Computational Hydraulics
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Lecture 25
Algebraic Equation: Gauss Elimination Method

Welcome tolecture number 25 of the course computational hydraulics. We were in module 2, numerical methods.And in this particular lecture class will be covering unit 21, algebraic equations and specifically we will talk about Gauss elimination. So Gauss eliminationis the first approach of direct solution of our algebraic equation. We already know that we have algebraic form of A phi equals to r. And we need to solve the thing using differentnumerical algorithms. So learning objective, at the end of this unit students will be able to apply Gauss elimination method for direct solution.

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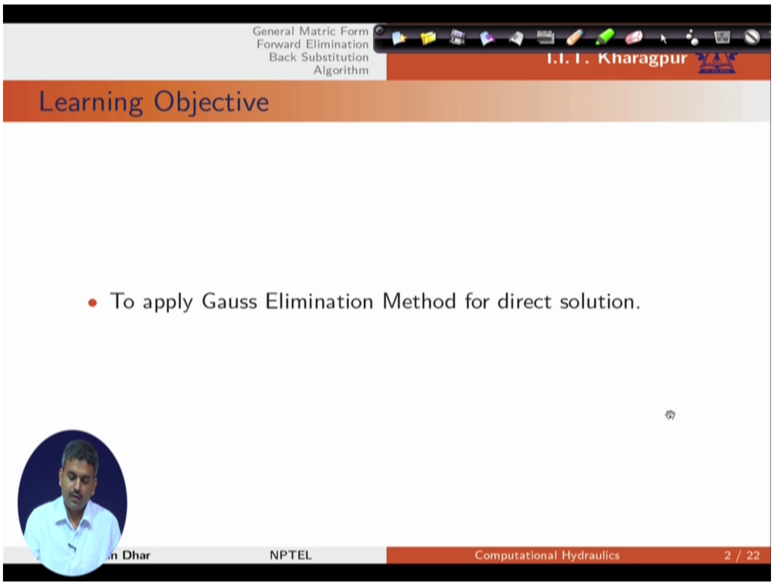

This is our general matrix form where A is ourcoefficient matrix, phi is our variable and r is right hand vector. Now in this case one important consideration is that A is constant coefficient matrix. That means each coefficient of this matrix A is constant for this solution process. These are not variables with respect to this phi.

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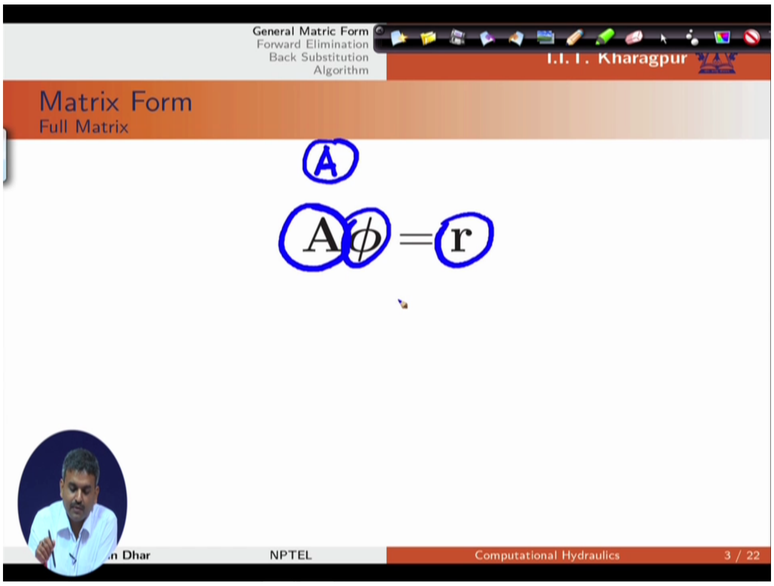

So if we write the complete matrix form we get this N by N matrix which is the square matrix. This A is square matrix,phistarting from 1 to N we have N into 1. So obviously in this case N cross N and N into 1, these two values are same. This is column and this is row.

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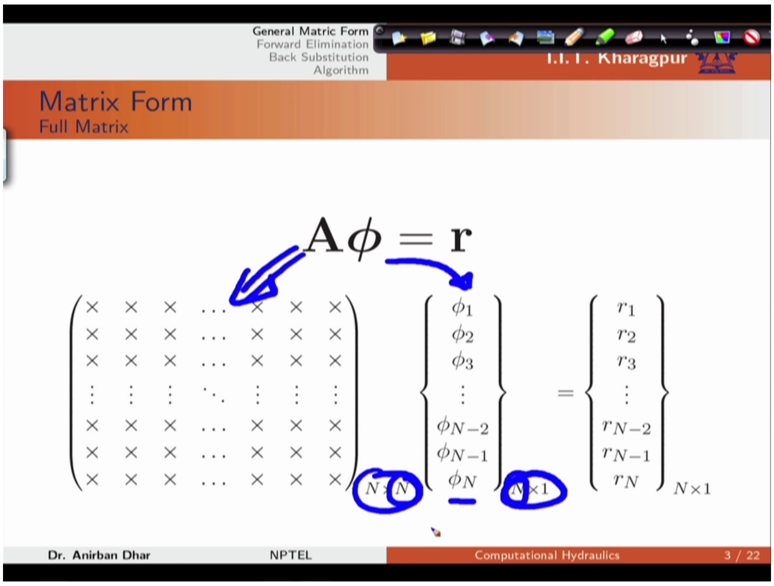

So we can say that these two are compatible matrixes from (multi) multiplication point of view. Now right hand side we have another N cross 1 vector here.

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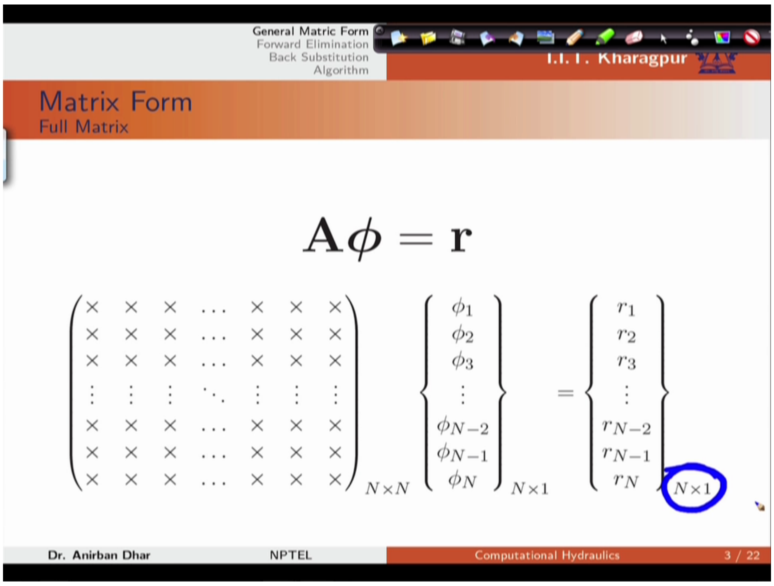

Now we need to apply the Gauss elimination. So what is this Gauss elimination?Let us explain the Gauss elimination process using this 5 by5 example matrix. And we have let us say, five variables,phi1,phi2,phi3, phi4 and phi5.

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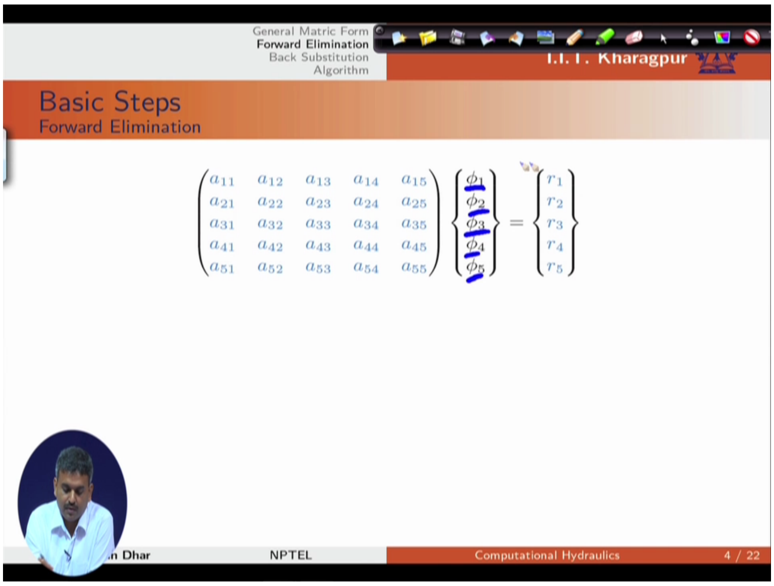

And right hand side this blue colored values or I have colored this A matrix and r matrix with blue color so that we can say that these values are already available. We have not changed this values. Now during thisGauss elimination process we need to change different entries in the A matrix to get the solution. So blue values or blue coefficients are actually undisturbed coefficients in your A matrix.

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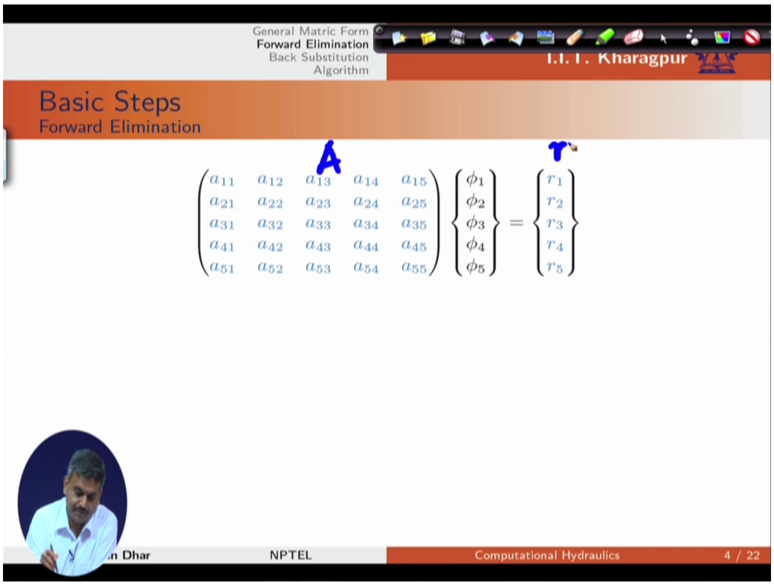

Now let us consider row 2 forour calculation. We will keep this row 1 as constant. So in this row 2 let us consider this gamma 12. Gamma1 2 gamma suffix 1, this is subscript 1 and superscript 2. This means that gamma is a factor which is multiplied with row1 for the calculation of row 2. It is multiplied with row 1 for calculation of row 2.

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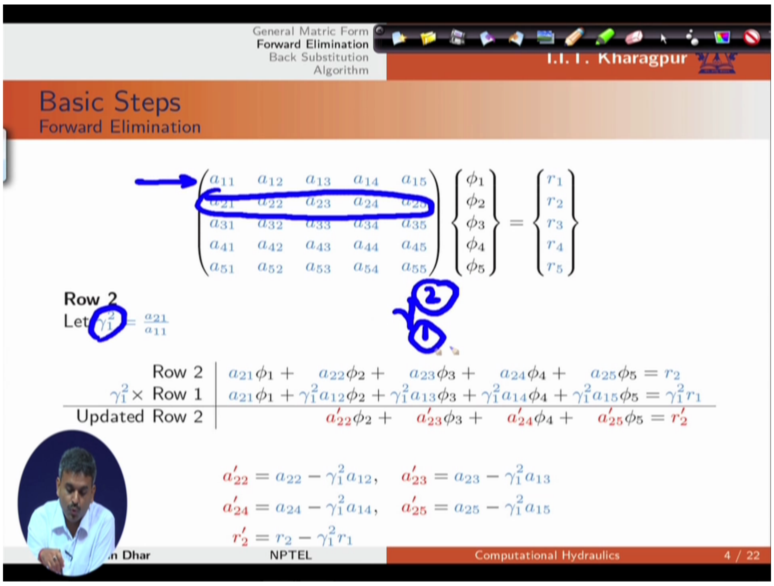

And we are multiplying the row1with this factor which is a21. A21 is essentially first entry in the second row divided by our first term which is ofrow 1. This is our row 1, this is our row 2.

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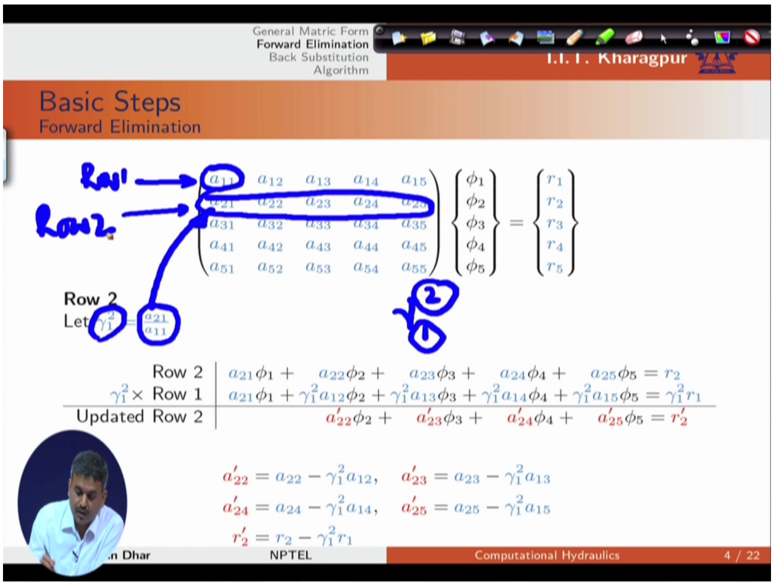

Now let us start the calculation process. As it is we will keep our row 2. So row 2 is with a21 phi1, a22 phi2, a23 phi3, a24 phi4, a25 phi5 for different available entries in the coefficient matrix.

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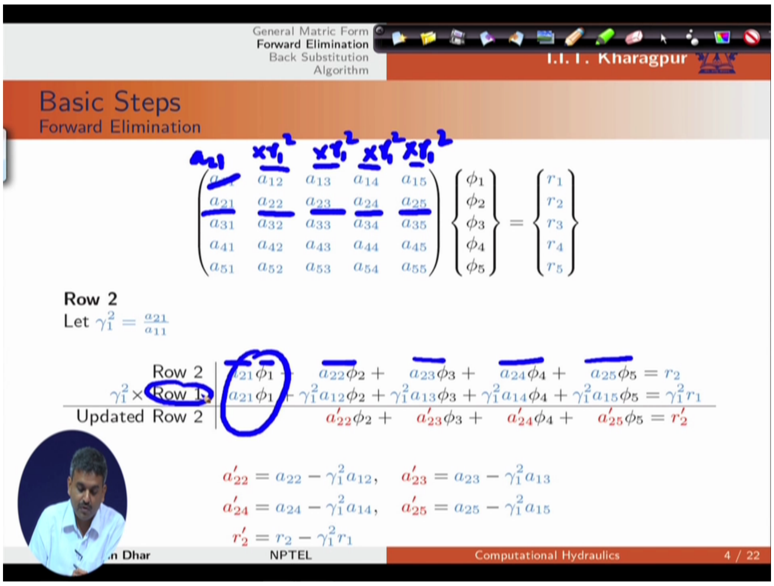

Now with the multiplication of our gamma 1 2 with row 1 we are getting a21phi1 and gamma 1 2 is multiplied with each term on the top. That means it is multiplied with each termhere. Now due to this multiplication this will change a21. So now if we subtract this value or row 1 from row 2 then the first term is getting eliminated from this process. So we will get zero value here.

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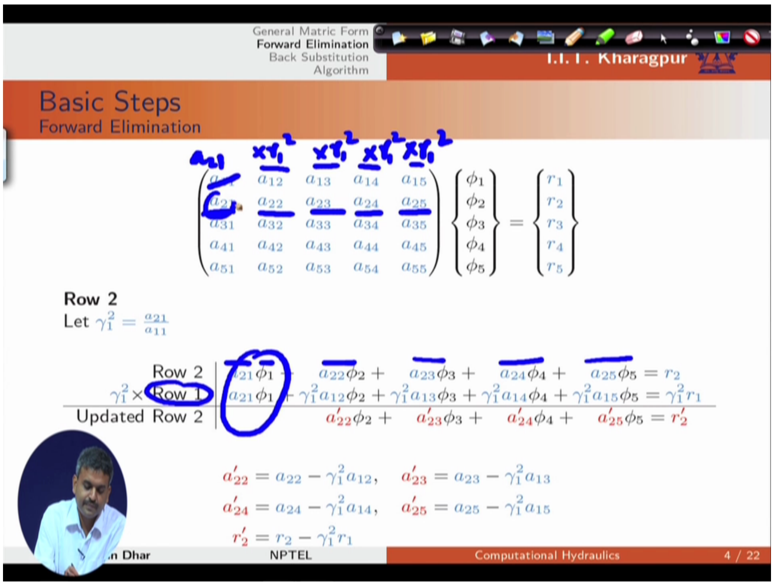

Obviously there will be change in the value of the second one, a22. So with a red color I am specifying the changed coefficients or representing the changed coefficients. So a23 prime,a24 prime, a25 prime and r2 also on the right hand side it is also changing. Because we are multiplying gamma 1 2 with row 2 directly in this case.

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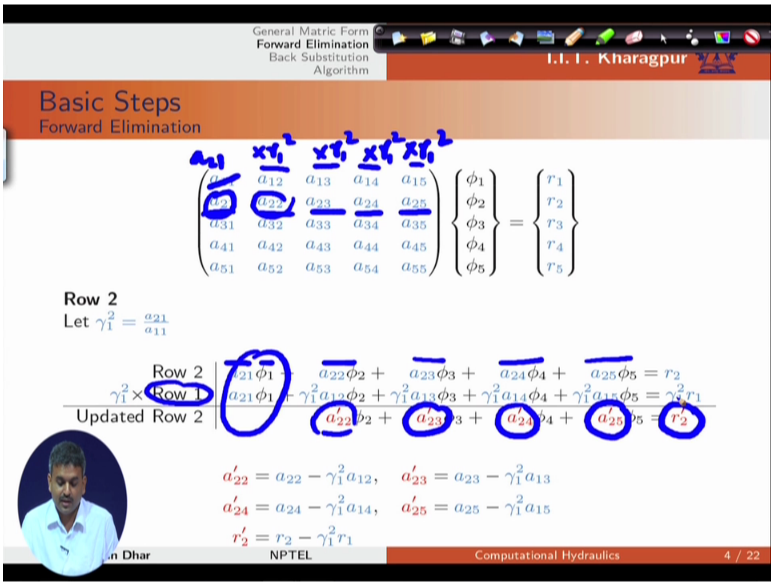

So individually this prime values are essentially a22 minus 12 minusinto a12, like that. So we have different values available here which are with respect to our original entries in the A matrix.

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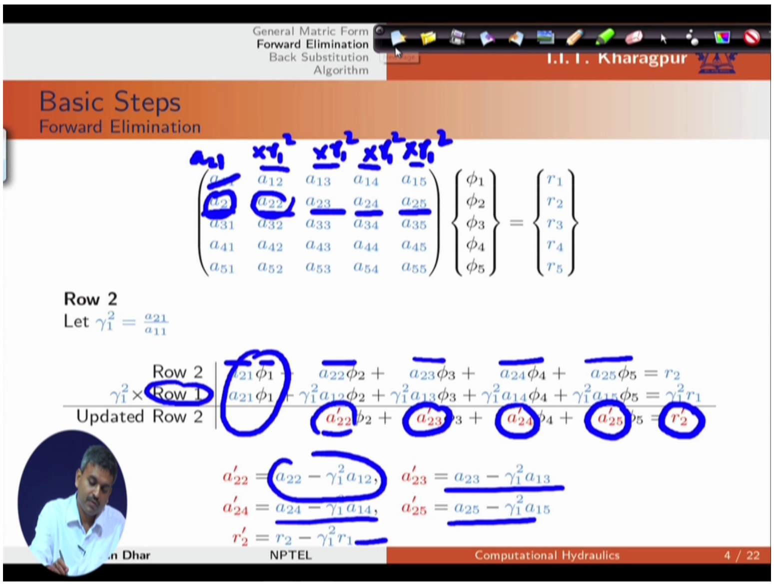

Nowdue to this calculation we have eliminated the first entry of our second row. So only four entries will be there in the second row. So red color represent the changed entries due to first step and if we continue with this for row 3, again we can select another gamma 1 3 which is we are multiplying a factor with the first row for row 3. So again this is the first entry in the third row, this is our third row, row 3.

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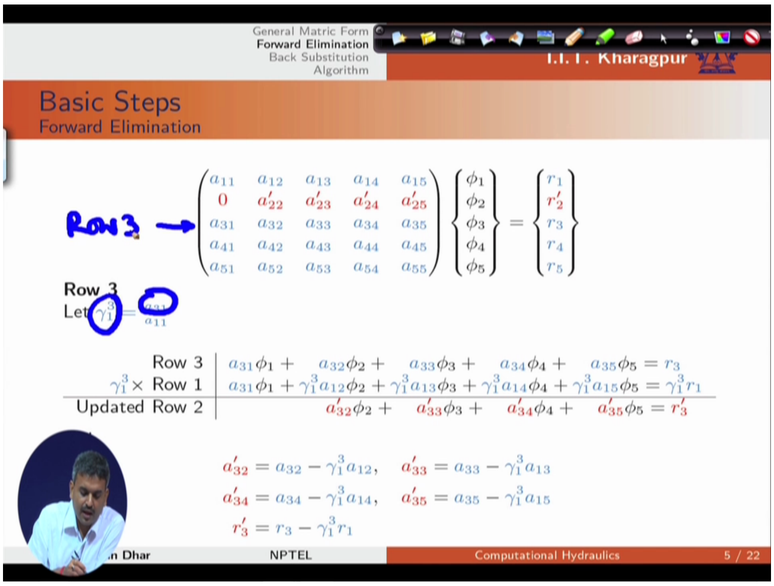

And divided by the diagonal term or first term in the first row. So with this case we can write our things. So this is 31, 31.So this is getting cancelled.

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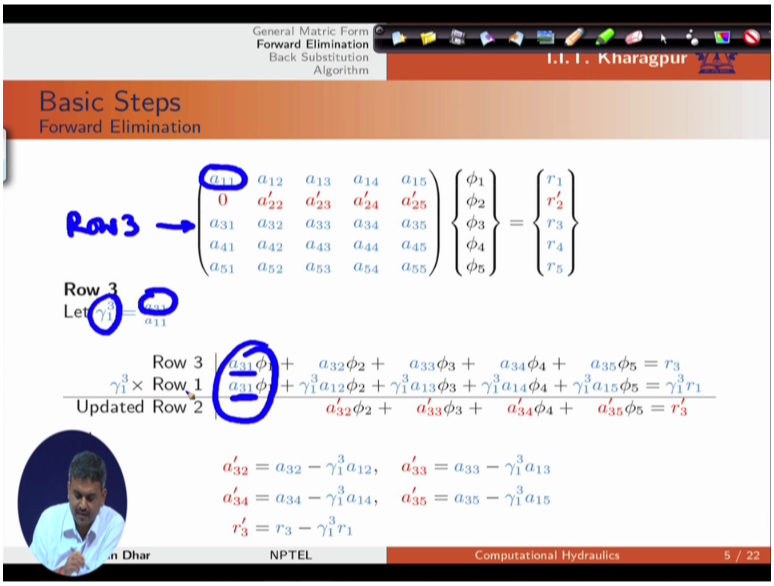

Again we are getting this red colored values due to first step. That means these calculations are based on row 1 only. Because we are multiplying some factor with row 1 to get changed values in row 3. So r3 is also changing. So different values we can represent like this.

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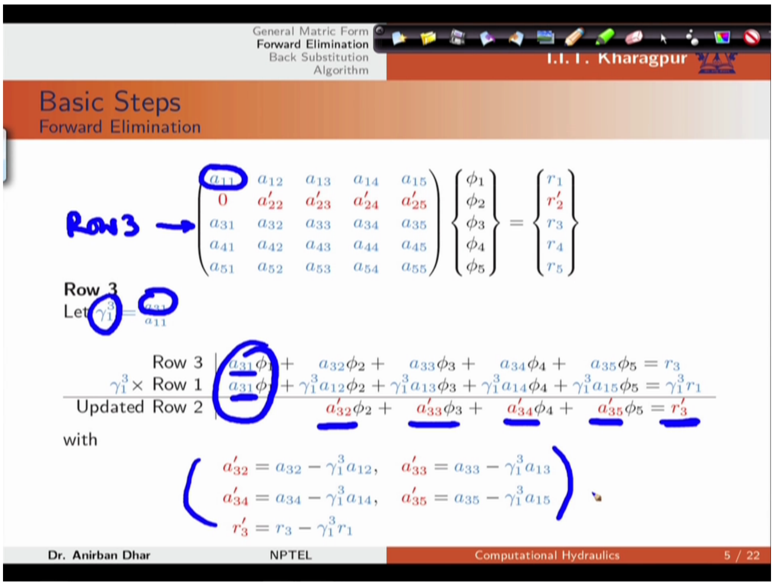

Similarly this process can be continued for next step. Again we are multiplying the first row with a41 divided by a11. Again first term we are eliminating. That means except the first row for first column we are inserting zero values here. Although we are changing therow values for that particular row.

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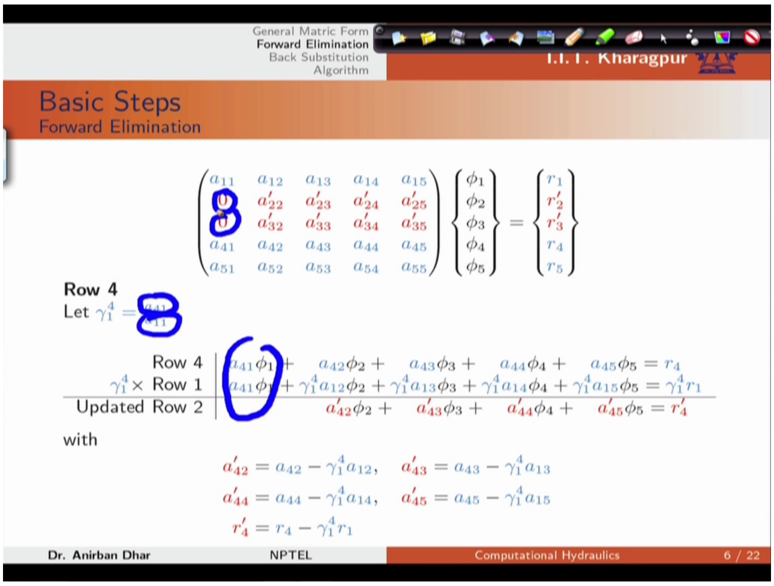

So in this case prime means our first iteration. So in this case again for 42 and similarly for gamma 1 5 we can generate the first level iteration in this case. So with the first look for our row 1 we can eliminate the first term from other rows. So we have eliminated the first terms from other rows. Next level in this case we can see thatthese values are also changed once, r1, r2 prime, r3 prime, r4 prime. These are not original entries. So original entry is only row 1. We are not disturbing the row 1 in this case.

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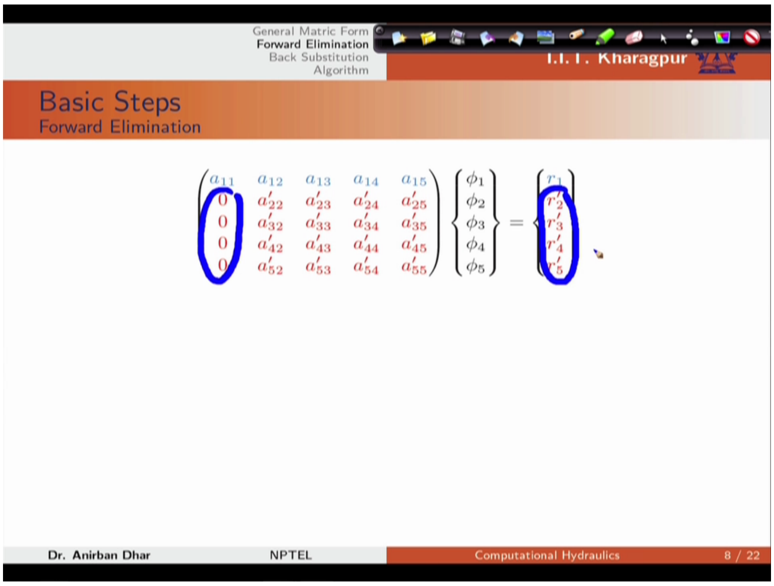

So in forward elimination process, let usstart with row 2. Now we will consider row 2 as our reference row and we will perform the same elimination process for other rows. So in first case we have considered row 1 as constant and we have eliminated the first term from the other rows.

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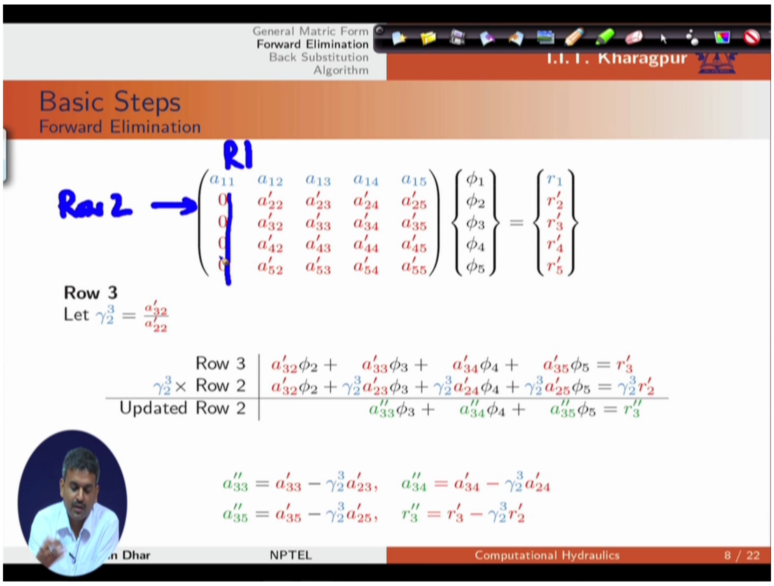

Now at the second step we are going to eliminate the entries here. That means in the second column other than the first and the second row, we will not disturb our first row. So second row is there a22 prime, a23 prime, a24 prime, a25 prime.

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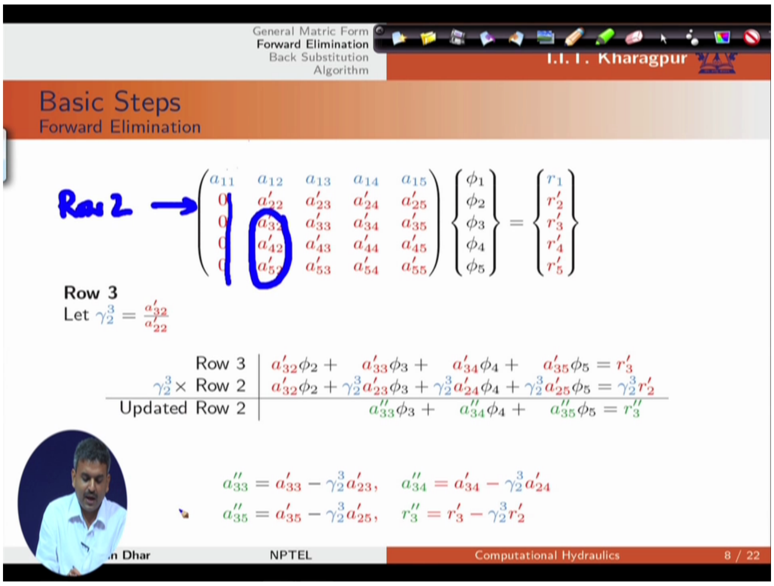

Now here let us multiply gamma 23. Gamma 23 is nothing but our first entry in the third row divided by again diagonal term for the row under consideration. Because row 2 is our reference row and we are performing calculation on row 3, row 4 and row 5. So row 3 is as it is.

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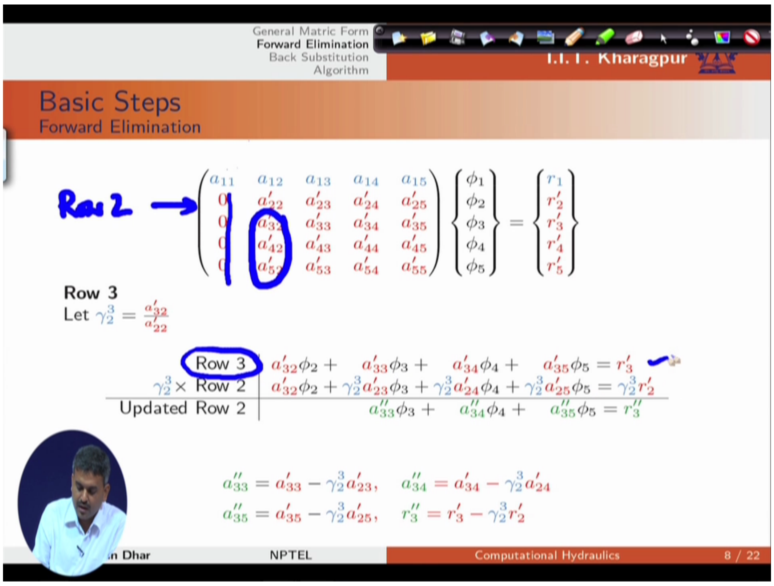

Now we are multiplying gamma 3 with row 2. If we multiply gamma 3 with row 2 again we will get these double prime values. This A double prime 33, A double prime 34, A double prime 35 and r3 in this case. So in this case again we are changing the changed values in the first iteration process. So in the first elimination step we have converted all blue colored values into red colored ones. Now we are introducing this green color to represent the next level iteration.

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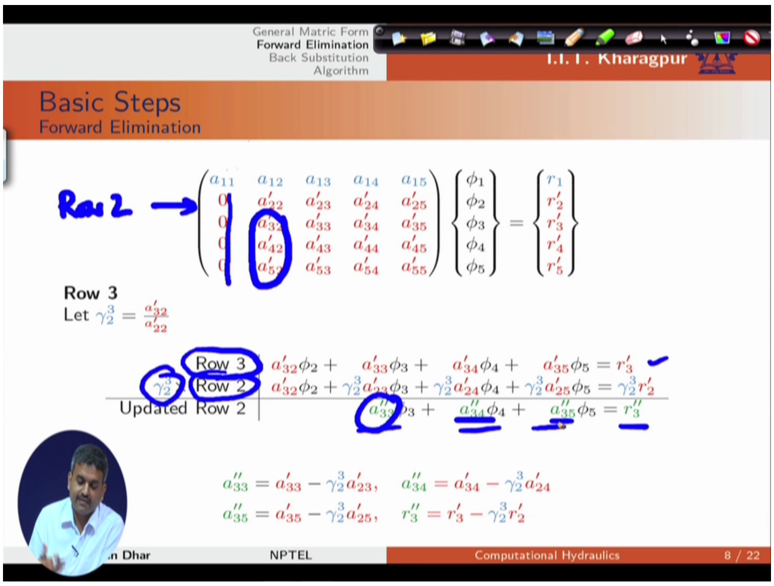

So in the process we have eliminated the first entry in the row 3 in row step 2. Again we have generated this r3 double prime.

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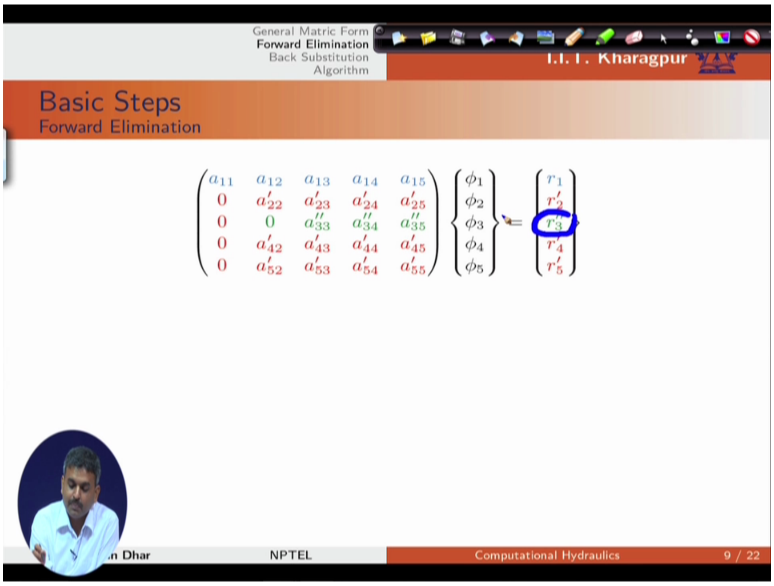

Now if we consider that row 4 again we can multiply a 42 with prime divided by a22. A22 is our diagonal term here. And we can again generate next level changed values for these 4 entries. Because the first entry will be zero in this case. Zero plus this one.

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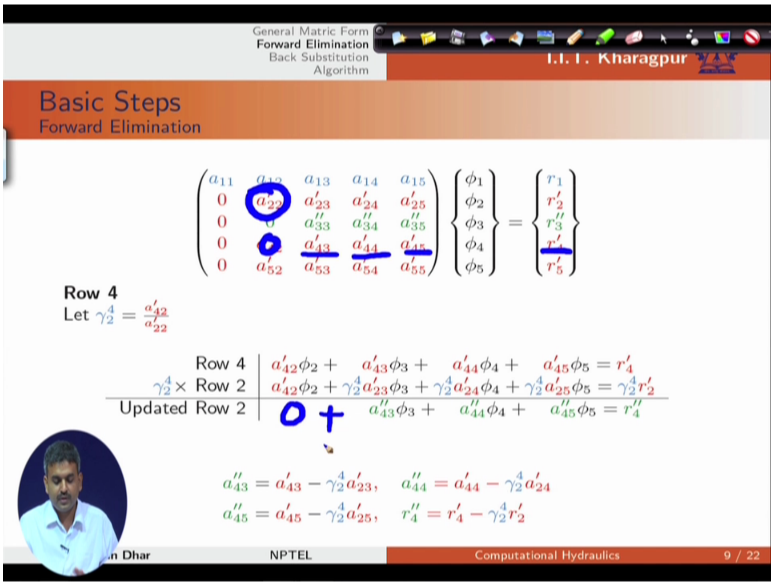

So in this process we have eliminated the next row. Similarly for fifth row with respect to row 2 we can multiply 52 in this case divided by a22. And we can get our desired multiplied value here. And in this case again green values are actually changed values.

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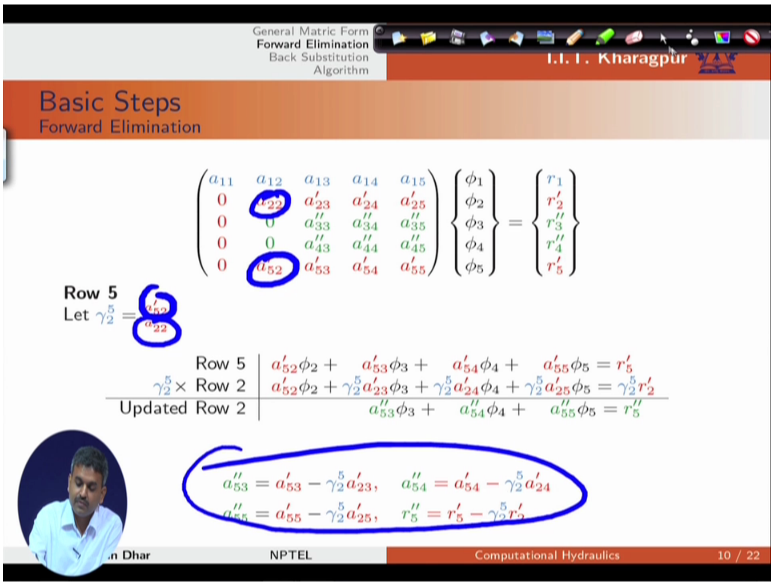

And finally after two steps, we have generated changed values for first the second, third, fourth and fifth rows. Okay. In this process we have eliminated the first column and we have inserted zero values here. Similarly for second column except first and second row, we have introduced zero values and we have changed other rows.

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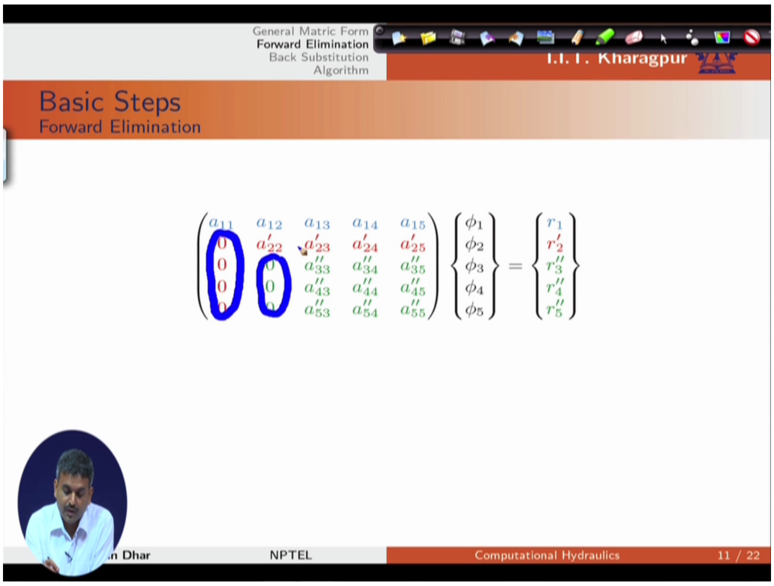

If we follow this process we can again eliminate the next row that is starting from fourth row up to fifth, we can eliminate the third column and we can introduce zero values here.

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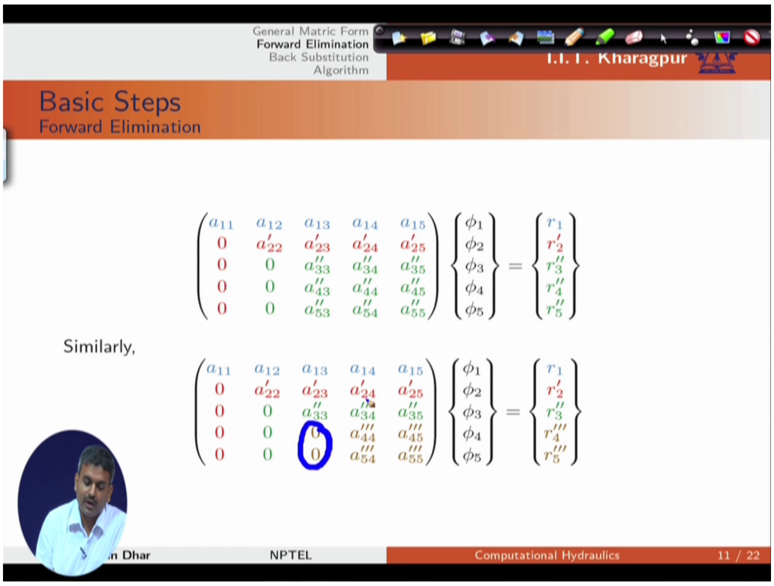

So our reference row is row 3 and we are operating on row 4 and row 5. Obviously in this case we need to multiply this gamma 34 and grammar3, this is 5, with row3 to get these desired values. Obviously this is third level or third step.So a triple prime is the representation for that. So we have represent it with a different color.

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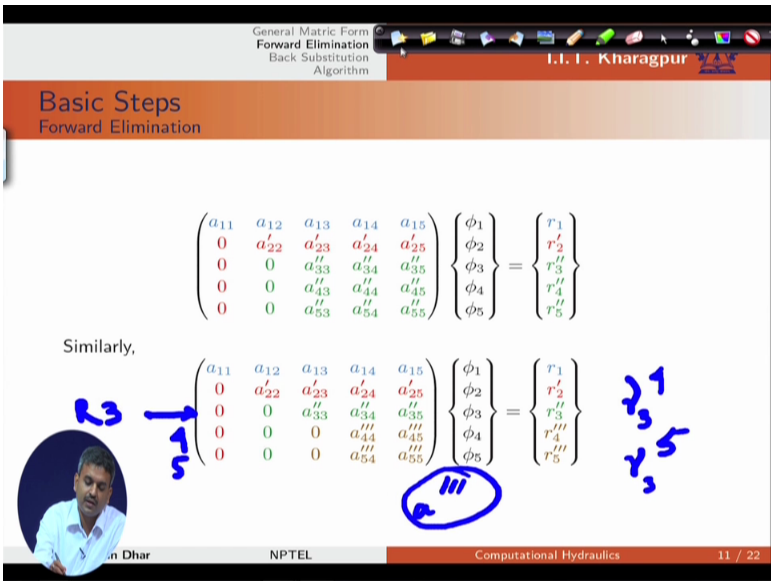

So in this process we have changed the second, third, fourth and fifth. So only change is there for second and third row. Fourth and fifth row is getting changed with every step. That means with respect to r1, r2, r3.

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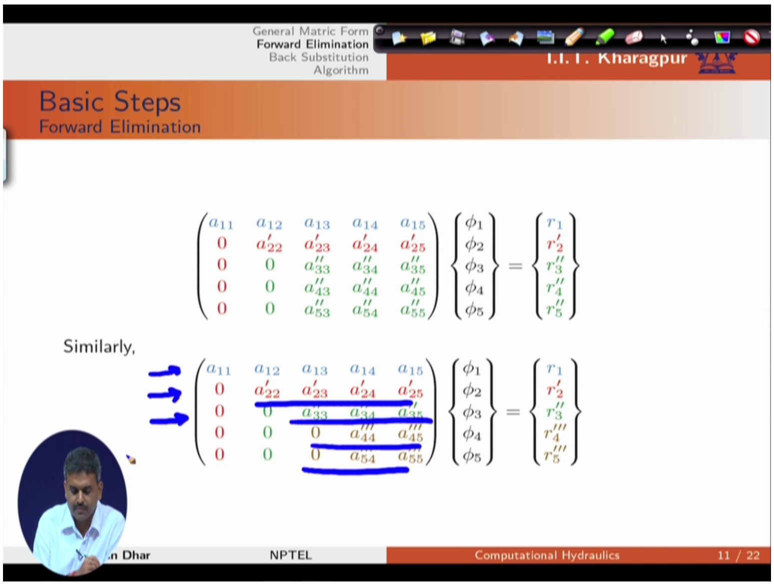

Now similarly we can consider this row 4 and we can eliminate thisfourth term. And our objective was to generate this upper triangular matrix including the diagonal term.

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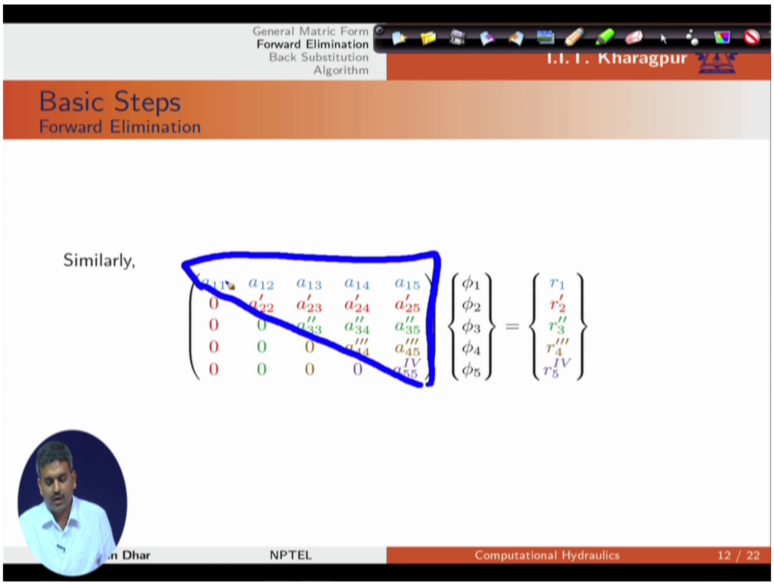

So what is the advantage? In the forward elimination process if we can generate this upper triangular matrix, finally we can directly get the last value. That means A this 14 is the value after fourth iteration multiplied with phi and this is r5 14. Obviously phi5 will be directly r5 4 divided by a 55 14. That means after fourth step. So directly we can calculate the value and we can use this value to calculatethe values phi4, phi3, phi2,phi1.

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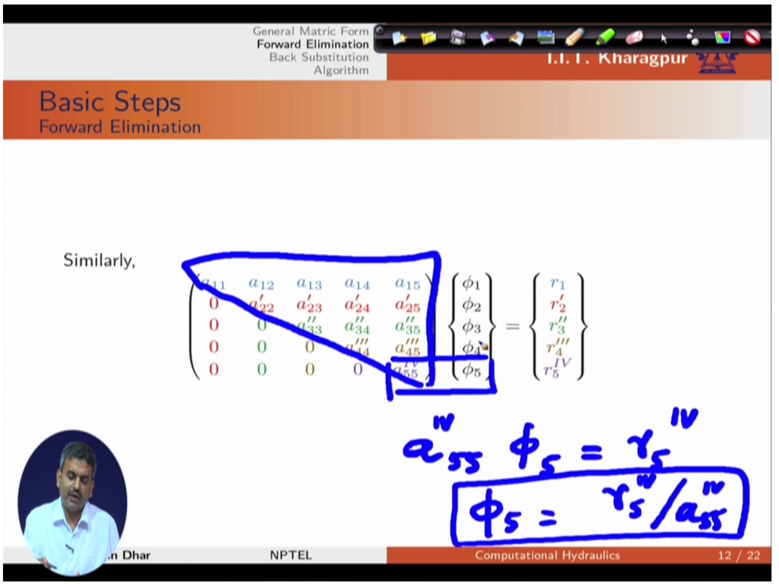

In the firstprocess we have moved from row 1, row 2, row 3, row 4 up to row 5. Now we will move in the backward direction with the back substitution process. Now in back substitution as I have discussed, this one is r 5. Now in this process we can directlyget the value here. And now this value is known.

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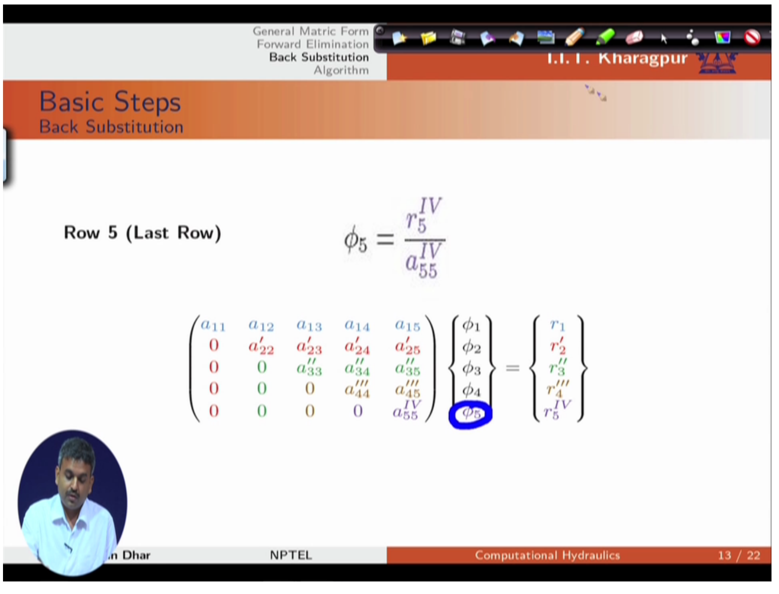

So in this case obviously this will be a 55 14. And in the next stepwe will have this triple prime 44, phi4, 45, triple prime r44. So we have already got the value of Aphi 5. Now we can insert that value here. So just we can transfer this in the right hand side and we can calculate phi4.

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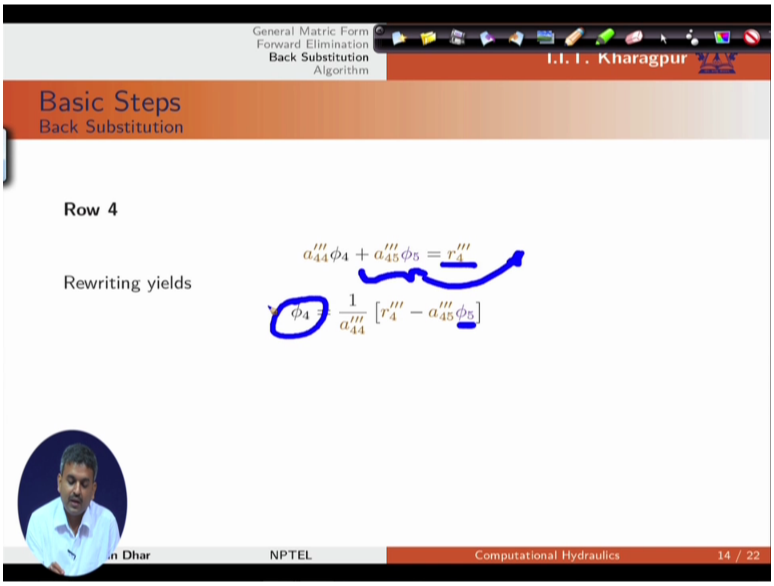

Similarly for other casewe can directly get the values phi3. So in this case we can get this phi3 here. So phi3 with the help of phi4 and phi5.

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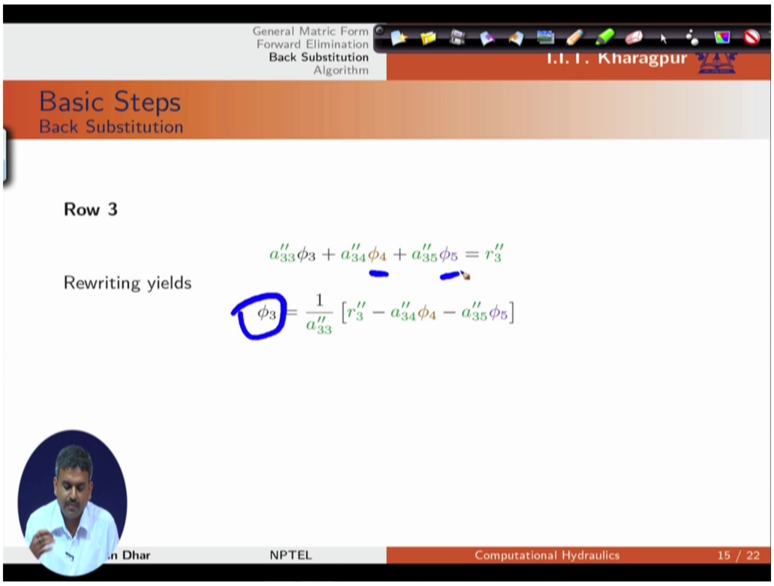

So we are (mokin) moving in the backward direction. So now these three values are known. With these three values we can get, because these coefficients are there and this is only unknown quantity with this coefficient.

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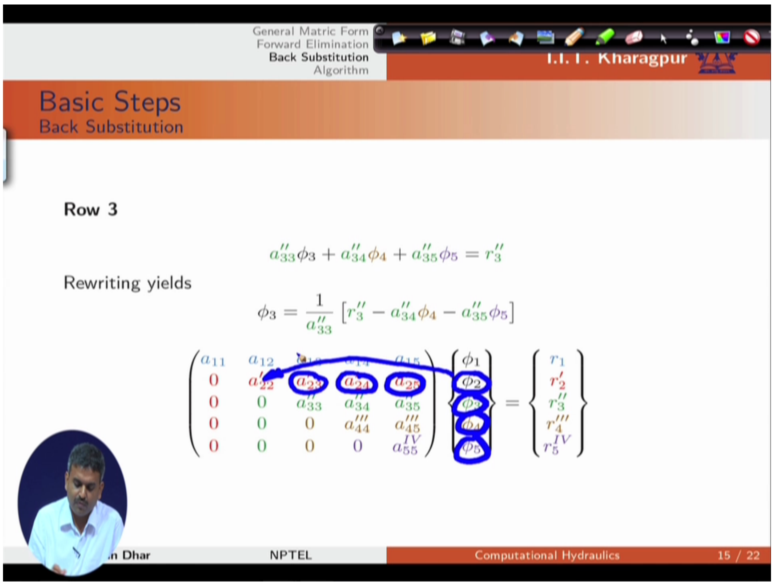

So we can get the phi2 value from row 2. Similarly phi1 from ultimate row 1 which is unchanged value from our original A matrix.

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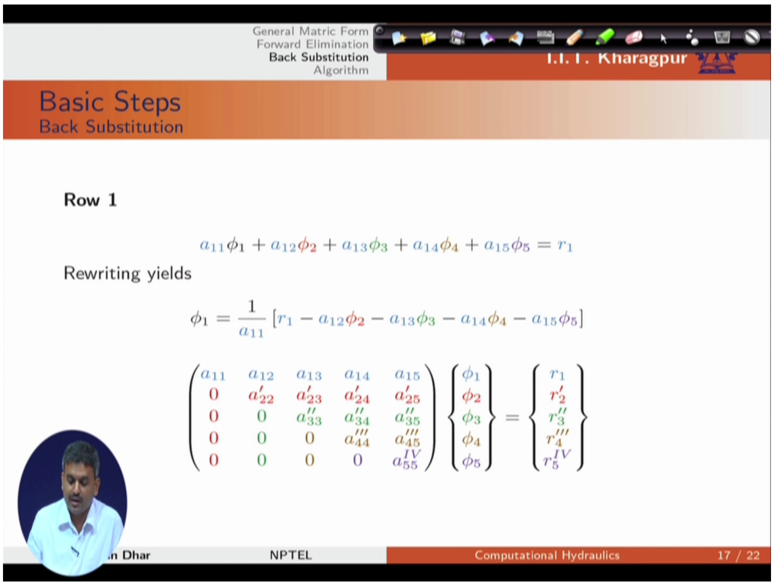

Nowif you see thisas algorithm, we need to input this A vector, r vector and phi is the result. So we are starting from k 1 to nminus 1. Because if we have 1, 2, 3, 4, 5. 1, 2, 3, 4, 5 square matrix 5 by 5. So in this case we are taking this row 1 as a reference. So we are starting from row 1. And we will move up to row 4. So that's why nminus1. And we will consider kplus 1 to n. K plus 1 to n means if you're considering row 1, we should consider k plus 1, 1 plus 1. That means 2 to 5. N is 5 here.So 2 to 5.

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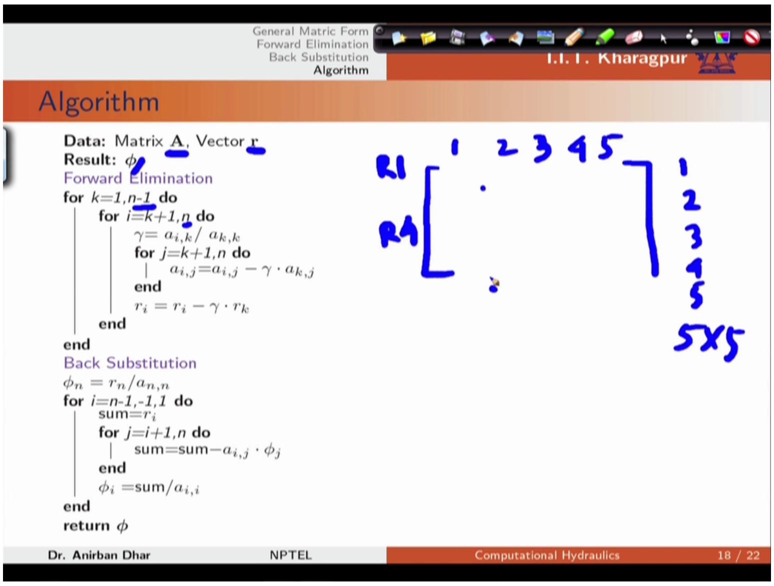

In the next step we will consider 2 to nminus1, this is 1 to nminus 1 here.So that means to R2. In case of R2 we will consider 3 to n. That means 3 to 5. That's what we have done for our previous calculations. Now this is our factor gamma.

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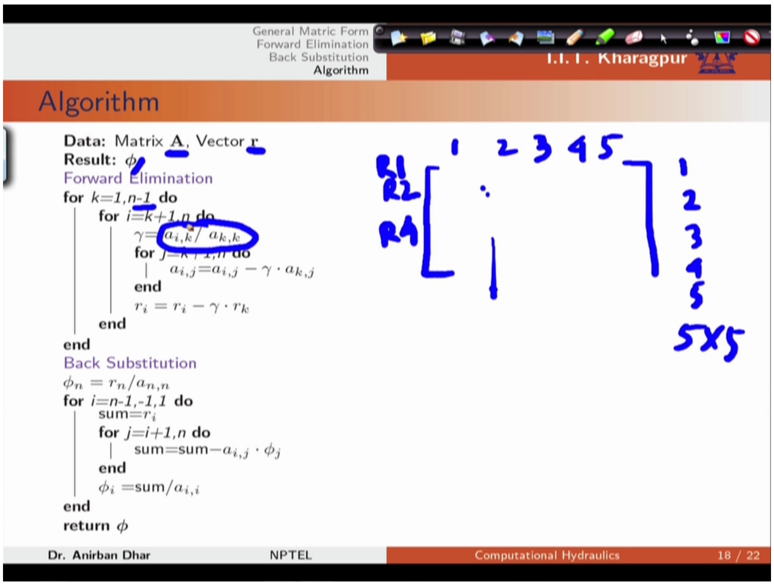

Now we need to subtract this from our aij. Aij we are keeping matrix as it isfor entries in the capital A matrix. And always weare updating those values. Similarly we need to update r values. So this is all about forward elimination.

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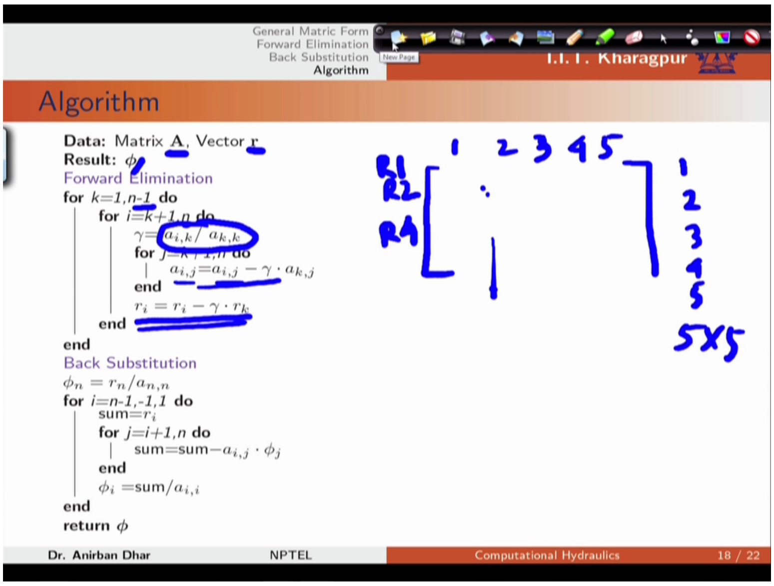

And in the next step we need to use this back substitution phi N, rN divided by a NN which is a55 in our case. Now we will start with n minus 1. So 4 to 1 again we started this sum is equal to r1. So r1 minus aij phi j. So whatever values available that is, i plus one, if we are starting with i. So i plus starting from i plus 1 to N numbers of values are already available or function values or phi values. So we will subtract this, we will consider this sum and this sum divided by a ii. This will give us the phi i value for a particular row.

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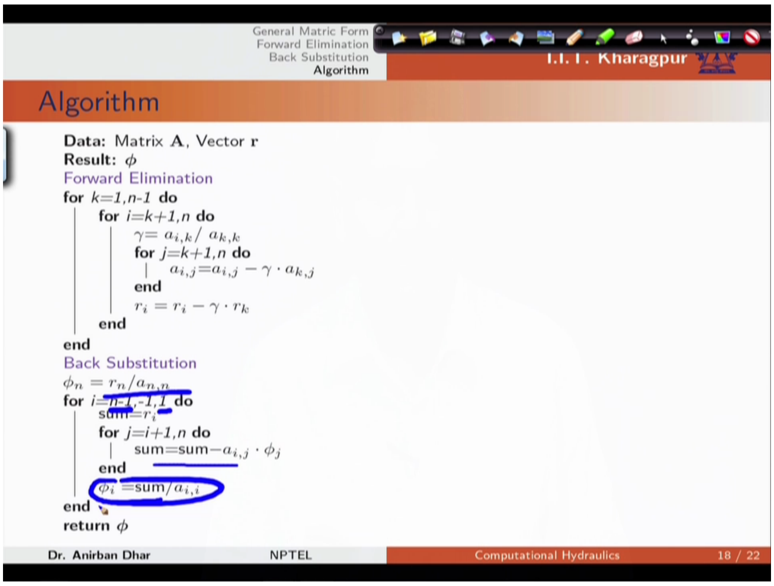

So row wise in the first step we have forward moment is there. That is why it is forward elimination. And next step we are moving in the backward direction to get thevariable values. That is why it is called as backward or back substitution.

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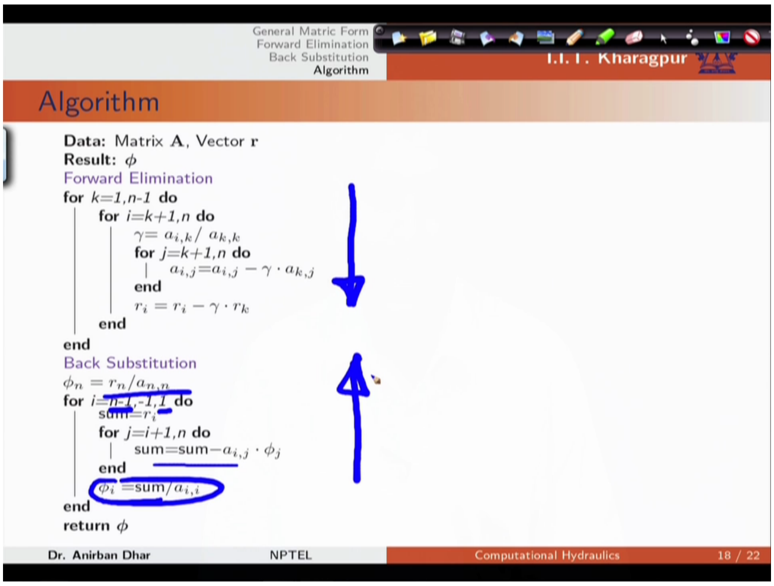

And at the end we can return this phi. So what are the problems?If the first term starts with zero, so division by zero is a problem. Then there will be associated problems with the round off errors and there maybe ill-conditioned systems. If we change a particular variable with a slight value, if there is too much change in the system, then we can call it as ill-conditioned system.

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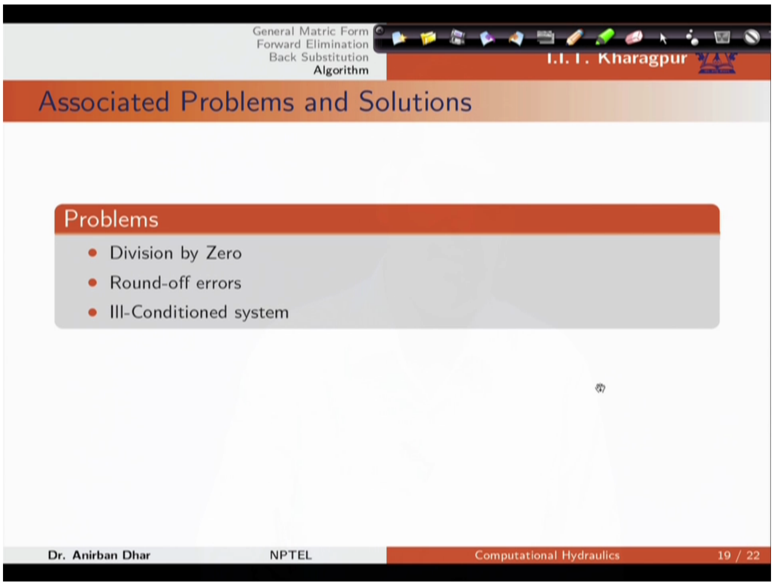

The solutions, pivoting. We can interchange the row or change the column to avoid these situations. And scaling, we can dividea particular (ra) row with a large value to maintain the order of the equations. So we can say that, that way we can avoid these problems.

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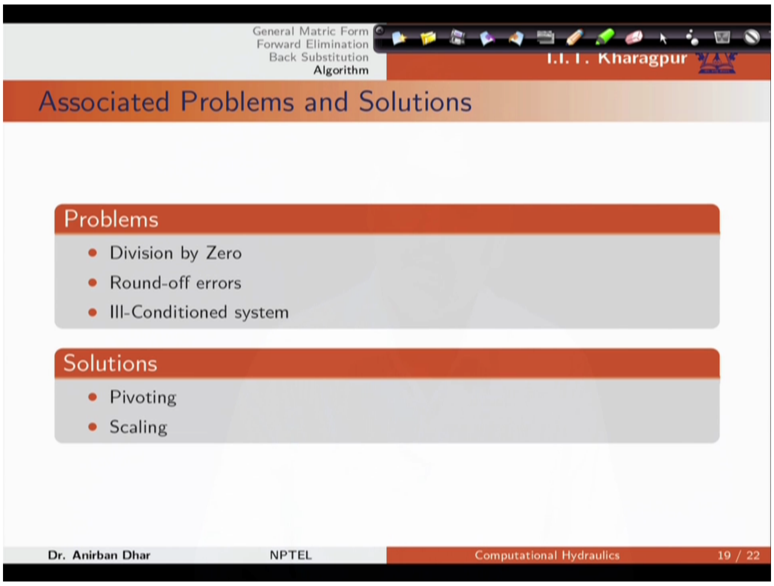

Now let us considerexample. This is one simple examplewhere phi1,phi2,phi3,phi4,phi5, these values are there. And the solution is 1, 3,5, 7, 9. Interestingly in this case only we have banded kind of structure. We have diagonal term available. And two of diagonal terms. And right hand side we have values.

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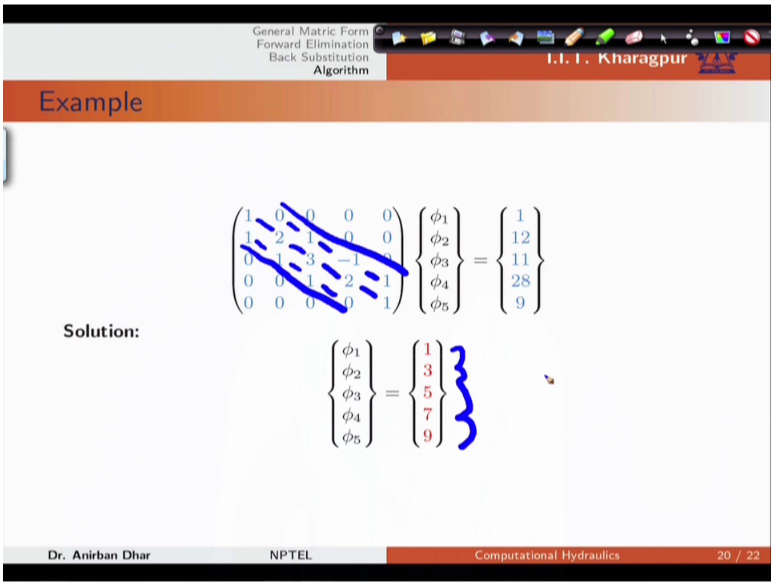

Now we can utilize Gauss elimination to calculate the solution. And second problem isbit complicated. We are considering again 5 by 5 matrix. And in this case we have number of rows with negative and positive coefficients. And the solution is 1, 2, 3, 4, 5. Now we will try tocode everything in scilab platform that we have already discussed, scilab.

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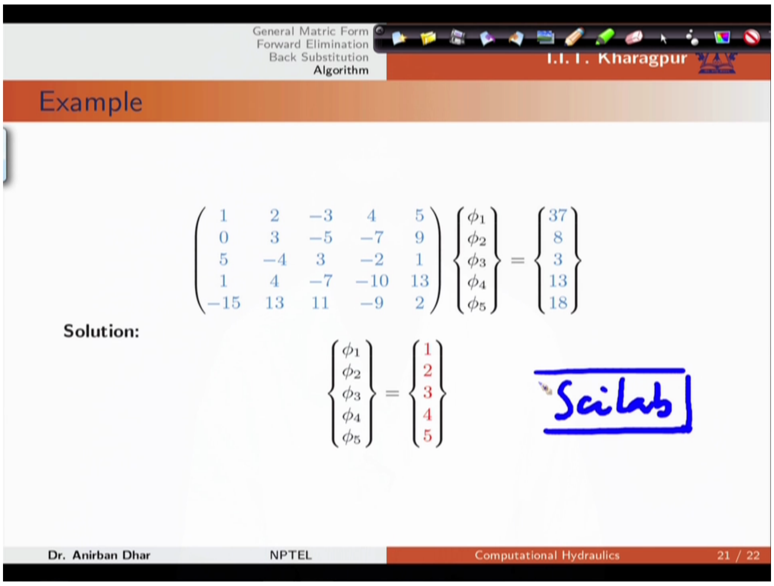

And we will try to get solutions for these two problems. So first problem is with banded matrix and next one is with the full matrix except one zero terminitially available.

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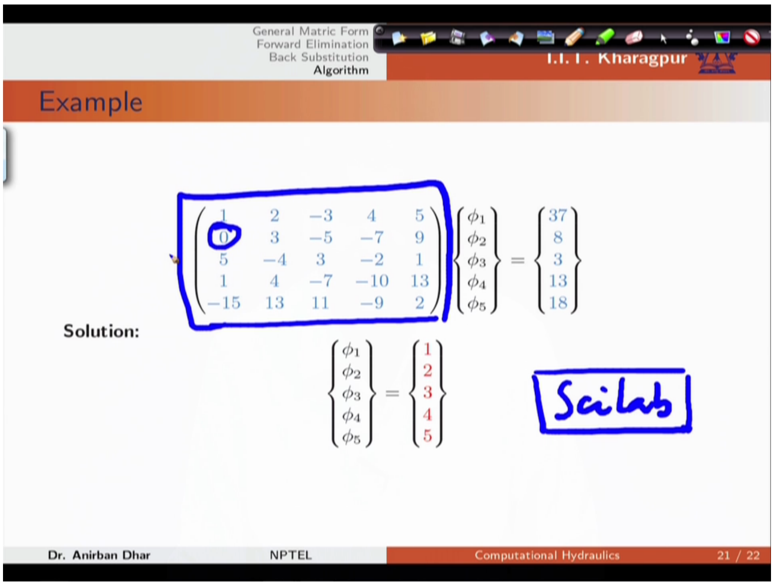

Sowe were discussing about this pivoting problem. Let us say that instead of this one as second row, if it is in the first row, then what will be the problem?

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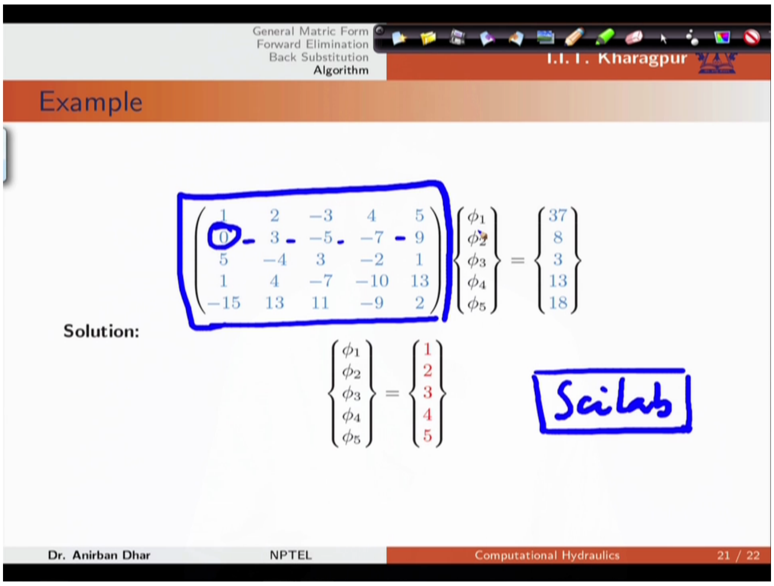

So problem will be that division by zero, we cannot do that thing. So we need to interchange the row to get the solution. Now with this slide we can finish our theoretical portion. Now we will try to implement this Gauss elimination algorithm in our scilab platform. Thank you.