

Computational Hydraulics
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Lecture 3

Classification of Problems based on Initial Condition (IC) and-or Boundary Conditions (BC)

Welcome to this unit number 3 and lecture number 3 that is classification of problems based on initial condition and or boundary conditions.

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Ordinary Differential Equation
Partial Differential Equation
References

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Module 01: Introduction to Computational Hydraulics

Unit 03: Classification of Problems based on Initial Condition (IC) and/or Boundary Condition (BC)

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So what are the learning objectives? Learning objectives first is to identify the initial and boundary conditions for hydraulic systems. Second, to distinguish between the problems based on initial and boundary condition.

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Learning Objectives

- To identify the initial and boundary conditions for hydraulic systems.
- To distinguish between the problems based on initial and boundary conditions.

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So what is this initial condition initial? Initial condition describes the initial state of the system in terms of dependent variables. Example initial water level and velocity in a channel network that should be defined for starting the simulation. And initial groundwater level in an aquifer system or aquifer region if we want to start groundwater simulation and that should be time dependent.

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Initial Condition

Initial Condition
Describes the initial state of the system in terms of dependent variables.

- Initial water level and velocity in a channel network
- Initial groundwater level in an aquifer region

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The types of boundary condition. Depending on the location we can define boundary conditions as external boundary condition defined for external or outermost location. Example is upstream and downstream locations of a river. We can define either inflow condition discharge or in the downstream section we can define depth for a particular river

system. River boundary for (aq) aquifer region we can define constant head boundary for aquifer region or we can vary that depending on the physical system.

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Types of Boundary Conditions

Location Based

External boundary condition
defined for external/outmost locations

- Upstream and downstream locations of a river
- River boundary for an aquifer region

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Internal boundary condition defined for internal locations operating conditions for hydraulic structure within channel networks. Sometimes these internal boundary conditions for channel networks are also called as junctions conditions. Constant water level maintained in the pond of aquifer region that is also our internal boundary condition if the pond is situated within the central portion of the aquifer region, then we can consider it as internal boundary.

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Types of Boundary Conditions

Location Based

External boundary condition
defined for external/outmost locations

- Upstream and downstream locations of a river
- River boundary for an aquifer region

Internal boundary condition
defined for internal locations

- Operating conditions for hydraulic structures within channel network
- Constant water level maintained in a pond of an aquifer region

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Then depending on the physical nature we can again classify this boundary condition. First one is called as Dirichlet or specified boundary. Example discharge specified at inlet or outlet in channel network. Neumann boundary that is flux boundary. Influx boundary example is low floor boundary near impermeable region in a aquifer system. Let us say that we have a rock boundary then obviously there will be no flow situation in the expert system and we can consider that as zero flux boundary.

Robin or mixed boundary is the combination of Dirichlet and Neumann conditions essentially weighted combination of Dirichlet and Neumann.

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The slide is titled "Types of Boundary Conditions Physical Nature Based". It lists three types of boundary conditions:

- Dirichlet/ Specified Boundary**: discharge specified at the inlet/ outlet in channel network.
- Neumann/ Flux Boundary**: no-flow boundary near impermeable region in aquifer system.
- Robin/ mixed Boundary**: weighted combination of Dirichlet and Neumann conditions

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So depending on the type of differential equation either it is ordinary or a partial differential equation we can classify that based on this initial and boundary condition. So in differential equation if you have ordinary differential equation then uh first type is initial value problem. In this one we need governing equation or g and initial condition. Next type is our boundary value problem. In this one we need governing equation plus boundary condition.

Now if we talk about partial differential equation then the types are boundary value problem there we need governing equation and boundary condition. And initial boundary value problem or IBVP where we need governing equation initial condition and boundary condition.

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Classification of Differential Equations

Differential Equation

- Ordinary Differential Equation
 - Initial Value Problem (IVP): GE + IC
 - Boundary Value Problem (BVP): GE + BC
- Partial Differential Equation
 - Boundary Value Problem (BVP): GE + BC
 - Initial Boundary Value Problem (IBVP): GE + IC + BC

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So let us talk about this initial value problems simple example is gradually varied flow in open channel. The governing equation is represented like this, y is the depth of flow, x is the longitudinal direction and s_0 is the bed flow, s_f is the friction flow, $1 - Fr^2$ square is the fraud number And in this case we can specify the initial condition, that means the particular location in the channel maybe we can consider that as x is equal to zero location we can define our water depth.

So y_0 is the water depth specified at that location. So this kind of problems where we have governing equations and initial condition we can call it as initial value problem for ordinary differential equation.

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Gradually Varied Flow in Open Channel

Ordinary Differential Equation

Initial Value Problem

Governing Equation:

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \quad (1)$$

Initial Condition:

$$y|_{x=0} = y_0 \quad (2)$$

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Now the example of boundary value problem for ordinary differential equation can be given by this steady one dimensional groundwater flow in unconfined aquifer.

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Steady One-Dimensional Groundwater Flow

Ordinary Differential Equation

Steady one-dimensional groundwater flow in unconfined aquifer

Figure: one-dimensional groundwater flow

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Let us say that we have this is x direction and this h_2 and h_1 these two are actually specified water levels on both sides. That means these two are Dirichlet kind of boundary condition or specified boundary condition and perpendicular to this that is y direction will be considered that there is no variation in that direction. So this is basically for $h(x)$ problem h is varying with x only. If we have constant levels of h_2 and h_1 with time.

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Steady One-Dimensional Groundwater Flow

Ordinary Differential Equation

Steady one-dimensional groundwater flow in unconfined aquifer

Figure: one-dimensional groundwater flow

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So this problem can be represented as boundary value problem where governing equation is, this is actually combined equations of mass and momentum.

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Ordinary Differential Equation

Steady one-dimensional groundwater flow in unconfined aquifer

Boundary Value Problem

Governing Equation:

$$-\frac{d}{dx} \left(T \frac{dh(x)}{dx} \right) = f \quad (3)$$

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Essentially q_x by dx is equal to f is the mass conservation equation and for q_x we can define this minus $T dh$ by dx as the equation and which is actually momentum equation. And we can combine this two to get this differential equation. Interestingly in this case h is varying with x only.

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Ordinary Differential Equation

Steady one-dimensional groundwater flow in unconfined aquifer

Boundary Value Problem

Governing Equation:

$$-\frac{d}{dx} \left(T \frac{dh(x)}{dx} \right) = f$$

$\frac{dq_x}{dx} = f$
 $q_x = -T \frac{dh}{dx} \quad (3)$

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Now if we consider the boundary condition that means on the left side if we have this h_2 as specified head boundary and l is equal to l_x that is on the right hand side h is h_1 , which is

again specified boundary. So we have two specified values for this problem. So this problem is actually boundary value problem for ordinary differential equation.

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Ordinary Differential Equation

Steady one-dimensional groundwater flow in unconfined aquifer

Boundary Value Problem

Governing Equation:

$$-\frac{d}{dx} \left(T \frac{dh(x)}{dx} \right) = f \quad (3)$$

Boundary Condition:

$$h|_{x=0} = H_2 \quad (4a)$$

$$h|_{x=L_x} = H_1 \quad (4b)$$

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Now let's talk about groundwater movement in aquifers. In this case that we are not considering the situation where there is no variation in the y direction.

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Groundwater Movement in Aquifers

Variable: $h(x,y,t)$

(a) Descriptive schematics of discretizations of global domain and two subdomains (Dogrul and Kadir, 2006)

(b) cross section of heterogeneous aquifer between two lakes and simulation grids (Dogrul and Kadir, 2006)

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Let us say it is constant by no flow condition within this ly distance then your problem is get slightly modified because we need to consider the qy component here along with qx and if your h2 and h2, h1 these values are not varying then we can get steady state situation because there will not be variation of water level within the system.

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Groundwater Movement in Aquifers

Variable: $h(x,y,t)$

(a) Descriptive schematics of discretizations of global domain and two subdomains (Dogrul and Kadir, 2006)

(b) cross section of heterogeneous aquifer between two lakes and simulation grids (Dogrul and Kadir, 2006)

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So essentially that is your boundary value problem we have no flow on these two sides and specified head in this two directions that means we have two Neumann boundaries and two Dirichlet boundaries.

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Groundwater Movement in Aquifers

Variable: $h(x,y,t)$

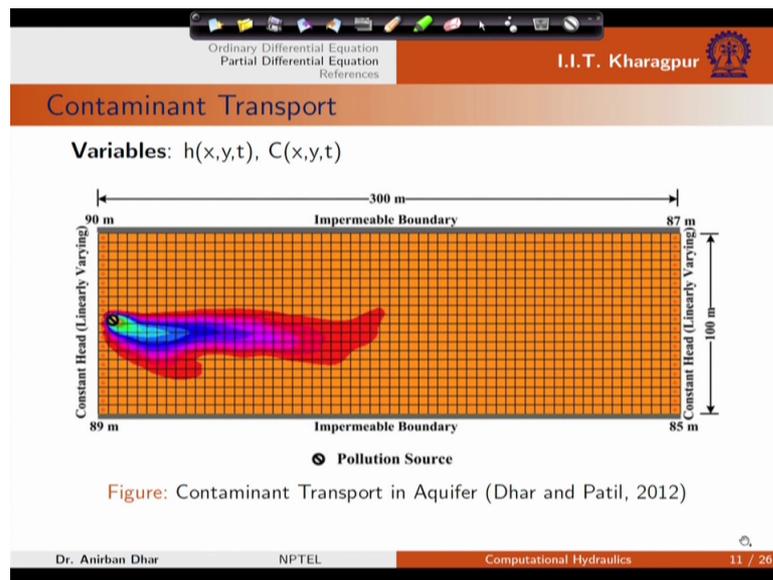
(a) Descriptive schematics of discretizations of global domain and two subdomains (Dogrul and Kadir, 2006)

(b) cross section of heterogeneous aquifer between two lakes and simulation grids (Dogrul and Kadir, 2006)

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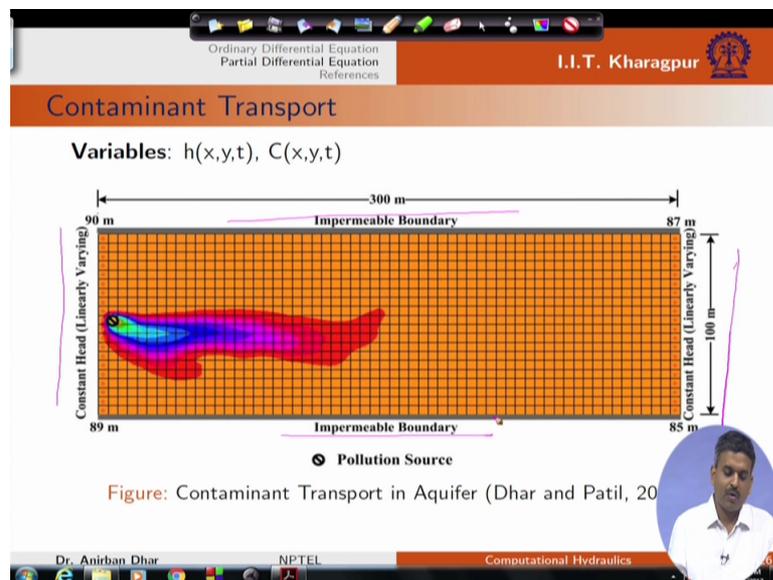
So if we talk about contaminant transport.

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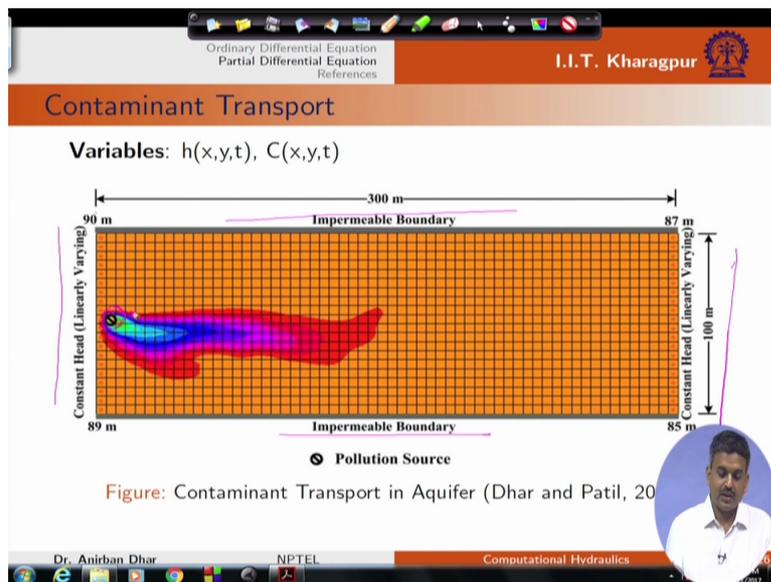
In that case again your left and right boundaries are constant head which is linearly varying from 90 to 89 and 87 to 85 and these two boundaries are impermeable or Neumann boundaries.

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And we have source sink term available at this location so water or contaminated water is injected within the system.

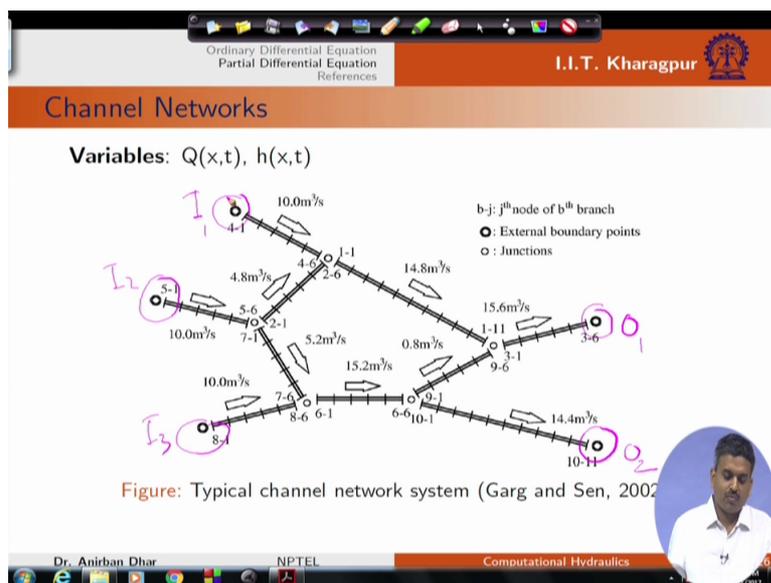
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So in this problem time varying injection is there so obviously time varying condition will be there. So with time there will be variation in head values. So we can consider this problem as initial boundary value problem for partial differential equations.

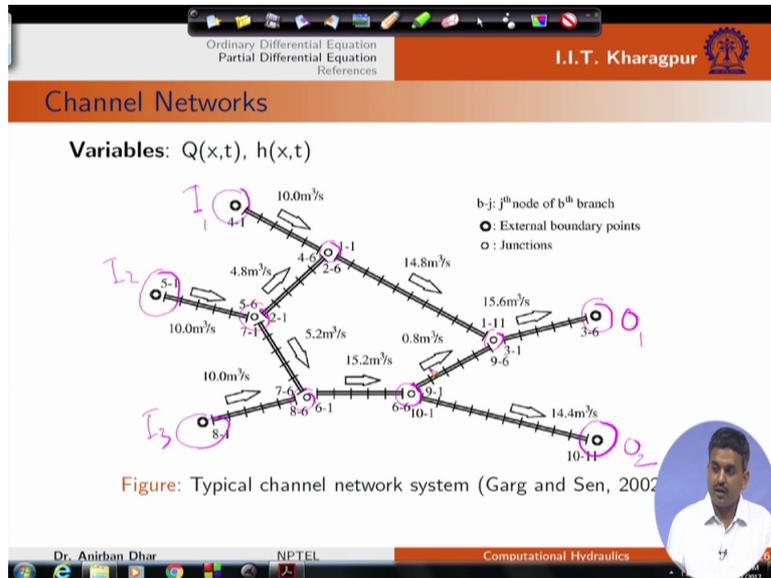
Again let us say that we are considering channel networks. Basically these are our inlet sections and these two are our outlet or o. This is i_1, i_2 and i_3 , these 3 are inlet. O_1, o_2 are outlet. So i_1, i_2, i_3, o_1, o_2 these are your in points or outer most points.

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In this case boundary conditions are specified however for internal points, internal points 1, 2, 3, 4 and 5 we generally define the junction conditions or these are actually your internal boundary conditions.

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So, junctions conditions your inflow and outflow in a particular junction that should be same and energy conservation if we do not consider this velocity head then we can simplify with this form.

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The junction conditions can be written as,

Mass conservation

$$\sum Q_i = \sum Q_o \quad (5)$$

where
 Q_i = discharge of channel bed at inflow branch [L^3/T]
 Q_o = discharge of channel bed at outflow branch [L^3/T]

Energy conservation

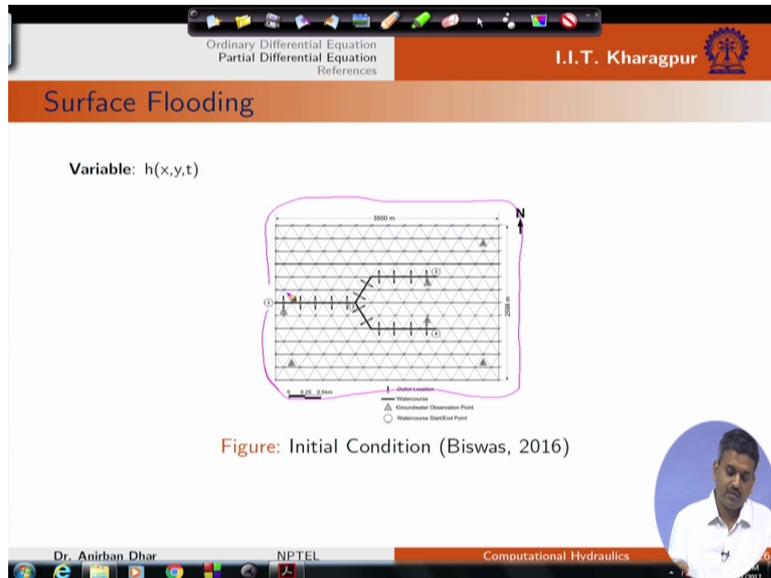
$$h_i + Z_i = h_o + Z_o$$

where
 Z_i = elevation of channel bed at inflow branch [L]
 Z_o = elevation of channel bed at outflow branch [L]

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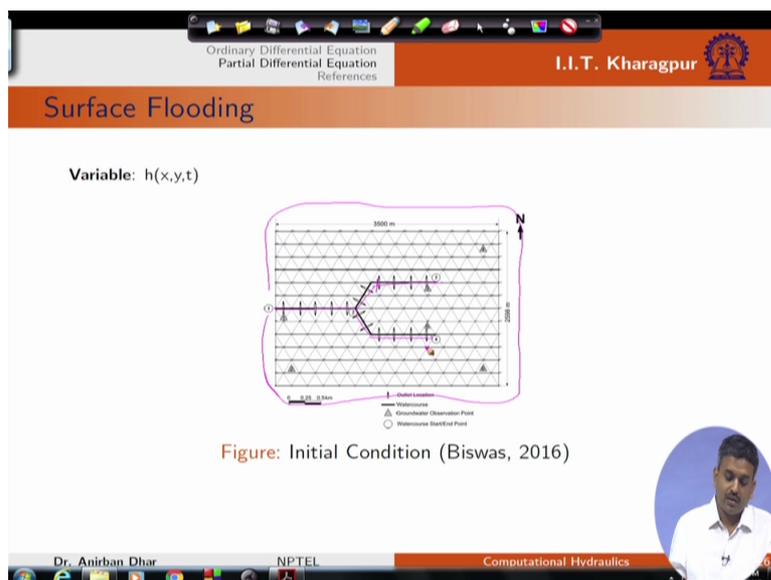
If we consider this surface flooding, so what are the boundary condition? So we need to define the outer boundary, so obviously there should be no flow condition or there should be no flow from this boundaries.

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And this is your internal channel condition. So obviously from channel there is lateral inflow to the surface area and water spreading is there because of this channel in flow or in flow from the channels.

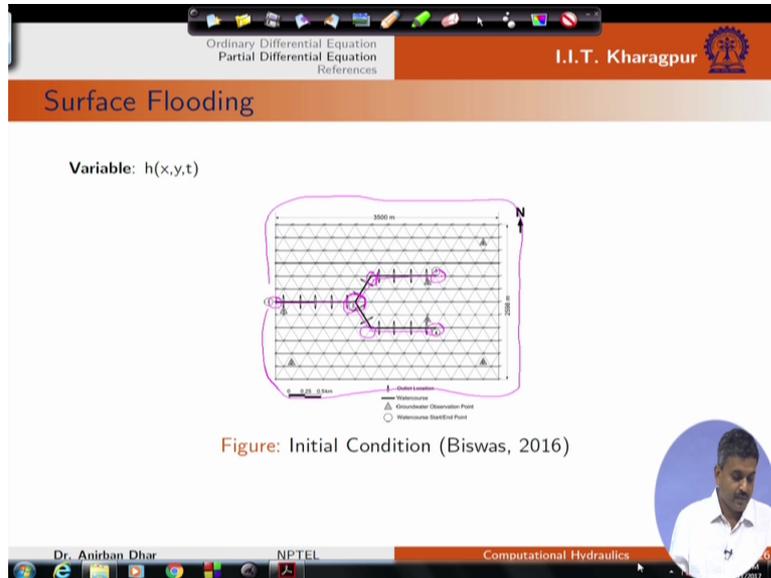
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And obviously in this case this is your junction condition position this is your inlet condition this is our outlet conditions, So 1, 2, 3 these three locations we need to specify the boundary

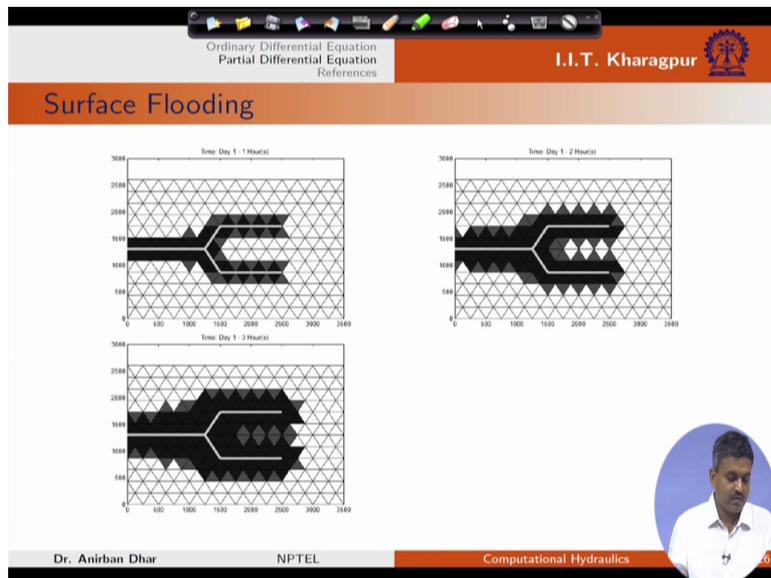
condition or external boundary conditions and for this internal node we need to specify the internal boundary condition. 1, 2, 3 these three locations.

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So we can get this kind of surface flooding situation. So again that is your time varying problem.

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And open channel flow in this case u , w these two are our variables. We have this initial condition that means u equal to u_2 and w equal to zero that means no velocity in the vertical direction and only the longitudinal velocity value is u_2 initially. And with time we can change the inlet velocity and we can get the desired hydraulic jump simulation.

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Open Channel Flow

Hydraulic jump

Variables: $u(x,z,t)$, $w(x,z,t)$

Initial Condition
 $u=U_2$
 $w=0$

Figure: Initial condition of hydraulic jump (Pahar and Dhar, 2017)

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Again for pressurized conduits for left and right hand case this fixed head is maintained. So obviously closure time of valve should be known to specify the condition. And again this is one dimensional system but this is again initial boundary value problem.

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Pressurized Conduits

Variables: $H(x,t)$, $Q(x,t)$

Figure: Connection between two reservoirs (Skific et al., 2010)

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For pressurized conduits again we have we have mass conservation and momentum conservation equation. We have internal conditions or junction conditions we have boundary conditions. So we can solve this problem as initial boundary value problem.

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Pressurized Conduits

Variables: $p(x,t)$, $q(x,t)$

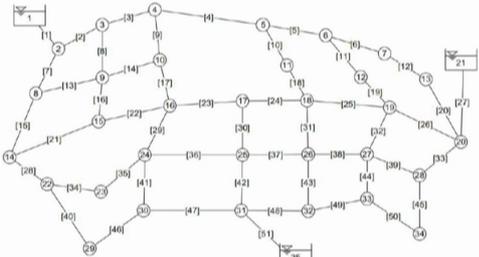


Figure: Pipe Networks (Zecchin et al., 2009)

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Surface water groundwater interaction so obviously the bottom boundary is impermeable. this two boundaries on the left and right hand side can be either specified or no flow boundaries and top surface as this is our depth average equations we generally use for this case. So we will have unique depth value or unique water level value will be there.

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Surface water-groundwater interaction

Variables: $\zeta^s(x,t)$, $p^s(x,t)$, $\zeta^g(x,t)$, $p^g(x,t)$

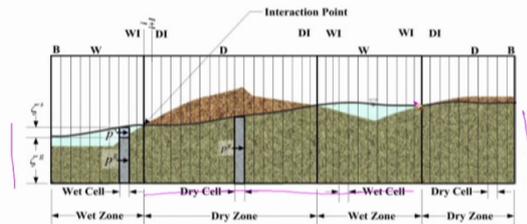


Figure: Conceptual representation of dry cell-wet cell theory (Pahar Dhar, 2014)

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Surface water groundwater interaction. Again for this case we need to define the interaction mechanism between the channel flow and our groundwater system.

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Surface water-groundwater interaction

Variables: $h_s(x, t)$, $Q(x, t)$, $h_g(x, y, t)$

(a) Coupled modeling domain (Gunduz and Aral, 2005)

(b) Stream aquifer interaction (Gunduz and Aral 2005)

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We have this h_g , h_r values for either river or ground water system.

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Surface water-groundwater interaction

Figure: Channel flow/groundwater flow interaction (Gunduz and Aral 2005)

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The interesting point is that in this groundwater equation solution we can either specify h_g or groundwater head value as specified head boundary or specified head boundary. And there can be situations where we can define it as flux boundary, n as a vector which is normal to the boundary or you need vector. And this q_n is some specified value of flux.

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Surface water-groundwater interaction
Boundary Condition

Specified Head Boundary

$$h_g(x, y, t) = H_D \quad (7)$$

Flux Boundary

$$-\mathbf{n} \cdot ((h_g - z_b)) \mathbf{K} \cdot \nabla h_g = q_N(x, y, t) \quad (8)$$

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There can be situation where we have mixed boundaries. Why this is called as mixed boundary or robin kind of boundary condition, because q_C is again dependent on this h_g value. So using this h_g and h_r we can change the flux condition. So although this kind of boundary condition is bit complicated but it combines both specified head and flux boundary condition.

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Surface water-groundwater interaction
Boundary Condition

Specified Head Boundary

$$h_g(x, y, t) = H_D \quad (7)$$

Flux Boundary

$$-\mathbf{n} \cdot ((h_g - z_b)) \mathbf{K} \cdot \nabla h_g = q_N(x, y, t) \quad (8)$$

Mixed Boundary

$$-\mathbf{n} \cdot ((h_g - z_b)) \mathbf{K} \cdot \nabla h_g = q_C(x, y, t) \quad (9)$$

$$q_C(x, y, t) = \begin{cases} -K_r w_r \frac{h_r - h_g}{m_r}, & h_g > (z_r - m_r) \\ -K_r w_r \frac{h_r - (z_r - m_r)}{m_r}, & h_g \leq (z_r - m_r) \end{cases}$$

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And if we are talking about 1D, 2D integrated system like our 1D channel flows situations we can define either internal or external boundary conditions. Internal boundary conditions in terms of junction conditions.

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1D-2D integrated system

Variables: $h_c(x, t)$, $Q_c(x, t)$, $h_f(x, y, t)$, $u_f(x, y, t)$, $v_f(x, y, t)$

(a) Integrated 1D-2D simulations with lateral and flow direction connections (Blade et al., 2012)

(b) Discretization of computational domain

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And surface flooding situations and we can define the boundary condition for this area either flux boundary or no flow boundary. And this condition is actually your channel flow condition we can specify the hydraulic structure corresponding internal condition for this system.

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1D-2D integrated system

Variables: $h_c(x, t)$, $Q_c(x, t)$, $h_f(x, y, t)$, $u_f(x, y, t)$, $v_f(x, y, t)$

(a) Integrated 1D-2D simulations with lateral and flow direction connections (Blade et al., 2012)

(b) Discretization of computational domain

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So depending on the physical situation within hydraulic system, we can change the boundary conditions and in the combined form along with governing equation we can try to solve our problem using this governing equations and boundary conditions. So either they can be ordinary differential equation or partial differential equation. Next lecture we will try to discuss the problem classification based on the nature of solution. Thank you.