Stability of Structures

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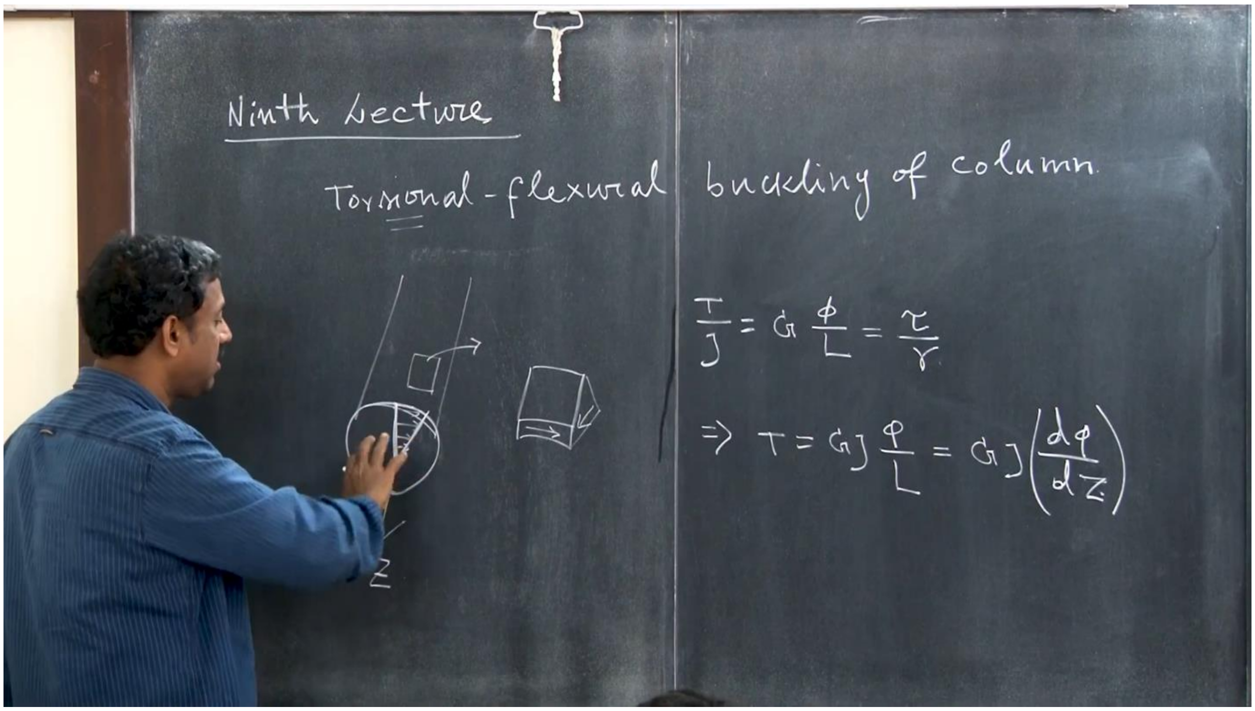
WEEK-05

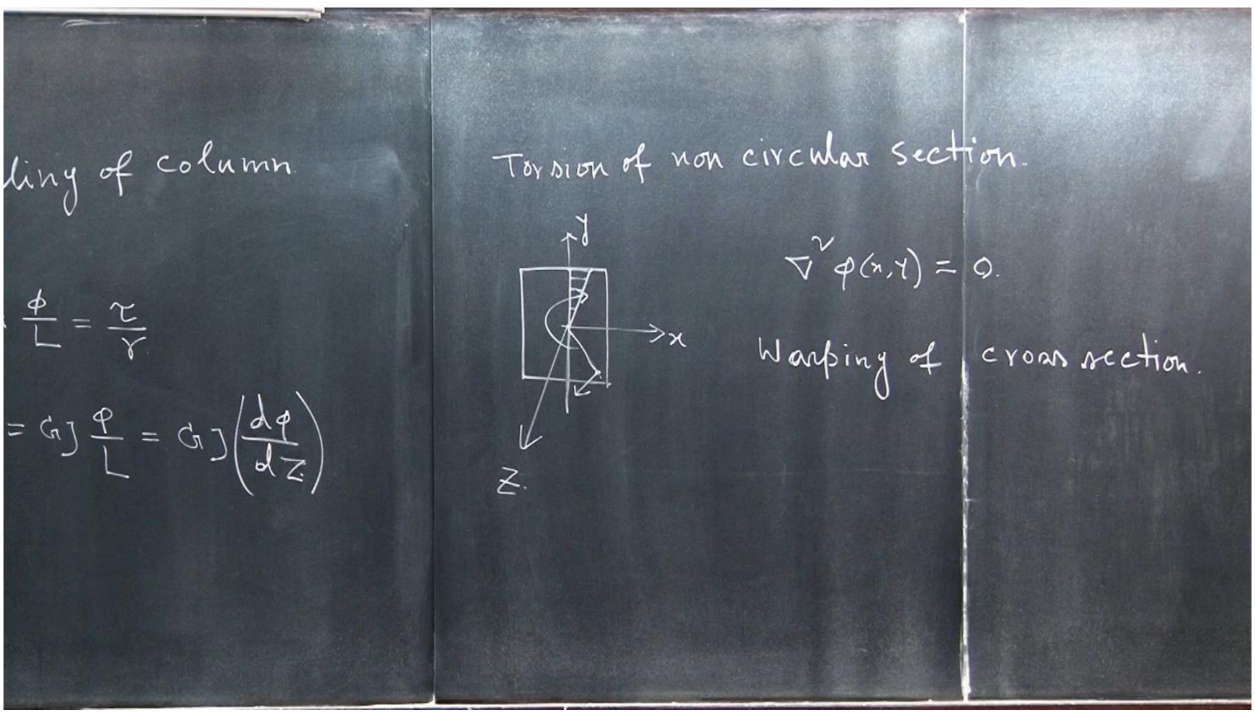
LECTURE 9: Torsional-flexural buckling of columns

Okay, welcome to the ninth lecture on the stability of structures. So let us briefly recapitulate what we have covered so far, especially in the second chapter. So, once we demonstrated the basic elements of structural stability and several classes of behavior, we started with this real example by taking a simple structural system that was a column. We have already learned about the buckling of columns in our undergraduate course, right? Because this is a little advance in stability. So, we have extended the discussion of the buckling of columns to the post-buckling regime and also several other issues we have introduced. For example, we have first demonstrated the amplification of displacement. So, amplification occurs in displacement as well as in bending moments due to axial compression force in a beam-column. The magnification factor depends on the ratio between the axial load and the critical load; the closer the axial compression is to the critical axial compression, the larger the amplification will be. Then we started investigating the influence of shear deformation on the critical load, and we saw that the critical load for buckling actually diminishes with the incorporation of shear deformation. The incorporation of this shear deformation is very important; otherwise, it may lead to a non-conservative design. So, the condition under which shear deformation will be significant is. For example, in a little deep beam and things with a deep, thick plate, shear deformation must be included, and then we started with this finite deformation. We didn't limit the deformation to be infinitely small, and that's what we referred to as elastica: the column undergoing finite deformation under axial compression. And then we saw that the governing equation can ultimately be expressed in terms of an elliptical integral of the first kind, and from there, using some kind of approximation of the elliptical integral, we demonstrated the symmetric stable bifurcation behavior. Which we have observed in a simple system, namely the rigid bar resting at one end with a rotational spring, and we have seen that this kind of thing is not imperfection sensitive. And then we have also demonstrated that it shows very little post-critical strength, even for a 30° rotation at the end, which can lead to only a 3% increase in the load-carrying capacity. So, for columns, we cannot rely on the post-critical reserve. Right.

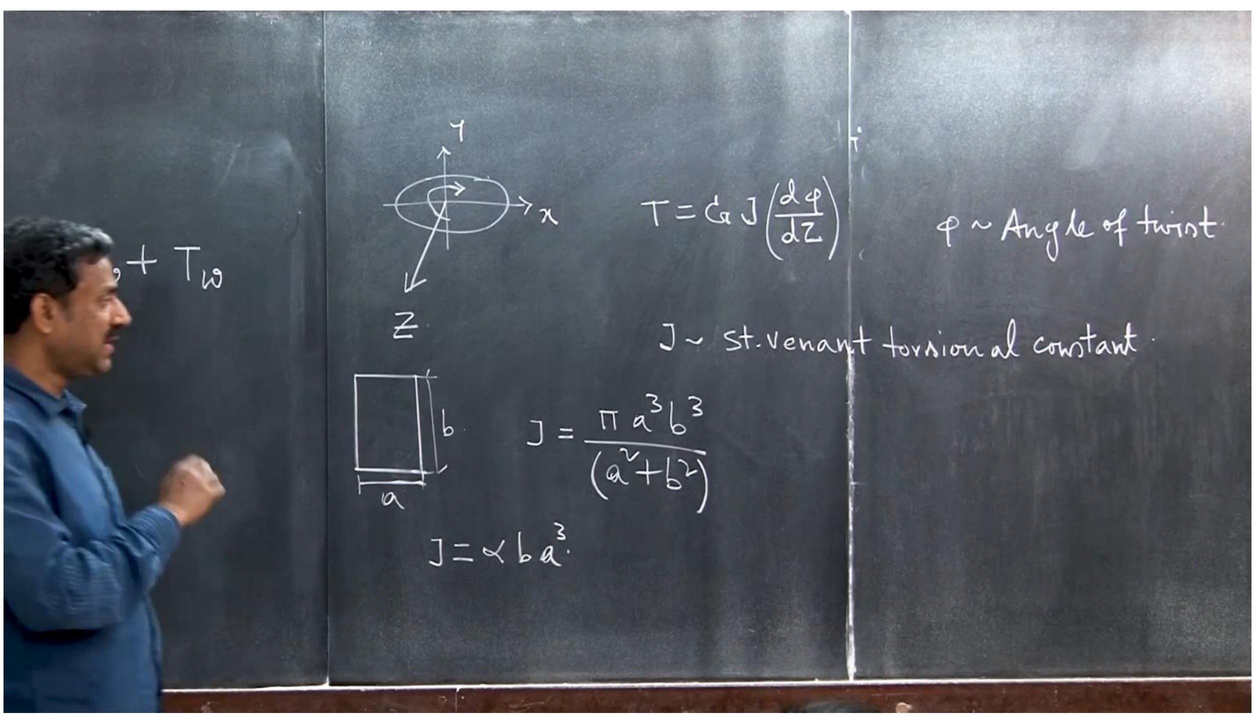
So as soon as a column buckles, it will not retain its capacity to load. It will basically deplete all the capacity of resisting axial compression. So, then we have also expressed lateral deflection as well as axial translation deflection in terms of this quantity of lambda, which is the square root of p by EI, right? And also, the elliptical integral of the second kind, okay? So now what we are going to start is, once again, we are going to discuss the buckling of a column.



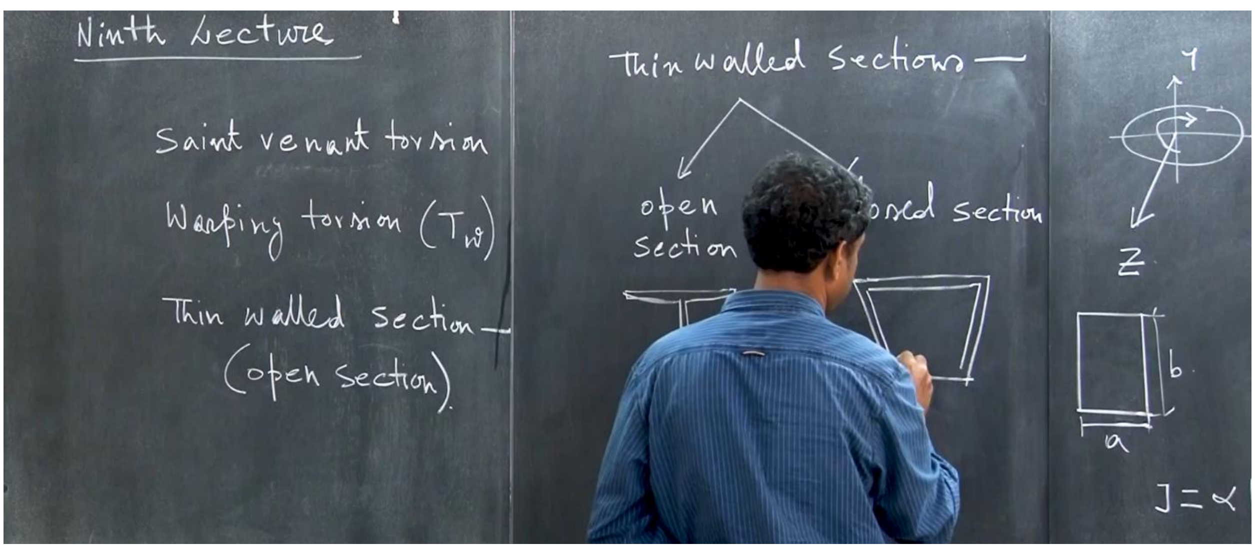
But we are addressing a slightly more advanced topic, right? So, what we are going to do now is describe the torsional-flexural buckling of a column. Please note that until now, everything we have learned is basically about flexural buckling. Why? Because when it goes out of plane deflection, it is through bending. We didn't consider torsion. So, where this is, mostly hot-rolled sections, especially wide flange, they essentially buckle by flexural buckling only, okay? But there are sections that are asymmetric, okay? In which the center of gravity doesn't essentially coincide with the shear center. All of you know what a shear center is, right? The shear center is nothing but the point through which, if the load is applied, it does not produce any torsion. So, how do you find the shear center for a section? If the shear center and CG don't coincide, then the bending will essentially be accompanied by torsion, right? So, we are going to consider those kinds of sections, okay? In that case, it will not allow susceptibility to losing stability, not only from flexure but also from a combination of flexure and torsion. So here is what we are going to consider: most civil engineering systems and even aerospace systems consist of thin wall sections or sections that are not circular. We have learned about the torsion of circular cross-sections. And based on certain assumptions, right? So, the assumption for a circular section is that plane sections will remain plane, just like bending. Here also, for a shaft with a circular cross-sectional area, plane sections will remain plane. Because there is no out-of-plane deformation, the torque is essentially registered by shear stresses and the relative shear between the different sections, right? Isn't that correct? So, if you can recall, there was this kind of shaft, right? And then if you know and plot this distribution of shear stresses, it will be linear, right? And then the formula was like ($\frac{T}{J} = G\frac{\varphi}{L} = \frac{\tau}{r}$) right? So, T is nothing but $G\frac{\varphi}{L}$, and then if you want to write it in terms of, we can write $GJ\frac{d\varphi}{dz}$ right?



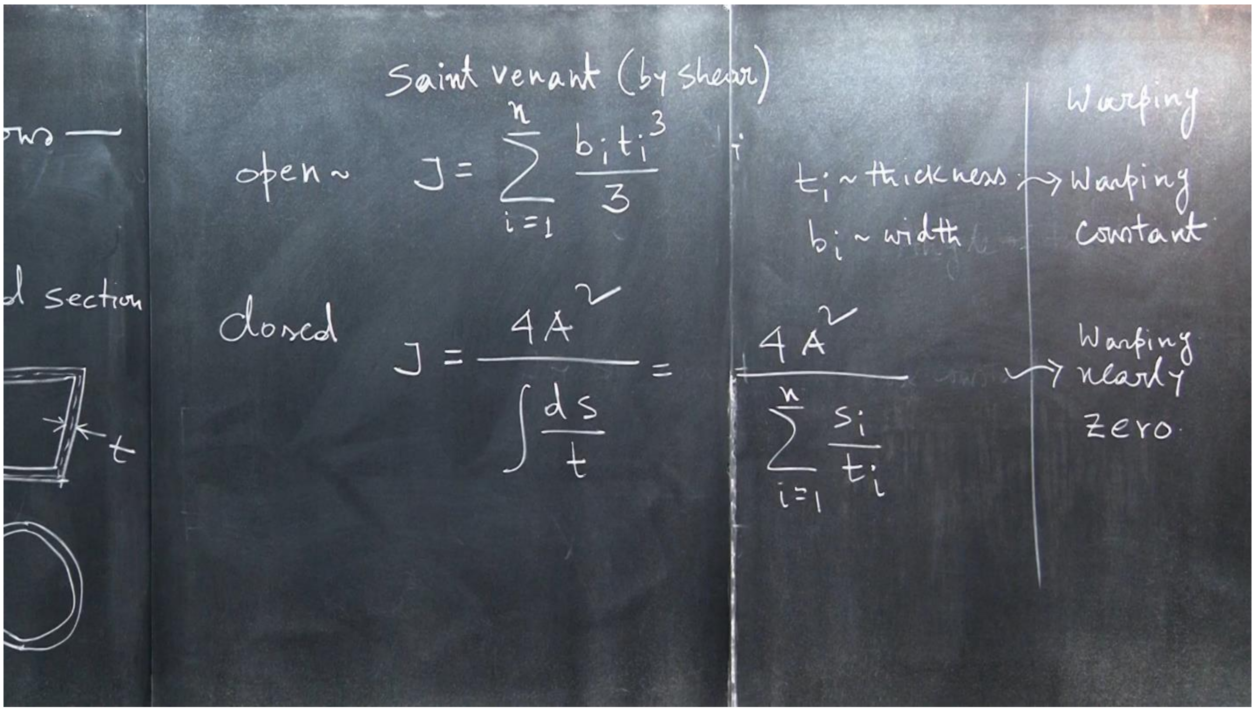
Where dz considers the longitudinal direction to be aligned with the Z-axis, right? If you consider this to be aligned with the Z-axis, then you see that Tau is basically proportional to r, so you know it is increasing. Please note that the stress, if you just consider a very small section here, is essentially something like this. And something like this right on the free surface, there will be no shear stresses, right? So, this, and then another assumption was also that any of this cross-section, this cross-section, when it is deforming, remains on a line, okay? So, basically, it is as if, upon twisting, when it is deforming under the twist, it is essentially rotating rigidly, right? Okay. As if it is rotating rigidly. So, this section will remain on a radial line, right? The problem is that these are all, you know, this was the torsion of a circular section, but as soon as it is a non-circular section, you know all these assumptions are no longer valid, right? For example, if you consider a rectangular section, then what will happen? If you have seen that, you can either have a displacement formulation or a stress formulation. So then, if you can recall, the governing equation was equal to zero, right? Stress function, right? Stress function formulation, things right. There are two alternate formulations, so I'm not going, and then you have solved the boundary value problem. We have seen that what happened to that plane section in the out-of-plane direction in the Z direction is that the plane section no longer remains a plane. Rather, there is warping. So, when it is subjected to some torsion, one of the quadrants comes out; it bulges out, another quadrant goes inside, another one comes out, and another goes inside, right? So, this is called warping, right? So, the warping of the cross-section is very important; then you also have to find out the warping magnitude, right? So, be it rectangular, be it elliptical, whatever, right? But one thing is very important: I think it's still valid that shear stresses vary basically. So, it is still with a radial line if you draw any radial line. Therefore, it increases linearly, okay? Right, isn't it? And at any point if you see, you have to draw a radial line, and the shear stresses here are basically proportional to this distance, right?



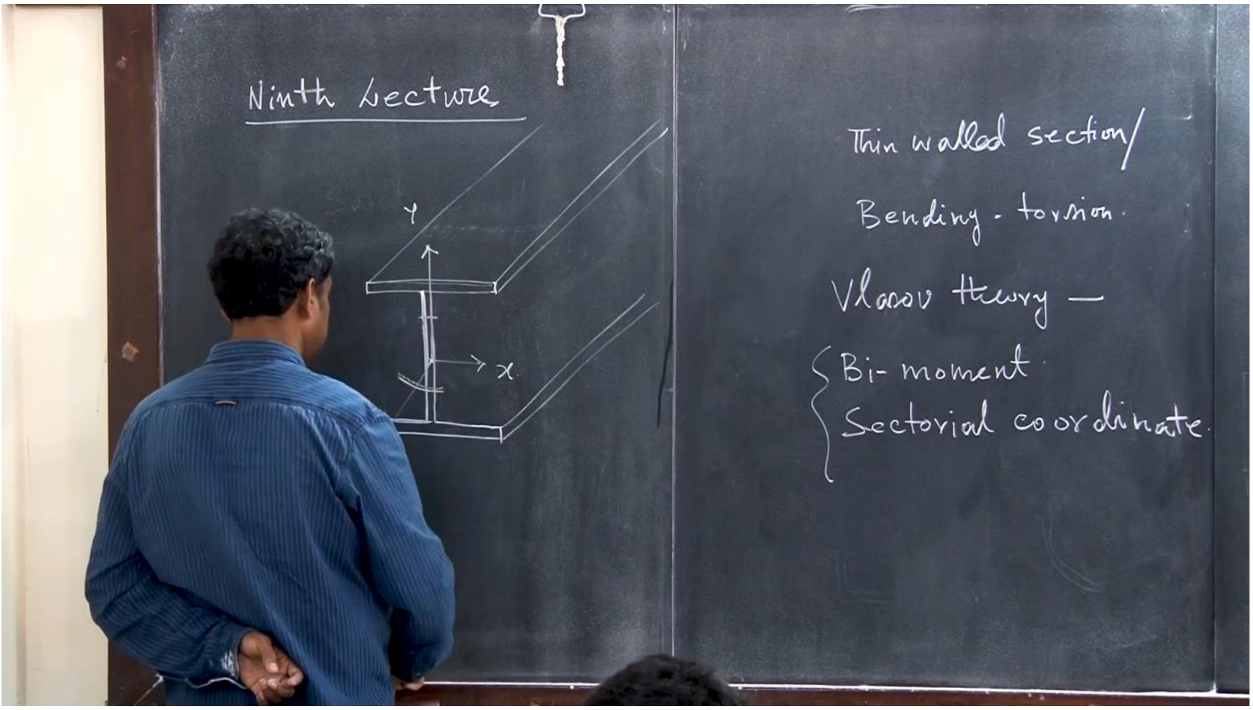
So, warping is one that is unlike the non-circular cross-section, and warping is extremely important. So, please note that this warping induces additional stresses. If this warping is prevented, if you prevent warping, then what we have learned, Saint, is that there will be two components, Saint Venant torsion (T_{sv}), right? There will be warping torsion. So, in torsion, the torque will be resisted by two components: one by shear stresses and the other when this warping is resisted. For example, take a cross-section, and the warping you are preventing, okay? No auto blend, then this will induce axial stresses in the fiber, right? If you prevent warping, that will induce axial stresses, right? So, in addition to shear stresses, there will also be axial stresses, right? So, the total torsion torque will be registered by one Saint Venant component, and the other is the warping torsion. Right? Two, isn't it? So, what we are going to consider here is mostly thin-walled sections. Thin-walled sections. Why? Because thin-walled sections will be more susceptible to that here. Okay, for example, thin wall sections, or I section, and then what we are going to consider mainly are open sections, not closed sections. We are going to consider open sections because, for the closed sections, what happens is that the warping torsional constant is too high. It is okay to have any significant warping. We have learned and found out about Saint Venant torsion for all rectangular and elliptical sections. For example, you know that for an ellipse, the same formula is used for an elliptical section. So, T (torque) will be GJ $\frac{d\phi}{dz,}$ right? So G is the shear modulus, J is the Saint-Venant torsional constant, and ϕ is the angle of twist, and $\frac{d\phi}{dz}$ is nothing but, if you want to define it as θ, that is twist per unit length. So, J, you have to find out for various areas; this is, for example, here it is $\frac{\pi a^3 b^3}{a^2+b^2}$, something like this, right?



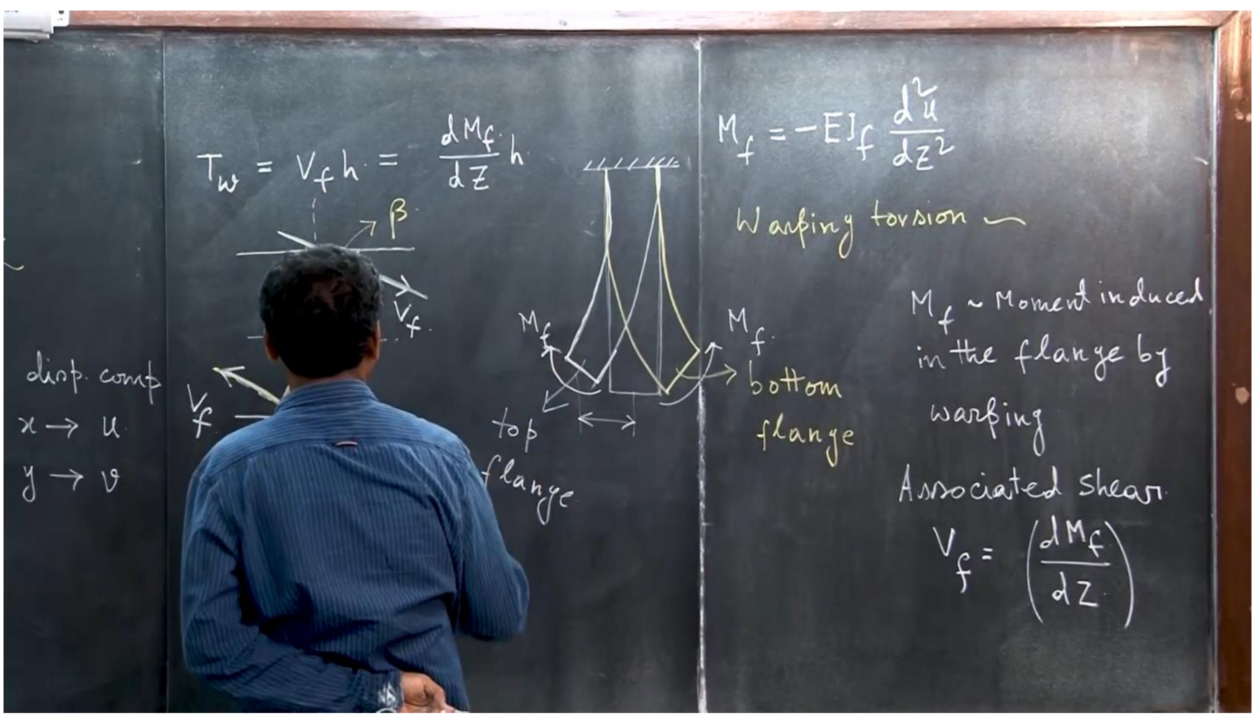
This was the expression for the rectangular section of the elliptical shape. And then, for the elliptical section, if it is subjected to this kind of torque, what will happen is that this quadrant will go inside, and this quadrant will come out, and then it will go inside and come out. So, bulging will happen, right? Then for the rectangular section, if you can recall, we consider this as a and this as b; then $J = \alpha ba^3$. What is that b into a cube? Whatever it is, a smaller dimension that will come is a cube, okay? And α is a constant for different things, and α will be different. Okay. So, you know how to find the Saint-Venant torsional constant for different sections that we learned about in the mechanics course, right? And these are for the solid cross-sections, but if you can recall the thin wall section. Now for the thin wall section, I don't know how we discussed it. There are more general formulations, but let me be clear about thin-walled sections. As you know, there can be two classifications: one is an open section, and the other is a closed section. So, what is an open section? An open section may be basically nothing, but maybe something like this, right? An open section may be, you know, this angled channel section is an open section because it is not closed, right? And what is a closed section? A closed section is maybe what it is if it is trapezoidal, you know, like what we use in box girders, right? Something like this, and maybe this circular section, you know? All these things you have learned in the mechanics course about thin wall sections and torsion. How did you find out about this torsion, and was this topic discussed in the mechanics class or not? So, you tell me what the torsional constant is for the open section. All of you have noted this down, right? I'm just briefly brushing up on your torsion in the non-circular and thin-walled sections. Okay. So, if you can recall, for the open section, the open $J = \sum_{i=1}^{n} \frac{b_i t_i^3}{3}$. Okay, where t_i is nothing but the thickness of the individual section and b_i is the width or length of the individual component. That means if you consider this as a rectangular section, this is the width B_1 and this is T_1. Then, from here to here, this is B_2 and this is T_2. Then, this is B_3 and T_3. So, for each individual section, you just sum them up, okay? So, I Is equal to 1 to the number of components in the section, so it's an open section, clear? Similarly, for the rectangular section, this is B_1T_1, okay? Now for the closed section. Closed one J = $\frac{4A^2}{\int \frac{ds}{t}}$ That means shear is $\frac{4A^2}{\sum_{i=1}^{n} \frac{s_i}{t_i}}$. A^2 means you know A is nothing but the area of this section. So, you draw the central line; whatever area is enclosed by this center line, this is the thickness t, and s is basically this length. Understand that$\frac{4A^2}{\sum_{i=1}^{n} \frac{s_i}{t_i}}$ .These are Saint Venant's torsional constants now. What about warping? The warping constant will be discussed later. Please note that for the open section, this one is for the Saint Venant component, which means the one contributed by shear. Which is registered by shear between the different cross-sections. Okay, now for the warping torsion, we will show you the open section. You know there will be some warping constants. Some warping constant will be there; I will tell you how to find it. Some warping constant, but for the closed section, the warping constant is nearly zero. This means warping doesn't really occur; basically, I mean that warping is nearly zero. So, we do not require, I mean, we can have the warping nearly zero.



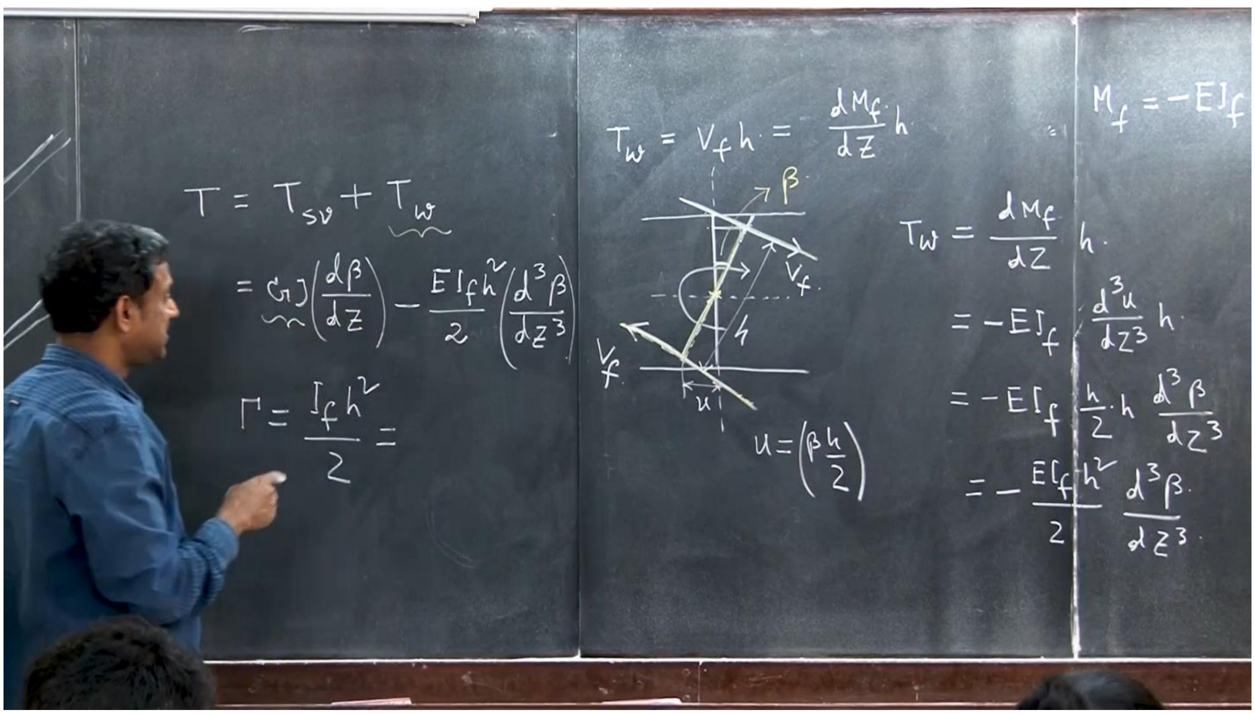
That means you do not need to consider the warping constant, you understand, because the closed section hardly warps, okay? It is very rigid against warping, okay? But closing the open section will work. Okay, so having said that, we'll try to, I'm just removing this part. I don't know how it was discussed, but if you can recall, for the thin-walled section, if you consider the flexure or bending and torsion theory combined, this is described by Vlasov's theory. And there you will see that a new concept is introduced called bi-moment, and there is an element called sectoral coordinate. I cannot go into details, but this is not simple stuff; bi-moment and sectoral coordinates, all of these are there, okay? It's not very similar in analogy to the bending theory. They have developed these things well, but we are not going to do it in a much simpler way, okay?



Now, for any theoretical section, I am specifically mentioning to you that I am going to consider a section, maybe something like this: you know I'm going to consider an I-section. So here is my high section. So here it is going, and then I know here. Huh? And then I'm choosing my coordinate systems. And of course, here you can see the shear center, and this is symmetric. Then this is X and the shear center, and then CG is the same right here. So, X and this fellow are Y, and then the out-of-plane direction is Z. Okay, there are two flanges, and this is the wave. Okay. Now and then, you know there is some torsion applied to it. Okay, or I can clearly define torque T. So now what happened is this? So, total torque is basically contributed by Saint Venant, and T is warping, right? We know that for Saint Venant, the expression is, $GJ\frac{d\beta}{dz}$ correct? That is the expression: z is the Saint-Venant torsion constant and β is the angle of twist. So, this is the way we write it, something like this. You know if it is like this. So, when it is twisting, it is something like this. So it will be, you know, that it is twisted, right? So, I'm just twisting, you know, the section will look something like this. Okay, so this is the angle, and then this angle is essentially β. Okay, clear? So, $\frac{d\beta}{dz}$, now we don't know how to treat it because we are all familiar with Saint-Venant torsion, but we are not familiar with warping torsion, right? So let us see what we will do about warping torsion. Okay, so now I am not going to introduce you to the grass-up theory bi-moment. I will mention what a bi-moment and a sectoral coordinator are. I'm not going to do that. Okay, I'm going to make it very simple. Okay. But those things are not in the present context. Okay. Because we are only concerned about torsional flexural buckling. Okay. That's why I'm not going into that. Huh? So now you tell me, when this fellow is getting twisted. Okay. So, whenever it is subject to, this kind of twist right. So, what is going to happen? See, along with the twist, you know, when you apply this twist, what will happen is that this twist will essentially be accompanied by bending in the flange, right? So, when you are twisting it, maybe this fellow, you know, this cross-section may bend like this. You see, whatever the bending in this cross-section is, the sense of bending here in this cross-section will be different. You understand what I'm trying to tell you, right? Because when it is basically twisted, it will be twisted. So, if this bends in this curvature, the bottom flange will bend in the opposite curvature, right? So, what I'm trying to say is, let me draw it. Okay, if the top plane I am considering is along the length, okay? So, if the bottom plane is going to deflect like this, you know? It's going to deflect like this, you know. Then the other one is going to deflect like that. You see that you understand what I'm trying to say, so maybe this is happening at the top flange and this is the bending of the bottom flange, right? And can you also understand why it is so?

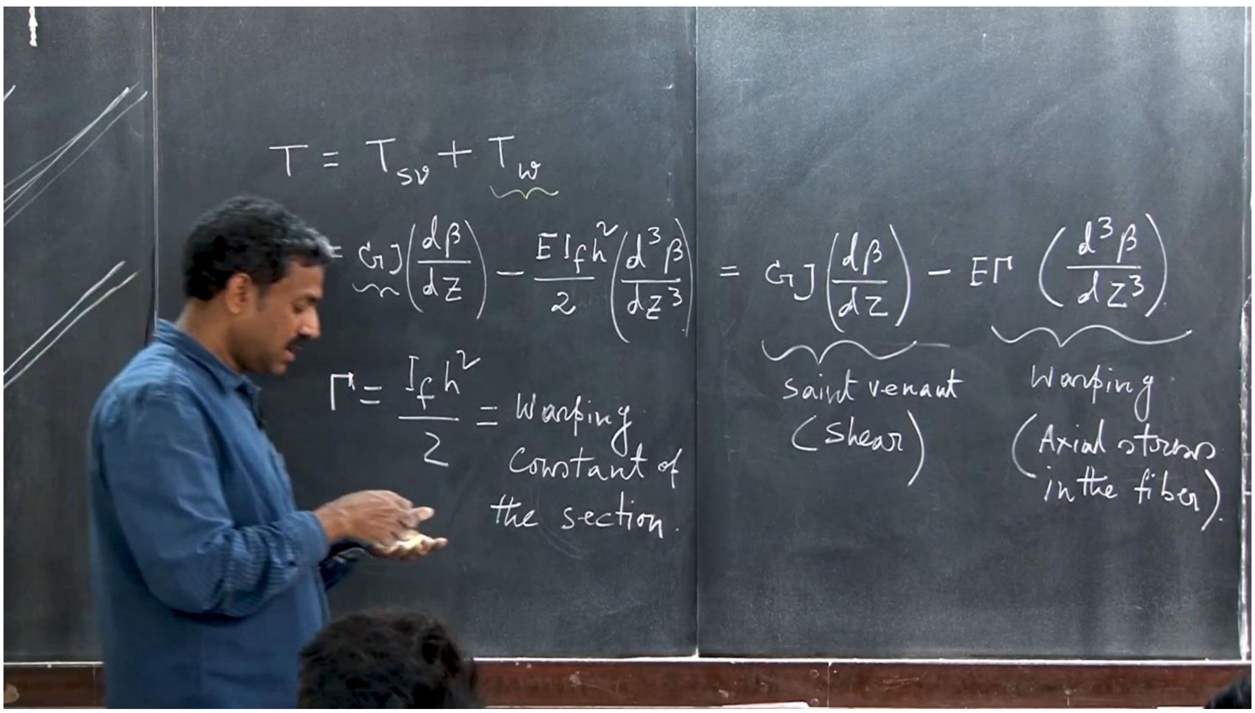


Because what is happening, you know it is trying to twist clockwise, so this fellow is moving in this direction, right? But at the fixed end, it is trying to fix itself like this So, it has to bend like this; it has to do it like this. See, whatever I am drawing is a little opposite, right? But consider, if it is bending like this, you know what is happening: this fellow is moving in this direction. So, when it is moving in this direction, this fellow is going to bend it like this. You see that it is happening like this, and then this fellow is going to move it like this. So, it will be okay. Do you understand that? So, if we consider this, the contribution of the web to the bending is negligible, right? You know the bending is essentially contributed to by the flange. The web is not going to bend; we can safely assume that this web will remain in place, right? Okay, so now when it is bending, I'm sure that there will be some hypothetical bending moment here that is causing the bend, right? So, this hypothetical bending moment I can define as moment M_f, and this will be equal and opposite, right? M_f and M_f, right? So, MF is nothing but warping; you know MF is basically a moment induced in the flange by warping. Right now, that's why when there is a moment MF, there is also an associated shear. and associated Shear, we can define as V_F right. And this $V_{F=-\frac{dM_F}{dz}}$ for bending moment expressed Right. Okay. Please note that. Right. So, if I want to define this V_F like this, how will this V_F occur? You know this V_F here; there will be a V_F. You see that because this shear is along the section of the flange. This thin flange section along that, right? Fine. Okay. Now, you can clearly see that if these are the two shear forces that are developed, they will basically consist of a resulting force. Both of these are going to cause a counteracting twist. $T_w = V_f h$. If we consider this depth of the flange to be H, then H times V_f will be nothing but some component of twist, and that twist is a contributor to the warping torsion. Do you understand that? Now you may ask why it is also coming in this direction; you know it's basically nothing. If it is applied externally in this direction, then it will result in something. Okay? So, you don't worry about that. So, you understand how I am getting here; you know the twisting component of the warping torsion, the warping torsion that is basically in action to counteract the twisting, right? You see that, so externally you know whatever torque is being applied is registered by Saint Venant and others due to the warping. The warping component is nothing but V_f, and when I'm writing V_f, V_f is nothing but, once again, $-\frac{dM_F}{dz}h$, or you can say, essentially, you can also write it to be positive here. Sorry, V is equal to dM_F or negative sign C; if this negative sign comes here, then, of course, M is equal to minus there in one place; it will be there depending on the choice of your coordinates, right? I'm not going into those things right now. Let it be like that. Okay, fine. So now I'm removing this one. Huh? What is H? Do you understand? H is nothing but the depth of the web. Okay. So, I'm going to put it there. And then I will once again write, now what I'll write you know $M_f = -\,\mathrm{E}\, I_f \frac{d^2u}{dz^2}$ Yes. What is E? mod elasticity I_f? If this flange is bending, that means the second moment of area of this flange with respect to this axis, the axis that is aligned along the web, right? So that is I_f; I_f is nothing but the second moment of area with respect to, you know, this axis, right? So, this is for the flange only, right? And what is D²u/dz²? It is nothing but this if this is deflecting. So, this deflection is occurring like this: this is the bending because of this lateral deflection, right? This lateral deflection is happening along the x direction; this direction is nothing but x. And we are defining that there is no displacement component along x; it is u along y, v along right, and w along z, okay? Whatever, so this is the displacement component you see.

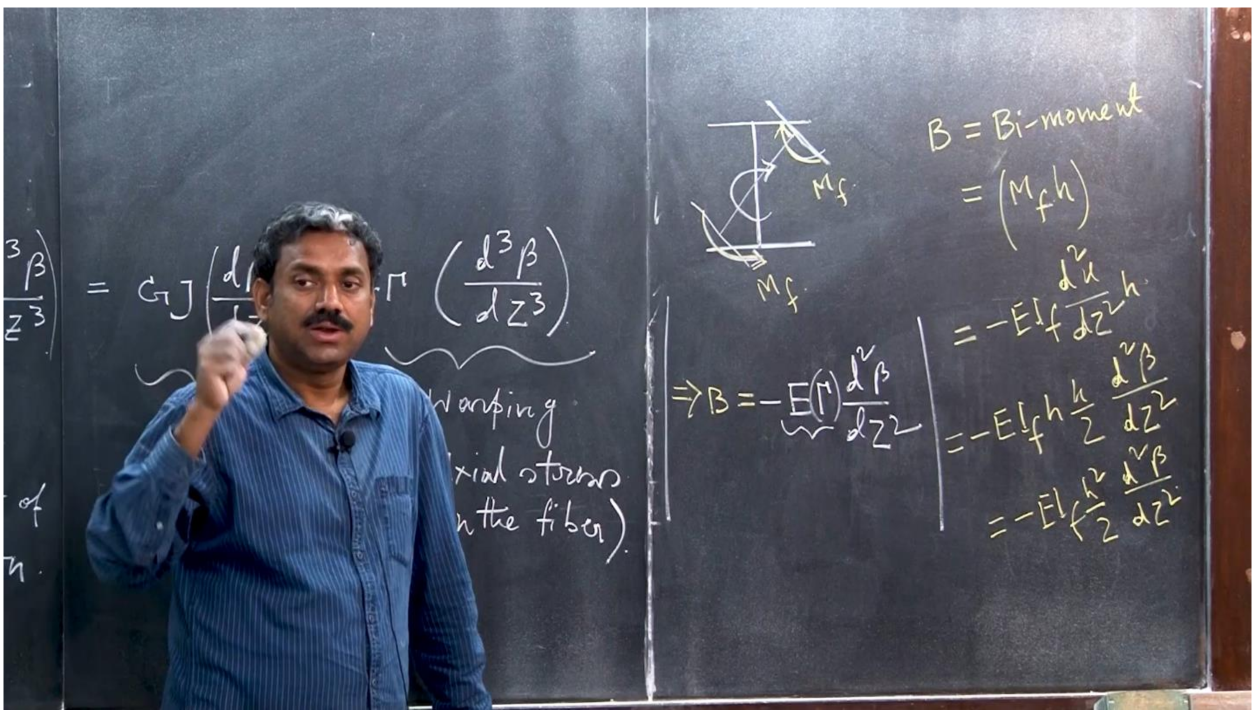


So, if that is $\frac{\text{D2u}}{\text{dz2}}$, why? Because z is along the longitudinal axis. So, do you understand? Is that correct? All of you understood why E_f is nothing but the second moment of area of the flange. Fine. So please try to understand that warping. Because of warping, what kind of differential moment and bending moment is sheared, developed in the flange, and how is it resisted? I'm just using a simple bending formula; I'm neglecting shear. So, just to write this clearly: what I'm going to do now is substitute over here. Should I remove this? I can remove this part at least, right? And what I can write is that T_w is nothing but $\frac{dM_F}{dz}h$, $\frac{dM_F}{dz}h$ is $-\,\mathrm{E}\, I_f\ \frac{d^3\beta}{dz^3}h$. Now you tell me, what is

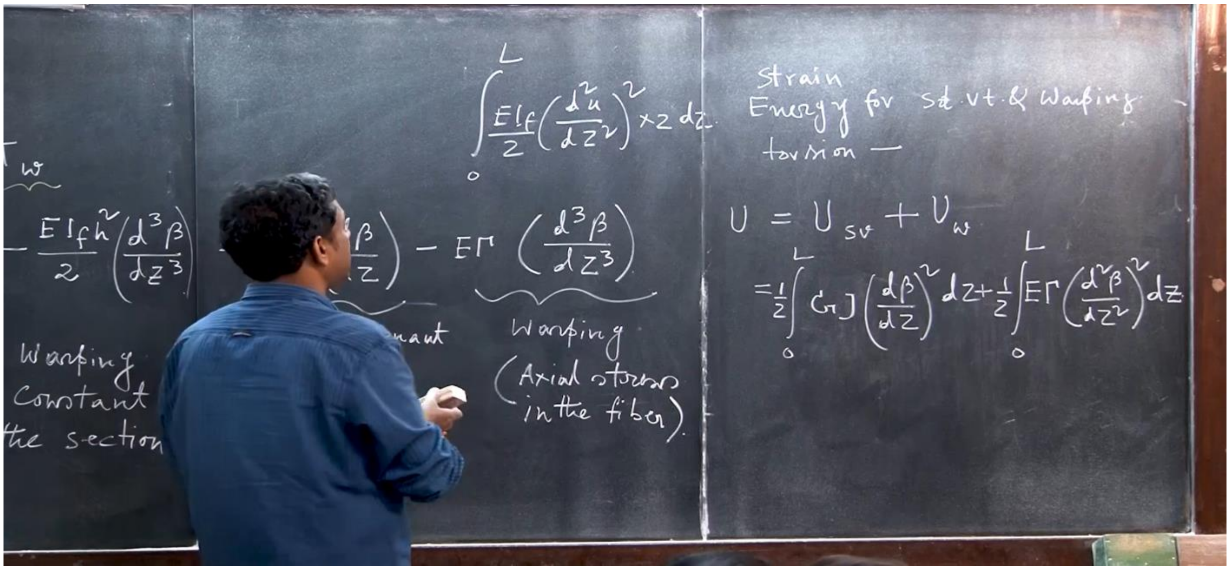
U? U is nothing but what? U is this deflection. Right. This U is what? U is what? U is this deflection. Right. This U is what? $\beta\frac{h}{2}$. Because this angle is β and this is $u = \beta\frac{h}{2}$, what I'll do here is subtract E if I'm going to H. Of course, you know β will change, but H divided by two, $-\,\mathrm{E}\,I_f\,\frac{h}{2}\,\frac{d^3\beta}{dz^3}$ So, it is going to what? $T_w = -\,\frac{\mathrm{E}\,I_f\,h^2}{2}\,\frac{d^3\beta}{dz^3}$. So, what I will write now, please see here, I will bring it once again. So, I'm just going to, T_W is nothing but, $GJ(\frac{\mathrm{d}\beta}{dz}) - \frac{\mathrm{E}\,I_f\,h^2}{2}\,\frac{d^3\beta}{dz^3}$ clear? So, here you can see that GJ and I_f H² divided by 2, which I am going to define as capital $\mathbf{\Gamma}$, okay, is $\frac{I_f \mathrm{H}^2}{2}$. We are going to define this as, you know, this is Saint Venant torsional rigidity, right? Here we can define it as the warping constant, right?



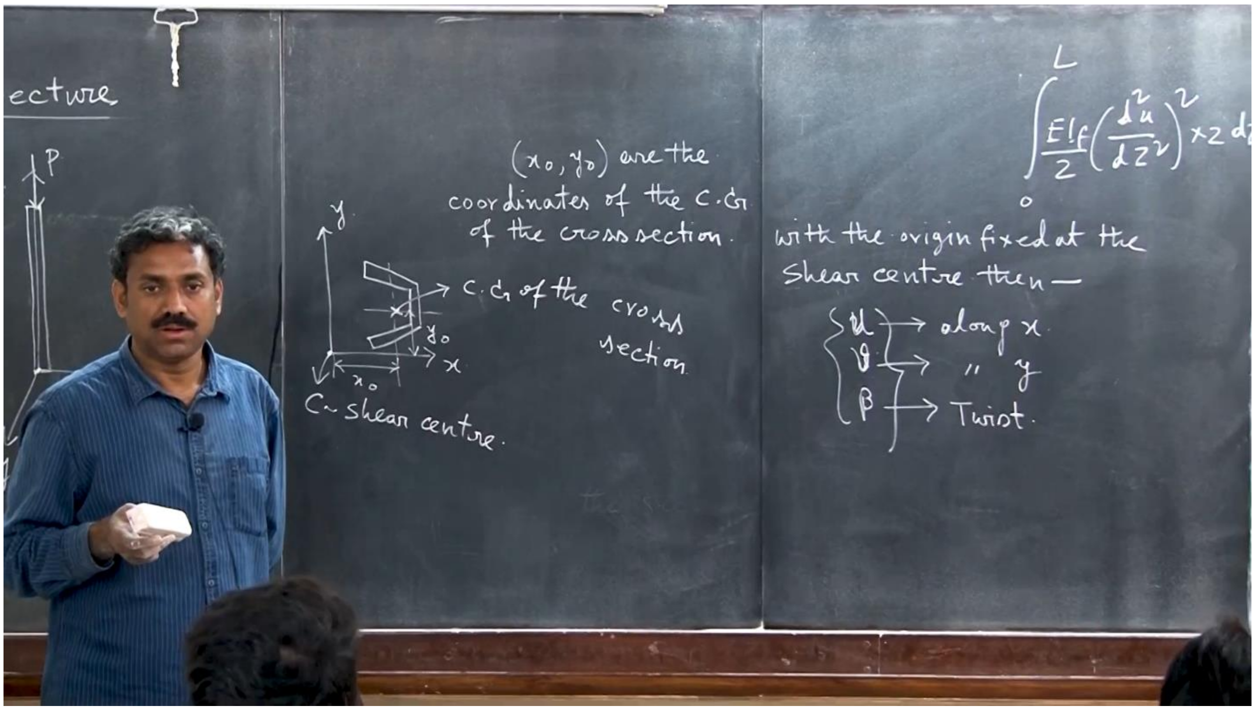
The warping constant of the section, I_f, is of course dependent on the flange dimension; H is simply the depth of the wave. I_f H squared divided by 2 is like a constant for the cross-section, similar to the moment area, and of course, it is of the higher dimension of the fourth order, right? So, I'll just simplify it further, and the way I'm going to do it, you see, is I'm removing all these things now, and so I'm writing, you know, $GJ(\frac{\mathrm{d}\beta}{dz}) - \mathrm{E}\;\mathbf{\Gamma}\;(\frac{D^3\beta}{dz^3})$. So total torsion, the torque is registered by Saint-Venant torsion and warping. This is Saint Venant's torsion, which is registered by shear, and this warping is resisted by the axial force developed. You know this is due to the shear stress between the different cross-sections, which is present due to the axial stress in the fiber. Of course, if it is at rest along the axial direction, right? Capital λ, you know, so I believe it is capital ω, right? If h^2 by 2 this one, you know, things capital ω is basically the warping constant, which is dependent on the section, okay? Now, before I go later, you know, I will just try to tell you something. What happens is that, you see, when these two, so this, you know, what is happening is that when this fellow is subjected to this torque, you know, this twist, then some differential thing is happening, you know, and then these two sections are subjected to bending. What really happens is that, you see, this fellow is subjected to a bending moment, right? So here, some bending is happening, and here, also some bending is happening, and they are different. So, there is a moment in the flange, a moment in the flange, right, and there is some distance, okay, which is called bi-moment, okay. You do not need to know this, but just to introduce it.



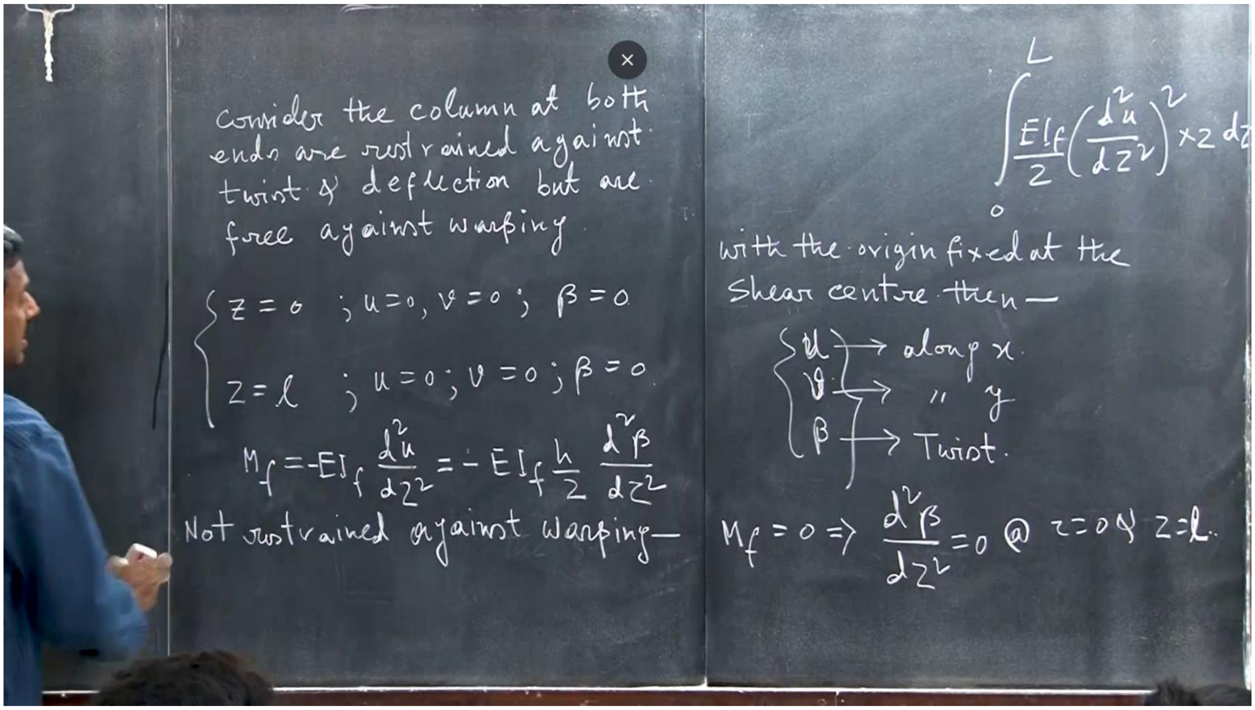
Bi-moment can be defined as Mf multiplied by h; this is called the bi-moment. And then, because it is a bi-moment, you see, that's why its dimension will be whatever the bending moment is, and then there is another additional h, okay? So, that's why you see that there is I_f h^2 by 2. So, it is not only related by; similarly, bi-moment can be related to the, you know, if you just put it mf into h, this mf is what? MF is nothing but - EI_f $\frac{d^2u}{dz^2}h$, and then you have written that $-EI_f$h, and this U is nothing but $\frac{H}{2}\beta$, right? So, $\frac{H}{2}\beta, \frac{d^2\beta}{dz^2}$, okay? So - $EI_f \frac{h^2}{2} \frac{d^2\beta}{dz^2}$. Do you see? We have a similar expression for the bi-moment. So, what we see is that this bi-moment is nothing but minus; this is nothing but capital $-E\mathbf{\Gamma}(\frac{D^2\beta}{dz^2})$.. Do you see the same expression, as M is equal to minus EI D^2W/DX^2? You see that EI, instead of I, this capital $\mathbf{\Gamma}$ is nothing but this is called I; you know, I mean in literature this is called I_ω, which means, you know, the torsion, the warping constant of the section. So, do you see how the bi-moment is related to torque? You know, the twist you see, and there is a separate formalism. You see, here we are doing it in a very simple way, okay? But please note that if we want to take the contribution of the wave along with the flange, then there is a technique; you know, there is a theory where you can find out the capital ω, and that's where some coordinate systems are defined, as you know, as sectoral coordinate systems. Sectoral coordinates mean nothing, but you know there will be a pole; I mean, you can take that Shear center to be okay, and then the perpendicular distance from that thing, you know, to the centerline of the sections, okay? When you have to do the whole thing over, you know, along the center line of the whole section, okay? And there are the sectoral coordinates, which are nothing but twice the area. For example, consider the section like this. And then from there, you see that sectoral coordinate. This is the area from here to here; when you move into this area, there will be some kind of sectoral coordinate. So, all these things are there, but you understand why we define bi-moment. Because whenever this twisting is occurring and this warping is happening, there is some quantity that is the moment of the moment, you see. So that's why it is called bi-moment anyway. Those things are not relevant here, but there are Vlasov theories you can learn about in depth; those are discussed in different books by Gara and Wavier, and others. Okay? So, we reached here. Now, if you want to, you know we have always followed the energy approach, right, for the stability analysis. So here, if you want to write the energy for strain energy for Saint-Venant torsion and warping torsion, you can do so. I do not need to derive it, but you can understand it as equal to Saint Venant and warping torsion. So, what will Saint Venant be? You all know what the integration from $\int_0^L$ GJ is. What is strain energy? $\frac{GJ}{2}(\frac{d\beta}{dz})^2$ it is the same as the axially loaded bar, right? You know $(\frac{du}{dz})^2 dz$, right? You know how to derive this, right? I do not need to derive it, right? All of you know, and what about the warping one? The warping one will be what? Try to just know it will be $\frac{GJ}{2}$. There will be half; we have to take it right. And here, what will also be? Plus, $\frac{1}{2}\int_0^L E\,\mathbf{\Gamma}(\frac{d^2\beta}{dz^2})^2 dz$. I am not deriving it.



I mean, you can do it right; it's not a big deal, right? Essentially, you just substitute the expression. So, $\frac{1}{2}\int_0^L E\,\mathbf{\Gamma}(\frac{d^2\beta}{dz^2})^2 dz$, right? $\int_0^L E\,\mathbf{\Gamma}(\frac{d^2\beta}{dz^2})^2 dz$ This is for an individual flange. For an individual flange, it is like this: it should be multiplied by two because there are two flanges. Okay. And then you just simplify that, and you will get it right. Fine. So when we derive, we are going to consider an arbitrary section, and in this section, we want to consider a column, okay? So, this is X, this is Y, this is Z, okay. And then there is this column, and this column is subjected to some axial force, maybe P, okay. And this fellow has some in the XY. If you consider this as X and this as Y, there will be a cross-section something like this. Okay. Uh, and then it will have some; please note that the CG of this XY is being considered as the shear center. This center, C, is nothing but the shear center; the origin is the shear center, and there is a reason for that. And this is the CG of the cross-section. Why is it? Because, you know, if we take the shear center, then here at this origin is fixed at the shear center; for the column, the origin is fixed at the shear center, whereas the CG of the cross-section is in a different place. Okay. So, from here to here, it is x_0, and from here to here, I'm defining it as y_0. So x_0 and y_0 are the coordinates of the cg. The CG of the cross-section X_0 Y_0 is okay, and the CG is fixed at the shear center. Why is the origin fixed at the shear center? Because if we fix the origin at the shear center, we can decompose the component into three components: U, V, and β. U is the translation along the x-axis, V is the translation along the y-axis, and B is the twist. You see U, V, and W; this is U, this is V, and this is W. Huh? Okay, β is the twist. We don't have W because, essentially, we are concerned about the two deflections and one twist rotation of the section's cross-section, right? That's what is essential. So here it is happening like this, but ultimately when it deforms, do you know what will happen? From this, you know it might; it will go to a—you have noted it down, right? So, what will happen is that I'll show you there. Right. So, what we can do at first is find out the strain energy. Okay, or we can do it later. Let us first do the work done.

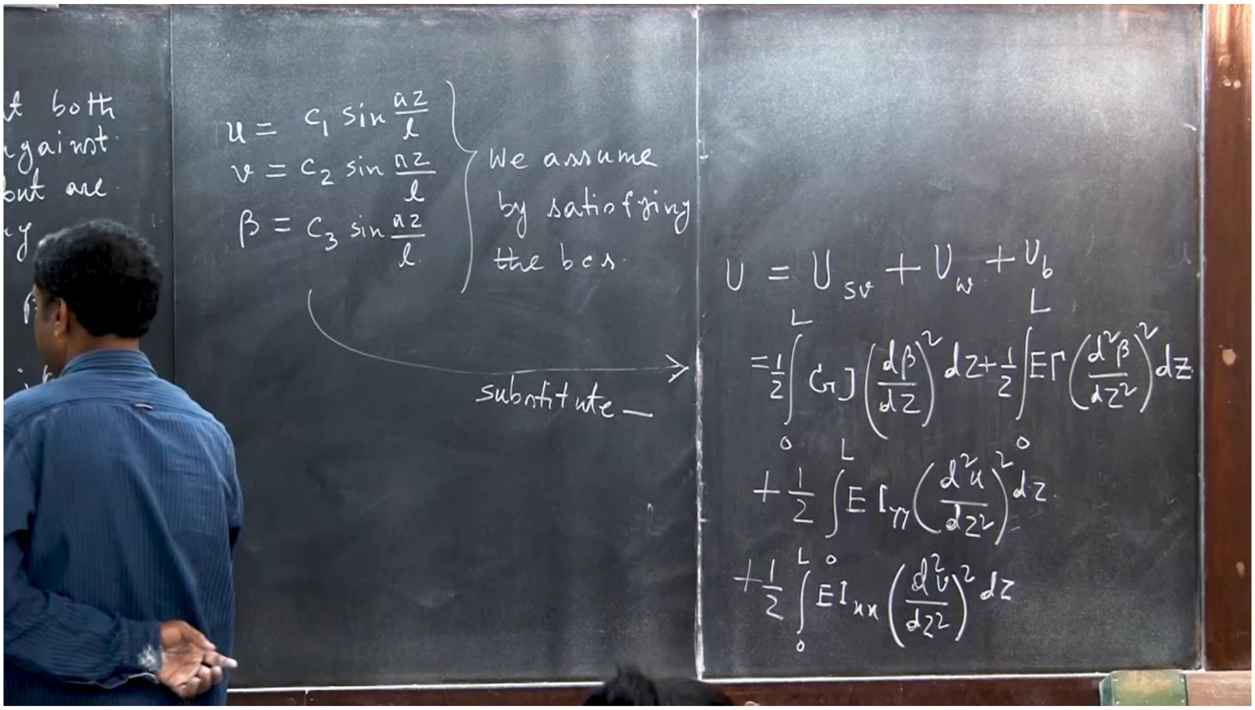


Okay, then we will go there. Huh? You see, let us consider the column at both ends that is restrained against twisting. So please note that these are for the strain energy due to torsion. Okay, but there will be strain energy due to bending also, right? Because along E and V, the bending will also be present, right? Okay. So along with that, there will also be other terms. I will come there. Restrained against twist and deflection but free against warping. What does it mean? That means at z = 0 and j = L, at z = 0, u = 0, v = 0, β = 0. Here, also at u = 0, v = 0, β=0, right? Because they are at rest, there is no deflection, and at rest against two, β=0, right? But when it is restrained rotation, please note that M_F is nothing but $-\mathrm{E}\, I_f\, \frac{d^2u}{dz^2}$, right? And then this is nothing but $-\mathrm{E}\, I_f\, \frac{h}{2}\, \frac{d^2\beta}{dz^2}$, right? So, when it is not restrained against warping, that means M_F must be zero; that is, M_F is zero. Then what? $\frac{d^2\beta}{dz^2} = 0$ at J=0 and at J=L. Do you understand what I'm trying to tell you? All of you understood what I'm trying to tell you. They are resting against deflection. So u_0, v_0, and β are equal to 0, restrained against twist, but they are not restrained against warping.



So, if they are not restrained against warping, that means mf should be zero at the end. If it is rest, then there will be some M_F, but I still need to be fine. Okay, could you please tell me why I require this condition? Any differential equation. Okay. Whatever the highest derivative exists in the differential equation, the boundary condition must be specified, you know, in terms of a derivative that is one order less than that. What was the twisting torque? $GJ(\frac{d^2\beta}{dz^2}) - \text{E}\,\mathbf{\Gamma}\,(\frac{D^3\beta}{dz^3})$. So, one order lesser means the third order will be the second order; moreover, what is the higher order derivative here? $(\frac{d^2\beta}{dz^2})$ right. So, $(\frac{d^2\beta}{dz^2}$)is okay. We can have some functions that satisfy, you know, by satisfying all these things we can assume some function, and then we can substitute there, and then you can do the minimization. Okay. Fine. When we assume that, if this is the case, then we can assume that u, v, and β can take $(C_1 sin\ \frac{\pi z}{l})$, $(C_2 sin\ \frac{\pi z}{l})$, and $(C_3 sin\ \frac{\pi z}{l})$,. Because this all satisfies that, right? Huh? Isn't it? All satisfy this double differentiation; the sign will be cosine, and the cosine will be sine. Fine. Okay. So, we can assume it the way we assume by satisfying the boundary conditions. Right. And now the total strain energy will be the strain energy from bending, huh?

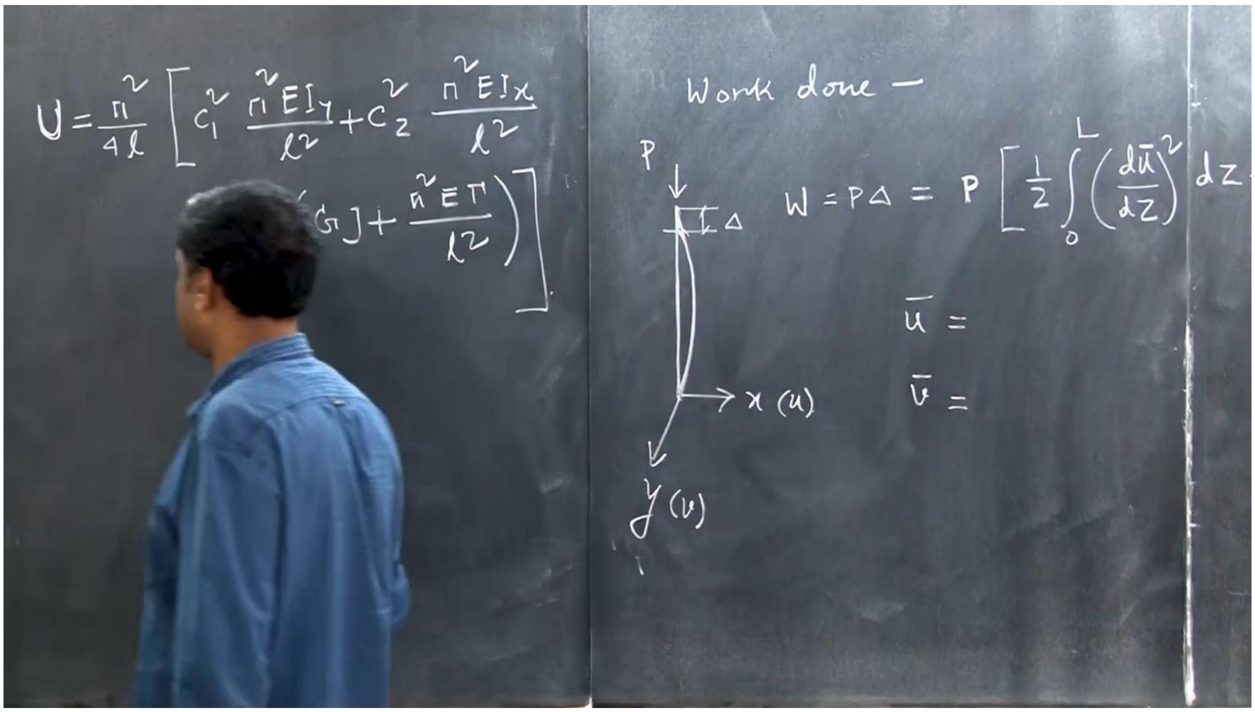
So, the strain energy due to bending will be, you know, $\frac{1}{2}\int_0^L EI_{yy}(\frac{d^2\beta}{dz^2})^2\,dz$. So, u means it is the deflection along a, which means the moment of the root area will be with respect to the y-axis, right?



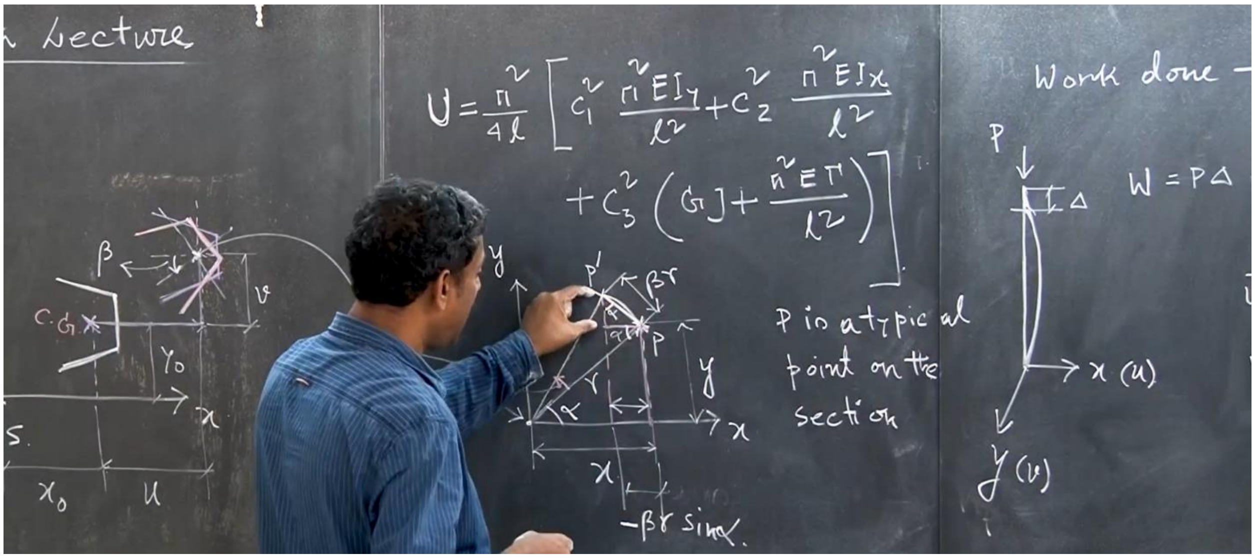
And then plus $\frac{1}{2}\int_0^L EI_{xx}(\frac{d^2v}{dz^2})^2\,dz$. Do you understand why? Because when it is bending with respect to the X-axis, U is along X, right? It is deflecting along X, which means the moment of inertia will be with respect to the moment of area concerning the Y-axis. And so, right, if it is deflecting along Y, that means along the X axis. Clear. So, this is for the bending, this is for the warping, and this is for the Saint Venant torsion, all components, right? So, we are discussing torsional and flexural buckling. That's what you have to consider: all the components. Okay. And so now you take and substitute. So, you substitute this. Taking a substitute is not a big deal, you know; it is basically that you differentiate and substitute, and then you integrate. I told you that for all these problems, recall the standard $\int_0^L sin^2\frac{\pi z}{l}\,dz$ equal to half and things like that, okay? May I write down the final expression? This integration is nothing; it's very simple. It's like a baby's job, right? Okay, so I'm removing this, and U will be the final expression. I'm going to write, and you will be, let me write, substituting this, it will be, you know,

$\frac{\pi^2}{4l}[C_1^2\frac{\pi^2 EI_y}{l^2}+C_2^2\frac{\pi^2 EI_x}{l^2}+C_3^2(GJ+\frac{\pi^2 E\Gamma}{l^2})]$.

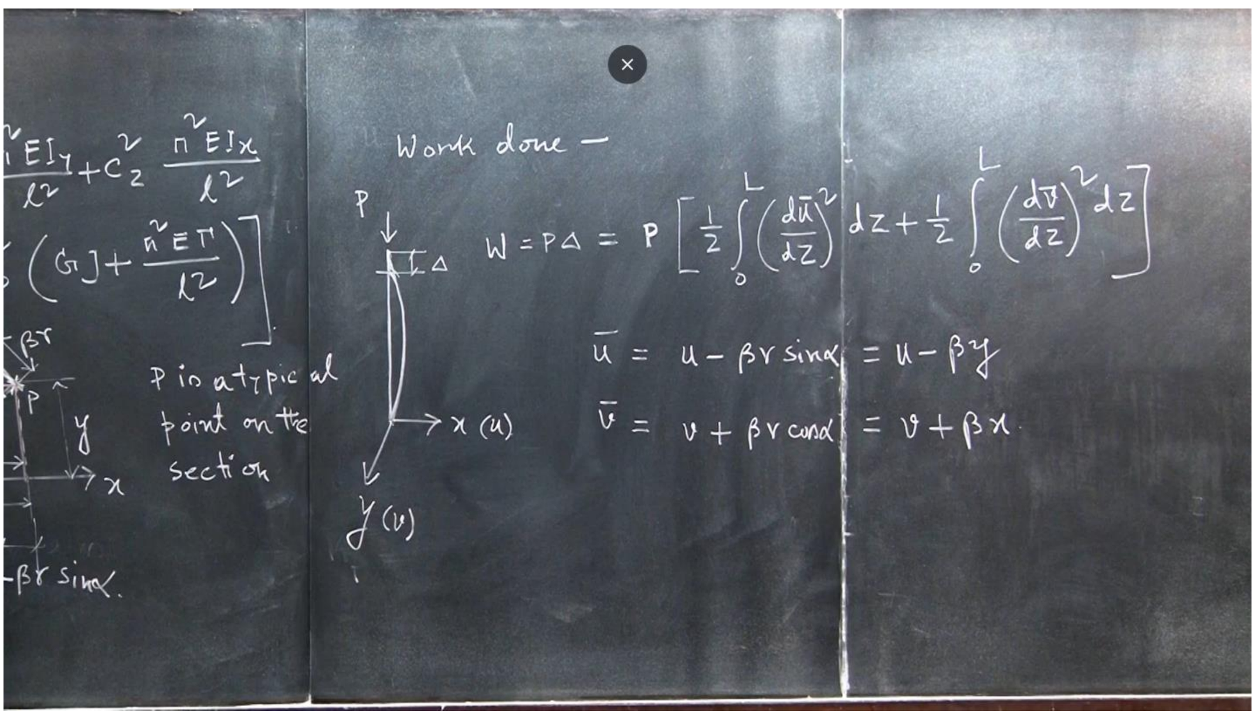
Please note that we are getting expressions that remind us of the expression for the critical load $\frac{\pi^2 EI_y}{l^2}$ Okay, we will come to that later, but these are basically I mean, of course, you can see c1, c2, c3, and then when they appear in squares, all the squared terms will be there; that's why c1 squared, c2 squared, and c3 squared are there, right?



Now for the bending, these two terms you know, u and v, will be two separate terms. That's why $\frac{\pi^2 EI_y}{l^2}$ and $\frac{\pi^2 Ex}{l^2}$ exist because of these two terms, along with the Saint Venant and the warping ones. For the Saint Venan GJC_3^2 and for the warping one, it is $\frac{\pi^2 E\Gamma}{l^2}$. Now, the interesting thing to see is that, look, for the GJ Saint Venant part, there is no π^2/L² term associated. Why is that? Because, see, in all other terms, it's bending. See when it is resisting torque by warping. Essentially, it involves the differential bending of the flanges. So that's where similar flexural rigidity terms come in: E capital ω, right? But when it is Saint-Venant torsion, it is analogous to what an axially loaded bar is; that's what in the strain energy expression it is? $GJ(\frac{d^2\beta}{dz^2})$, all other second-order derivatives, but here is a first-order derivative, right? So that's what it is, only GJ. These are all dimensionally consistent. Understand that, right? Okay. Now we are going to concerns, so this is strain energy. Okay. Now we are going to consider work done. So, how we will consider the work done. So, this fellow, this column is there. Now we don't care about the cross-section; we will consider it. So here is this P, right? And here, whatever. So, this fellow is going to come down now, right? So, it will basically deflect, you know, somewhat. So, this is the deflection, right?



So, the work done is PΔ right? PΔ is what? U bar and V bar are nothing but what? The von Karman nonlinear term, right? Please note that here we have x and y. So, we have deflection along X and component U along Y; it is V. So, von Karman, both the components along x and v must be considered; that's what I'm taking du bar dz squared dz, yeah, right, both, okay. And then that should be multiplied by p, right, to have the work done, right, w fine. Now there is another thing: see why I'm putting u bar and v bar? U bar and V bar I am putting because you see that U, V, and W are what? Here, the deflection for the CG, right? That's what we have taken, right? But here, what will happen is that another thing will come into the picture, so please note it down, okay? So, what will happen, you know, is that the section is also undergoing a twist. Because of the twist, there will be some U and V. U and V will be modified; that's why U bar V bar. There will be an additional thing with U and V. So how is that? That I'm just basically explaining to you, okay? So here I am, you know, consider this: here is the point I was considering. This is the point, right? This is the point, huh? Now I mean, when it is being twisted, assuming that it is coming here, right? So, it is P, and it is P dot, right? And, of course, you know this angle is β; β is very, very small. So, I mean, we can essentially replace the arc with this one, right? So, what I'm going to do, and then this angle I'm assuming is as if this is, you know, let me show you what is essentially happening. Okay. So, you see, when I'm plotting the coordinates of my axes, this is the shear center; the shear center is my origin, right? So, I have this, you know, maybe this section here. I have this section here, right? And it has hit CG. So, from here, this is CG. So, this is x_0, and from here to here, this one is y_0. Right? Now from this CG, it is going to some other point, you know. So, from here, it is going somewhere else. Okay. So, what is that? From here to here, what is that? This is U. And what is this? From here to here, what is this? This is V. So much to see; from here, I'm amplifying this infinitely small deformation, right? But I'm amplifying the deformation. This CG is moving U, and then along this, it is also moving this. But along with that, it is also going through some twists, right? So, it is like an anticlockwise. Basically, along with that, you know this fellow is, you see. So, along with that, you know, this is happening here; you see that, β. So, what is happening in the three stages of deformation? From here, this fellow is then coming here. Because of translation, and then finally. because of the rotation it is attaining this configuration.



So, initially it was X_0 Y_0; here it was the CG of the section, and this is, you know, the shear center. So, from X_0 Y_0, this CG is moving to X + U and Y + V. In addition to that, there is a twist with respect to the Z-axis, β, right? So, U and V are the pure translation components. What is initially? But when there is a translation component contributed by β as well, that's what I am defining as

U bar V bar. Do you understand what the difference is between UV and U bar V bar. U bar V bar is nothing but the final component of the deformation Y, which contributes both the pure component and the component induced by this twist β right. So, this fellow was here, and this fellow is going, and then this angle is β. What am I doing? I am going to drop this here, you know. So, from here to here, I'm assuming there is a coordinate x_0, okay? And then this fellow, its coordinate is the y, basically not x_0. Any point on the cross-section P is a point on the cross-section; P is a typical point on the cross-section of the section, right? So, this fellow, I mean we are defining this as X and this as Y, and there will be a set of X and Y, right? So, P is going to, so now you see that, from here what I'm going to do is P is moving to P dot, and this is because there is an anticlockwise rotation β. Okay. Now what I'm going to do is drop a perpendicular here, and I'm also going to drop a perpendicular here, and then I'm perpendicular here. Okay. This fellow, this fellow is like this. So, this angle, I am going to say this angle is α. This angle is α. So, you know this. I am assuming from this origin and through this distance you know, so please note that when I'm rotating it, it is basically this point, right? It has already translated, and please note that this origin is nothing but this one. Okay. So, this fellow is going to be the origin here. Okay. Now I'm assuming that this rotation, this one, this radial line is nothing but what? R. Okay. So, then you know, I'm putting this in R. Huh? Okay. So then, what is this distance? This distance is nothing but what? β*R, huh? So, if that is β*R, then what is happening? Because of this, what is this distance? This distance is nothing but what? β R, this angle is α. This is α. So, $\beta r sin\alpha$. Therefore, this one is nothing but $\beta r sin\alpha$. And if it is positive β, anticlockwise β, then this fellow is moving in the opposite direction, which means $-\beta r sin\alpha$, right? So, what I can say is u bar is $U - \beta r sin\alpha$, right? So along with u, whatever u was. because of this rotation u will be reduced. But what about V? This V and this are nothing but what? $\beta r cos\alpha$ this one. And this is in positive Y direction, right? So, this is $V + \beta r cos\alpha$, right? So, this is $U - \beta$. You tell me, what is $r sin\alpha$? $r sin\alpha$ is nothing but this. This is R. This sin α is what? This is x; y, $r sin\alpha$ is nothing but the y-coordinate, and then $r cos\alpha$ is nothing but x. All of you understood what I'm trying to do; the displacement component will be changed. There will be some components to the displacement in relation to the pure translations U and V. because of the you know twist of the section and that's what essence you'll have. So, this UV it defined on each and every point on the cross-section right. So, this fellow is going to happen like this, right? So, I will tell you what else you know: this is P into, you know, we can write σ DA. P will change; P is what? It is nothing but $\int \sigma dA$. I will come back there later. Okay, in the next class, we'll continue that. Thank you very much for today's class.