**Stability of Structures**

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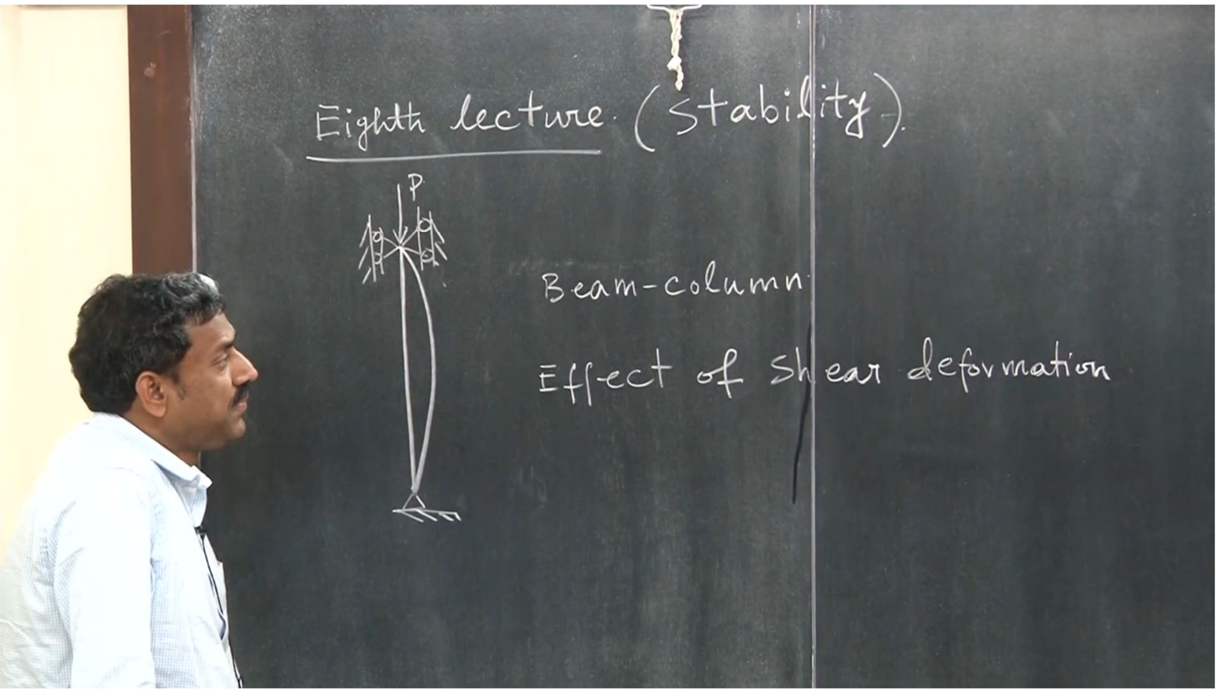
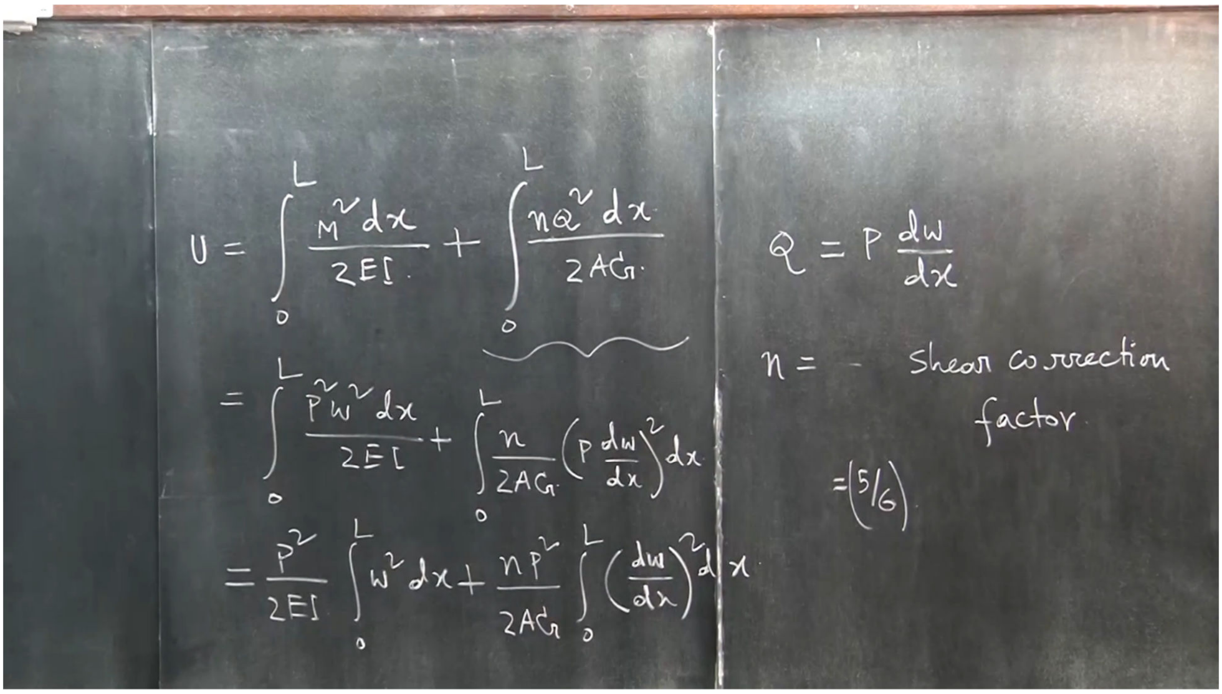
**WEEK-04**

**Lecture 8: Effect of shear deformation on Beam-column system and Elastica**

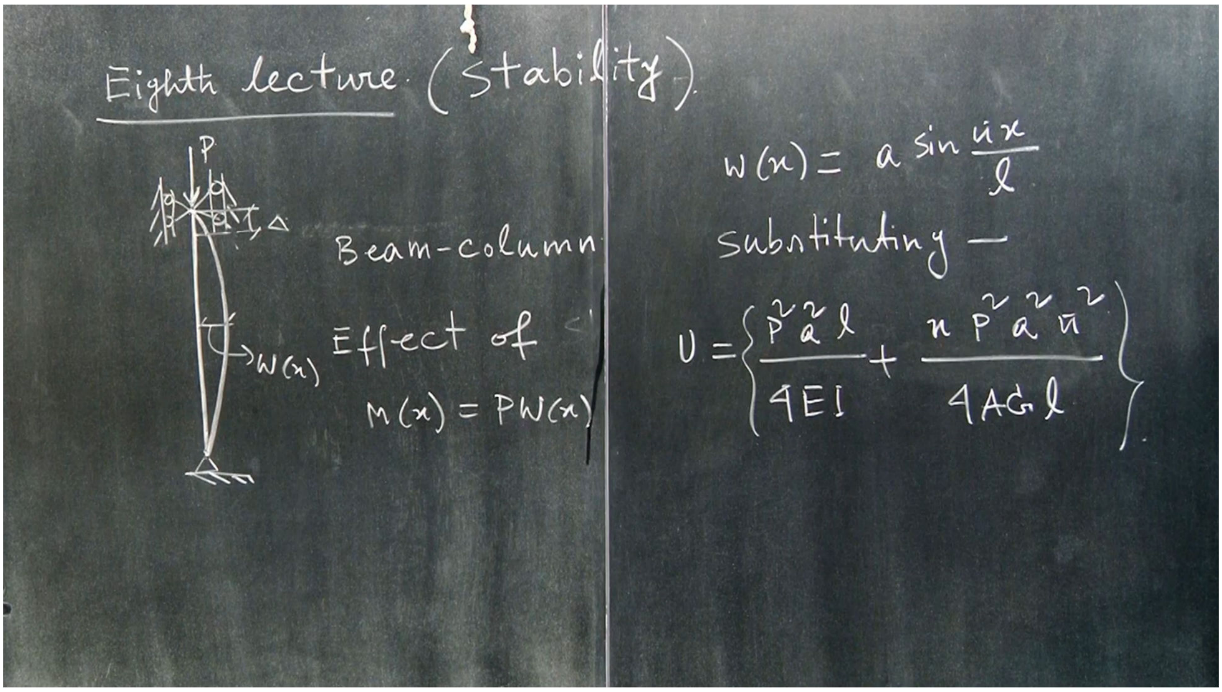
Welcome to the 8th lecture on the stability of structures. So, what were we discussing? We were discussing the effect of shear deformation on the critical load, right? We will demonstrate that knowing shear force and shear deformation basically helps you understand how it reduces the critical load, and if it reduces the critical load, it leads you to the non-conservative side, right? So, it must be accounted for in the design. If we neglect this shear deformation, then it might lead to a non-conservative design, you know, right? Of course, you know shear deformation is significant, as you all know that it is significant when in a beam, you know a shell.



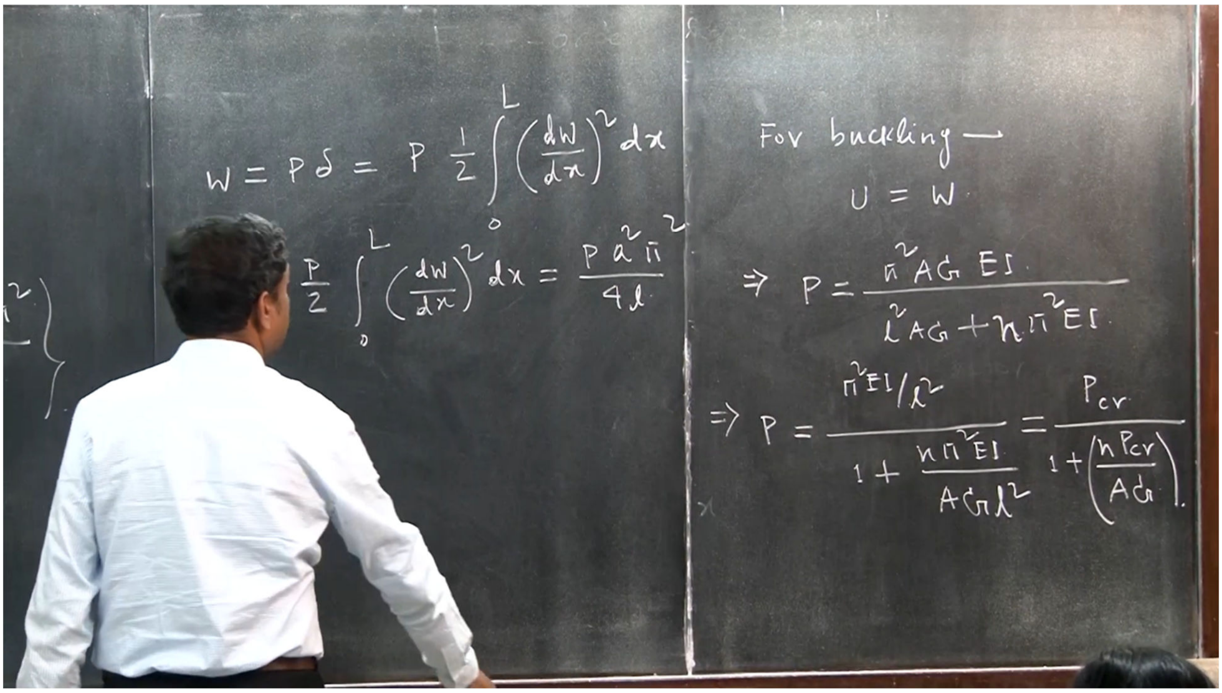

I mean a plate and shell when it is span by depth is what? Its depth, you know, span by depth ratio basically is smaller, right? So, generally, if it is less than—I mean, less than 10, less than 8—you know, then we consider the shear deformation to be significant. The way you have considered the shear deformation in the beam is right. We have learned about Euler and Bernoulli bending of a beam, right? Beam theory, but then if you include shear deformation, that is where you give rise to Timoshenko beam theory, right? Similarly, in plates, you have.... Thin plate theory is correct. What you are learning in plates, if you consider this, which is Kirchhoff plate theory, is that in thin plates where you neglect shear deformation, the deformation is purely bending, as you see. So, plane sections remain plane, you know, after deformation, right? That is what the assumption is. But in thick plates, you have to consider shear deformation, and that is why the assumption that plane sections will remain plane is no longer valid; you need to consider the additional rotation of the cross-section.



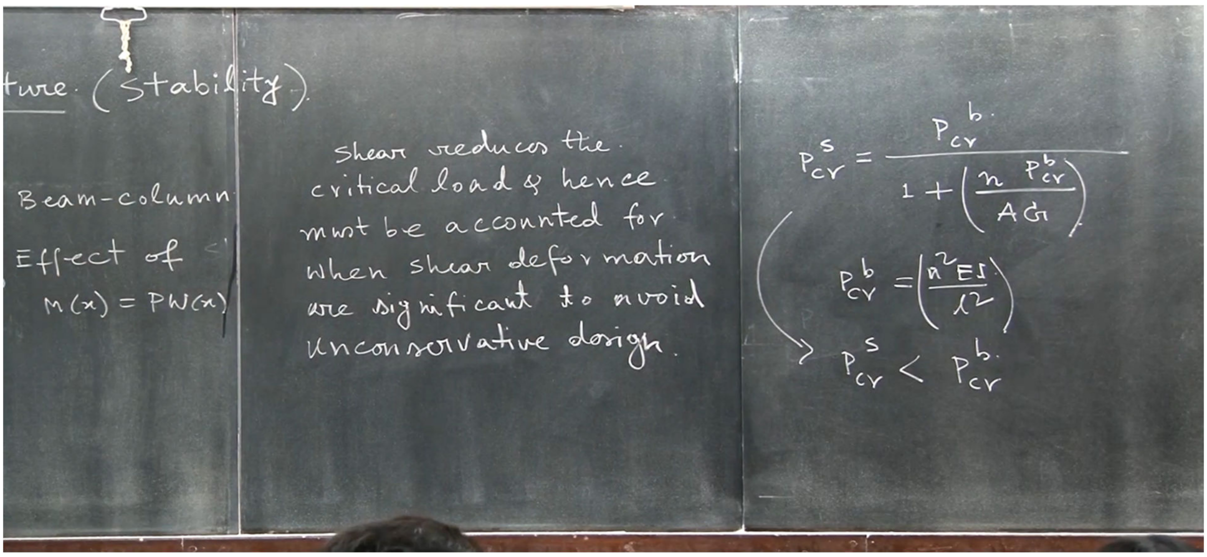

So, that theory which includes the shear deformation is the Reissner–Mindlin theory. Similarly, in thick shell theory, you will see the Nagadi theory for thick shells, including shear deformation. So, here we will first demonstrate the influence of critical shear deformation on the critical load for buckling. So, for that, once again we are adopting the energy method, which is very intuitive and easy, right? So, here is a, you know, we are considering a, you know, column, right? And the subject is subjected to load; you see that here it is hinged at one end and the other end is also hinged, but axial deformation is allowed, right? Okay. So, this way, we can consider the shear deformation, you know, the strain energy of bending; the strain energy of bending is contributed by the bending moment as well as the shear, right? So, the bending moment can also be expressed in terms of, you know, deformation, like integration $EI\frac{d^2W}{dx^2}$ in terms of the curvature term. But we are next considering it in terms of the bending moment, right? And then the additional term is coming because of shear; this term is basically the additional one that we neglected earlier. Now, we include this thing, and we will see what the influence is. And for that, you see that for the shear force, I am considering skew. So, $v^2/2he$ and then is nothing but the shear correction factor; the shear correction factor varies from 2, depending upon the geometry of the section. But $\frac{5}{6}$; is typically used for this rectangular section, right? And then shear force is $P\frac{dW}{dx}$; you know where this is coming from, right? You know $\frac{dW}{dx}$, right? So, this is the contribution of the axial force to shear. So, now and then, the bending moment can also be expressed for this case, at least, as far as this thing is concerned; you see that this is out-of-plane deflection $w$ as a function of $x$. So, I can write the bending moment as $p$ times $w$, right? This is the bending moment; let us substitute it there, okay? So, $\frac{p^2wdx}{2EI}$. Plus, you know integers from here, $\int_0^L \frac{n}{2AG}$, and $q$ is nothing but $p\frac{dw}{dx}$ squared $dx$, right? This is the way, right? Now we can simplify it further. So, that is not a problem. So, I just take this $\frac{p^2}{2EI}$ outside. $\int_0^L w^2dx + \frac{np^2}{2AG}\int_0^L(\frac{dw}{dx})^2dx$.



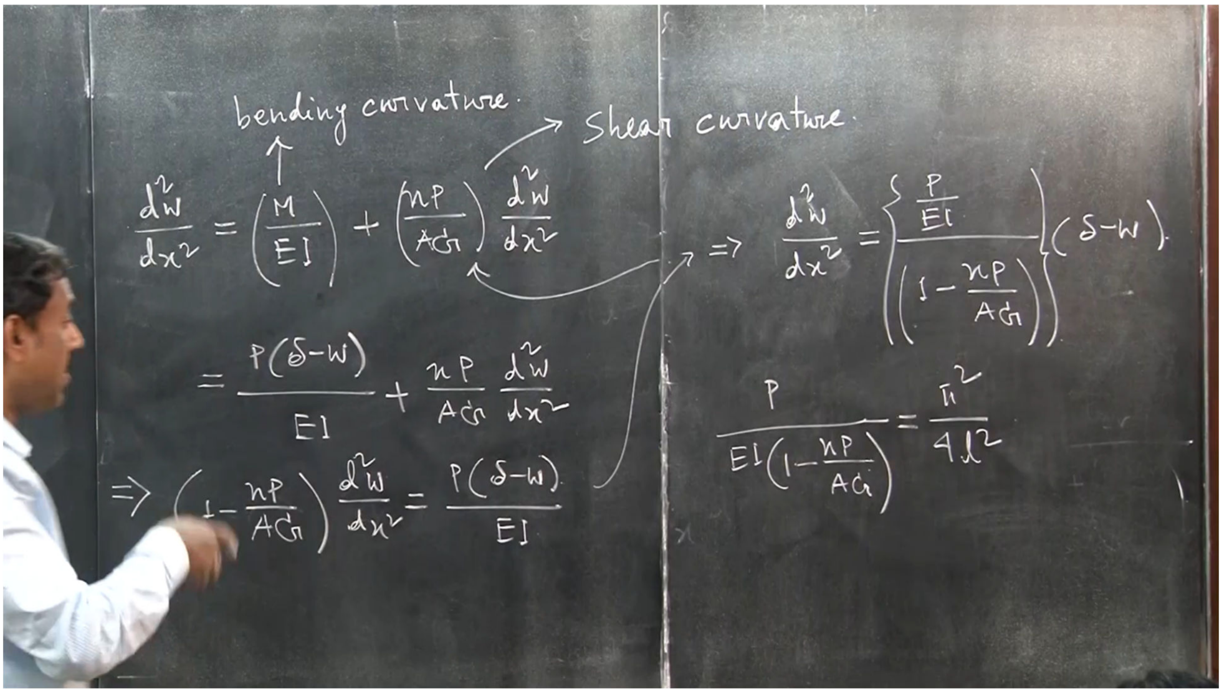
So, now we assume this out-of-plane deflection; you can apply a procedure similar to the LHT to assume some displacement profile satisfying the essential boundary condition. That displacement boundary condition is simply supported at this end; the displacement here is 0. So, we can assume $W$ is equal to something like, you know. And let us assume that only a single mode is governing. Okay, let us find out the single mode. So, it is fundamental that we can have other, you know, terms also included, but I am considering a single, you know, approximation, okay, single-term approximation. So, if you take and substitute, we integrate $w^2 dx$ and $(\frac{dw}{dx})^2$. So, here you will see that I am just expressing the "u," okay. I will just express the $u$, and you will see that final expression of $u$. You can integrate $w^2$ into $\sin^2(\pi x/l)$, right? The integration of $\sin^2(\pi x/l)$ is nothing but $l/2$. Similarly, the double derivative, single derivative is $\cos(\pi x/l)$, right? So, I am just writing the final expression. This is plus $\frac{P^2A^2L}{4EI} + \frac{nP^2A^2\pi}{4AGL}$, right? Now, this is the strain energy of bending, right? We equate this one with the work done right, you know, in order to find the critical load. Equated with the work done, the work done that this axial force is doing is going through out-of-plane deformation, right? Out of plane deformation is this point will move right; this point will move and what is this move? Δ is an irrigation von Karman non-linear strain, right? To get the in-plane deformation due to out-of-plane displacement, right? So, what was that? So, I am removing this.



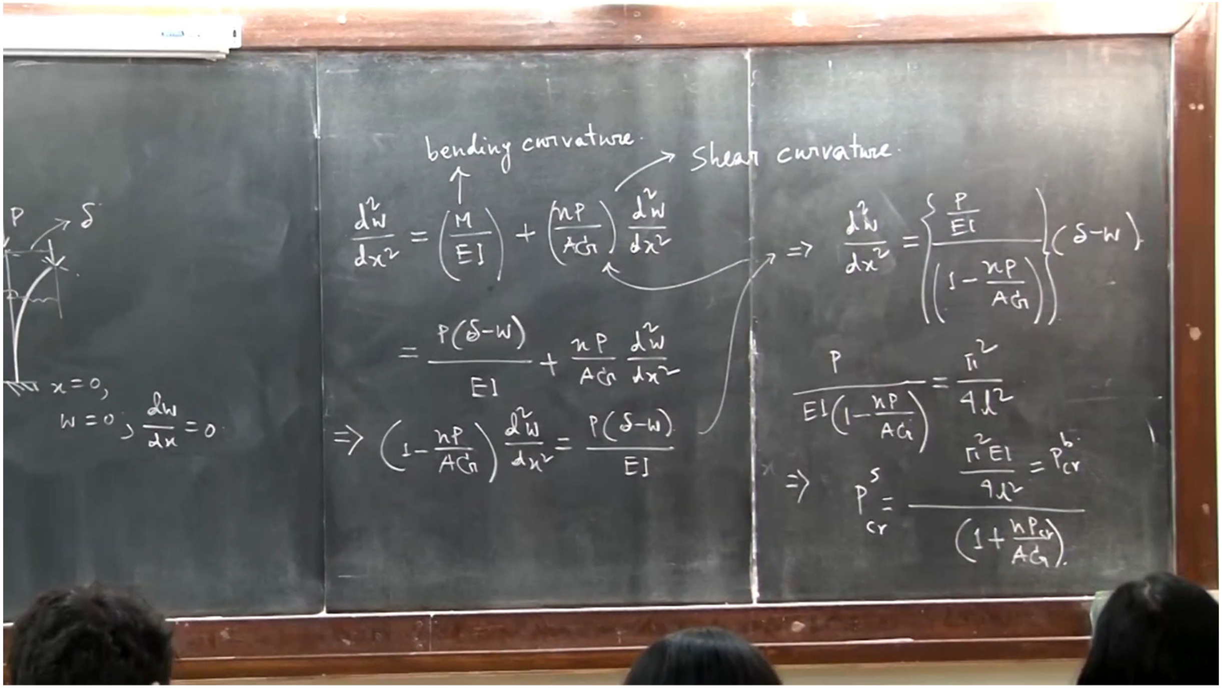
So, if you write $W$, nothing but, you know, $P$ into $\Delta$, and this is $P$, and $\Delta$ is nothing but, you just $\int_0^L (\frac{dw}{dx})^2\, dx$, okay, and then $\frac{P}{2}\int_0^L (\frac{dW}{dx})^2$. So, once again, we assume $W(x)$ to be $a\sin(\pi x/L)$, right? Substitute it, and then you will see; if you integrate correctly, then you will get $\frac{Pa^2\pi^2}{4L}$. Now for buckling, you know about the critical load, right? For buckling, what we have seen is that, energetically, what happens is that I am standing here, right? So, during buckling, whatever axial force is present is applied; that axial force is going to work on me, right? And then that will allow for out-of-plane deflection, right? The buckling is happening, right? So, I mean instead of writing $u$ and $w$, you can write $\Delta u\ \Delta w$. So, I am just writing $u$ to $w$, okay. And then if you equate these two terms and simplify, that is why I am going to write the final expression, which will be $P$, you know, $\frac{\pi^2 AGEI}{I^2 Ag + n\pi^2 EI}$, okay, that is the expression, and then we will further simplify it. So, if you further simplify it, then it is $\frac{\pi^2 EI}{l^2}$; then you know $1+\frac{n\pi^2 EI}{AGL^2}$ and $\frac{\pi^2 EI}{l^2}$ is what? This is nothing but $P$ critical load, so this is $P_{critical}\left(1+\frac{nP_{critical}}{AG}\right)$. So, what we see here is that the critical load, which I am going to write as $P$, is nothing but the critical load in the presence of shear deformation, right? So, I can write that $P_{critical}$ here; you know, in the presence of shear deformation, it becomes $\frac{P_{cr}}{1+\text{n}\frac{P_{cr}}{AG}}$. That was for bending and then $AG$. You see dimension; this is also a dimensionless quantity because $P_{critical}$ bending was nothing but $\frac{\pi^2 EI}{l^2}$, okay. Critical load is okay. So, $A$ into $G$ is the same dimension as force, right?



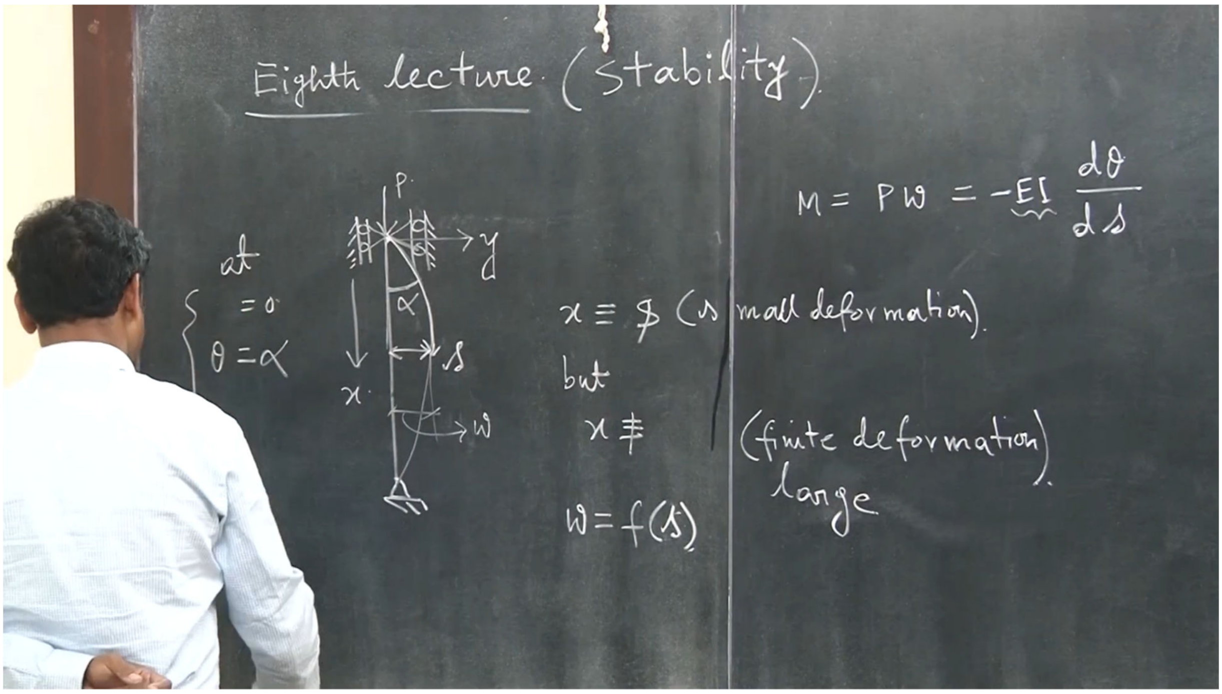
The unit is the same, so it is a dimensional quantity, right? So, $1 +$ some number. So, you see this denominator is always greater than 1, right? So, $P_{critical}$, we can clearly understand from here that $P_{critical}$ in the presence of shear is less than $P_{critical}$ in the presence of bending, right? So, shear deformation is here; you know that shear deformation helps in reducing the effects. Let us write down what we have learned in the lesson: shear helps me. Shear reduces the critical load and hence must be accounted for when shear deformation is significant to avoid unconservative design. In fact, I mentioned to you that there was a deep, you know, structural failure, you know, and that was due to our lack of understanding of shear deformation, okay. So, what we have learned is that shear deformation helps reduce the critical load. In fact, in the late 90s, you know, I mean not the late 90s, around the 90s, and even before that, the late 80s, you know, lots of shear deformation theories were actually coming out for plates and things, including this Reddy's, Jane Reddy's shear deformation theory. Now, what we are considering here, you know, for most of the shear deformation theory, is that the first-order shear deformation theory is by Timoshenko. Right in beam, and in plate is raised and mined correctly. So, when we consider one additional rotation of this cross-section, these are mostly, you know, thoroughly discussed in finite element or mechanics class, in your mechanics class, okay, and in the theory of plates and such. But you see that the deformation of the shear force distribution is parabolic, right? So, there is an inconsistency. So, for that People introduce higher order shear deformation theory. That means beyond first order shear deformation theory. And that is why it gives rise to Reddy, Jane Reddy's higher order shear deformation theory, and Professor Kant's shear deformation theory. And all this theory is not asymptotically consistent. There is an asymptotically consistent shear deformation theory by Babuskas, okay. So, all this theory they demonstrated the influence and it has been a very important you know, because very very easy I mean topics for PhD, many PhDs has been guided on this. However, later people realized that it is not that significant in many cases. I mean, you do not need to consider the shear deformation unnecessarily, okay? If it is significant, then it was particularly significant, because we were using laminated composite, right? In laminated composites, you have a series of lamina that are glued together to form a laminated composite because they have low shear resistance. So, there the shear deformation will be significant, and that is why all refined shear deformation theories are important, okay? You see. You know higher-order shear deformation theory when you include it here in buckling. So, there have been lots of studies there, okay. So, shear deformation becomes important in $L/D$, as you know, when it is less than 10 or maybe less than 8 or so, okay. So, that means you know it is a deep beam, okay? Deep beam, or deep, you know, shell, and then deep, you know, plate, and things like that, okay? Thus, we can also find out the same things in another way.



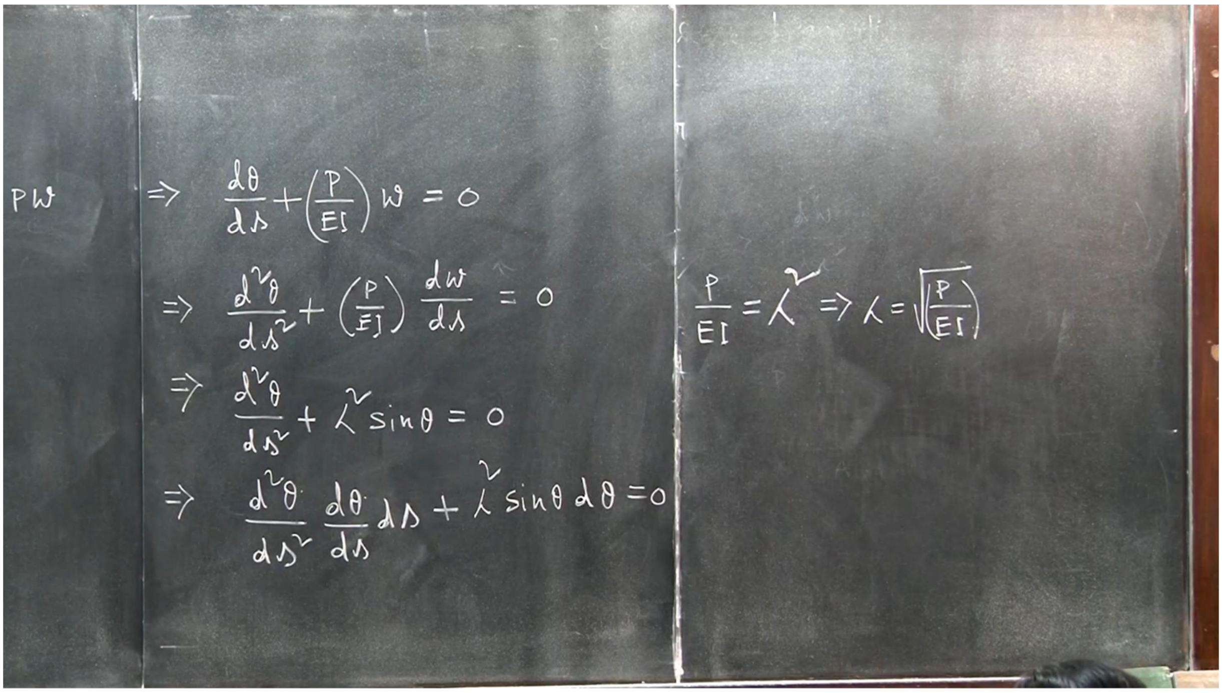

It is very simple. So, we have used the energy method, but we can use another method. I will just briefly talk about it, you know. So, I mean, if you consider that, you know, consider a cantilever, okay, and this fellow is deforming, right? So, here it is subjected to $P$, you know, and then it is basically deforming, you know. So, there is some deflection, you know. Now, you see that what we write is that the curvature $\frac{d^2w}{dx^2}$ will be bending right $\frac{dw}{dx^2}$. So, it is the bending moment by $EI$; the first is the same $M/EI$, right? And what is the bending moment? Bending moment here in this case, at least you know, what will the bending moment be? If this deflection is $w$, assume that you know this distance was nothing but $\Delta$. So, whatever the bending moment is, it is nothing but $p$ times $(\Delta - w)$, right? This is the bending moment, right? So, this is the bending moment, but then in addition to that. So, you see that this is the bending moment, but in addition to that, because of the shear, there will be additional curvature. So, what is the shear? Shear, what is shear deformation? Shear deformation is $\psi$; you know I am assuming that this is shear deformation. Shear deformation, and let us assume that first-order shear deformation is valid. Well, what will this shear deformation be? This shear deformation is nothing but $V$ divided by $AG$, and then $n$ will be there, of course, $n$ right. $V/AG$, because that is the additional rotation, right? $V/AG$ is the deformed. I am assuming that this is another rotation of which you know, the transverse normal right. And then this $v$, you know we have learned that this $v$ is nothing but. That $Vn/AG$ this $v$ is nothing but $P\frac{dw}{dx}$; that is what we have. We know how this $P\frac{dw}{dx}$ is coming, right? So, now this is the, you know, bending curvature, right? This is the bending curvature, and if this is the shear deformation, that is rotation. So, shear curvature will be what? What will shear curvature be? $\frac{d\psi}{dx}$ the shear curvature; that will be nothing but what? $\frac{NP}{AG}$, this is sheer curvature, right? So, let us bring it there as well. Is the curvature due to shear, shear curvature? Once again, we are assuming that you know first-order deformation is valid, and that is why we need to enforce the consistency between shear stress distribution and shear strain distribution. We are putting in the $n$ that is the shear correction factor right. So, all of you understand, you know, you are looking back and should not have a vacant look. You understand what I am talking about, right? Okay, good. So now I am removing this part. So, okay, $p$ is equal to $(\Delta - w)\ EI$ that I am going to put. So, $\frac{p(\Delta-w)}{EI} + \frac{nP}{AG}\frac{d^2w}{dx^2}$. So, I will take this out. So, here is what I will write: $\left(1-\frac{nP}{AG}\right)\frac{d^2w}{dx^2} = \frac{p(\Delta-w)}{EI}$. Now, you see that this one, I mean, you can. You can solve this equation; I mean, you just take this thing and also take this thing. So, first of all, this is the homogeneous for inhomogeneous, but I am not solving it; you can solve it, right? And if you solve it, then you will see that one of the eigenvalues that you are going to get. So, you will see that here the way I will just take it. So, here I will just write $\frac{d^2w}{dx^2} = \frac{P}{EI\left(1-\frac{nP}{AG}\right)}$, okay. So, this is basically the term that will be eigenvalue, okay. I know. So, I am going to write. So, if you just enforce the boundary condition, maybe for this, you know, he fixed it and he should write that $x = 0$, you know, $w = 0$, and $\frac{d^2w}{dx^2} = 0$, right, something like that. Then you will see that if you solve it, you will be reinforcing this. I am not doing it, but you can see that $\frac{P}{EI\left(1-\frac{nP}{AG}\right)} = \frac{\pi^2}{4L^2}.\ 4L^2$, why is it coming? Because you see that 4 is the effective length, which is $2L$, right? Okay. So, now you see, what you are going to get here, you know $P$, this $P$ is nothing but $\frac{\pi^2 EI}{4L^2}$, and if you just multiply it, It will be $1-\frac{nP}{G}$, and then, if I take it to the bottom by polynomial expansion, it will be approximately $nP$. Do you see the same expression? The same expression we are getting. So, $P$ critical load in the presence of shear is nothing, but this is nothing but $\frac{P_{cr}}{1+\frac{nP}{AG}}$. Do you see that $1 + nP$ and this $nP$, $nP_{critical}$ in bending? Ok.




We are getting the same expression. So, both the energy using approach and the energy equilibrium approach, by writing down the governing differential equation considering the curvature contributed by shear deformation, lead us to the same expression. So, you see that in the presence of shear deformation, the critical load reduces, right? So, it is consistently okay. Now, what we are going to do is that I am going to do the, you know, so all of you understood, right? So, the effect of shear deformation on buckling is significant. So, shear deformation reduces the critical load. Now we are going to consider it. Large deformation of the column is okay, so we are going to consider one second, you know. I am considering it; it is going through large deformation, and I am just here. I am going to consider the x-axis and the y-axis here, okay? And then this angle is $\alpha$, and here I am going to consider this arc; there is another curvature, which I am denoting as $s$. $s$ is along this arc, okay? $x$, $y$ arc, and then this is the origin. You know, $P$, and then this deflection, of course, this deflection, this one is basically $W$. Now, what we are going to do is write that a bending moment is nothing but $P$ times $W$; that is what we can write: $P$ times $W$, right? The $P$ is the axial force; whatever the lateral deflection $P$ is, $W$ is the bending moment, right? For this, $W$ is a function of $X$ or a function of $S$, whatever, right? Now we can write it; you know that this is nothing but $-EI$. See $EI\,\frac{d^2w}{dx^2}$, but we cannot write $\frac{d^2w}{dx^2}$ because we are considering the finite deformation of the column. So, instead of $\frac{d^2w}{dx^2}$, we can write $\frac{d\theta}{ds}$, right? $\frac{d\theta}{ds}$, that means we are distinguished; you see, in small deformation theory, finally, in small deformation theory, $X$ is the same as $S$, right? But in this case, $X$ is not the same as $S$. Do you understand, or am I going to write $S$? $S$ is basically the curvature; you know $S$ is the arc length coordinate. You understand what I'm trying to say, right? This is for small deformations, right? And here we are considering the finite deformation, right? Finite deformation, large deformation, or rather, large deformation. Large deformation.

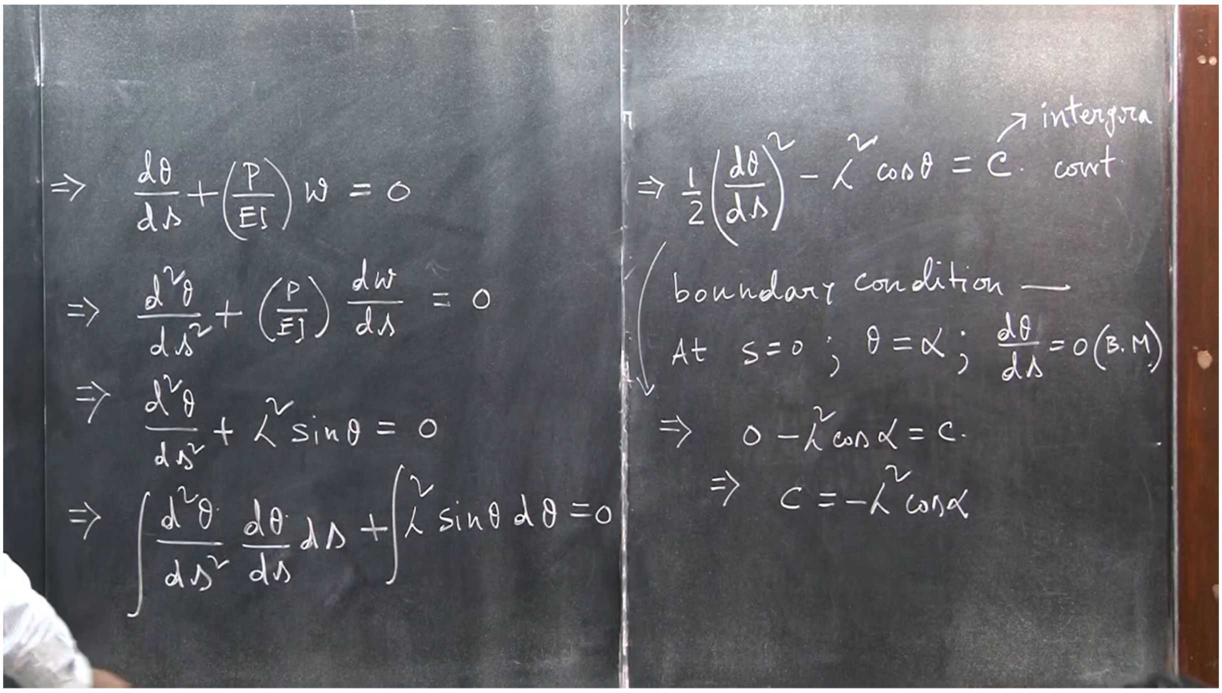



So, we cannot write $X$ the same as $S$; you understand what I mean. Whatever we were writing $W$, we can write $W$ as a function of $X$, I mean we can write it as a function of $S$, right? We can write it as a function of $X$ as well; there will be some mapping between $X$ and $S$. So, when I am writing, $EI\,\frac{d^2w}{dx^2}$. So, $EI$ is the flexural stiffness and $\frac{d^2w}{dx^2}$ is the curvature. Curvature is nothing but a change of slope, a change in slope. So, a change in slope means that if I consider this slope to be $\theta$, the slope is $\theta$; you know at $\theta$, basically at $s = 0$, we are writing $\theta = \alpha$, right? That is one of the boundary conditions, right? I am considering. We are distinguishing between the $x$ and $s$ coordinates, which are different. So, the change in slope is nothing but $\frac{d\theta}{ds}$, $\frac{d\theta}{ds}$, right? Let us write this small "$S$" okay. So, $\frac{d\theta}{ds}$, you understand this is perfect, right? So, then this one is nothing but $p$ into $w$, right? So, now what we will do is, from here we will $\frac{d\theta}{ds} + (\frac{P}{EI})W = 0$. That is the governing equation, right? Isn't it? Let us do one thing, and you will understand what I am going to do. So, we will differentiate it to one second. So, $\frac{d^2\theta}{ds^2} + (\frac{P}{EI})\frac{dw}{ds} = 0$; if you differentiate it once again, right? So, you see that here, now what we are going to do is see if, when I consider the small, you know, segment here, it is deforming, right? So, this is $ds$, and then this is, you know, so this is $ds$, this is $ds$, and this is $dw$, right? $Dw$, and this angle is $\theta$. So, $\frac{dw}{ds}$ is nothing; this is nothing but $\sin\theta$, right? So, I can write this as $\frac{d^2\theta}{ds^2}$ + I will write $\frac{P}{EI}$; let us define $\frac{P}{EI}$ as $\lambda^2$, okay. Now, earlier, we saw $\lambda^2$, and I am just going to write $\sin\theta = 0$, okay? Then I will do further simplifications. So, what I will do is $\frac{d^2\theta}{ds^2}$, and then I am going to write $\frac{d\theta}{ds}\frac{ds}{ds} + \lambda^2\sin\theta\frac{d\theta}{ds} = 0$. So, see $\frac{d\theta}{ds}ds$ is nothing but, you know, I am just saying there is nothing but $d\theta$.

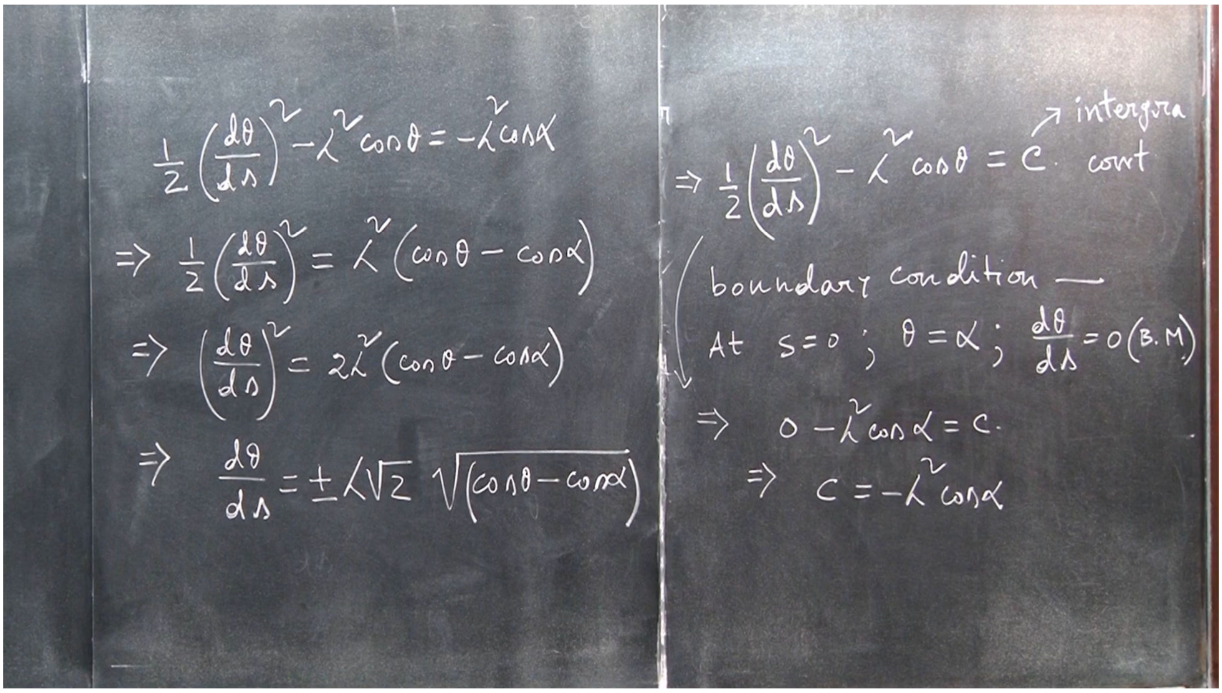



I am multiplying both sides, right? Fine. Why I am doing that is because it will lead to some simplification of the equation. Okay, and please note that $\lambda$ is the parameter; $\lambda^2$ is nothing but $P$, so $\lambda$ is nothing but the square root of $\sqrt{\frac{P}{EI}}$. Okay. $EI$ is the flexural stiffness, $P$ is the axial force.

So, until now, you follow how I am climbing here; nowhere have I highlighted the small violation of the large deflection assumption, right? There are no assumptions of small deformation. This is a large deformation theory of beams, and this of column beams, and that is why this is called, you know, when we consider the large deformation of a column, this is called elastica. So, a little fancy name, Elastica, okay? Elastica. There is a band, right? So, you see that I do not get confused.

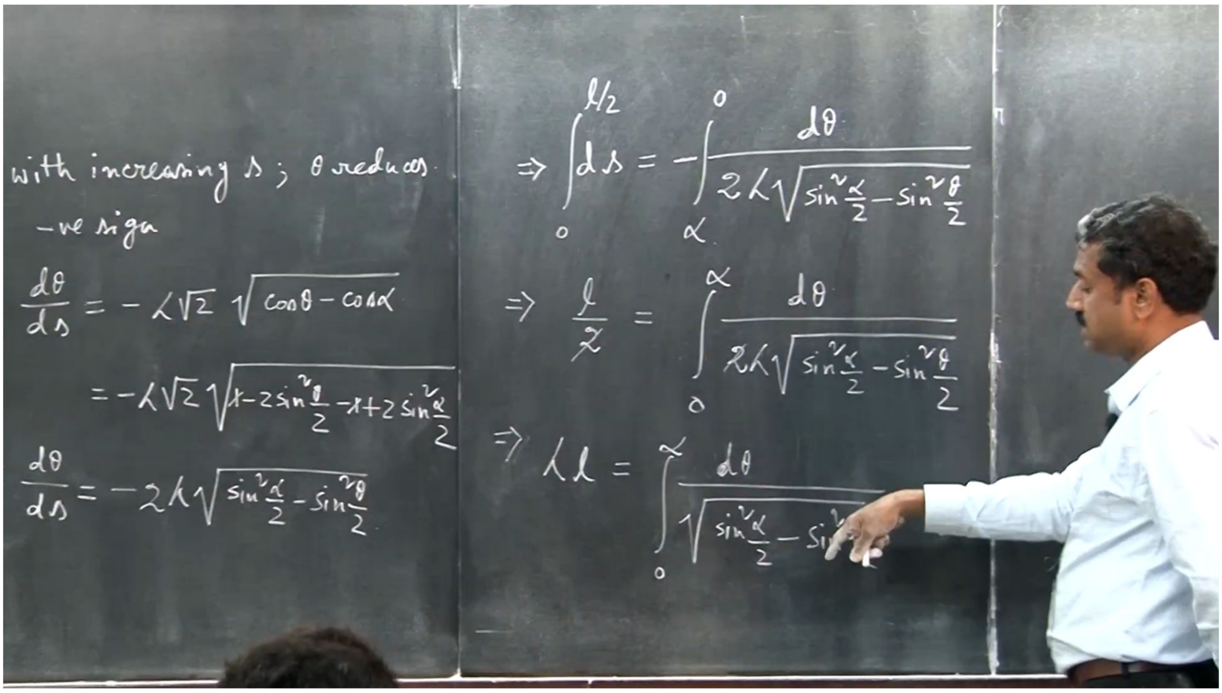



So, this elastic is nothing but this guy. This deforms because it is elastic. We are still assuming that it is an elastic formation, but the material is good enough that it is still going to remain elastic. So, the non-linearity is only coming from the geometry, not from the material. Do you understand that? Okay, Elastica. Okay, good. So, what I will do here now is integrate it. I will integrate it for $\theta$, okay? So, what first, what I am going to get, $\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 - \lambda^2 \cos\theta =$ maybe some constant, see constant of integration. Is that fine? The integral of $\sin\theta d\theta$ is nothing but $-\cos\theta$, right? And is that fine? Because $\frac{d\theta}{ds}$ square, if you differentiate it twice, $2\frac{d\theta}{ds}$, you see that, $2\frac{d\theta}{ds}$ and then $\frac{d^2\theta}{ds^2}$. Okay, $\frac{d^2\theta}{ds^2}$. So, 2 and 2 will cancel out. Is it fine by integrating? Right, this is the integration constant, right? Integration constant, okay. Now you see that we have to find out this constant; we have to enforce the boundary condition, right? So, what is the boundary condition we are going to write? Enforcing boundary conditions. So, the boundary condition is that, at $s = 0$, $\theta = \alpha$, and not only is $\theta = \alpha$ at $s = 0$, which means at this point I am considering this the origin at $s = 0$. The bending moment is also 0, right? So, what is a bending moment? Bending moment is nothing but $\frac{d\theta}{ds} = 0$. So, that means here, $\frac{d\theta}{ds} = 0$. Why? Because the bending moment is 0. Here you write the bending moment as 0. The bending moment is 0, right?

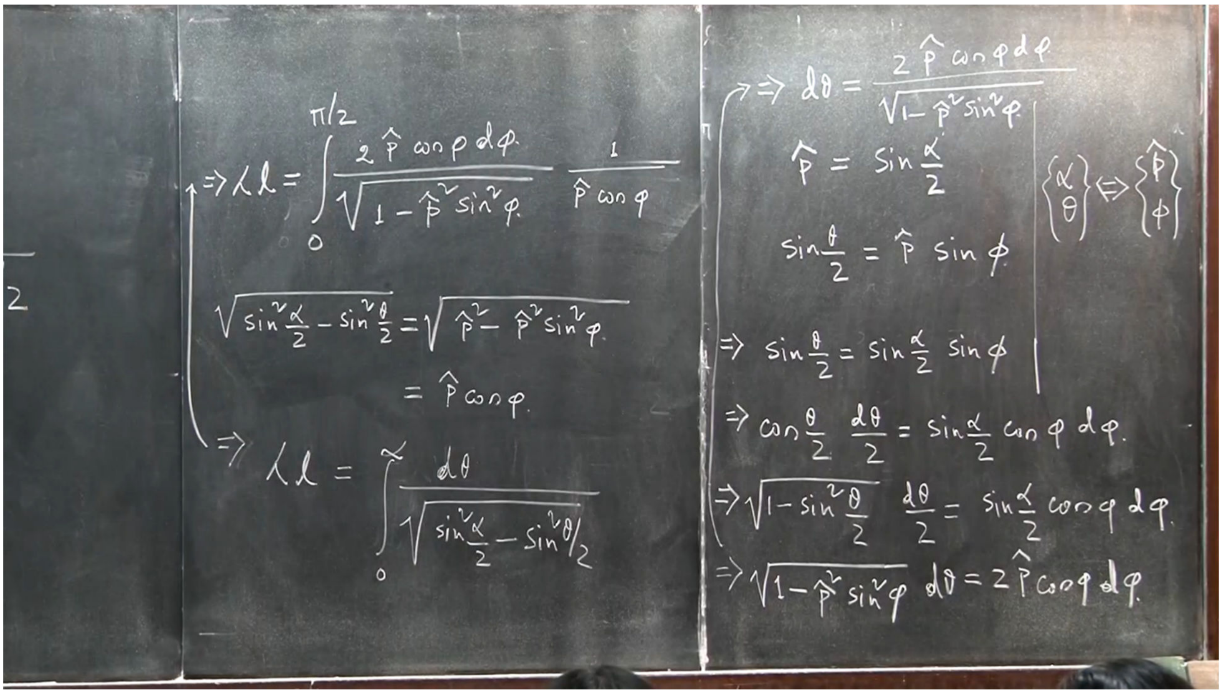



So, if that is then enforce this condition here. So, if you enforce this condition then let us write. So, $\frac{d\theta}{ds} = 0\ -\lambda^2\cos\alpha = C$, and then $C$ is nothing but $-\lambda^2\cos\alpha$. Okay, fine. So, now I am just removing this equation; please concentrate on that, okay? So, what we ultimately obtain is that $\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 - \lambda^2\cos\theta = C$, which is $\lambda^2\cos\alpha$, and then $\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = \lambda^2(\cos\theta - \cos\alpha)$, and then I am going to write, you know, $\left(\frac{d\theta}{ds}\right)^2 = 2\lambda^2(\cos\theta - \cos\alpha)$. Now until, so please note that, So, you see here that, which sign to take, plus or minus, you see that with increasing $S$, with increasing $S$, this is the positive direction, right? Increasing $S$ $\theta$ reduces, right? Is it not true that as $S$ increases, $\theta$ reduces? That means this must be negative, right? Why is it so? It is because, you know, the bending moment and curvature are opposite; that is why the minus sign is present, right? You know, with increasing $S$, $\theta$ reduces, so let us take the negative sign; the negative sign is valid. So $\frac{d\theta}{ds}$ I am going to write $\pm\lambda\sqrt{2}\sqrt{(\cos\theta - \cos\alpha)}$, right? Okay, and here if you do a little simplification

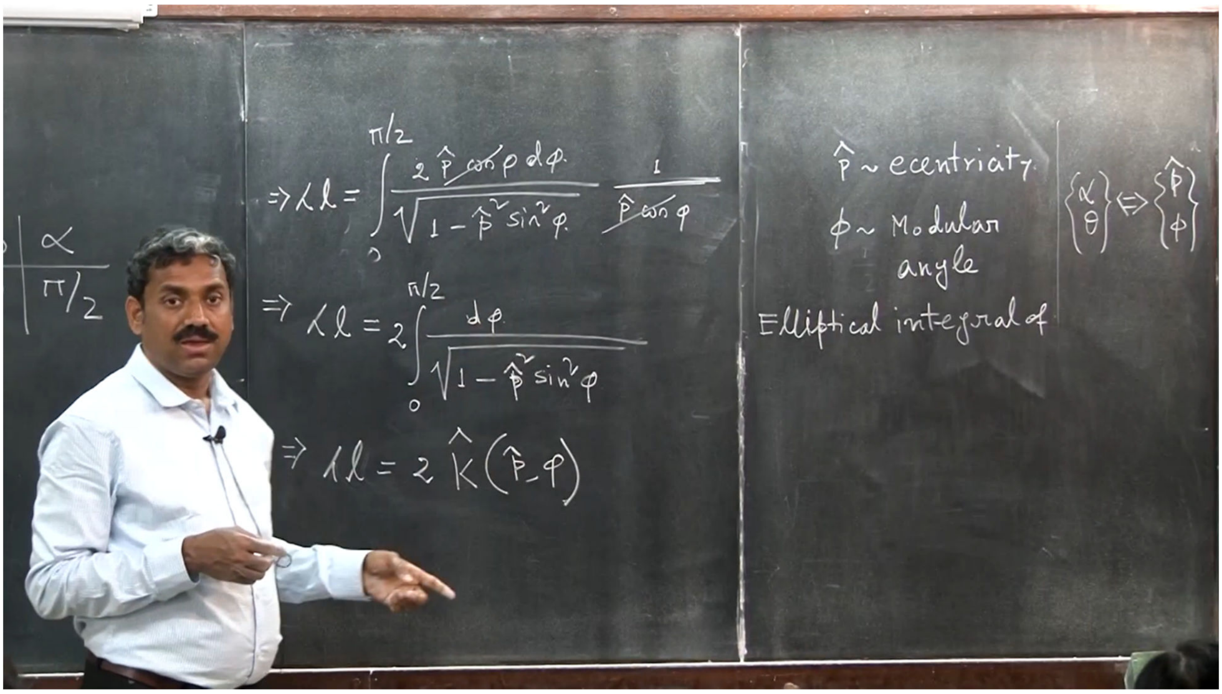
$-\lambda\sqrt{2}$, so $\cos\theta$ is $1 - 2\sin^2(\theta/2)$, and this is $-1 + 2\sin^2(\alpha/2)$. Okay, this will cancel out. Minus 0 to root 12 taken out, so $2\lambda\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}$, ok.




That is what we obtain, right? To write it further. You see what I am doing, so from here, I am doing it like that. Right now, I am going to integrate. What am I going to integrate? I am going to integrate from here to here; it is symmetric, right? From here to here, why? From here, when $s = 0$ and $l$ is divided by 2, so the integration from 0 to $s$, $L$ divided by 2 is nothing but half the length, right? You know the length of the right, is it not? So, this length is going to, you know, deform in the form of an arc, right? So, $L/2$ is fine, and then here $\theta$ is 0, and here, sorry, here, $\theta$ is nothing but $\alpha$, right? This angle is $\alpha$; that is what, but here $\theta$ is 0 because the slope is 0, right? So, is that fine, right? So, now that I can write here, here it is: $L/2$ is equal to what I am going to do. So, minus, here I can write 0 to $\alpha$, right? So, negative sign I can take out. You see that, or I can cancel out these two. So, I will write $\lambda L$; this I will take as $\lambda L$, you know they went from in, $\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2(\alpha/2)-\sin^2(\theta/2)}}$, right? Now I will simplify this integral. This integral, what I will do, I think somewhere, you know, since this expression, I am missing half and half that I missed somewhere, okay, $\alpha$ 0, that is perfectly fine. Because $\alpha$, I am taking $\alpha$ from here, and here it is basically 0, right? So, that is what it must be: 0 to $L$, so $L$ divided by 2, okay. Okay, that is fine. So, I will change this integral now. You see that this integral will be simplified to a, you know, will be simplified. So, how can we simplify this integral? This integral will lead to an elliptical integral. So, how? So, we will assume that this peak, $\sin(\alpha/2)$, will be denoted as $\hat{p}$; there is a parameter $\hat{p}$, or you can also write $p$, which is not a problem. $p$ or write $p$, $p$ is a parameter. That $\sin(\theta/2)$ we will write as $\hat{p}\sin\phi$. Now, so that means what we are essentially doing from $\alpha$ $\theta$ is that we are basically transforming this as $P$ and $\phi$, $\hat{p}$ and $\phi$. Why is that? Because you will see that if you do this substitution, it will lead to a standard integral, okay. So, if we do, then what will happen? $\sin(\theta/2)$, $\hat{p}$ is nothing but $\sin(\alpha/2)$, and this is $\sin\phi$. Then you see that here I am differentiating $\cos(\theta/2)\frac{d\theta}{2} = \sin(\alpha/2)\cos\phi d\phi$, right? So here, $\cos\phi$ is nothing but the square root of $1 - \sin^2(\theta/2)d\theta$, which is equal to $\sin(\alpha/2)\cos\phi\frac{d\theta}{d\phi}$, right? Let us see that. For $\sqrt{1-\sin^2(\theta/2)}$, $\sin^2(\theta/2)$ is $\hat{p}^2\sin^2\phi$, okay, $\hat{p}^2\sin^2\phi$, right, and this is $d\theta$. This is $2\sin(\alpha/2) = \hat{p}$; you see that, and this is $\cos(\phi)d(\phi)$, okay. So, from here you see that I can, please see, there is no space coming here; that means $\frac{d\phi}{d\theta}$, $d\theta$ is nothing but. But $d\theta$ is nothing but $\frac{2\hat{p}\cos\phi d\phi}{\sqrt{1-\hat{p}^2\sin^2\phi}}$. Clear? So, do you see from $\alpha$ $\theta$ how we are going to $\hat{p}$ and $\phi$? Now, these two parameters have some meaning that I will tell you later. Okay? That means that what I am going to do here will give you a very beautiful structure of the integral. So, I am removing everything. I will only do $\lambda L$ this. I am going to show you that this is $\lambda$; $l$ is nothing but integration, and now, you see that $\alpha$ goes from 0 to $\alpha$, right? So, $\theta$ is 0 to $\alpha$, okay. And then we are going to make it $\phi$. So, when $\theta = 0$, what is $\phi$? So, if $\theta = 0$, you see that $\theta = 0$; what is $\phi$? You know if $\theta = 0$, what is $\phi$? 0 right is 0 right, but if $\theta = \alpha$. $\theta = \alpha$; $\theta = \alpha$; this is $\alpha$. So, $\sin\phi$ is nothing but 1. So, $\phi$ is $\pi/2$. So, $\pi/2$, right? So, from here, this integral, you know, I am going to write it here. So, from 0 to $\alpha$. So, 0 to $\alpha$ instead of, here I am going from 0 to $\pi/2$, right. And $d\theta$ is nothing but, what is $d\theta$? $d\theta$ is nothing but this one, $\frac{2\hat{p}\cos\phi d\phi}{\sqrt{1-\hat{p}^2\sin^2\phi}}$ right. And what about this one? In the denominator, whatever is there, what is that $\sin^2(\alpha/2) - \sin^2(\theta/2)$? $\sin^2(\alpha/2)$ means, you know, $\hat{p}$. Square minus $\sin(\theta/2)$ is nothing but what? $\sin(\theta/2)$ is nothing but $\hat{p}\sin^2\phi$, so $\hat{p}\cos\phi$, right? So, in the denominator, I am just $1/(\hat{p}\cos\phi)$; is that fine? All of you, please note that. I am transforming the integral. So, $d\theta$, the differential, I have expressed in terms of $\phi$ and $\hat{p}$ and $\phi$, right? And then the denominator is the square root of $\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}$. I am simplifying as $\hat{p}\cos\phi$ clear. All of you please note it down. All of you understood, right? Note it down, okay? So, now I have obtained this.

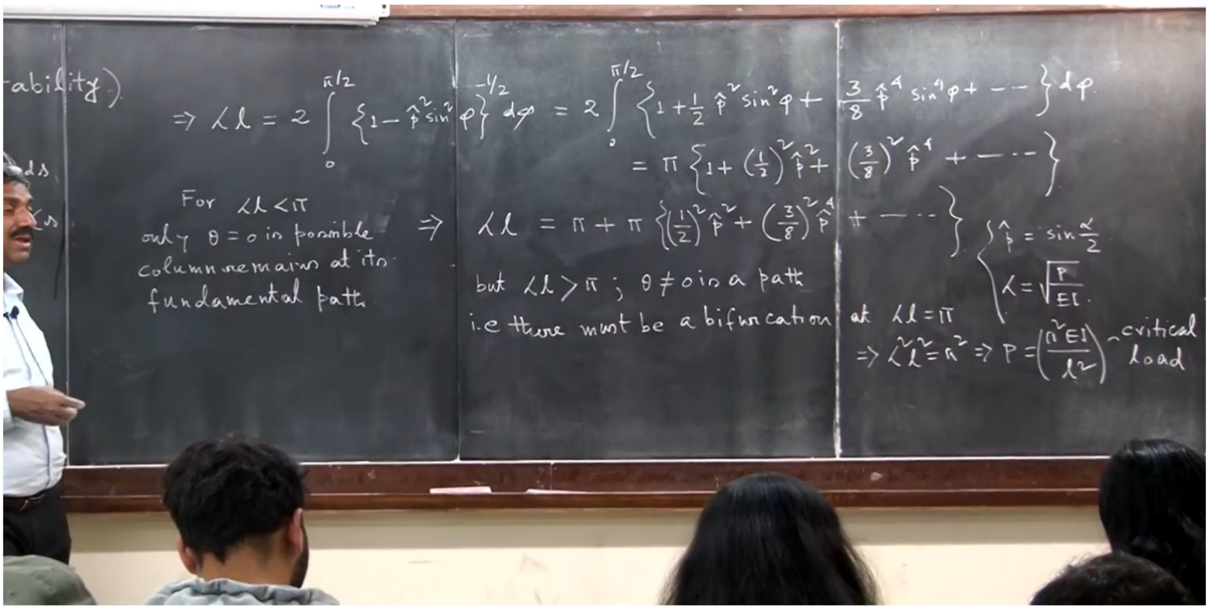



I am removing all other things, right? You have noted it down, okay? This integral, I mean, some of you may recall, what is this integral? Is this form of the integral familiar to you, anyone? No, this integral is nothing but the elliptical integral of, I think, the first or second kind. Okay, elliptical integral of, I think this is the second kind. Okay. So, see when I am converting $\alpha$ and $\theta$ to $p_k$, this is a standard integral; that is why they are called elliptical integrals. Because this is integral, it was developed in the mid-17th century by Euler and another fellow, Galileo, or something like that. Not that you know astronomical this one, okay? Euler and one of his co-workers. So, they are trying to find out the arc length of an ellipse, and that is what they have found in this integral. This integral appears to find the arc length of an ellipse; this $\hat{p}$ and $\phi$: $\hat{p}$ is nothing but this parameter called eccentricity, $\hat{p}$ is used to describe eccentricity, and $\phi$ is called the modular angle. Is the angle subtended at the center of the ellipse, okay? Not cent. I mean there are, you know, so many modular angles and eccentricities. So, this integral, the elliptical integral of the first kind, can be used to describe this. What is $\Lambda$? $\Lambda$ is $\sqrt{P/EI}$, right? So now from here we will see what we can do, okay. So elliptical integrals are available as built-in functions in all this mathematical software, including MATLAB.

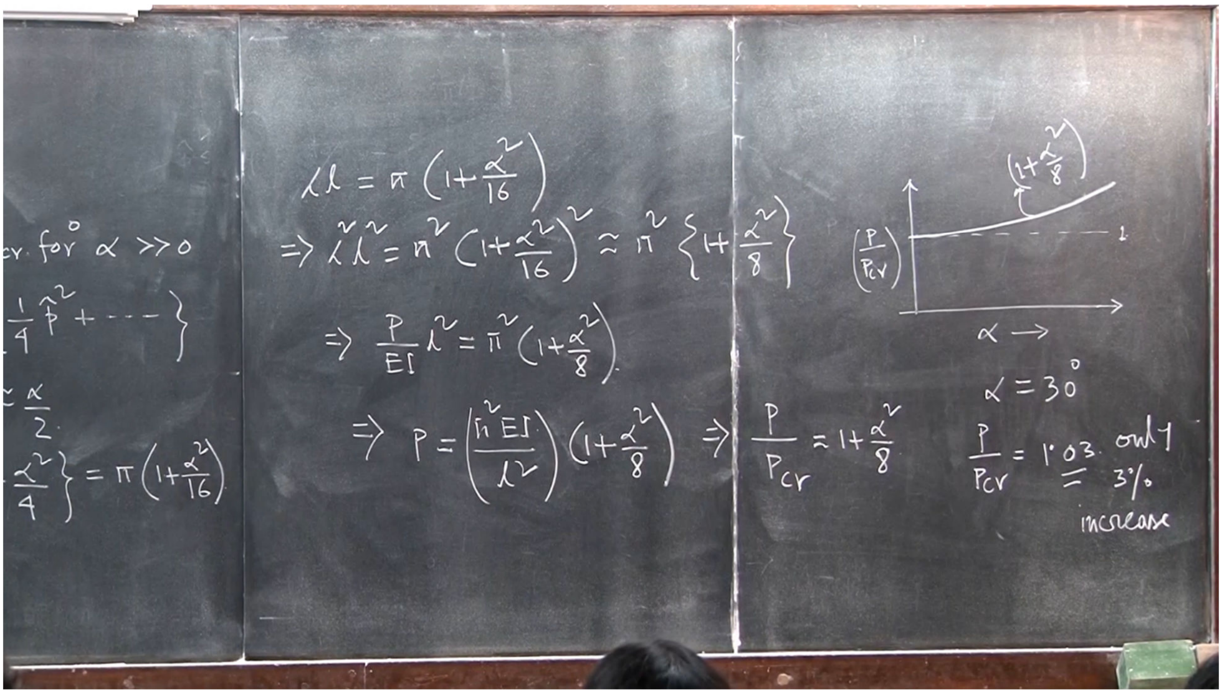



If you give an elliptical integral, it is generally defined here; you see that this is defined as 2, and this is defined as elliptical $k$ cap, $p$ cap, and 3. So $k$ cap is defined as an elliptical function. So, this elliptical function, you can give the comment for the given $p$ cap and $\phi$, be given the value of this eccentricity and $\phi$ modular angle; you can find out the value, and if you find out the value, you will be able to find out $\lambda l$, right?

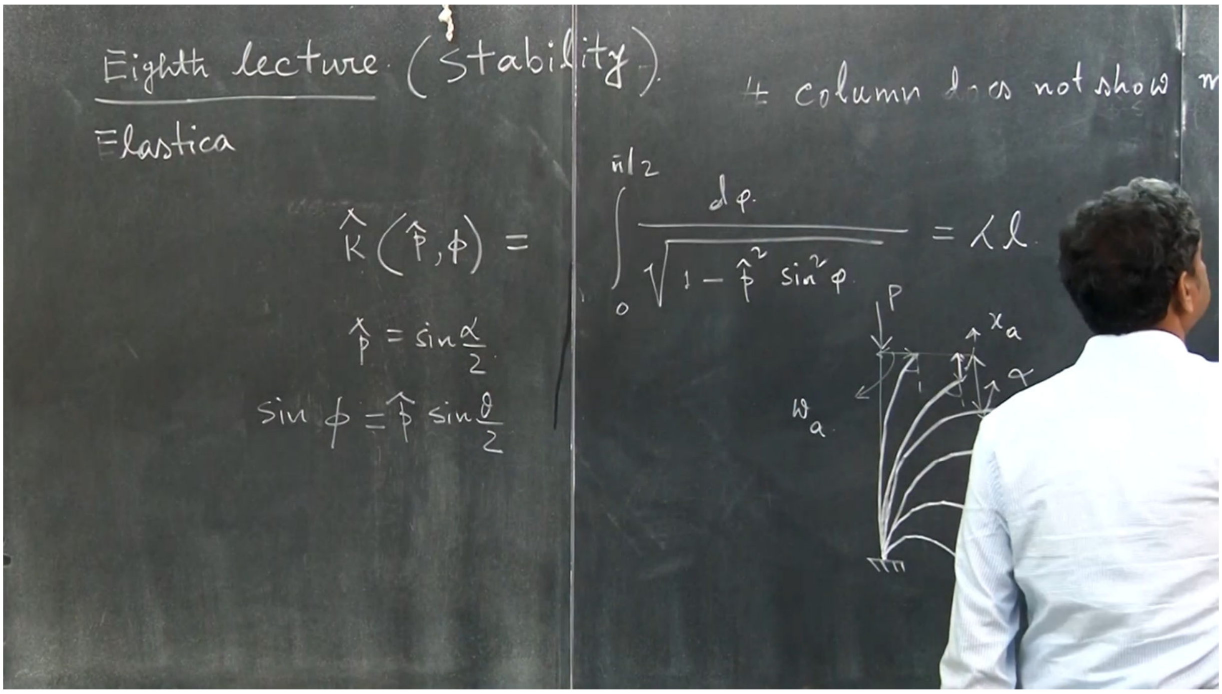
So, for and $p$ cap depend on $\alpha$, this $\alpha$ is this rotation; this $\alpha$ basically defines, you know, designates how much deformation it has, and it can have finite deformation. We will try to simplify it a little bit. So, what we are going to do is I will just simplify this for all of you, okay? What I will do, you see, is this: let $\lambda l$ be equal to the integral from 0 to $\pi/2$. I will take this denominator: $(1 - \hat{p}^2 \sin^2\phi)^{-1/2} d\phi$, and then it is 2 integration from 0 to $\pi/2$. So, $1 + \frac{1}{2}\hat{p}^2$, right, $\sin^2 p$ plus $\frac{1}{2}$ of $-\frac{1}{2!}$, and then you will see that is, I think, $\frac{3}{4}$. Okay, that will be $\frac{3}{4}$. Let me write it down: okay, $\frac{3}{4}\hat{p}^4 \sin^4\phi$ plus okay. So, we expand it using the binomial, knowing that $\sin\phi$ is less than 1 and $\hat{p}$ is also less than 1. Since this is a small number, when appearing as a square term, we can expand it using this expression. For the time being, I am just keeping it, but you can do this term by term; you can integrate it, right? Not a problem, so when you integrate this. To last what will be $\pi/2$, integration $d\phi$ from 0 to $\pi$, right? $\pi$ plus, then $\frac{\hat{p}^2}{2}$ integration from 0 to $\pi/2$ $\sin^2\phi d\phi$, you will see that here, there will be another half, okay then. It will be, plus this can be simplified as $\pi\left(1+\frac{1}{2}\right)$; if you do this integration, you will see this: $\hat{p}^2+\frac{3}{8}\hat{p}^4$, and then I will talk about this later. Okay, for the time being, leave it. And then I am just writing it: $\lambda L =$ here I will just write $Y + Y$; you see that is the expansion we are ultimately getting. Taken, and please note that $\hat{p} = \sin(\alpha/2)$. This is, please keep in mind, that $\lambda$ is nothing but $\sqrt{P/EI}$. So, what we see here is that, for $\lambda L < \pi$, only $\theta = 0$ is possible, right? It is possible that the column will remain at its fundamental path; this means the column remains at its fundamental path when $\theta = 0$. But as soon as $\lambda L > \pi$, and $\lambda L > \pi$ $\theta \neq 0$ is a path, right? That means, right. So, there is a bifurcation occurring at $\theta$. So, what we have is that $\lambda L < \pi$ fundamental path is possible. But when $\lambda L > 0$ and $\theta \neq 0$, it is possible that there must be a bifurcation. So, that means there must be a bifurcation at what $\lambda L = \pi$. So, $\lambda L = \pi$; there must be a bifurcation. Do you understand why we are just doing that? Because $\lambda L < \pi$, no non-trivial solution is possible, right? But when $\lambda > \pi$, there is a solution for $\theta$ that is equal to a non-zero value. So, $\lambda L = \pi$ must be a point of bifurcation. Which is leading to this bifurcation, $\frac{\pi^2 EI}{L^2}$, that is the value we know; we all know the column, which is hinged at both ends, right? The critical load is this, and that indicates the bifurcation.



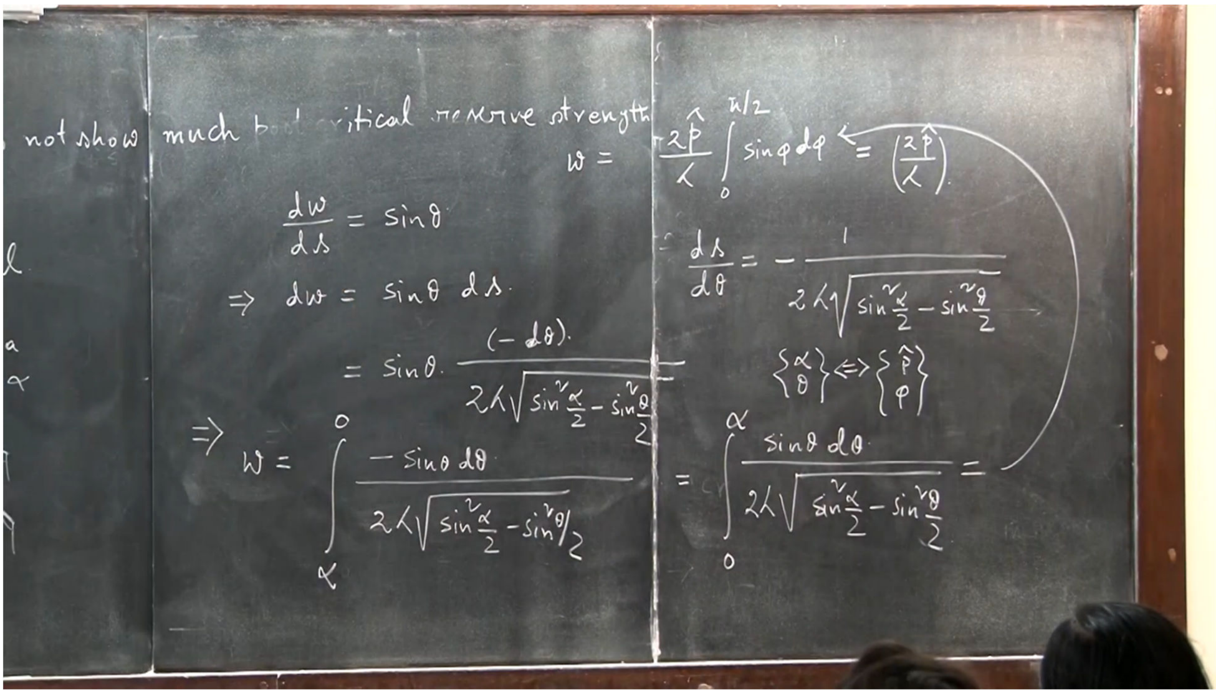

Why did you consider the finite deformation of the elastic column and not the infinitely small deforming column? Because now we can really gain insight into what happened beyond the critical load. So, in order to do that, please note that all of you have noted down all these things, right? So, what will happen when you see that? But for $P > P_{cr}$, that means for $\alpha$ is a reasonably higher value, right? It is a little okay. What is happening? So, $\lambda n$, let us take a first-term approximation, okay. So, the first term is an approximation. So, then I can write $\lambda l = \pi + \pi$ and then $\frac{1}{4}\hat{p}^2$; another term I am neglecting, knowing the fact that $p$ is small. So, what I mean is that $\hat{p}$ is nothing but $\sin(\alpha/2)$, and I am assuming that maybe $\alpha$ is not that 30 degrees, okay? I am assuming $\alpha$ is maybe a little higher. So, when this assumption that $\sin\theta = \theta$ is valid and $\theta$ is less than or equal to 4 degrees. So, I am just approximating is with $\alpha/2$, okay. So still, then what we can write is $\lambda L = \pi + \pi\frac{1}{4}$, and $\hat{p}$ is because $\alpha^2/4$, right? Do you see that? So that means, here is what we can write: this is $\pi$, you know, $1 + \frac{\alpha^2}{16}$, right? Okay.



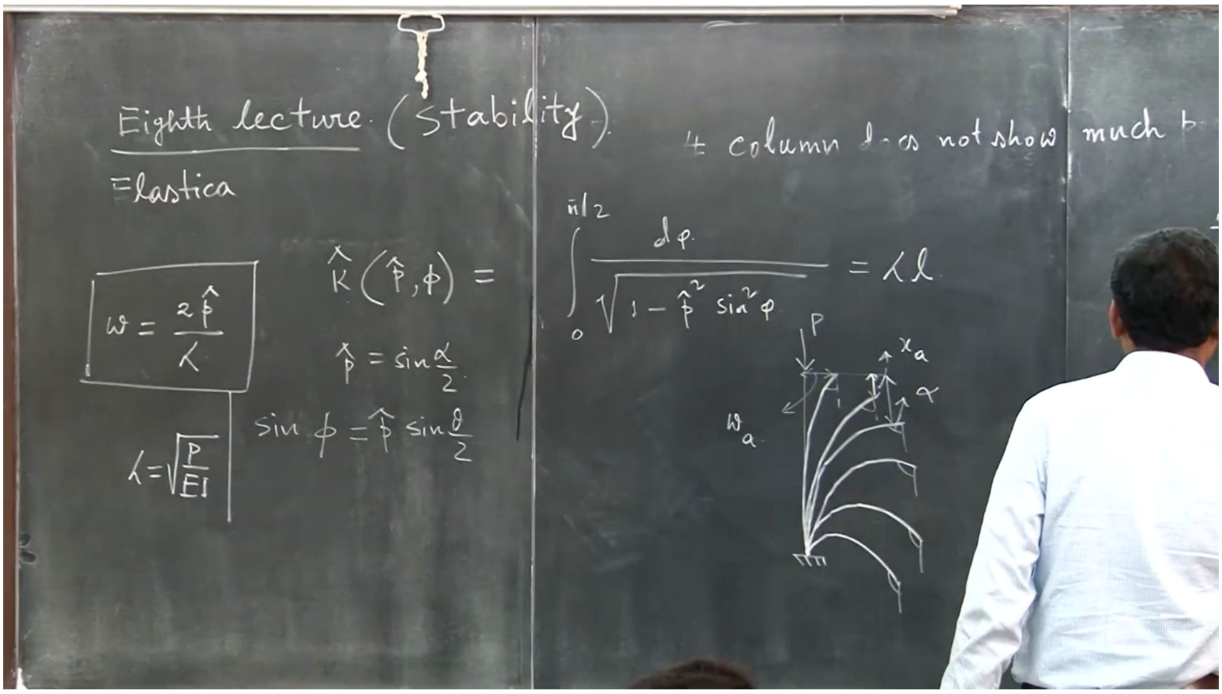

And I will just remove all these things. So, I can see that $\lambda l$ is being $\pi$ and then $1 + \frac{\alpha^2}{16}$, and I will just write $\lambda^2 l^2 = \pi^2\left(1 + \frac{\alpha^2}{16}\right)^2$. And then I will take this $\pi^2 + 1$; I will not take the square; I will take this $\frac{\alpha^2}{8}$, and here. $l^2 = \pi^2\left(1 + \frac{\alpha^2}{8}\right)$, and $P$ is nothing but $\frac{\pi^2 EI}{L^2}\left(1 + \frac{\alpha^2}{8}\right)$. That means I can write $\frac{P}{P_{critical}} = 1 + \frac{\alpha^2}{8}$. What does "$P/P_{cr}$" mean, you know? So, with this $P/P_{cr}$, that means, you know, with varying $\alpha$, you know, varying $\alpha$ $P$, see, $P/P_{cr}$, it is 1 if the 1 is buckling. There is a very slight increase here, okay. And this curve is $1 + \frac{\alpha^2}{8}$. Very slight increase; very slight. What does it remind you of? Symmetric stable bifurcation, but very little rising. So, you will see that if $\alpha$ is considered to be 30 degrees, then $P/P_{cr}$ becomes 1.03, which means only a 3 percent increase in the load. See, $\theta = 30$ degrees indicate a huge deformation; you understand that $\theta = 30$ degrees mean there is a huge rotation at the end. But even at that, there is only a 3 percent increase in the load; the post-critical rising path will only rise by 3 degrees. So, it is definitely a very stable bifurcation, but there is very little post-critical reserve, very little post-critical reserve, okay, very, very little post-critical reserve. So, now to understand the post-critical, the column does not show many post-critical results. So, what is the takeaway from this? The column does not show. This is possibly simply supported, but any of it does not yield much post-critical result. Critical reserve strength, I mean reserve, okay? Load whatever it can get. So, as soon as the column reaches the critical load, it almost depletes its load-carrying capacity. So, we cannot rely on the post-critical rise for the design purpose, but we can clarify that. So, what kind of imperfection sensitivity will it show? Stable postcritical bifurcation, stable postcritical two-thirds power, right. It will be imperfection, mild imperfection sensitive, two-thirds power law, okay. And then, in fact, it will be imperfection insensitive, not imperfection, because the stable postcritical path shows what? This is with increasing load; that is why there is a rise in postcritical load, right? So, critical load will increase with increasing $\alpha$. So, it will be two-thirds power load, but imperfection-insensitive. Do you see a real example of those toy systems that we have demonstrated? We have demonstrated this whole essential behavior by taking rigid model in spring, right. But do you see how that kind of qualitative behavior is exactly represented by the real structure column? Do you see? So, that is the beauty. There lies the beauty. Now, for the elastica, I will note down all these things correctly. One more thing, one or two more things I am going to show you. So, one thing is that we have seen that $k$ cap. $p$ cap $\alpha$ $\phi$, which is the integral from 0 to $\pi/2$ of $\frac{d\phi}{\sqrt{1-\hat{p}^2 \sin^2\phi}}$, is the elliptical integral of the first kind, right?




From here, you can directly find out that there was nothing, but you know $\lambda l$. So, from different sources, you must recall that $\hat{p}$ is nothing but $\sin(\alpha/2)$ $\alpha$, and $\phi$, you know, and then you know. $\phi$ you know, $\phi$ you have decided right what was the $\phi$. So, $\sin\phi$ is nothing but $\hat{p}$ and $\theta/2$ something like that right. So, from there you can find out if there are any critical loads. Now, another small thing I am going to mention here, you know that. So, the way this elastica will deform is something like this; you see that? Here, if you take simply, say, like it is $P$, and then here it will be maybe 4 degrees, then it will be something like this degree, you know this, you know. This is for a different $\alpha$ you will see. So, this is some angle; this will be some angle. See how $\alpha$ is increasing, you know, $\alpha$. And with this formulation, you know, $\alpha$ does not really change, okay. This is the $\alpha$. So, this is why this is called elastic, okay. For different $\alpha$s, you can find out what $P$ is. Because $\lambda$, you can convert that $\alpha$ to $\hat{p}$, and $\phi$, you can find out the elliptical integral, and from there, you can find out what $p$ is. And how much it is different from $P_{CR}$ clear, okay? Now, if somebody asks you what the deflection is, what is the deflection, this Δ I? What is the deflection of the tip Δ I, and also, somebody might ask you not only about deflection but what this is? What is this? This is $X_a$. This one, this one, or this one, you know, for that I am writing it, okay. Now, for $w$, it is easy because what we have seen with $ds$ is nothing but what, $\sin\theta$, right. So, $dw$ is nothing but $\sin\theta$ $ds$, and what is $ds$? What is $ds$?




We have noted that $ds\ d\theta$ is nothing but we have noted that $\frac{ds}{d\theta} = -\frac{1}{2\lambda\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$, right? So, this is $\sin\theta$, and $ds$ is $-\frac{d\theta}{2\lambda\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$. Now, you can integrate it $w$ is nothing but you know just integrate it integrate how will you integrate it $\alpha$ to 0 right. So, $\alpha$ to 0, you know, $-\frac{\sin\theta d\theta}{2\lambda\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$, and here, then you just write $\int_0^{\alpha} \frac{\sin\theta d\theta}{2\lambda\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}$, and you will see that if you simplify, it ultimately will lead to a very simple expression. You will see that here you will be; once again, you just substitute that in, and you make everything in terms of $\hat{p}$ and $\phi$. So, from $\alpha$ $\theta$, you do a variable substitution. From $\alpha$ $\theta$, you just go to, you know, $p$ and $\hat{p}$ and $\phi$, okay? Then you can also convert it into an elliptical, say, standard integral. You will ultimately see that it will be something like this. I am just taking it like this. I am taking it like this. Please take note of it. Please note this. So, ultimately, $w$ will be from this integral; ultimately it will be $\frac{2\hat{p}}{\lambda}\int_0^{\pi/2} \sin\phi d\phi$, and then if you do, then you will see $\frac{2\hat{p}}{\lambda}$. So, from here you can calculate what the deflection is because $\hat{p}$, you know, this is a very simple expression, right? $W$ is equal to, you know. So, a very simple expression. So, $W = \frac{2\hat{p}}{\lambda}$, you know. So, for any $\alpha$, you can calculate $\hat{p}$ and $\lambda$; $\lambda$ is nothing but the $\sqrt{\frac{P}{EI}}$.




For any level of $P$, you can find out the lateral deflection $w$, okay, clear you see that. Now, $X_a$, you can also find out what $X_a$ is. So, this is lateral deflection. Now, if you consider what the axial deformation will be, $X_a$ will be, I am just writing without deriving it, I am writing $\frac{2}{\lambda}\int_0^{\pi/2}$. Please note that this is the integral of the first kind, $\sqrt{1-\hat{p}^2\sin^2\phi}\,d\phi - L$ ok. So, this is it. Elliptical integral of the second kind, this one. So, here's how you know this minus $L$ is coming: please note that what we are doing is essentially integrating the arc length and then multiplying it. So, that arc length, basically, this total length is nothing but is mapped into that arc, right? So, if we are in a straight line. So, in whatever this mapping is, now you take the projection of that, okay. So, see, this thing was mapped into this. So, from this one, the projection of this curve is given by this first term, okay. and you just subtract $L$ from that. So, that is what you are going to get. $X_a$ understands how we are getting along, okay? So, this and this are nothing but the elliptical integral of the first kind, okay? The elliptical integral of the second kind, this is the elliptical integral of the second kind, okay? This elliptical, once again, this parameter is $\hat{p}$ and $\phi$; $\phi$ is the modular angle, $\hat{p}$ is the eccentricity, and $L$ is the initial length. So, how do you find out what the level of critical load is, and what the level of axial force is that it can carry for a given level of $\alpha$ deformation, right? That elastica can carry you can find out; then next, you can find out the lateral deflection $W$ as a function of $\alpha$. Also, you can find out the the compression in the deformation along the axis direction is okay. If you do not consider the elastic, you consider infinitely small deformation. We do not care that $w$ is essentially 0 tending to 0, $x$ was tending to 0, and $\alpha$ was less than 4 degrees here. $\alpha$ can go to 30, 60, 90, 120, 150, you know, 180; even it can go to, you know, something like this. You know, you see that, that is why it is elastic. Thank you very much for your class today. Thanks.