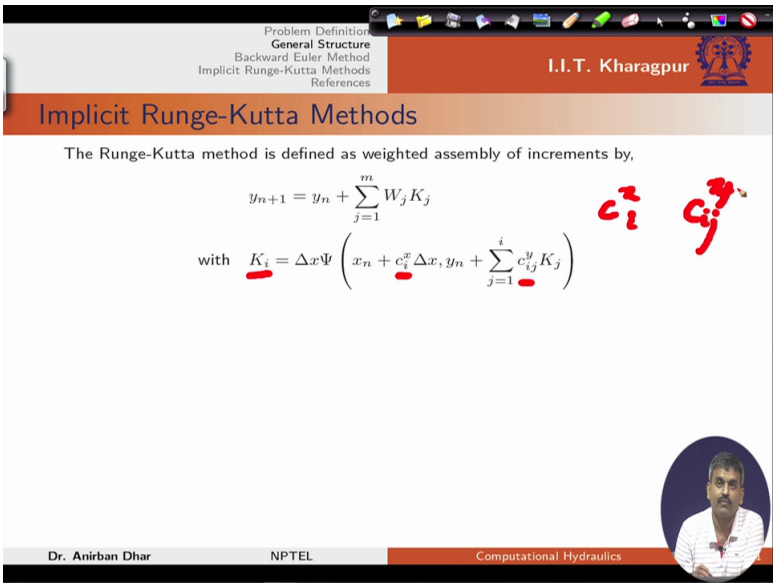

Now in this case if we consider this Butcher Tableau, this can be expressed in general terms with this format. So we have let us say m terms or m increments. Now for each increment we need increment in case of x. So this is corresponding to increment for x or xn and on the right hand side, this is for x. On this side we have increment for y and the bottom portion we have different weights. So if we consider the portion which is for y that is one matrix and size is m by m matrix.

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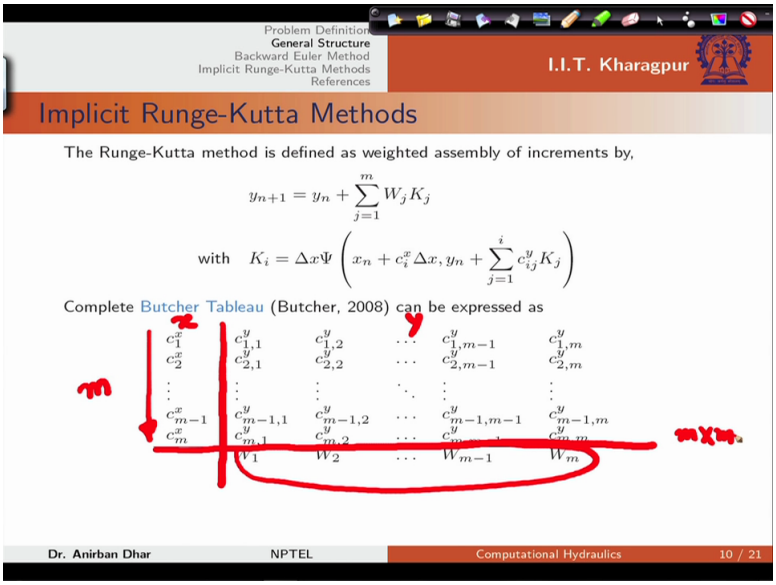

So each row is representing the coefficient of the corresponding increments that are required for the calculation of flow depth y. And the first term in this row, this is corresponding to increment for x. So this is corresponding to x.

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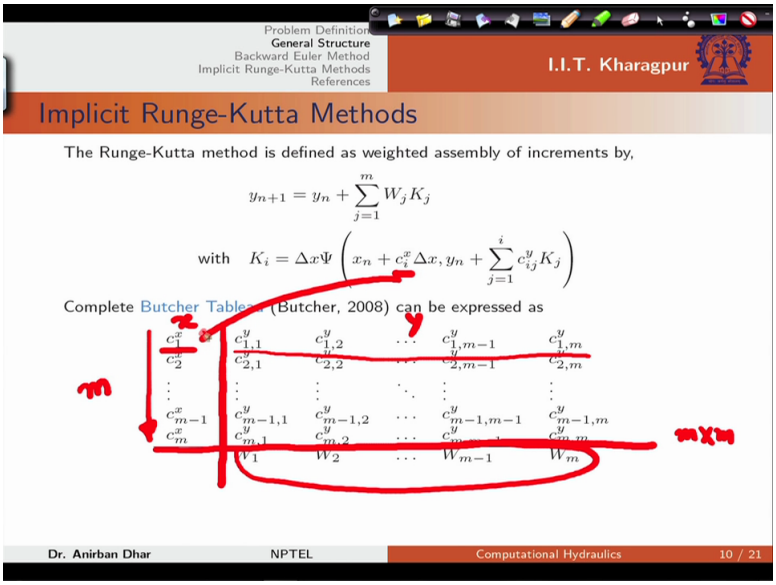

And these values are corresponding to different coefficients that are required for the calculation for coefficient here.

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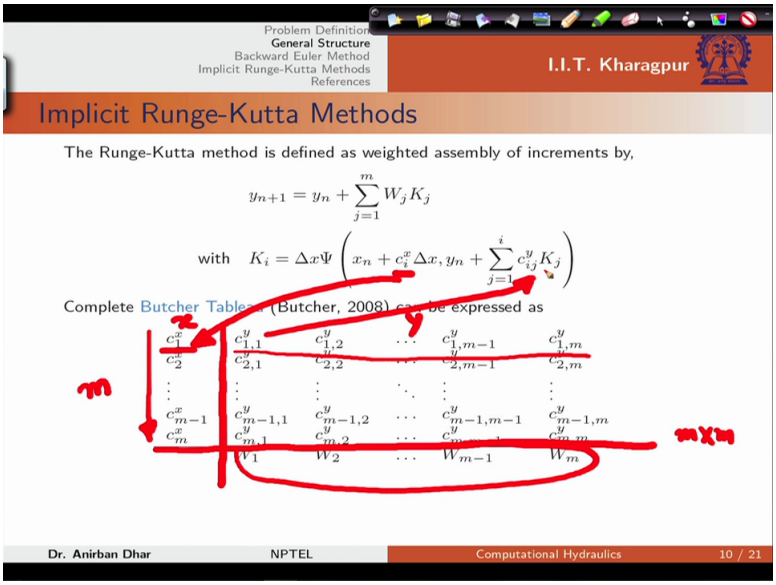

And these weights are related to the final calculation.

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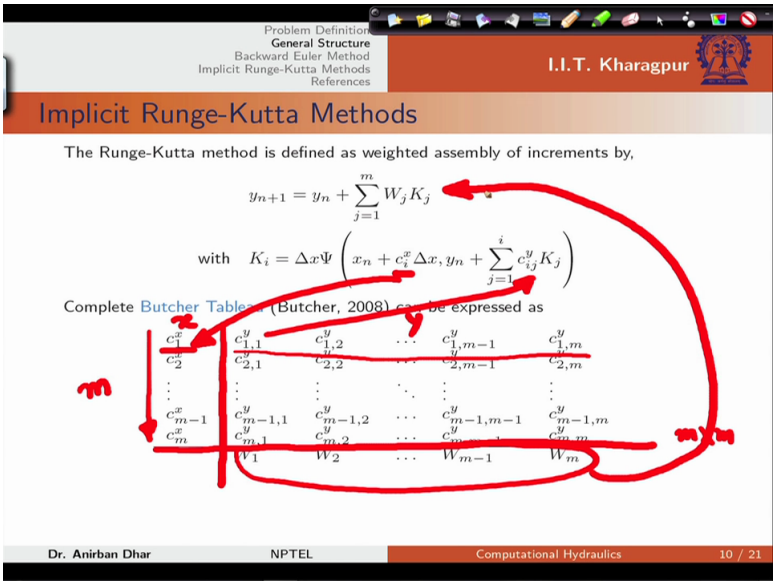

Now if we express it in terms of reduced matrix form, we can write this as column vector. First one is one column vector and this is a row vector. So in general we can express this as column vector and we can take this transpose here and capital Cy, this is the a full matrix here. So this is the general representation of using Butcher Tableau.

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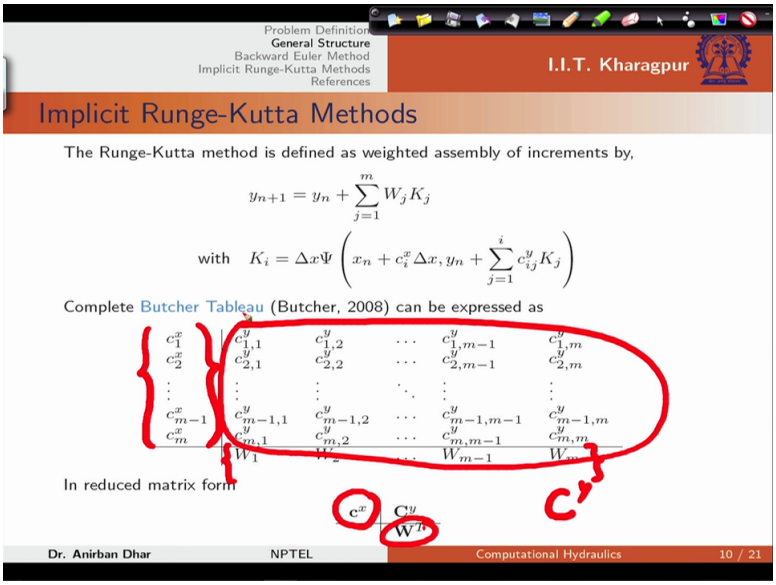

Now using this tableau we can represent our Runge Kutta methods. And if we start from our well known basic approach which is backward Euler. In backward Euler let us say that this is Cx 1 and this is Cy 1-1. That means only one term is there. And on the bottom we have the W1 which is again equal to 1. So what will be the structure of our increment? Increment will

be K1 equals to del x psi, in this case xn plus, this is one so directly we can add that del x comma we should have yn.

Again the coefficient is 1, so in this case we can write it as K1 and this is our case and which is a coefficient of this K1. And finally we can add this yn equals to yn plus W1 which is again K1 in this case because W1 equals to 1. So from this process we have identified the backward Euler method.

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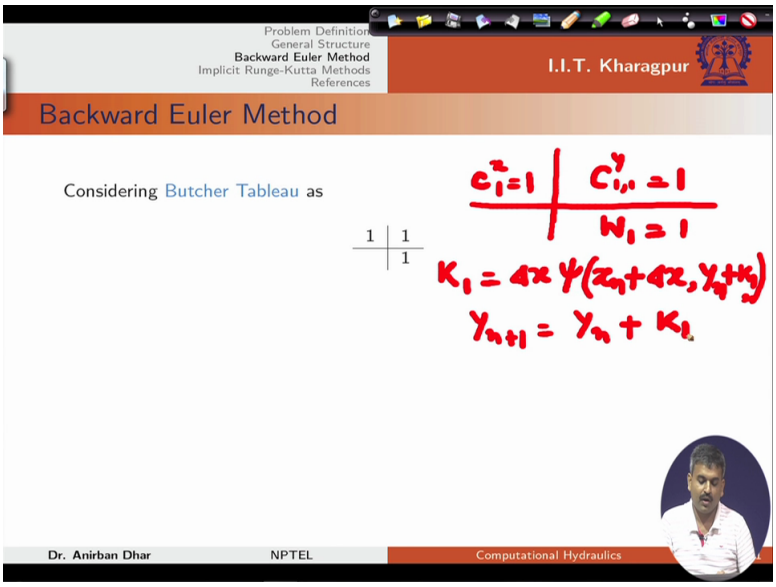

So in general terms if we write this is xn plus 1, yn plus 1. This is yn plus 1 but yn plus 1 is unknown thing. So we cannot solve it using our usual explicit approach. We need to solve this equation here.

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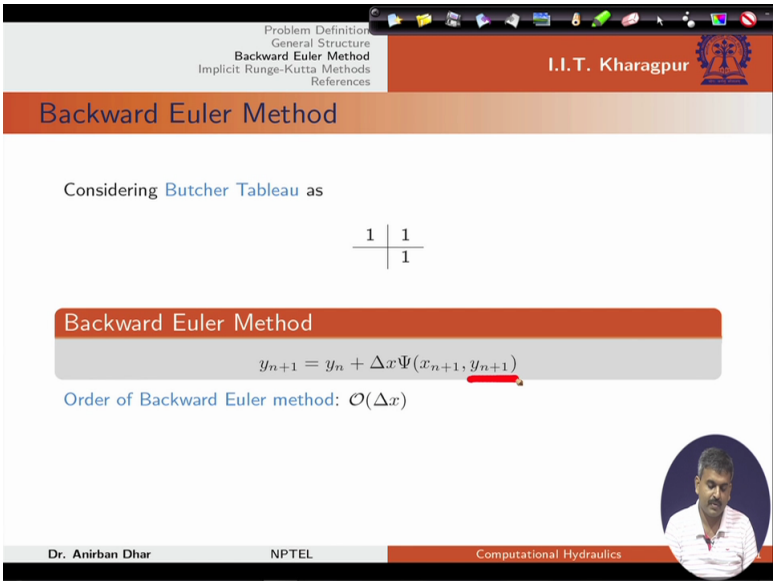

Now in this case again we can write a general function because yn plus 1 is unknown so I can transfer this yn plus 1 and other terms on the left hand side. So yn plus 1, this is coming from here. This is my second term and the third term which is unknown quantity. So this part is unknown and this is known quantity in this case.

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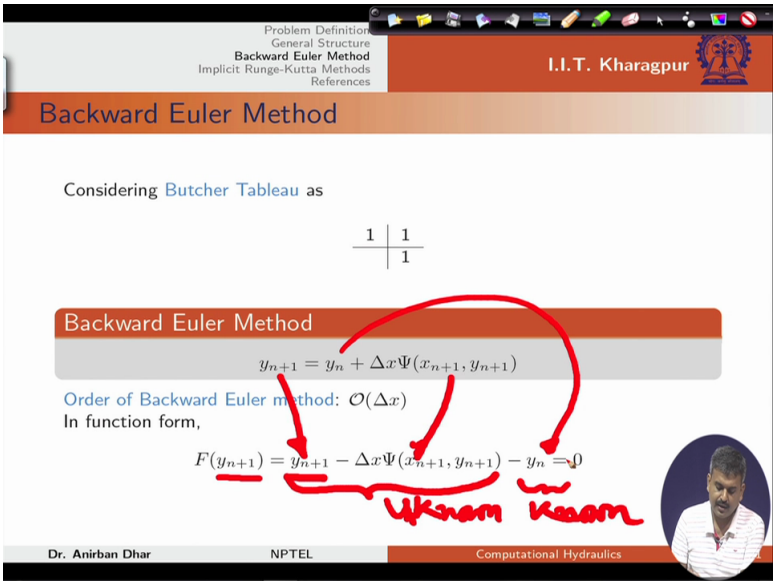

So again this is non linear equation and we can solve it to get this yn plus 1. Now in this process if we write this general equation which is yn plus1 P, yn plus 1 P minus 1 which is again value which is coming from previous iteration. And these are the function values and derivative of the function value.

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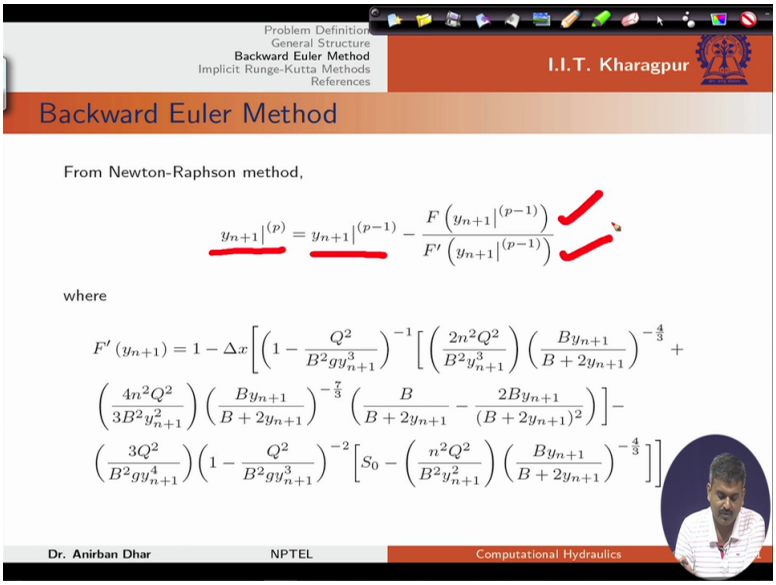

So initial was yn plus 1 minus del x into psi xn plus 1 comma yn plus 1. Obviously if we differentiate this one with respect to yn plus 1 which is the unknown quantity here, this is yn plus 1 and minus yn. So initially this quantity is not significant here because if you differentiate it with respect to yn plus 1 this will be zero, this is 1. from the second term we are getting this big expression for rectangular channels.

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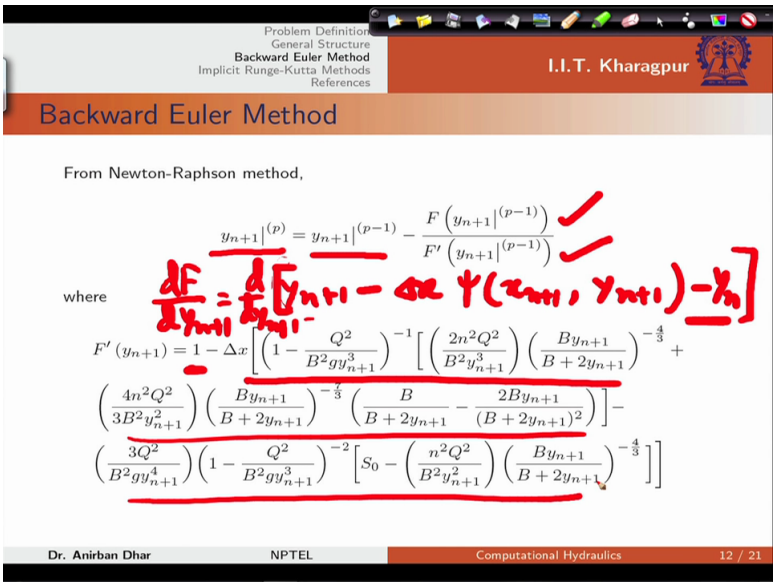

Now if we write the code for this one we can get the solution out of this. So in this case if we start this scilab so we have GVF implicit. In GVF implicit, backward Euler. Now given data from our problems statement Q is 20 metre cube per second, S not is point 0008, n is point

015, bed width for rectangular channel that is 15 metres, G is 9 point 81, Lx is 200, y is point 8 metres.

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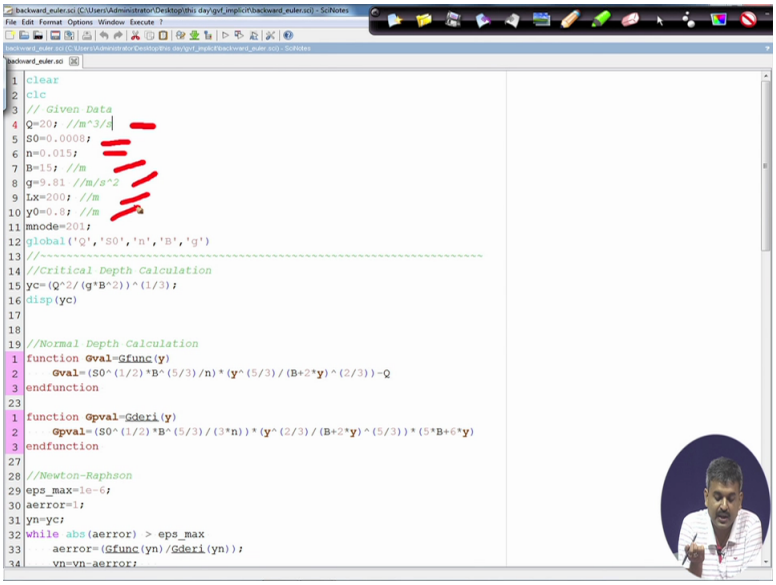

Now with this let us say that mnode or nodes in the m or l x direction that is 201 and these values are global, Q, S not, n, B and g. In this case first step is critical depth calculation. Yc is Q square g B square and whole to the power 1 3rd. We can get the yc value. Then normal depth calculation Gval and G prime val. So G and G prime, these two values are required.

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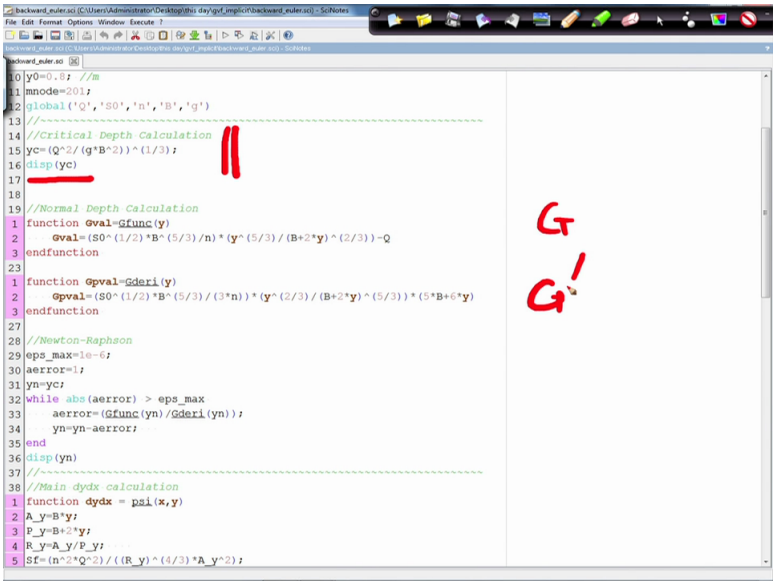

So using Newton Raphson here we can get the solution and we can get this yn. So after getting this yn and yc we can comment on the flow condition.

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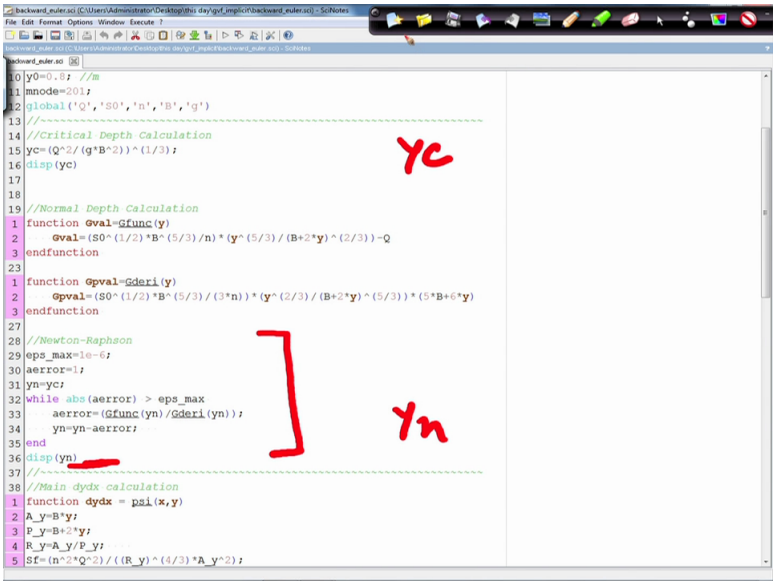

So obviously if your yn is greater than yc we have mild slope condition. And if yn is less than yc we have steep slope condition.

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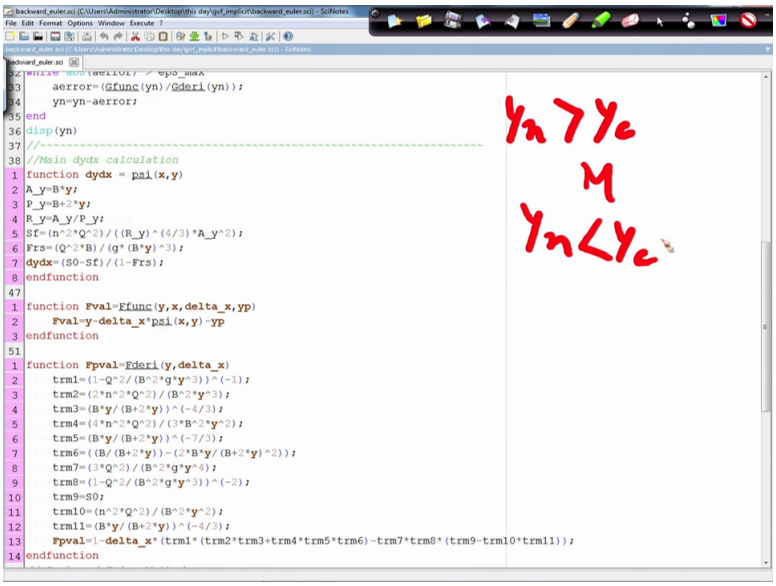

Now first case we have dy by dx equals to psi xy, our general governing equation in this case.

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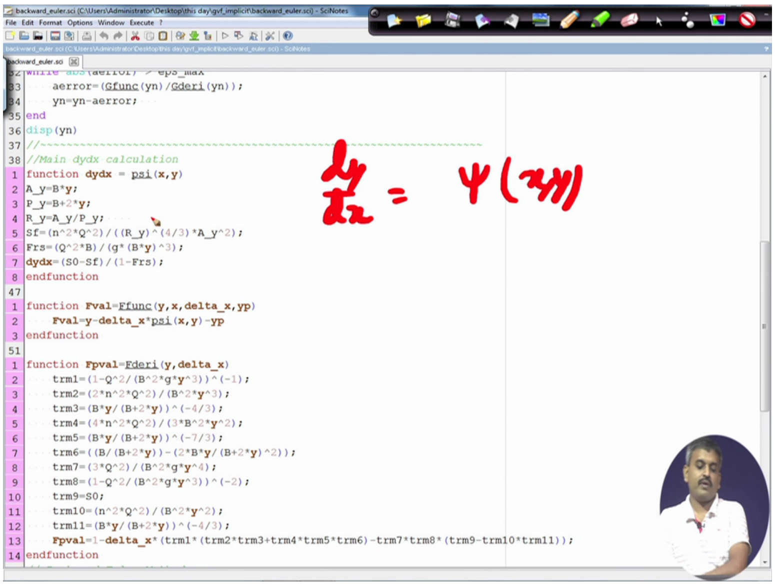

So this is By, B plus 2y, R is Ay divided by Py. Now we can calculate this dy by dx.

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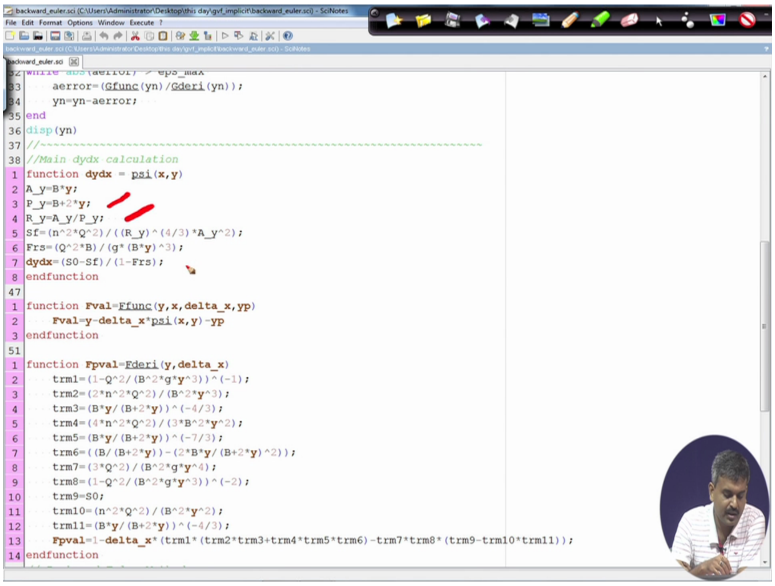

Next step is Fval because we need that function val. Y is actually yn plus 1. This is delta x into psi x comma y. And at this step we have this minus yp. Yp is previous time (val) level value.

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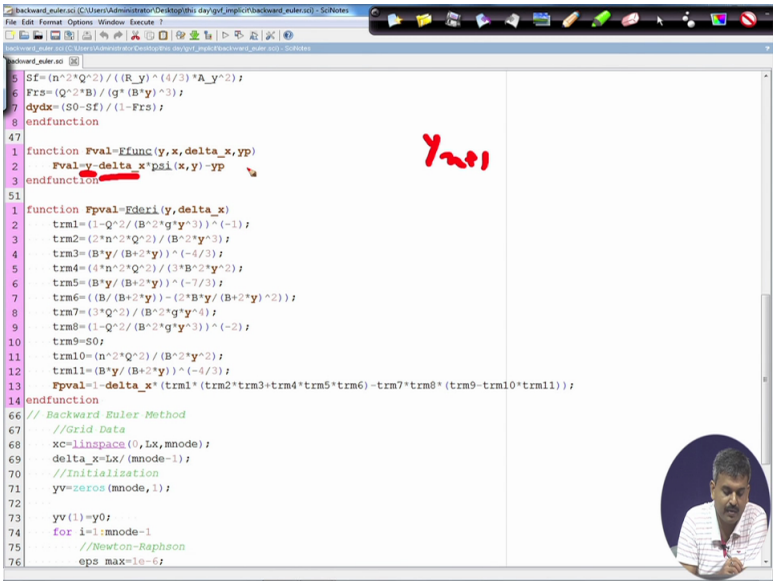

Now we need to calculate this term by term Fp value or derivative of F with respect to yn plus 1. Now if we check the first term here so this is 1 minus Q square divided by B square gy cube n plus 1.

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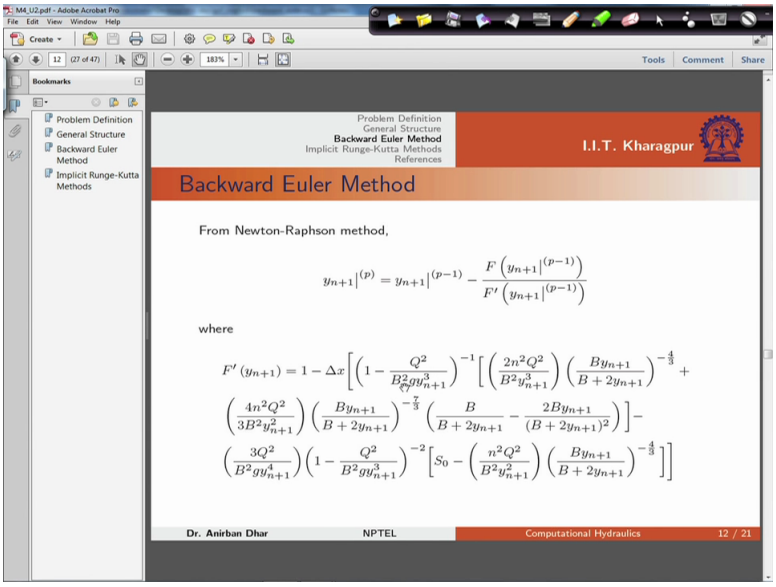

So same thing I have written here to the power minus 1. Then second term 2n square Q square B square y cube which is the second term here.

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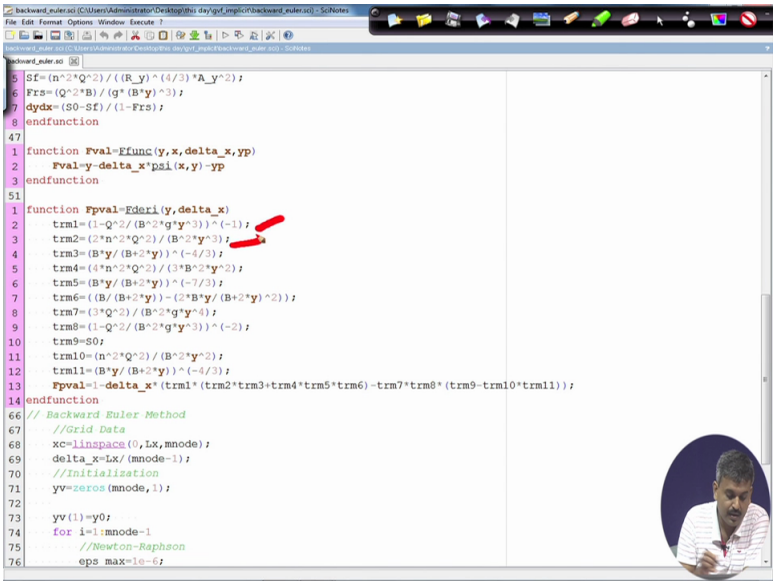

Third term is B yn plus 1 B plus this 2 yn plus 1. So same thing if I write here this is the thing. Now like that I have written all the terms in this F prime calculation. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 terms here.

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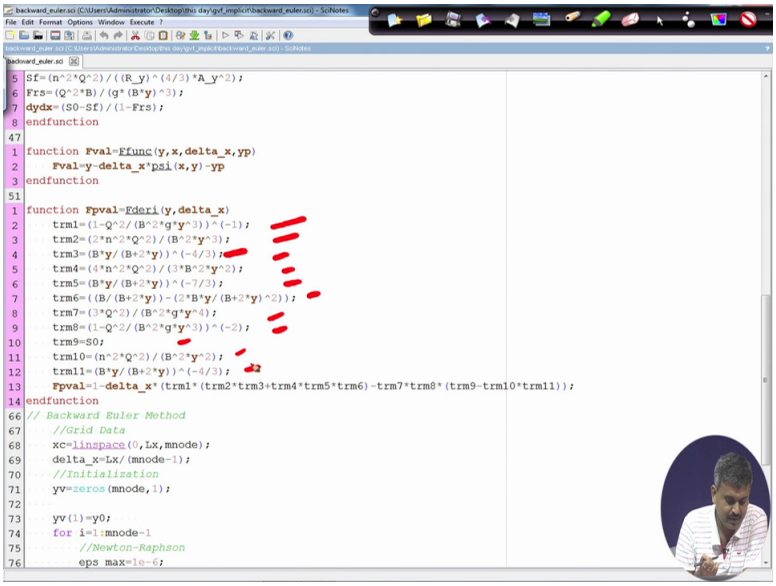

Now after getting this terms we can simply, this is term 1 into term 2 into term 3 plus term 4, term 5, term 6. So we can get this Fpval or F prime value from this derivative calculation.

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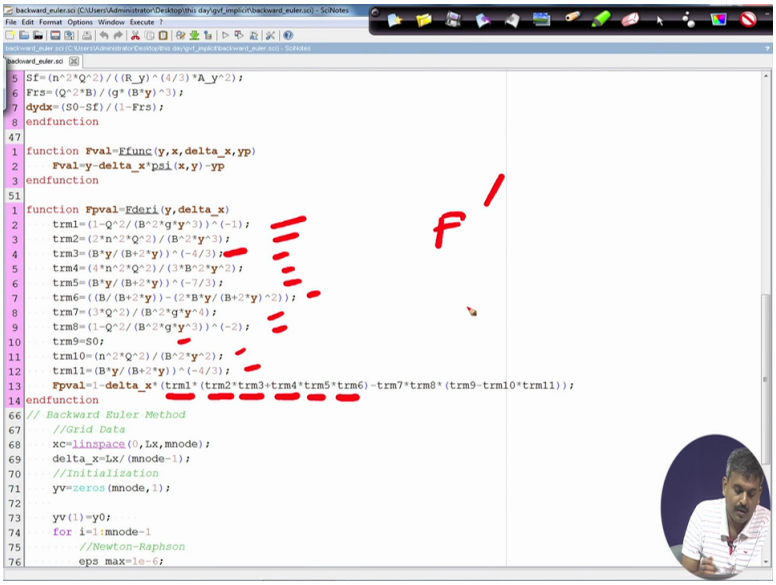

Now we need to use this one in our backward Euler method. So first of all we need to have this grid which is xc. Xc is running from zero to Lx. We have all total mnode, number of nodes which is 201 in our case. In this case Lx is 200 metres and delta x is Lx divided by mnode minus 1. So this is individual del x we can calculate. Now in this case yv is our variable. We need this y value.

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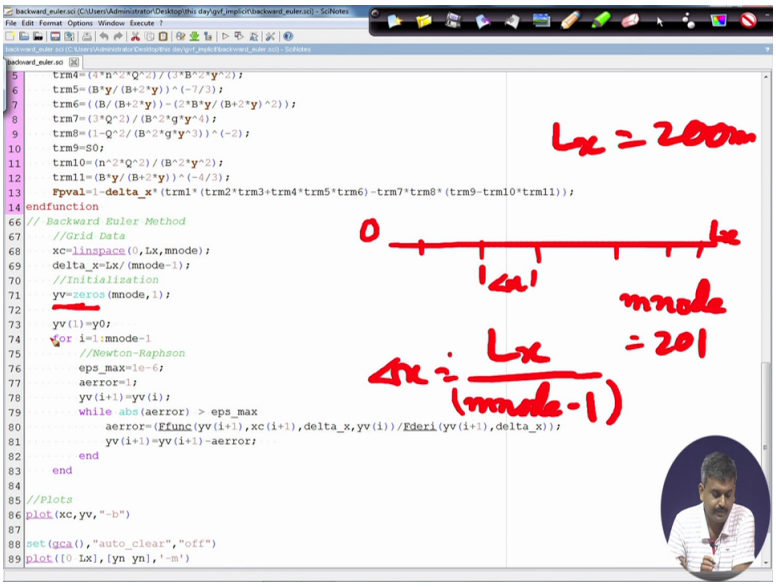

At the starting we have yv 1 equals to y not. Yv 1 which is specified at the first node, this is y not.

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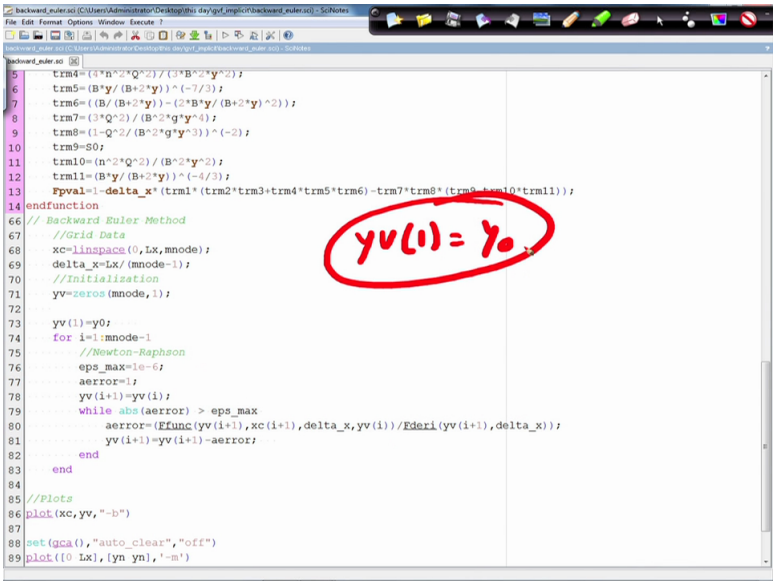

Now we can start our calculation for i equals to 1 to m node minus 1. So epsilon max, this is the iteration parameter 1 into 10 to the power minus 6. Aerror, this is 1 or absolute error in this case. Now in this case we are starting from the guess value yv i plus 1 equals to yv i.

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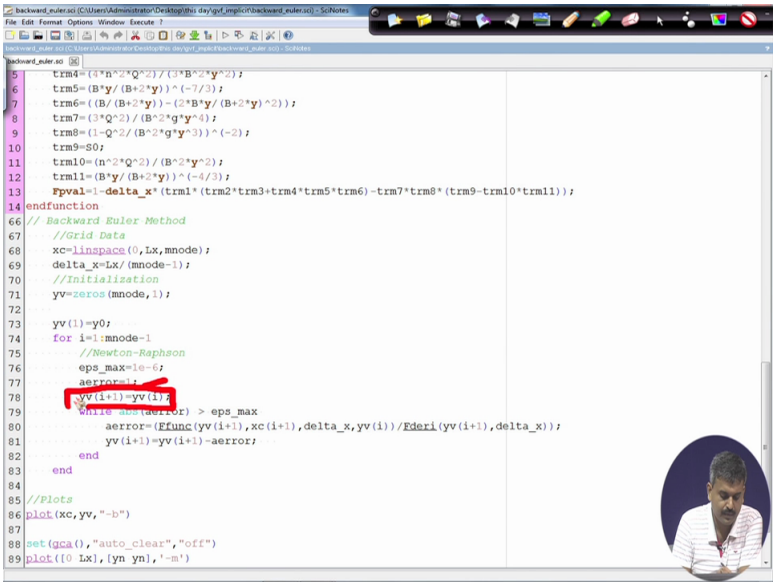

Now we should iterate if absolute error is greater than epsilon max. So what is this absolute error? Absolute error is absolute value of aerror. Aerror is f func that means function value at yn plus 1 for previous iteration divided by F prime yn plus 1. That is also calculated based on the previous iteration. Now this is minus and this is yn equals to yn plus 1. So from this one we can get the solution.

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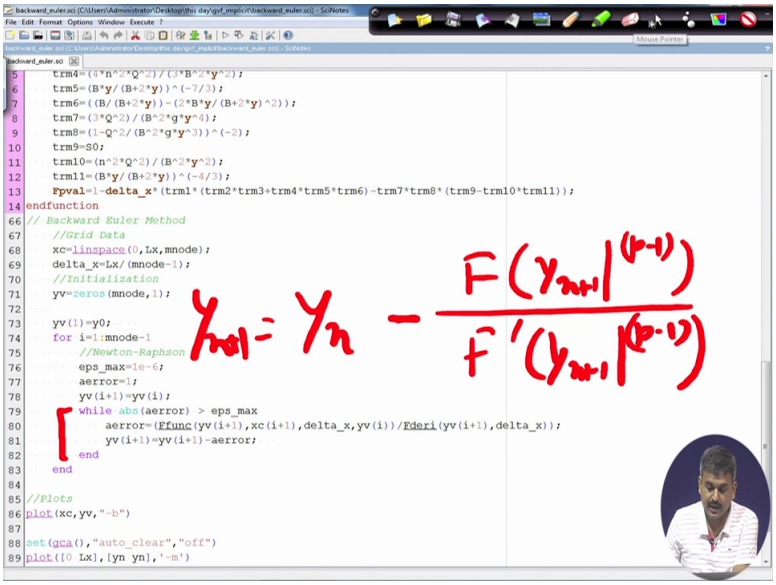

So in this case we have this yn plus 1 which is coming from iteration P and this is yn plus 1 which is coming from iteration P minus 1 and minus F by F prime and this was yn plus 1, not yn.

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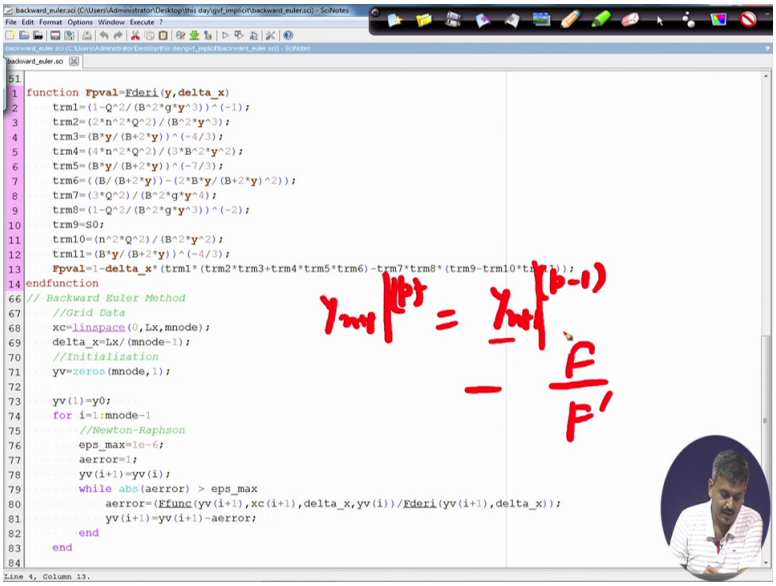

So we can use this for calculation of Newton Raphson. Now this is minus error, this is xc, y. Now if you run it, similar to our explicit method we are getting M2 curve in this case. This is NDL or normal depth line. This is CDL or critical depth line.

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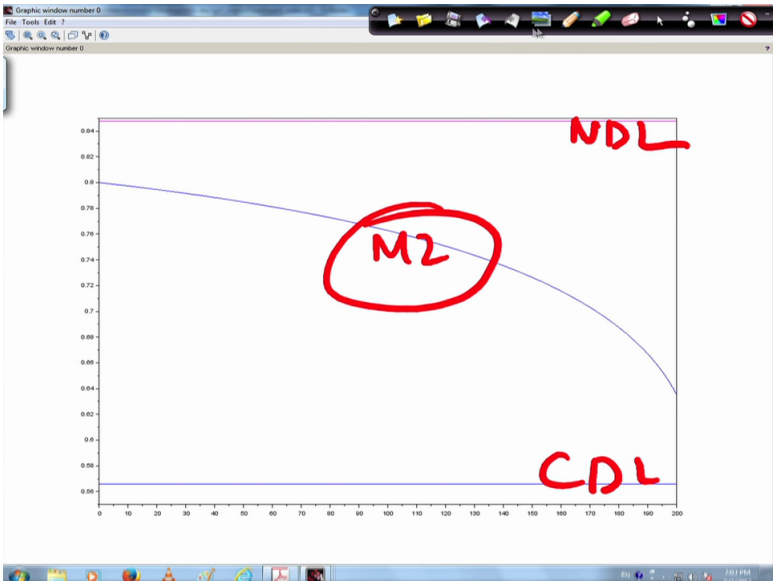

From our calculation point of view this yn is point 8478 and yc is point 5658. Now if we try to utilise the higher level methods or higher order methods in this case then if we have implicit Runge Kutta in this case, please remember that we have this Ki and i is there. That is why it is implicit.

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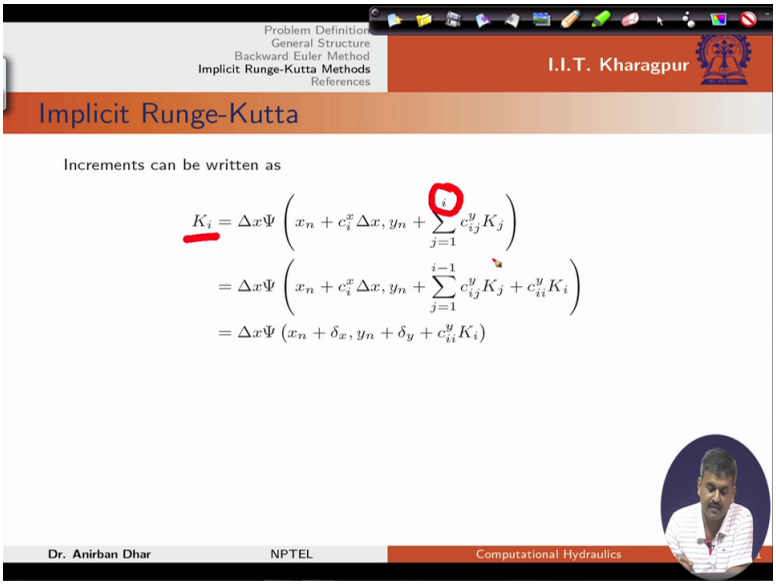

Now what we can do, we can split it into explicit part plus some increment. This C ii y K i, this is the increment part. So we can write this quantity, this is again explicit in nature. So this is explicit, this is explicit, only implicit is this quantity. This is C y ii.

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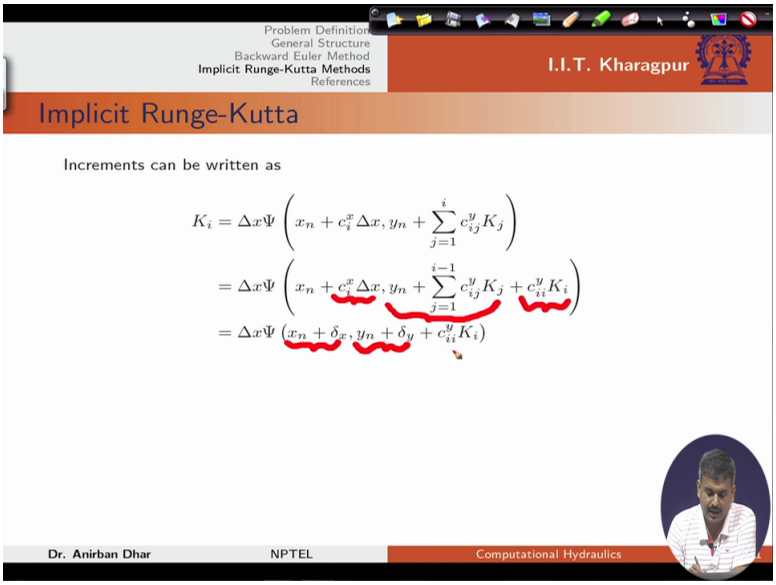

Now in this case we have this delta x which is only Ci del x and delta y which is our j equals to 1 to i minus 1 C ij Kj. This is explicit quantity.

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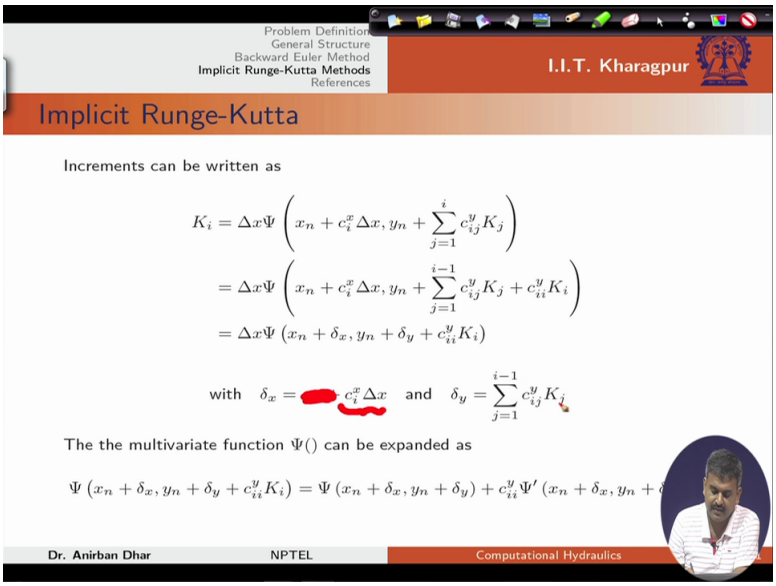

Now in this case we have this psi xn plus delta x yn delta y C ii y Ki. So this is increment part. So we can expand it with Taylor series and this is a first order expansion. So, on the right hand side we have xn plus delta x yn plus delta y and C y ii3. This is xn delta x yn delta y and this quantity is multiplied there and we are differentiating with respect to y here.

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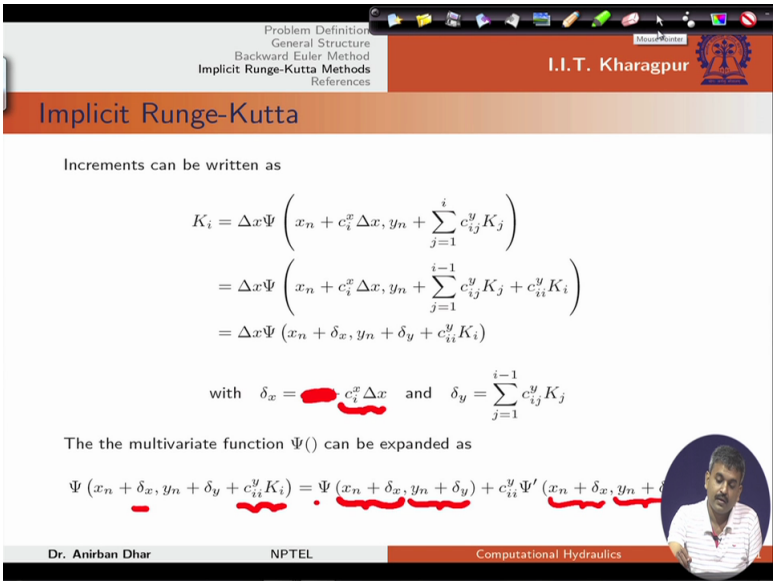

So in this case if we combine the terms so this is Ki delta x into psi which is our known quantity and this Ki is unknown.

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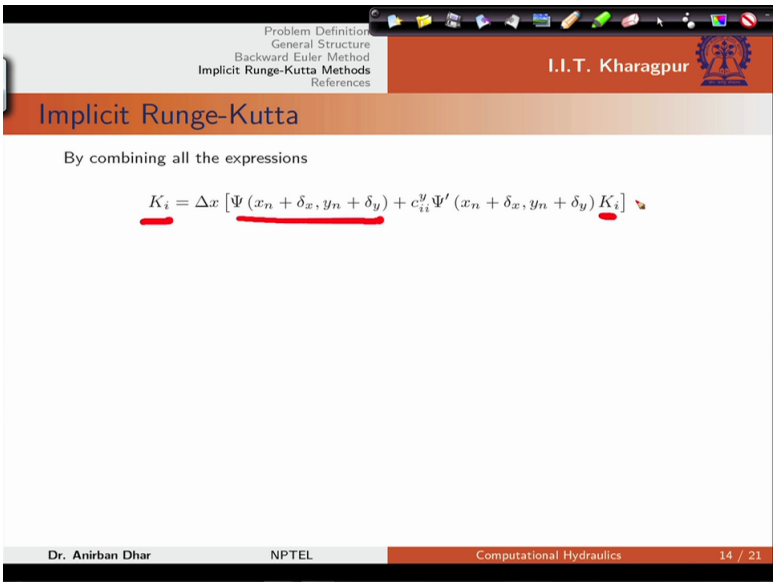

Now if I transfer this thing on the left hand side so I can write this as 1 minus C ii and y. This is delta x pi plus prime into Ki in this case and this is delta x into psi xn delta x yn plus delta y. So I can transfer this on the right hand side. This is inverse term del x.

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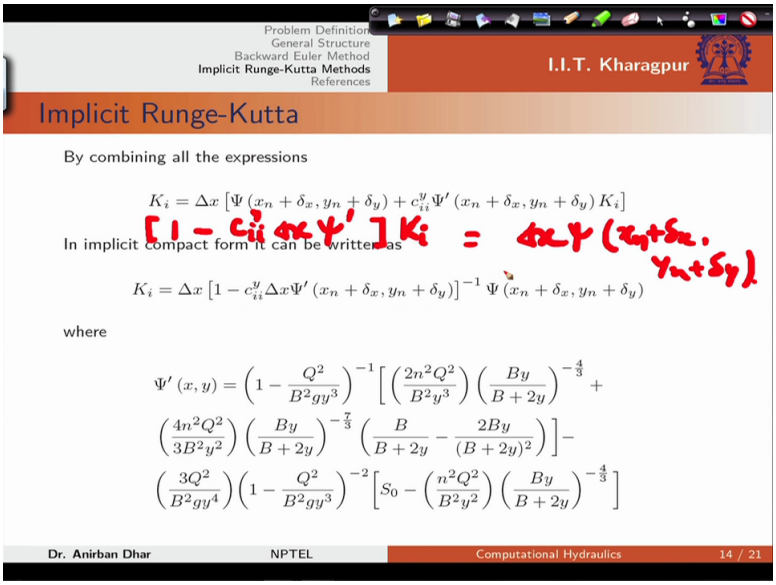

Now from this case we have this psi prime. Psi prime value we can calculate and we can directly get the Ki value here. So Ki value we can get in terms of known values because xn del x yn del y and psi and psi prime, these quantities are known. And for a given Runge Kutta method C ii y, this quantity or this parameter is known and del x is fixed for a particular problem.

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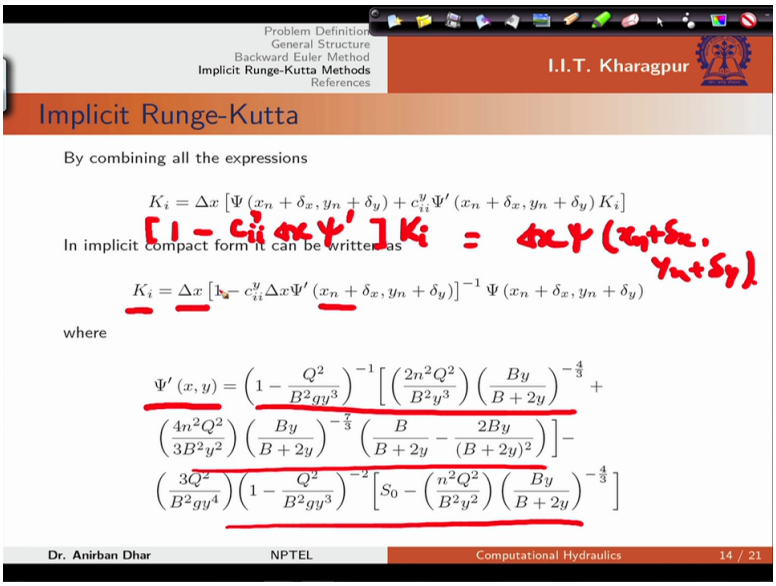

Now in this case if we utilise it for second order RK method or RK2 then considering the Butcher tableau we have half, half. That means coefficient of x is half, coefficient of y is half.

So this is nothing but K1 equals to del x and we have psi xn plus half del x comma yn plus half K1. And final thing is yn plus 1 equals to yn plus K1.

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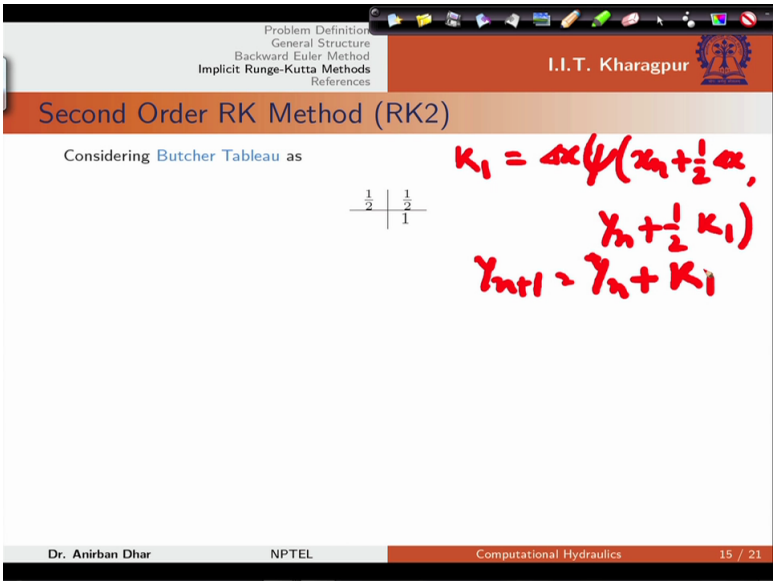

So if we utilise this, this is second order method. Now what should be our approach? We can expand this and expansion is in terms of our known quantities. Now in this case we can expand this part using Taylor series as psi this is xn plus half del x. This quantity is yn which is known, plus we can write psi prime and we can multiply our half into K1 in this case. If we multiply again del x outside and equate it with K1 we can get this particular format.

So, on the right hand side we have all known quantities. Psi, psi prime, these are known quantities on the right hand side. So we can get this K1 value.

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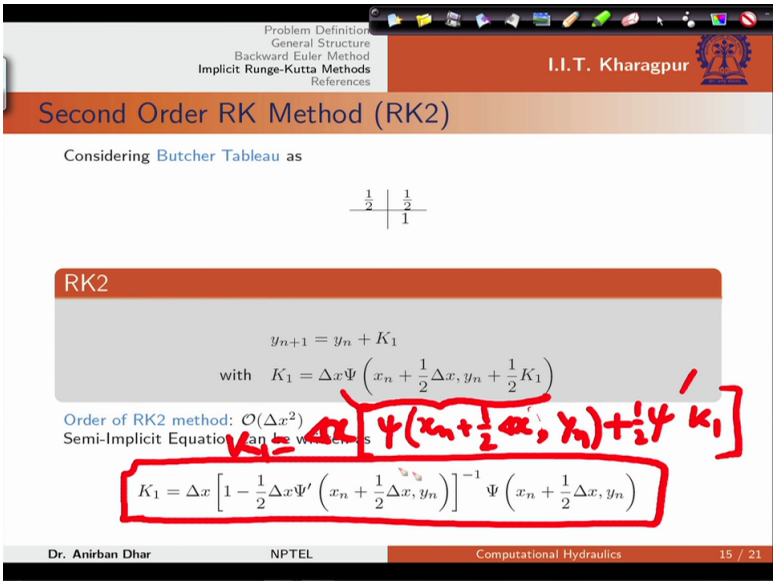

Now if we see the code for this one, this is RK2 implicit. First portion is same. In this case we have RK2 method. We need grid data in this case also. Xx, delta x, yv, this is yv 1 equals to y not. And then we are calculating the K1. Now K1 in this case this is coming from this known quantity. This is 1 minus half delta x, psi or psi derivative, these are known quantities.

This is psi derivative, this is Psi value directly. And we are considering inverse of that value and we are adding this K1 with our previous time value to get the final plot.

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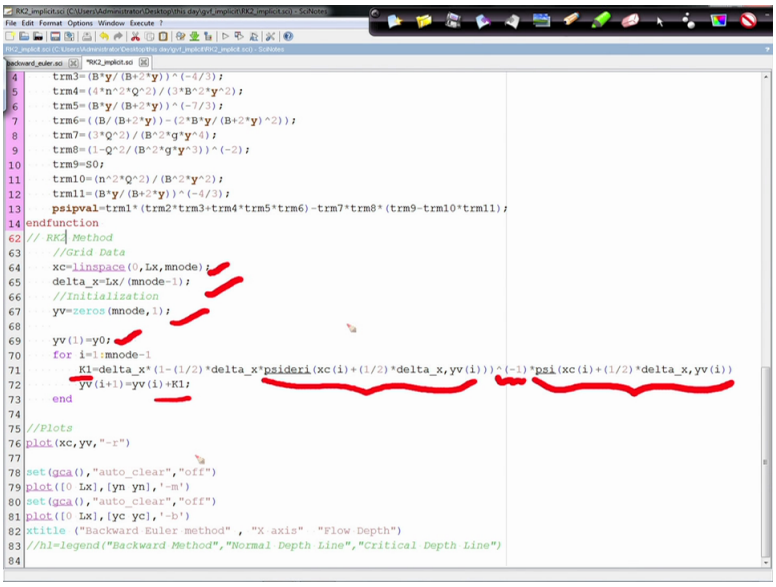

Now in this case we have RK2 method. Now let us try to plot this thing on top of our previous solution. So this is with red colour. Now if I compel this, so it is coming here. Now

it is interesting to see that red line which is RK2 method is somewhat on the upper side. That is it is giving higher value compared to the blue line which is our backward Euler method.

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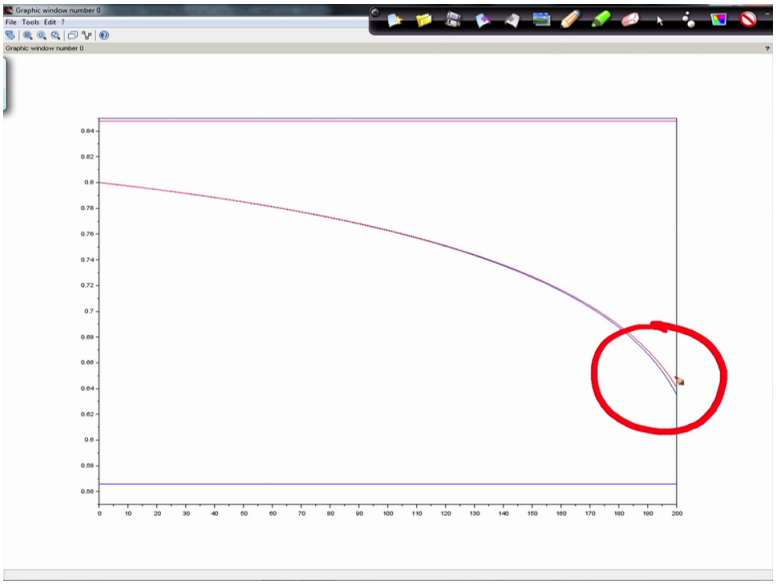

Now if we further increase the order of RK, second order RK approach we have this thing. Now considering the fourth order RK method which considers these many points that means we have two entries here and 2 by 2 matrix for Cy. This is for Cx, this is for Cy and this is for W1 and W2.

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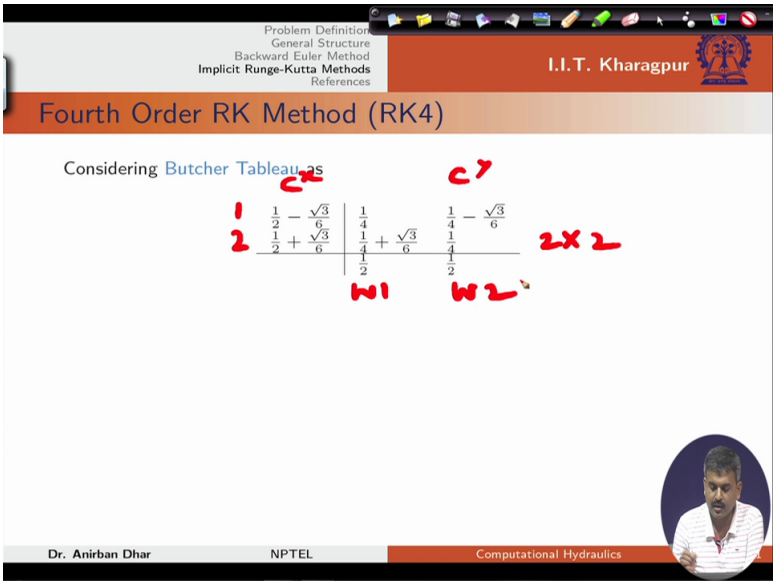

Now with this if I write this RK4, this is RK4 in this case. So RK4 is half-half because this is W1, this is W2 and we have these terms. This K1 is equal to del x into C1 which is coming from here, the first quantity. This C2 is coming here, then C 1-1, this is the first one, C 1-2, then this quantity is C 2-1 and last one this is C 2-2.

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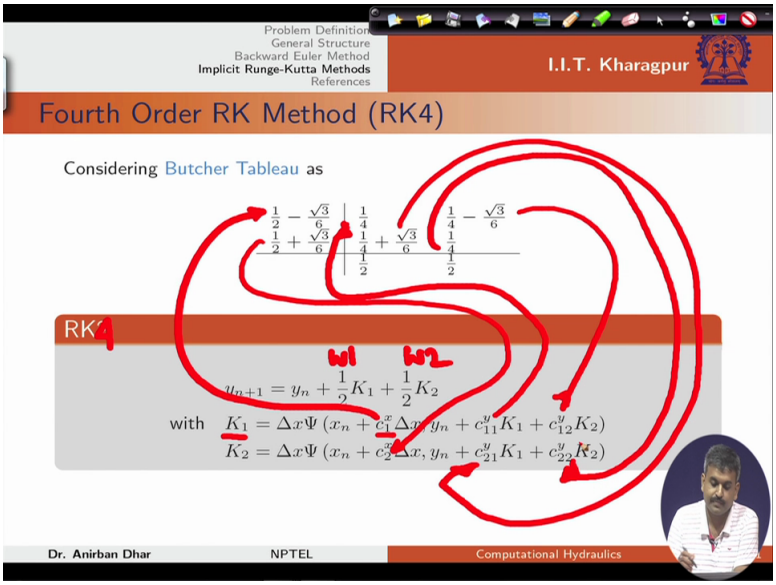

Now with this if we try our method or approach we can get the solution. This is fourth order RK. Now in this case we have K1 and K2. Now for K1 and K2 for both the cases we can expand it. Now expansion is in terms of both the terms. Only yn is known, K1 and K2 both are unknown. So this is our increment, increment. So psi prime, psi these are known quantities now. Now this is coming from Taylor series.

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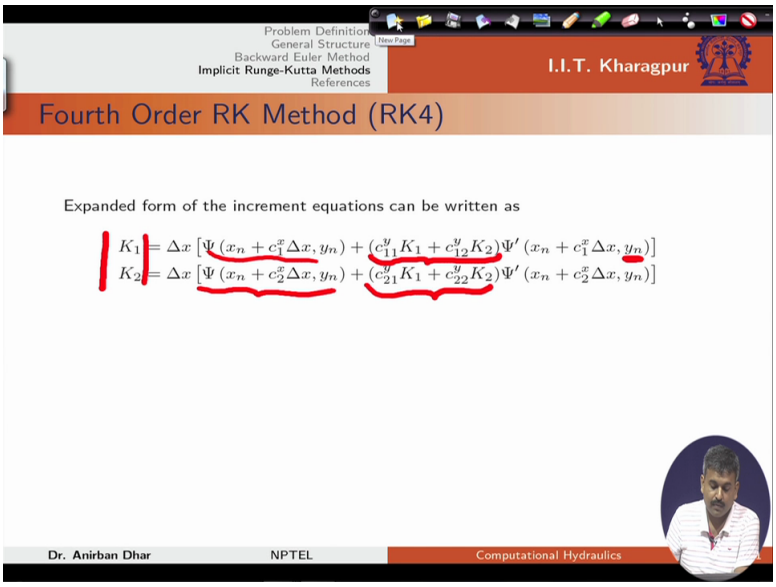

Our next step is calculation of K1, K2. Now K1, K2 can be calculated by solving this system where we have 2 by 2 matrix which is the coefficient matrix. On right hand side also we have

known quantities. Now we can solve it using our standard approach and utilise it for the solution of fourth order RK.

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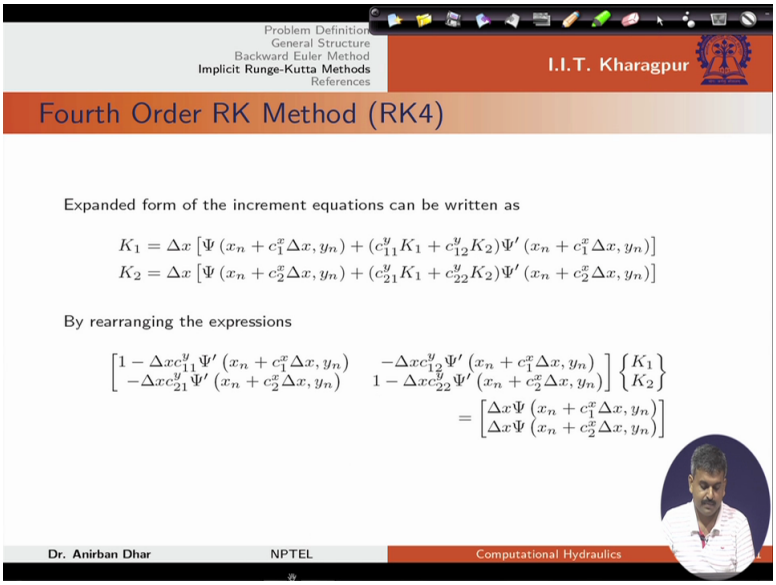

Now if I open this fourth order RK, the first portion is same. Now in this case this is RK4 method. Now I have created this AK which is coefficient a or coefficient matrix for K, this is 2 by 2 zeros.

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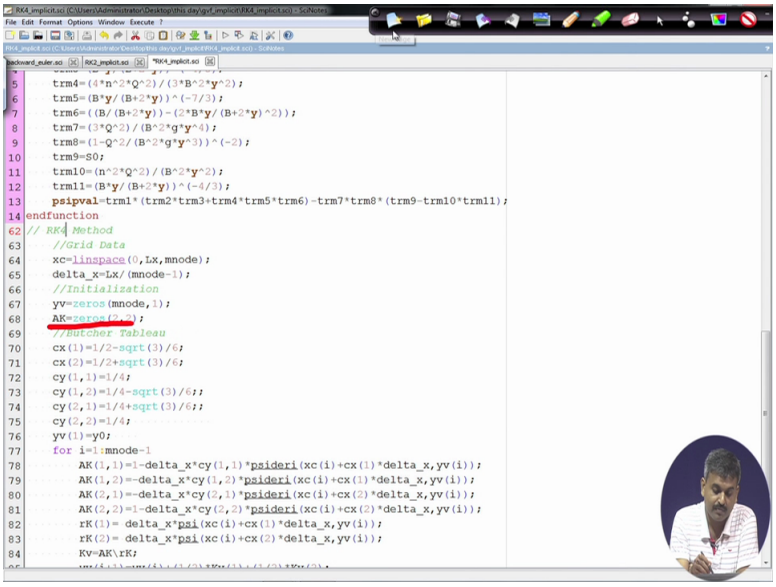

Now we have this cx1, cx2, cy 1-1, cy 1-2, cy 2-1, cy 2-2 and y1 equals to y not. These values are available.

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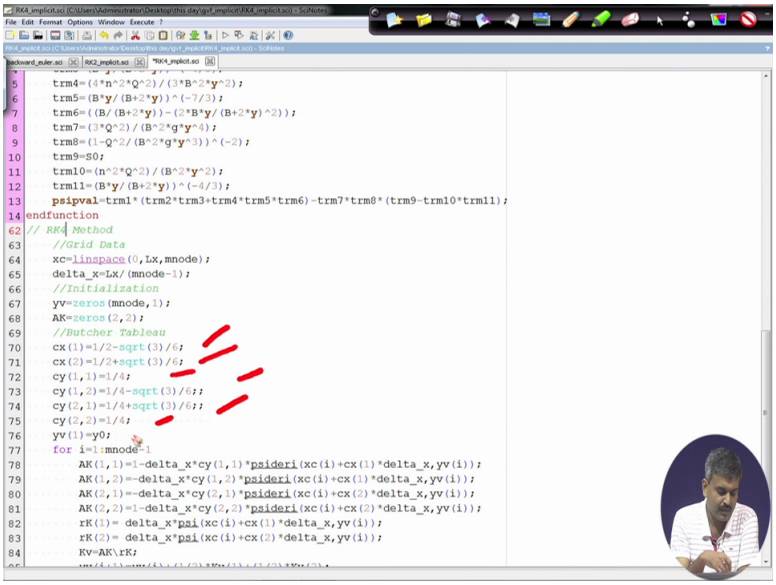

Now within this calculation process starting from y equals to 1 to mnode minus 1, we can create that AK or coefficient matrix then calculate this K and RK which is on the right hand side. So AK is coefficient matrix. So in this case we can solve it using internal function this KV which is K value. K value we are directly getting from this backslash, this RK. Now from this one we have K1 and K2. So half of K1 and half of K2 if we add with the previous section value then we can get y i plus 1 in this case.

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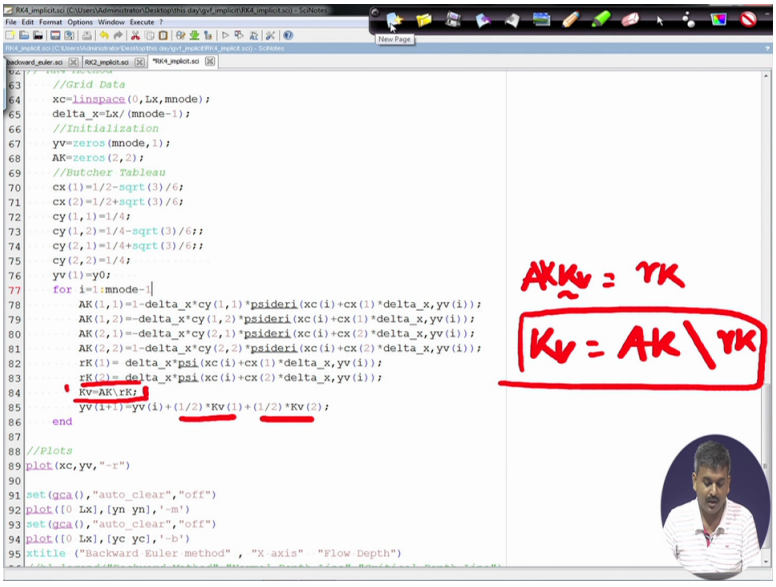

Similarly this approach is our RK4 approach. Now I can plot this one, this is again red one or magenta. I can change the colour in this case.

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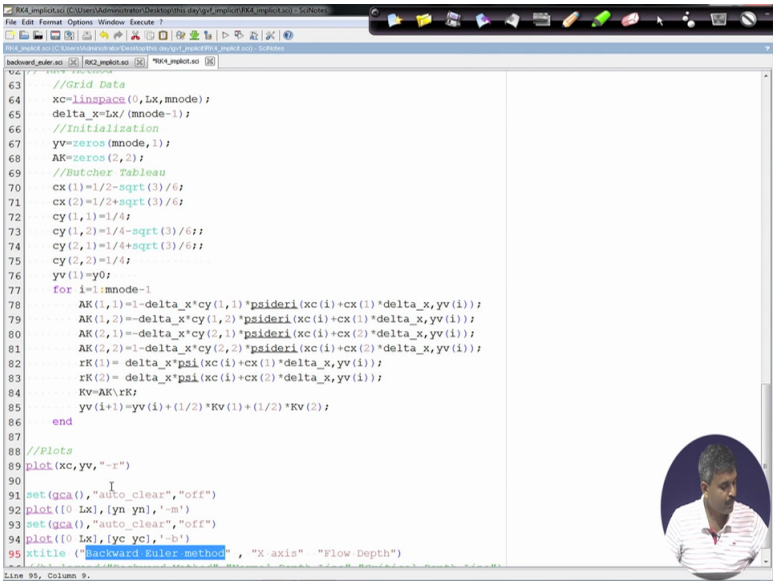

And if I run it, again it is coinciding with our second order or it is near about value of our second order method. If we zoom it, obviously we will get some difference for this one.

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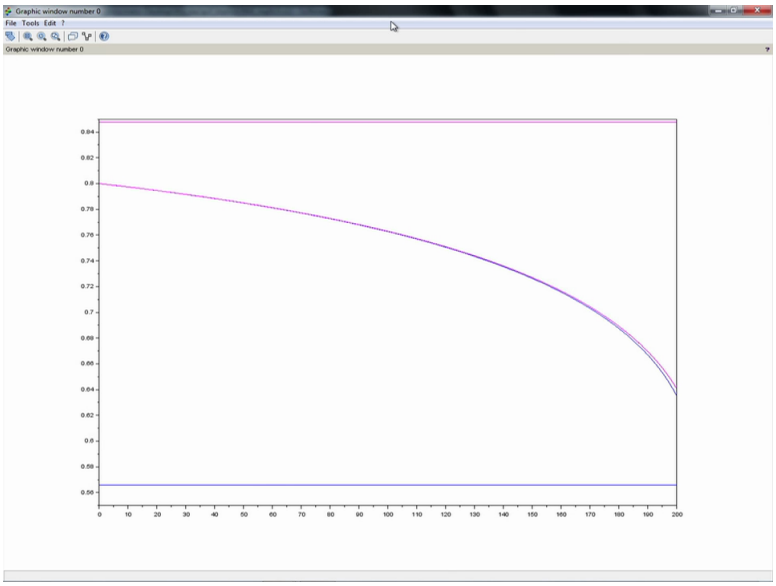

Now in this case we have seen that using backward Euler second order RK and RK4 we can solve our GVF profile equation that is the first order initial value problem with implicit equations.

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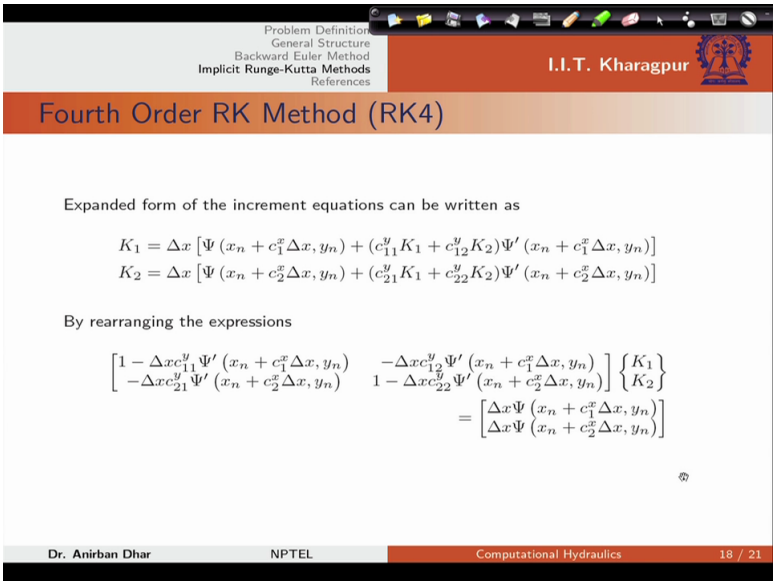

And we have discussed this backward Euler approach, RK2 approach, RK4 approach. Now you can utilise these codes for running and creating the GVF profiles. And you can change different values and get different kind of GVF profiles out of this.

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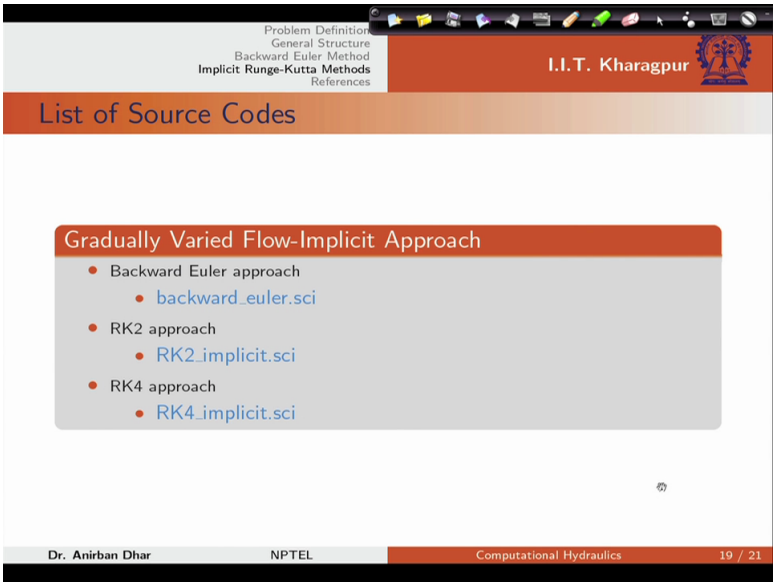

Thank you.