

**Probability Methods in Civil Engineering**  
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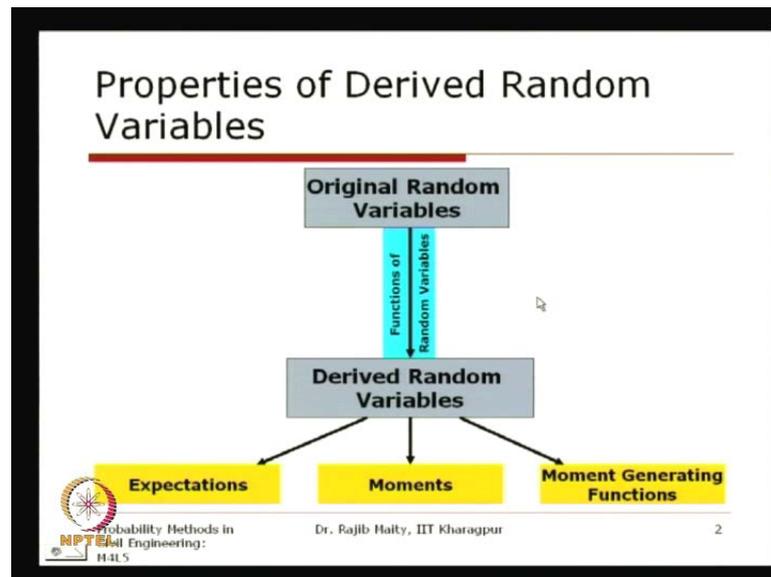
**Lecture No. # 18**

**Expectation and Moments of Functions of RV (Contd.)**

Hello and welcome to this 5 th lecture of this module 4. You know in this module, we are discussing on this functions of random variable. Today we are basically continuing from our last lecture where we have started with that computation of expectation and moments directly for a particular function of this random variable without getting its PDF or PMF explicitly. So, now we have discuss in the last class that, we will be miss why this is useful, because sometimes for some transformation that getting that explicit form of this PDF or PMF may not be that easy particularly for those non-linear function where the roots are more than 1.

So, in those cases if we can get the estimation of these moments and expectation then that will be useful if we can get that directly. So, our point of discussion for from this last class as well as this class is this issue, and so what we are discussing in today's class that this expectation and moments of this functions and random variable this will continuing from this last class.

(Refer Slide Time: 01:36)



So, what we discuss in the last class is that, this is just quick a recapitulation that, there is that original random variable is there. And from this random variable through some functional correspondence with this random variable to some other the function itself is another random variable. So, this random variable we can just call it to be the derived random variable and all. So, if we get this one in the initial part of this module, we have discuss how we can get the PDF and PMF and corresponding CDF as well. From those derived random variables which are the functions of another random variable. It is true that if we get that PDF directly and after that we can get whatever the properties of that random variable we want.

But, here in this one we are discussing that how we can get this expectation, moments and this moment generating functions as well directly from those functional dependence. In the last class, we mostly covered this expectation, and these moments parts with one problem. So, this class we will start with that some more discussion of those problems and mainly today we will focus on this moment generating function. So, you know this that moment generating function what we are looking for if we know moment generating function of a particular random variables then, its derivative about its derivative of a particular order and its value at that origin we will give you that moment of that order.

So, once we know that moment generating function all the, that all the moments, **all the moments** means moment of all order is known to us. So, if we know this information,

that means, it is equivalent to know everything of all the properties of that particular random variable. So, this we will discuss today mainly our focus is that, now for a functional correspondence. If you get one moment generating function, and if you see that that moment generating function has similarity with some of the known distribution then, we can directly say that this distribution. So, the distribution of that particular random variable is following approximately or exactly to that known that distributional form. So, this will be focusing in today's lecture, when we are talking about this moment generating function before that we will see some more problems that we, **that we** did not discuss in this last class on these moments of this random variable.

(Refer Slide Time: 04:39)

## Moments of functions of RV

- In general, the  $r^{\text{th}}$  moment of the function of discrete random variable  $g(x)$  is given by:
 
$$E\{[g(X)]^r\} = \sum_{\text{all } x_i} \{g(x_i)\}^r p_X(x_i)$$
- And the  $r^{\text{th}}$  moment of the function of continuous random variable  $g(x)$  is given by:
 
$$E\{[g(X)]^r\} = \int_{-\infty}^{\infty} \{g(x)\}^r f_X(x) dx$$



Probability Methods in  
Electrical Engineering:  
H4L5

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3

So, quickly we will just see what we have seen in this in the last class that we first discuss about the moments of the function of random variable in general, where the  $x$  is one random variable, and its that functional form is a  $g(x)$  which is equals to  $y$ . So, the  $y$  goes to  $g(x)$  is that functional form, and so if we want to know what is the  $r$ -th moment of that function itself. Then, we know that this will be the expectation, this moment when we are taking, we are taking the moment with respect to the origin. Now, if this  $X$  is a discrete random variable, then we know and we discuss that it should be the summation of that particular value of that function power  $x$  multiplied by the probability mass concentrated at that point.

Means, the probability mass for that particular outcome  $x_i$ . So, in this way if we just add of all such possible  $x_i$  that is all possible outcome, then we will get its expectation and that is the in general form that is the  $r$ -th moment of that function  $g(x)$ . This is for the, **this is for the** discrete one, because we are taking this summation. If we, if the random variable is continuous, then also we have seen that it will be the integration, and integration of the **of the** function of PDF multiplied by the  $r$ -th power of that functional form. And that integration should be done over the entire support of the random variable  $x$ . Here it is shown that minus infinity to plus infinity. So, we will get that moments, but this moments are with respect to the is about the origin. Now, you know that the mostly from the second moment onwards it is meaningful, if we take it with respect to the mean.

(Refer Slide Time: 06:40)

### Moments of functions of RV about its mean

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- $r^{\text{th}}$  moment of functions of discrete RV  $Y=g(x)$  is expressed as
 
$$E \{ [g(x) - E[g(x)]]^r \} = \sum_{\text{all } x_i} [g(x_i) - E[g(x)]]^r p_X(x_i)$$
  
- $r^{\text{th}}$  moment of functions of continuous functions  $y=g(x)$  is expressed as
 
$$E \{ [g(x) - E[g(x)]]^r \} = \int_{-\infty}^{\infty} [g(x) - E[g(x)]]^r f_X(x) dx$$



Probability Methods in  
Electrical Engineering:  
H4L5

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4

So, that is why now if we want to know that it is for that with respect to its mean then this. So, this function minus the expectation of that function that is the, that is the mean of this one. So, that power  $r$ , and if we take the  $r$ -th moment of a discrete variable with a respect to its mean then, this is the general form that is  $g(x_i)$  minus expectation of  $g(x)$  power  $r$  and  $p_X(x_i)$ . So, this we multiply and add it of our all this possible outcome ( $x_i$ ). And then, we will get that, this  $r$ -th moment with respect to the origin. This is for the discrete one. similarly, for the **for the** continuous random variable also the functional form remain same. But we have to do the integration over the entire support integration of  $g(x)$  minus expectation of  $g(x)$  power  $r$  multiplied its PDF. So, this integration is

done entirely over its support to get that, r-th moment of that function with respect to its mean.

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### Variance of functions of random variable...Contd

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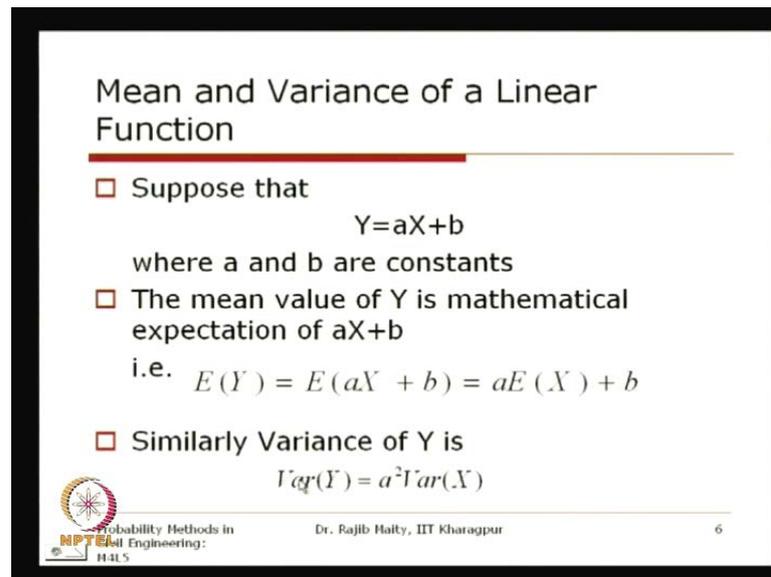
- Variance of discrete functions  $Y=g(X)$  is expressed as
$$\text{Var}\{g(x)\} = \sum_{\text{all } x_i} [g(x_i) - E\{g(x)\}]^2 p_x(x_i)$$
  
- Variance of continuous functions  $y=g(x)$  is expressed as
$$\text{Var}\{g(x)\} = \int_{-\infty}^{\infty} [g(x) - E\{g(x)\}]^2 f_x(x) dx$$

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Electrical Engineering:  
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Now, so, this we have seen and we also know that the variance of a random variable. Here the variance of that function which is also a random variable as well. So, variance of that one is nothing but the second moment with respect to the mean, so in place of the r, if we just write that 2 that is the second moment. So, second moment if we take then we will get the variance. So, in case of the discrete random variable we have to do it for the all possible outcome of this x i. And we have to sum it up this product we have to sum up to get that variance of that function g (x). Now, similarly if this g (x) is for the continuous one, we have seen that the variance of this continuous function which is y equals to g (x).

Then, this is also that second order moment with respect to its mean and mean is this expectation of that function g (x). So, g (x) minus that expectation if whole square multiplied by its PDF and taking this integration over the inter support this will give you that variance.

(Refer Slide Time: 09:11)



Mean and Variance of a Linear Function

- Suppose that
$$Y = aX + b$$
where  $a$  and  $b$  are constants
- The mean value of  $Y$  is mathematical expectation of  $aX + b$   
i.e.  $E(Y) = E(aX + b) = aE(X) + b$
- Similarly Variance of  $Y$  is
$$Var(Y) = a^2 Var(X)$$

 Probability Methods in  
Electrical Engineering:  
H4L5

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6

We have also discuss the issue related to one linear function the common linear function. Here the linear function that, we discuss is your  $Y$  equals to  $aX$  plus  $b$ , this  $X$  is what random variable and  $Y$  is your that another random variable, which is this which is related to  $X$  through this linear function. Where this  $a$  and  $b$  are the constant. Now if this is the linear relationship then, the expectation of this  $Y$  can be expressed in terms of the expectation of the  $X$  that we have seen last class that, expectation of  $Y$  is equals to the expectation of its **of its** function. For this function the constant is can be taken out that is expectation of this  $aX$  can be written that  $a$  multiplied by expectation of  $X$ , and expectation of  $b$ .  $b$  means this is a constant against expectation of  $b$  will be equal to  $b$ . So, the expectation of  $Y$  is equals to  $a$  multiplied by expectation of  $X$  plus  $b$ .

Similarly, if we see the variance of this  $Y$  then, this variance of this of a constant is equal to 0, because this is constant. So, there is no question of any variability. So, and the constant if the constant is multiplied with the random variable, when we take out that variance from the variance then it will be the square that also, we have seen in the last class. So, the variance of  $Y$  for this case for this linear transformation the variance of this  $Y$  is equals to the  $a$  square times the variance of  $X$ . So, if the random variable is multiplied by a scalar factor then, it variance of its multiplication should be the, **should be the** that square of that constant times the variance of the original random variable. So, this is the relationship for. So, this what is shown here is for the first two moments, one is the mean, and second one is for the variance.

(Refer Slide Time: 11:20)

**Example**

Q. The random variable  $X$  has a pmf  $p_x(x) = 1/3$ , for  $x = -1, 0, 1$ . Find the variance of the function  $Y = X^2$

Sol.:

The variance of the function is:

$$\begin{aligned} \text{Var}(Y) &= \sum_{\text{all } x_i} [g(x_i) - E(Y)]^2 f_x(x_i) \\ &= \left[ (-1)^2 - \frac{2}{3} \right]^2 \frac{1}{3} + \left[ (0)^2 - \frac{2}{3} \right]^2 \frac{1}{3} + \left[ (1)^2 - \frac{2}{3} \right]^2 \frac{1}{3} \\ &= \frac{2}{9} \end{aligned}$$

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So, here we will take one problem and this problem is similar to that one what we took in the last class and we discuss about what is its mean and that. So, mean of this of one, of one random variable which is the function of another original random variable that we have discuss in the last class. And we have seen that without knowing the PDF of this one how we can use this relationship and we can get that mean directly. So, in this problem what we are discussing that how we can get variance directly. So, we will be showing the both of methods here may be we will show that, how we can get it directly and also how we will get it from through this PDF. Then, we will show that this two results are same.

So, here the we are taken one that discrete random variable like example. So, this discrete random variable is  $X$  which that PMF is shown as this are there are three possible outcome that  $x$  is equals to  $-1, 0$ , and  $1$ . For all these three outcome the these are all equal probable. So, this  $p_x(x)$  that is, PMF is equals to one-third for all such cases. Now, so this, everything about this random variable  $X$  is known. So, what we have to, what we have to find out that, if the functional relationship is like this  $Y$  is equal to  $X$  square then, what is the variance of this, of this function of this function  $Y$ . So, now in the before I come to this how we have solve it here, what we have seen that in the in the in the last class is that, first of all this one we that for this  $Y$  is equals to, here that.

(Refer Slide Time: 13:35)

$$Y = X^2 \quad \text{Var}(Y) = \sum_{\text{all } x_i} [y(x_i) - E(Y)]^2 P_X(x_i)$$

$$P_X(x) = \frac{1}{3} \quad \text{for } x = -1, 0, 1$$

$$P_Y(y) = \begin{cases} \frac{2}{3} & \text{for } y = 1 \\ \frac{1}{3} & \text{for } y = 0 \end{cases}$$

$$\sum_{\text{all } x_i} (y - \mu_y)^2 P_X(x) = \left(1 - \frac{2}{3}\right)^2 \times \frac{2}{3} + \left(0 - \frac{2}{3}\right)^2 \times \frac{1}{3}$$

$$= \frac{1}{9} \times \frac{2}{3} + \frac{1}{9} \times \frac{1}{3}$$

$$= \frac{2}{27} + \frac{1}{27} = \frac{3}{27} = \frac{1}{9}$$

For this Y is equals to **Y is equals to** X square. If you see it here on this paper that, Y equals to X square in this relationship that we are talking about. So, this X is equals to one-third for x, sorry, that P X (x) it is, is equals to one-third for this all this x that is -1, 0, and 1 and this functional relationship is this. So, from this fundamental also we have **we have** shown it the last class that is that P Y of this y is equals to is coming together two-third, for the sorry, for y is equals to 1, and equals to one-third for y equals to 0. So, the possible outcome of this x are three that 1, 0, and 1. And following this relationship possible outcome for this Y are only two that 0 and 1.

What we have seen that here for this 1, there are the set that X, it will take that is that r X that we discuss in while discussing the fundamental theorem the possible set is both this minus 1, and plus 1. So, this is more probable that y equals to 1, this is one-third. Now, if we know this one, now are in the original question what we are looking for, what is the variance of this Y? So, if we want know that, what is the variance of this Y? And we know that, if I know this PDF then it will be equal to that. So, this possible outcome that is we know that y minus that mean of that expectation of this Y, that is mu y. I can write that mu y. So, this one square multiplied by that its PMF that is what is this problem for all, such x i if I just add up then, I will get that **get that** variance.

Now, this y when we are taking about then this is the 2, the first outcome is the 1. So, 1 minus this mean is two-third the square multiplied by this P X at here is your two-third

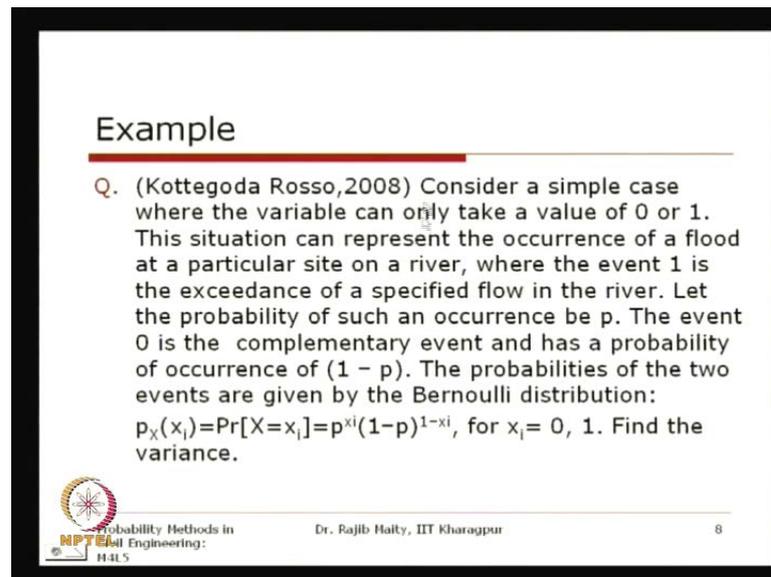
plus another outcome is your 0. So, 0 minus again this is your two-third whole square multiplied by its PMF is one-third. So, if we just do this one then it will come as this, 1 by 3 which is becoming that 2 by 27 plus, 4 by 27 which is equals to 6 by 27 I can write it 2 by 9. So, I know this complete form of PMF for the function and then I am getting this variance. Now, if I want to know that one that directly for this function using this functional relationship then, we know that I have to add that functional form that is  $g(x_i)$  minus that expectation of this random variable  $Y$  the square this multiplied by this  $P_X$  of  $x_i$ .

So, directly I am using this functional relationship minus its means, and I am just adding it up for this all  $x_i$ . So, for this one I need to know this mean of this random variable. In the last class, we have discuss that how directly we can get that mean as well, that mean that in the last class you can refer to the last lecture that mean have from the directly from this function, we have seen that it is that mean is two-third. So, using, here also you have seen that this  $\mu_y$  is your **is your** two-third you can just multiply this and this then, we will get that what is after this PDF and also from the functional relationship. You can get that mean is two-third. Now from this functions evaluated at those outcome directly we can also get its variance, and that is what is explained here.

So, here if you see that when we are taking that variance of this  $Y$  then, this is the summation for this all  $x_i$   $g(x_i)$  minus  $x_i$  minus their mean that square are multiplied by this will be  $p$ , you know that this is PMF. So, this  $p_X$  a  $x_i$ . So, this first outcome that  $x_i$  that the original random variable we are referring to. So, this is minus 1. So, this minus 1 **minus 1**. So, this is the functional form the square  $g(x)$  is equal to  $x$  square. So, this minus 1 square minus two-third this is the mean square this multiplied by what is the probability at  $x_i$  equals to minus 1 which is one-third. Second outcome is 0. So, 0 square that is a functional relationship minus two-third square multiplied by its probability mass at this outcome is one-third. Then, the third outcome is 1 that, 1 square minus two-third power this square multiplied by its probability at 1, which is one-third again.

So, if you just do this one you will also get that 2 by 9 which is the variance of this new of this random variable  $Y$ . So, this is we are getting from this direct from this functional relationship, and what just what we discuss, what we have evaluated here is that after obtaining that, what is that PMF of this  $Y$ . So, both we have seen here.

(Refer Slide Time: 19:32)



**Example**

Q. (Kottegoda Rosso,2008) Consider a simple case where the variable can only take a value of 0 or 1. This situation can represent the occurrence of a flood at a particular site on a river, where the event 1 is the exceedance of a specified flow in the river. Let the probability of such an occurrence be  $p$ . The event 0 is the complementary event and has a probability of occurrence of  $(1 - p)$ . The probabilities of the two events are given by the Bernoulli distribution:

$$p_X(x_i) = \Pr[X=x_i] = p^{x_i}(1-p)^{1-x_i}, \text{ for } x_i = 0, 1. \text{ Find the variance.}$$

 Probability Methods in  
Civil Engineering:  
H4L5

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8

So, next we will take another problem. This is basically if we know this is basically the direct we are calculating the variance of a **of a** random variable. So, you know that when we are talking about that we are calculating the variance of a random variable there what we can say that, the function itself is that is the random variable. So, if that **if that**  $g(x)$  itself is the  $x$  then what we will get? We will get the **get the get the** properties of that random variable. So, here we have taken one problem from Kottegoda and Rosso, 2008. So, there we will we have given one PMF and for that I have to find out the variance. So, problem states, consider a simple case, where the variable can take only a value, only 2 values that is 0 or 1.

This situation can represent the occurrence of a flood at a particular site on a river where the event 1 is the exceedance of a specific flow level in that river. And so, if it exceeds then it is 1 and if it does not exceed then it is 0. Let the probability of such occurrence be  $p$ . So, again if you just see it from the **from the** concept the Bernoulli's distribution that we discuss in the last module. So, here if I just already say that when the random variable taking the value 1, it is the success. So, the probability of a success here is  $p$ . Now, the event 0 is the complementary event; that means, when it is not 1. So, 1 is not exceeding that flow level and has a probability obviously, this probability of occurrence will be the total probability that is 1 minus that  $p$ .

The probabilities of two such events are given by this Bernoulli distribution and this Bernoulli distribution here the PMF is shown that probability of X equals to x i is that probability p power xi multiplied by 1 minus p power 1 minus xi. Obviously, that factor here is that there are two situation how you can select one that is 2c 1 to choose 1 is equals to 1. So, that is why the multiplication factor is 1. So, this is one PMF that is shown. The outcome that is x i value can take only two possibilities one is 0, and another is 1. So, we have to find out what is the variance of this function? Now, as we were discussing that this when we are directly looking for this variance itself; that means, that function itself is that random variable. We will see now how we can get this one from this relationship that, what we have discuss just now.

(Refer Slide Time: 22:38)

### Example...contd

Soln.:

- If we take the second moment about the origin, then from

$$Var(X) = E\{(X - \mu_x)^2\} = \sum_{all\ x_i} (x_i - \mu)^2 P_X(x)$$

$$E[X^2] = 0^2 p^0 (1-p)^1 + 1^2 p^1 (1-p)^0 = p.$$

- Let p=0.1. Since  $\mu_X = p$ , the variance is given by

$$Var[X] = E[X^2] - 2E[X]E[X] + (E[X])^2 = E[X^2] - (E[X])^2$$

$$Var[X] = \sigma_x^2 = p - p^2 = p(1-p) = 0.09$$


Probability Methods in  
Electrical Engineering:  
H.4.5
Dr. Rajib Maiti, IIT Kharagpur
9

If these that, if we take the second moment about the origin then, from that variance of this X is equals to that expectation of this X minus mu x square variance of X is equals to expectation of this X minus mu x whole square. So, this is the second moment that we are taking with respect to the mean, and so as this is the discrete variable. So, x i minus mu this is whole square that multiplied by P X (x). Now, there are, I can use this relationship I can also solve with whatever that outcome is here and from here we can directly solve this particular problem. So, this one what we will see here now, first before I come to this what is the technique it is shown here.

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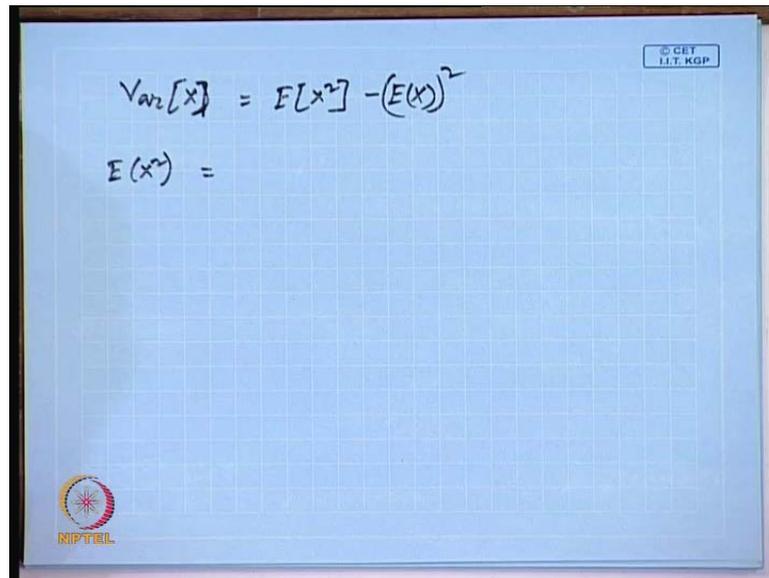
$$\begin{aligned} \text{Var}(X) &= E\{(X - \mu_x)^2\} \\ &= \sum_{\text{all } x_i} (x_i - \mu_x)^2 P_X(x_i) \\ \mu_x &= E(X) = \sum_{i=0,1} x_i P_X(x_i) = 0 \cdot (1-p) + 1 \cdot p = p \\ \text{Var}(X) &= \sum_{\text{all } x_i} (x_i - \mu_x)^2 P_X(x_i) = (0-p)^2(1-p) + (1-p)^2 \cdot p \\ &= p^2(1-p) + (1-p)^2 p \\ &= p^2 - p^3 + p - 2p^2 + p^3 \\ &= p - p^2 \end{aligned}$$

So, what we have seen here if you just see it here on this paper that, expectation of. So, first of all that variance of this X which is as I told that it is the expectation of that X minus mu x that square if I take this one then this is the **this is the this is the** variance. So, which is again that for this discrete random variable this is that x i minus mu x this square multiplied by this probability mass function at that x i and in this way all such x i if I just add up I will get. So, first of all what we have to get, we have get this mu x, **mu x** again we know that, this is the expectation of this X, and which is equals to for this discrete random variable. This is that i is, i for that for all such i so, 0 and 1.

So, if I take this one then, this is x i probability mass function at this point if I just add it then this is the first one it is 0 probability, 1 minus p plus another outcome is that 1 and its probability is p. So, its mean is equals to p. Now, the using this one if I just take this one then, variance of this X is equals to that summation of x i minus mu whole square multiplied by this P X (x i). For all i, for all x i which is equals to the first one is the outcome is 0 minus this mean P which is square multiplied by for this 0 outcome this probability is 1 minus p. This one plus this one again that x i it can now take 1 minus this mean is p whole square this one multiplied by, its probability is p.

Now if i do this one then it is p square multiplied by 1 minus p plus 1 minus p square multiplied by p which is equals to, p square minus p cube plus p minus 2p square plus, this p cube equals to that, p minus p square. That is what we get here.

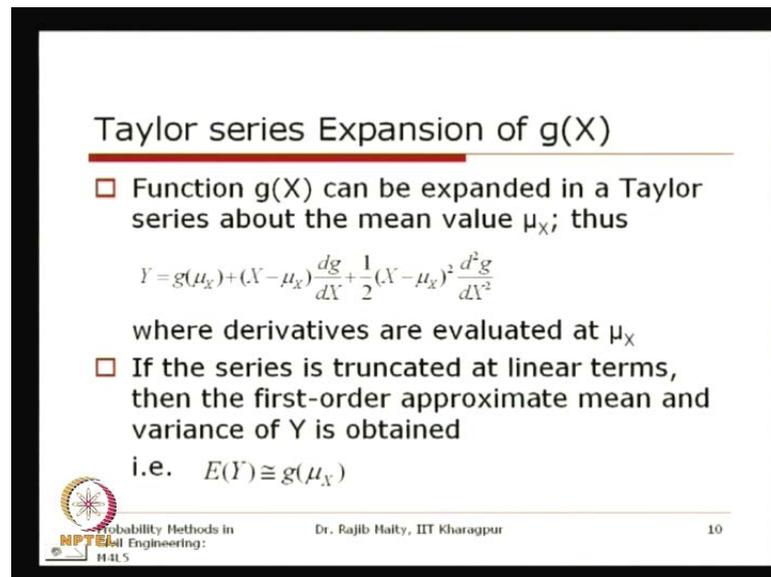
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$$\text{Var}[X] = E[X^2] - (E(X))^2$$
$$E(X^2) =$$

Now, if we are talking about that if we are what the other way of this the same problem if I just take that is that, variance of this X can also be **can also be** represented at that, expectation of this X square minus that expectation of X whole square. This we have seen in the earlier lectures as well. So, from this relationship if I just want to know now this expectation of X we have already got this your p, now if we see that what is your expectation of X square then, if I get this value then, we will **we will** also get that what is the variance of this X, which is solved it here in this procedure is that this expectation of. So, here on this screen if you see that this expectations of X square is, if I take that again that all this outcome square and their then, their probability mass, if you just add up for this to such outcome then, this also comes as p.

So, this variance now is equals to the expectation of X square minus X square. So, now, so, this is your p minus this your p square. So, that. So, the variance here also it comes that p minus p square which is here it shown that variance of X is equals to p minus p square. One special case is shown here, if the p is equals to 0.1, then the variance just we put this values we will get that variance will be equals to 0.09 that is 9 percent. So, we have seen that using this relationship and as well as from that basic one we have shown that this relationship is coming to be **to be** same.

(Refer Slide Time: 28:35)



**Taylor series Expansion of  $g(X)$**

- Function  $g(X)$  can be expanded in a Taylor series about the mean value  $\mu_X$ ; thus

$$Y = g(\mu_X) + (X - \mu_X) \frac{dg}{dX} + \frac{1}{2} (X - \mu_X)^2 \frac{d^2g}{dX^2}$$

where derivatives are evaluated at  $\mu_X$

- If the series is truncated at linear terms, then the first-order approximate mean and variance of  $Y$  is obtained

i.e.  $E(Y) \cong g(\mu_X)$

Reliability Methods in Mechanical Engineering: H415 Dr. Rajib Haitly, IIT Kharagpur 10

In the last class we have also seen that, if some time when this functional relationship is not linear we have seen that, this Taylor series expansion about of, that function about that point very briefly. Here, also again we have just put this one and this will be as I as I mention that this will be mostly useful in case of when we are talking about a non-linear function. So, when we are talking about a non-linear function then, if we just see this one this may be some time very useful to us and that is why this Taylor series expansion is just discuss very briefly. So, this is states that, here that about that mean that is the  $\mu_X$  if I just describe that, if I do that Taylor series expansion of, that function  $g(x)$  then, it comes to be this one.

$Y$  equals to  $g$  of  $\mu_X$  plus  $X$  minus  $\mu_X$  multiplied by that by that derivative the first derivative of that function about  $X$  about  $X$ . This is this should evaluated at this at this point itself. So, plus that half of this  $X$  minus  $\mu_X$  square multiplied by the second derivative of this one plus obviously, the higher order things also will come here. Now, it says that if we just. So, if we ignore this non-linear part if we just curtail it here at this at this linear terms itself then, the series if the series is truncated at the linear terms then, the first order approximate mean, and the variance of this  $Y$  can be obtained as the expectation of  $Y$  is equals to this is now a constant, because we have already get got the value at some point. So, this is a constant.

So, it is approximately equal to this constant, because this  $X - \mu_x$  when we are taking this expectation obviously, this will become 0. So, basically we are considering this function up to this **up to this** linear term and this while taking the expectation becoming 0. And we are just getting that this expectation of  $Y$  is approximately equal to the value of the function at  $\mu_x$ . Now, if the functional relationship is linear then, we have seen in the last class that this will be exactly equal to this **to this** value. But even when it is taking that non-linear form this will be this can be said that this is approximately equal to.

(Refer Slide Time: 31:21)

### Taylor series Expansion of $g(X)$ ...Contd.

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- The Variance  $\text{Var}(Y)$  is:
 
$$\text{Var}(Y) \cong \text{Var}(X - \mu_x) \left( \frac{dg}{dX} \right)^2$$

$$\cong \text{Var}(X) \left( \frac{dg}{dX} \right)^2$$
- Note: If the function  $g(X)$  is approximately linear for the entire range of value  $X$ , then the above two equation will yield good approximation of exact moments



Probability Methods in  
Electrical Engineering:  
H4L5

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11

Similarly, for the variance of this  $Y$  this also can be shown that this is now is this variance of  $X - \mu_x$  multiplied by to that first derivative square. So, this first order derivative evaluated at mean that is basically a constant. So, that constant when we are taking out of this variance we have seen that should be square. So, that is why this value is getting square multiplied by with this variance. Now inside this one variance of  $X - \mu_x$  now  $\mu_x$  again is the constant for whose that variance is 0. So, this basically the variance of  $X$  multiplied by this first derivative of that function with respect to  $X$  evaluated at mean that square.

So, this should be multiplied with this variance of  $X$  to get that variance of  $Y$ . So, this is what we are getting as a as approximation from this that Taylor series expansion. If the

function  $g(x)$  is approximately linear for the entire range of the value  $X$ , then the above two equations will yield good approximations of exact moments.

(Refer Slide Time: 32:42)

### Example

Q. The maximum impact pressure,  $p_{max}$ , of ocean waves on coastal structures is determined by:

$$p_{max} = 2.7 \frac{\rho K U^2}{D}$$

where  
 $\rho$ =density of water;  
 $K$ =length of hypothetical piston;  
 $D$ =thickness of air cushion and  
 $U$ =horizontal velocity of the advancing wave

 Probability Methods in  
Civil Engineering: H4L5

Dr. Rajib Maity, IIT Kharagpur 12

This will seem to be a problem, that is why we have just briefly mentioned about this theory again. So, here the problem is taken on this maximum impact pressure at this maximum. This problem is taken from Aang and Tang, and 1975 that book. So, this is the relationship between this maximum impact pressure and its relationship with other parameters like that density of water, length of hypothetical piston, thickness of the air cushion, and horizontal velocity of the advancing wave. So, this is best on that this air cushion model and sometimes it is used for this to calculate the impact of that sea wave on a structure, that structure may be vertical or inclined.

So, here this relationship is given that what should be the, **what should be the** maximum impact pressure? So, this relationship seems like this that  $p_{max}$  is equal to  $2.7 \rho K U^2$  by  $D$ . Now these are expressed in that **in that** APS units. So, we are taken those some one example with that APS unit itself.

(Refer Slide Time: 33:56)

**Example...Contd.**

Suppose that the mean crest velocity is 5.5 ft/s with a coefficient of variation ( $C_v$ ) of 25%. The density of sea water is about 1.96 slugs/cu.ft, and the ratio  $K/D=20$ . Determine the mean and standard deviation of the peak impact pressure.

Sol.:

We have  $E(Y) \cong g(\mu_x)$

i.e.  $E(p_{\max}) \cong 2.7(1.96)(20)(5.5)^2$   
 $= 3201.66 \text{ psf} = 22.23 \text{ psi}$



Probability Methods in  
Civil Engineering:  
H4L5

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13

If that the mean crest velocity **mean crest velocity** at a particular point if we say that it is 5.5 feet per second with a coefficient of variation. You know this coefficient of variation is the ratio of that standard deviation and mean. So, that coefficient of variation is that 25 percent. So that means, that standard deviation divided by mean is equals to 0.25. The density of the sea water is about 1.96 slugs for cubic feet, and the ratio of K by D is equals to 20. We have to determine the mean and standard deviation of the peak impact pressure. So, now you see we even have not given about anything about this the propriety of this U. We have just got a mean of this particular value and what we are and this relationship is again we can see the non-linear, because U is square is given here.

So, what we are supposed to know is its mean and standard deviation of this peak impact. What is supplied to us is, that mean of that velocity and the standard deviation of that crest velocity, because standard deviation is supplied to this coefficient of variation. Now, this is these one. So, if we want to know so, you know that for a function expectation of this Y that is the mean. We can calculate this is that approximately can be equated to that value of that functional relationship evaluated at this  $\mu_x$ . So, expectation of the  $p_{\max}$  that is the maximum impact pressure is equals to the that 2.7 that is the constant that is multiplied in that equation, that multiply by this rho. Rho means the density of the sea water which is supplied as 1.96 slug by cubic feet.

That multiplied by that K by D ratio is 20, and that U is the mean U that is mean crest velocity is 5.5 square. So, if we do this one, we see that 22.23 psi is the maximum impact pressure that, mean of the maximum impact pressure that coastal structure can experience.

(Refer Slide Time: 36:24)

### Example...Contd.

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Similarly

$$Var(Y) \cong Var(X) \left( \frac{dg}{dX} \right)^2$$

i.e.

$$Var(\rho_{max}) \cong Var(U) \left( 2.7 \rho \frac{K}{D} \right)^2 (2U)^2$$

$$= (.25 \times 5.5)^2 (2.7 \times 1.96 \times 20 \times 2 \times 5.5)^2$$

$$\sigma_{max} = 2.7 \times 1.96 \times 20 \times 2 \times .25 \times (5.5)^2$$

$$\Rightarrow = 1600.83 \text{ psf} = 11.12 \text{ psi}$$



Probability Methods in  
Civil Engineering:  
H.4.5

Dr. Rajib Maity, IIT Kharagpur

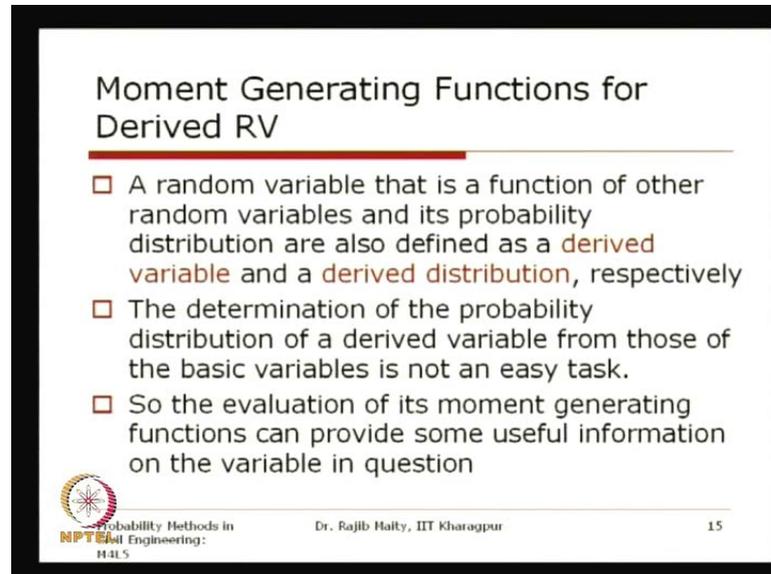
14

Now if we calculate that variance now, variance of that maximum impact pressure. So, we have seen from this Taylor series expansion, and its approximation that variance of Y is related to, the variance of the X multiplied by the square of its first derivative evaluated at its that mean. So, if you take this one then variance of this is p max not the rho max this is p max, the variance of this p max is approximately equal to this variance of this U that is that original thing multiplied by that derivative. Now this derivative is again the 2.75 rho K by D square multiplied by 2U square. So, this one this mean velocity now we have to put here that is that 5.5 this what we have put.

This variance of U is again that we get it from the information of the coefficient of variation. So, the coefficient of variation multiplied by this mean 5.5 this will give you the standard deviation of that U. That square will give you the variance of that variance. So, standard deviation square is gives a variance of this U. So, using this variance and using this derivative form evaluated at this mean we will give you this all this full information. So, this the standard deviation of the maximum impact pressure is equals to after doing this calculation 11.12 psi. So, we have seen that from this approximation how

we can get some information of this the first moments of this function of a random variable where the function is that  $p$  max and this original random variable is your  $U$ .

(Refer Slide Time: 38:24)



**Moment Generating Functions for Derived RV**

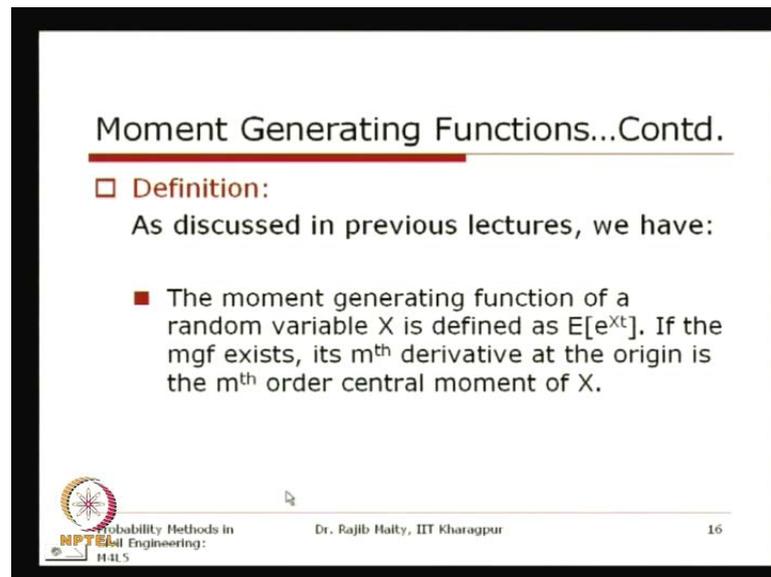
- A random variable that is a function of other random variables and its probability distribution are also defined as a **derived variable** and a **derived distribution**, respectively
- The determination of the probability distribution of a derived variable from those of the basic variables is not an easy task.
- So the evaluation of its moment generating functions can provide some useful information on the variable in question

Probability Methods in  
NPTEL  
Electrical Engineering  
M4L5 Dr. Rajib Maity, IIT Kharagpur 15

Now, we are. So, far what we have discuss is this moments and moments are generally up to that second moment mostly that our point of discussion is. Now, if we just take that if you want to make it more generalized that is if we **if we** want to know what is the moment generating function? Then, using that moment generating function you know that if we, **if we** obtained its derivative any order, or derivative of any order first order, second order, third order. So, in that accordingly we will get that, particular moment if evaluated at that **at that** particular point. So, we have to first of all know that what is the moment generating function for this derived random variable?

So, as we have told at the beginning that the random variable that is the function of the other random variables and its probability distributions are also known as this derived variable and derived distribution. The determination of the probability distribution of a derived variable from those of the basic variables is not an sometimes it is not the easy task, but we have shown it from the fundamental theorem how to do, how to obtain this one. So, here if we just get what we are our point of focus here is now we will get that directly that moment generating function and that will give us some useful information about the variable in question.

(Refer Slide Time: 39:54)



**Moment Generating Functions...Contd.**

□ **Definition:**  
As discussed in previous lectures, we have:

- The moment generating function of a random variable  $X$  is defined as  $E[e^{xt}]$ . If the mgf exists, its  $m^{\text{th}}$  derivative at the origin is the  $m^{\text{th}}$  order central moment of  $X$ .

Reliability Methods in Mechanical Engineering: H415 Dr. Rajib Maity, IIT Kharagpur 16

So far, this moment generating function this is discussed in the previous lecture. That is, the moment generating function of a random variable  $X$  is defined as the expectation of that exponential of the  $Xt$ . So, if we take the expectation of this **of this** function now how to get the expectation of a particular function that we have we know. So, if you if we get the expectation of this particular function that is that  $X$  is here is a random variable the e power  $Xt$ . Then, this is giving you that moment generating function. Then, if the moment generating function exist with this condition that, moment generating function exist. Then, its  $m$ -th derivative at origin is the  $m$ -th order central moment of  $X$ .

So, this is very important. So, what should we do with this moment generating function? So, what we will do is that, the this moment generating function will take that derivative of some order  $m$ -th order in general. If we evaluate that derivative form at the origin then, the central moment should be that. So, at that point if we evaluated the origin then that will give you the central moment of that random variable  $X$ .

(Refer Slide Time: 41:28)

**Moment Generating Functions...Contd.**

□ **Some Notes:**

- The basic concept is that if two random variables have identical moment generating functions, then they possess the same probability distribution.
- The procedure is to find the moment generating function and then compare it to all known ones to see if there is a match.
- This is most commonly done to see if a distribution approaches the normal distribution as the sample size goes to infinity.

Probability Methods in  
Electrical Engineering:  
H4U5

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17

Now, some notes on this moment generating function is that, the basic concept is that if the two random variables have the identical moment generating function, then they possess the same probability distribution. In the others other way also we can say that, the random variable which are having the same probability distribution obviously, should have the same moment generating function. Now, if we get if we have some two random variables and we see that their moment generating functions are same then obviously, it is **it is** quit straight forward that this that their probability distributions are identical. Now, the procedure is to find out the moment generating function, and then compare it to all known ones to see that if there is a match.

So far a particular random variable if we just take that moment generating function and then, we compare it to all known ones and see if there is a match. So, this is one usefulness is that if we have a random variable and we calculate that what is its moment generating function? That moment generating function, if we see that yes it is the known moment generating function and that we have seen it for this standard distribution, this normal distribution or for this gamma distribution. Then, what we can say that, the random variable that we have analysing is following that particular distribution. Some time it is also again, also help to know that if we have some functional relationship and then, we have calculating the derived random variable what is its moment generating function?

And that moment generating function we have we got some similarity with some other known distribution then, we can conclude that after taking this function, then the derived random variable is following this distribution. One such example we will show, the depending on that for from a stranded normal distribution to a normal distribution. How the moment generating function? Through this moment generating function we come to know that this is also **this is also** the normal distribution that we will see. The most commonly done, the most commonly it is done to see that if a distribution approaches to the normal distribution as the sample size goes to infinity. This is another application this is generally done that when the sample size goes to infinity sometimes, some distribution we can say that this can be approximated to some normal distribution.

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### Method of Moment Generating Functions...contd.

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□ Theorem: Let  $F_X(x)$  and  $F_Y(y)$  be two cumulative distribution functions all of whose moments exist. Then

- a. If  $X$  and  $Y$  have bounded support, then  $F_X(u)$  and  $F_Y(u)$  are equal for all  $u$  if and only if  $E(X^r) = E(Y^r)$  for all integers  $r = 0, 1, 2, \dots$
- a. If the moment generating functions exist and  $M_X(t) = M_Y(t)$  for all  $t$  in some neighborhood of  $0$ , then  $F_X(u) = F_Y(u)$  for all  $u$ .



Probability Methods in  
Electrical Engineering:  
H.4.5

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18

So, here the theorem says theorem states that this let this  $F_X(X)$  and  $F_Y(Y)$  be two cumulative distribution function all of whose moments exist. Then, if  $X$  and  $Y$  have bounded support, then  $F_X(u)$  and  $F_Y(u)$  are equal for all  $u$  if and only if, their moment their  $r$ -th order moment with respect origin for both the random variables are equal for all integers that means, for the all the moments. So, expectation of this  $X$  power  $r$  and expectation of the  $Y$  power  $r$  are equal for all this  $r$  **this r** obviously, integer here that is the  $0$  moment, first order moment, second order. All this expectation if they are same, then this their cumulative distribution  $F_X(u)$  and  $F_Y(u)$  are equal.

Second, if the moment generating function exist and that moment generating function  $M_X(t)$  and  $M_Y(t)$  are equal for all such  $t$ . So, this is the moment generating function for the  $X$ , and this is the moment generating function for  $Y$ . Then, if this two are same for all  $t$  in some neighbourhood of  $0$ , then  $F_X(u)$  is equals to  $F_Y(u)$  that is the that cumulative distribution of this  $X$  and  $Y$  is for all  $u$  these are equal.

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### Method of Moment Generating Functions...contd.

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- If  $x$  is a random variable taking integer values, then its moment generating function is by definition:
 
$$m_z(t) = E\{z^x\} = \sum_{n=-\infty}^{+\infty} p\{x=n\}z^n = \sum_{n=-\infty}^{+\infty} p_n z^n$$
- and for continuous RV  $m_z(t) = E[e^{tz}] = \int_{-\infty}^{+\infty} e^{tz(x)} f_X(x) dx$
- **Moment theorem:**
- Differentiating the foregoing equation  $k$  times, we obtain:
 
$$m^{(k)}(z) = E\{x(x-1)\dots(x-k+1)z^{x-k}\}$$



Probability Methods in  
Electrical Engineering:  
M-41.5

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19

So, here we are saying that if the  $X$  is the discrete random variable, so how to get that moment generating function both for this discrete as for the continuous random variable. That  $z$  is the function of that random variable  $X$ . Then the  $X$  is a random variable taking the integer values then its moment generating function is by definition, this  $m_z(t)$  is equals to expectation of  $z$  power  $X$ , which is equals to summation of all such  $n$  that is  $n$  equals to minus infinity plus infinity. And is equals to probability of  $x$  equals to  $n$ . So, for those specific outcome, because this the discrete random variable multiplied by  $z$  power  $n$ . So, this summation is gives you the moment generating function for this random variable  $z$ .

If we take the continuous random variable, then this moment generating function is equals to the expectation of  $e$  power  $tz$ . This is the function and this  $z$  equals to your  $g(x)$  if this is the functional form. Then the integration of this  $e$  power  $t$  multiplied by  $g(x)$  multiplied by their PDF  $f_X(x) dx$ . This integration is taken over the entire support; we will give you the moment generating function of that **of that** function  $g(x)$  that is  $z$  here.

What we have also seen that if we take that derivative, then derivative generally takes the forms like this. That is the general derivative form we have to **we have to** obtain that is, if I take that k times that if I differentiate this moments in k times.

Then  $m(k)(z)$  is equals to expectation of this  $x$  multiplied by  $x$  minus 1 multiplied by  $x$  minus 2 up to that  $x$  minus  $k$  minus 1 that is  $x$  minus  $k$  plus 1. This multiplied by  $z$  power  $x$  minus  $k$ , so this full term if I take that expectation. So, this is basically that  $k$ -th order derivative of that moment generating function.

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### Method of Moment Generating Functions...contd.

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□ Some Facts

1. If  $Z = aY + b$ , then
 
$$m_Z(t) = E(e^{tZ}) = E(e^{aY+bt}) = e^{bt} E(e^{Y(at)}) = e^{bt} m_Y(at)$$
2. If  $Y_1, Y_2, \dots, Y_n$  are independent and  $Z = Y_1 + Y_2 + \dots + Y_n$ , then
 
$$\begin{aligned} m_Z(t) &= E(e^{tZ}) = E(e^{tY_1 + tY_2 + \dots + tY_n}) = E(e^{tY_1} e^{tY_2} \dots e^{tY_n}) \\ &= E(e^{tY_1}) E(e^{tY_2}) \dots E(e^{tY_n}) \\ &= m_{Y_1}(t) m_{Y_2}(t) \dots m_{Y_n}(t) \end{aligned}$$



Probability Methods in  
Civil Engineering:  
H. H. S.

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20

There are some facts for some linear function as for some additive function if you know then this as generally very useful, when we are dealing with this moment generating function. Then if there are two random variable here one is that related to this linear function linear relationship that  $Z$  is equals to  $aY$  plus  $b$ , then the moment generating function of that  $Z$  is equals to we know that expectation of  $e$  power  $tZ$ . Now, this  $Z$  can be replaced by this here this  $aYt$  multiply, plus  $bt$ . So,  $aY$  plus  $b$  multiplied by  $t$ . So, this expectation now this  $e$  power  $bt$ . So, this is a constant. So, if this is a constant we know that this we can take out. So,  $e$  power  $bt$  multiplied by expectation of  $e$  power  $Y(at)$ .

Now this  $Y(at)$  when we are talking about some this is a, **this is a** this is again we can say that this is the moment generating function of the of where the variable is that  $at$ . Because this is a **this is a** general form of this moment generating function. So,  $e$  power  $bt$  multiplied by  $m_Y$ . So, this is the moment generating function for  $y$  multiplied by  $at$ .

So, if I just compare with this linear relationship and this relationship of this moment generating functions. Then if the constant, if there is any constant that constant is now here is multiplied in the form of that  $e^{bt}$ . So,  $e^{bt}$  is there, whatever the constant is here that is multiplied. This is multiplied with that moment generating function of that whatever the constant is here that  $a$ , that  $a$  of that random variable original random variable  $Y$ .

So, this is the relationship between this moment generating function of this between two random variable which are related to, related through a linear relationship. Another one is that if there are  $n$  numbers of independent random variable. Here it is  $Y_1, Y_2$  up to  $Y_n$ , and a function we are talking about is the summation of all such independent random variable that is  $Z$  is equals to  $Y_1$  plus  $Y_2$  plus up to this up to  $Y_n$ . Then that moment generating function of this  $Z$  is equals to  $e^{tZ}$  which is now  $Z$  is replaced by this  $Y_1$  plus  $Y_2$  plus up to  $Y_n$ . So, this are the now the moment generating function we can just write it their multiplicative form. And these can be replaced by this expectation of this individual random variable. We can write this one, because they are independent to each other.

So, now each entity is now the moment generating function for that random variable. So, this results that multiplication of the moment generating function of all such  $n$  original random variable. So, if we know that there are **there are**  $n$  numbers of random variable which are obviously, independent. Then the multiplication of their moment generating function we will give you the moment generating function of that new random variable  $Z$ , which is obtained through the addition of this functions.

(Refer Slide Time: 51:20)

### Examples

Q. Let  $Z \sim N(0, 1)$  and  $Y = \mu + \sigma Z$ , Find  $m_Y(t)$ , given

$$m_Z(t) = \exp\left(-\frac{1}{2}t^2\right)$$

Sol.:  $m_Y(t) = \exp(\mu t) m_Z(t)$

$$= \exp(\mu t) \exp\left(-\frac{1}{2}(\sigma t)^2\right)$$
$$= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Probability Methods in  
Mechanical Engineering:  
H415Dr. Rajib Maity, IIT Kharagpur21

So, we will take one example here that, this is a normal distribution with 0 and 1 where the and the functional relationship here is the linear relationship  $\mu + \sigma Z$ . We know that, this is a if this is a standard normal distribution by this discussion in our last module. So, this  $Y$  will follow a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ . Now, will see it through this moment generating function for this normal standard normal distribution the moment generating function is exponential of half  $t$  square. Now, if you take the  $m_Y(t)$ , and using that relationship this compares expectation of  $\mu t$  plus half  $\sigma^2 t^2$ .

(Refer Slide Time: 52:02)

### Examples...Contd.

- Here we obtained

$$m_Y(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

- So  $Y \sim N(\mu, \sigma^2)$
- i.e.  $Y$  has a normal distribution with mean  $\mu$  and variance  $\sigma^2$

Probability Methods in  
Mechanical Engineering:  
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Now, this one we know that this one is the moment generating function of the, of a normal distribution which is having the mean  $\mu$  and sigma square. Thus,  $Y$  has a normal distribution with mean, so that  $\mu$  and variance sigma square.

(Refer Slide Time: 52:19)

### Examples...Contd.

**Q. (Kottegoda and Rosso, 2008):** The impact pressure of sea waves on coastal structures may be evaluated as  $Z=cX^2$ , where  $X$  is the horizontal velocity of the advancing wave and  $c$  is a constant. Because of the uncertainty involved in the evaluation of  $X$ , we consider this to be a random variable;  $Z$  is thus a derived variable from  $X$ .

Assume  $X$  has mean  $\mu_X$  and standard deviation  $\sigma_X$  and has the normal pdf.

$$f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-5[(x-\mu_X)/\sigma_X]^2}$$



Probability Methods in  
Civil Engineering:

Dr. Rajib Maiti, IIT Kharagpur

23

We will take one problems on Kottegoda and Rosso he states that, the impact pressure of the sea waves on a coastal structure the same expression we have taken. But on this constant here now is club with a new constant  $c$ . So,  $Z$  is equals to  $X$  square. Now, what is given here supplied here is that this  $X$  is following a normal distribution which is the given like this. We have to know what are the properties of these  $Z$ .

(Refer Slide Time: 52:50)

### Examples...Contd.

Evaluate the second-order statistics of the impact pressure on the coastal structure.

Sol.:

Let  $Y = (X - \mu_X) / \sigma_X$  with zero mean and unit variance and pdf

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-0.5y^2}$$

Now  $Z = cX^2 = c(\mu_X + \sigma_X Y)^2$

$$= c\mu_X^2 + 2c\mu_X\sigma_X Y + c\sigma_X^2 Y^2$$


Probability Methods in  
Civil Engineering:  
H4L5

Dr. Rajib Haitly, IIT Kharagpur

24

So, if I want to do it from this moment generating function, then we are just we have to stating another new random variable X minus mu X by sigma X and obviously, this is that 0 mean and unit standard deviation and its PDF is equals to 1 by square root 2 pi e power minus 0.5 y square. Now, this Z is equals to c X square, c multiplied by this function. So, this is if you just expand this one we will get the c mu square plus 2 c mu X sigma X Y plus c sigma square X, Y square.

(Refer Slide Time: 53: 27)

### Examples...Contd.

□ Representing  $W=Y^2$  and by substituting  $f_Y(y)$  in moment generating equation

$$M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{t(x)} f_X(x) dx$$

i.e.

$$M_W(t) = \int_{-\infty}^{\infty} e^{ty^2} f_Y(y) dy = \int_{-\infty}^{\infty} e^{ty^2} \frac{1}{\sqrt{2\pi}} e^{-0.5y^2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5y^2(1-2t)} dy = \frac{1}{\sqrt{1-2t}}$$

(using the transformation  $Z = y\sqrt{1-2t}$  and also the area under the  $f_Y(y)$  curve is unity)



Probability Methods in  
Civil Engineering:  
H4L5

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25

Representing the Y is equals W is equals to Y square and by substituting that F Y in this moment generating function. We will get that M Z of this t is equals to this function, **this function** relationship this one we have seen from this general thing. Now if you fit this one for this our these case then this that f y we are just replacing with this standard normal distribution. Now, if we just do this integration and if we use this transformation that Z equals to y square root 1 minus 2t. Then we can show after some eligible calculation this that, this is 1 by square root of 1 minus 2 t.

(Refer Slide Time: 54:08)

### Examples...Contd.

- Taking the first derivative of the mgf at the origin, we can obtain the mean of W as:
 
$$\mu_w = \frac{\partial M_w(t)}{\partial t} \Big|_{t=0} = [ (1 - 2t)^{-3/2} ]_{t=0} = 1$$
- Similarly taking the second derivative, second-order moment of W is obtained as:
 
$$E[W^2] = \frac{\partial^2 M_w(t)}{\partial t^2} \Big|_{t=0} = [ 3(1 - 2t)^{-5/2} ]_{t=0} = 3$$
- Hence the variance of W is:
 
$$\sigma_w^2 = E[W^2] - E^2[W] = 3 - 1 = 2$$


Probability Methods in  
Civil Engineering:  
H4L5
Dr. Rajib Haitly, IIT Kharagpur
26

Now if I just want to know that its different moments then this mu t we know that we have to do the first derivative and evaluated t equals to 0. And which if we do we will get the this is equals to 1 and this obviously, W square also for taking the second derivative and evaluated at 0 this will be equals to 1, and so the coefficient of variance again the expectation of W Square minus, this expectation of Y whole square. So, 3 minus 1 this is equals to 2.

(Refer Slide Time: 54:38)

**Examples...Contd.**

- The mean of the required impact pressure Z follows immediately from the linear property of expectation as

$$\mu_Z = c\mu_X^2 + 2c\mu_X\sigma_X\mu_Y + c\sigma_X^2\mu_W = c(\mu_X^2 + \sigma_X^2)$$

- Thus the variance is

$$\begin{aligned}\sigma_Z^2 &= (2c\mu_X\sigma_X)^2 \text{Var}[Y] + (c\sigma_X^2)^2 \text{Var}[W] \\ &= 2c^2\sigma_X^2(2\mu_X^2 + \sigma_X^2)\end{aligned}$$

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Civil Engineering:  
H4L5

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27

And so, if I just see that mean of this required impact pressure then this is becoming from this equation c equals to this and sigma Z square is equals to this equation.

(Refer Slide Time: 54:52)

**Examples...Contd.**

- $C_v = \sigma_X / \mu_X$  represents the coefficient of variation of horizontal velocity of the advancing wave X, then
- The mean of Z can be obtained as:

$$\mu_Z = c\mu_X^2(1 + C_v^2)$$

which equals  $g(E[X])$  augmented by a factor  $(1 + C_v^2)$

- The variance of Z is :  $\sigma_Z^2 = 2\left(1 + \frac{2}{C_v^2}\right)c^2\sigma_X^4$

which equals  $g(\text{Var}[X])$  augmented by a factor  $2\left(1 + \frac{2}{C_v^2}\right)$

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Civil Engineering:  
H4L5

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28

And from these two, if we just evaluate after some algebraic that calculation that C v the coefficient of variation equals to standard deviation by mu, then we see that this mean is augmented by a factor 1 plus C v square, and this variance is augmented by a factor that 2 into 1 plus 2 by C v square. So, these things we can see from there, the basic relationship that movements those that in the first moment and variance here we have

seen. This also we can take it from general expression of this moment generating function. And that derivative their respective derivative when evaluated at a particular point that is at origin. So, we have discussed here how this moment generating function can be useful to get the properties of this function of this random variable. So, this is basically the conclusion of these full functions of random variable. And in this module we have discuss about only a single random variable multiple random variable will be taken in the next module. Thank you.