Stability of Structures

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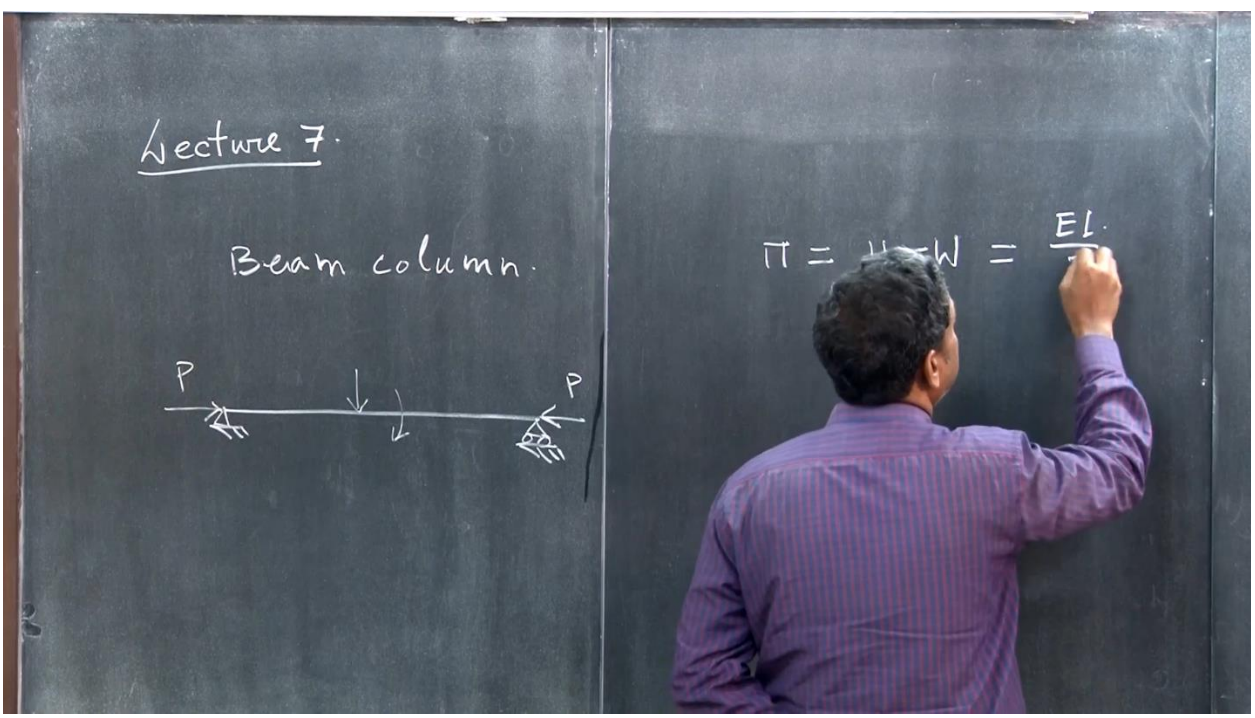
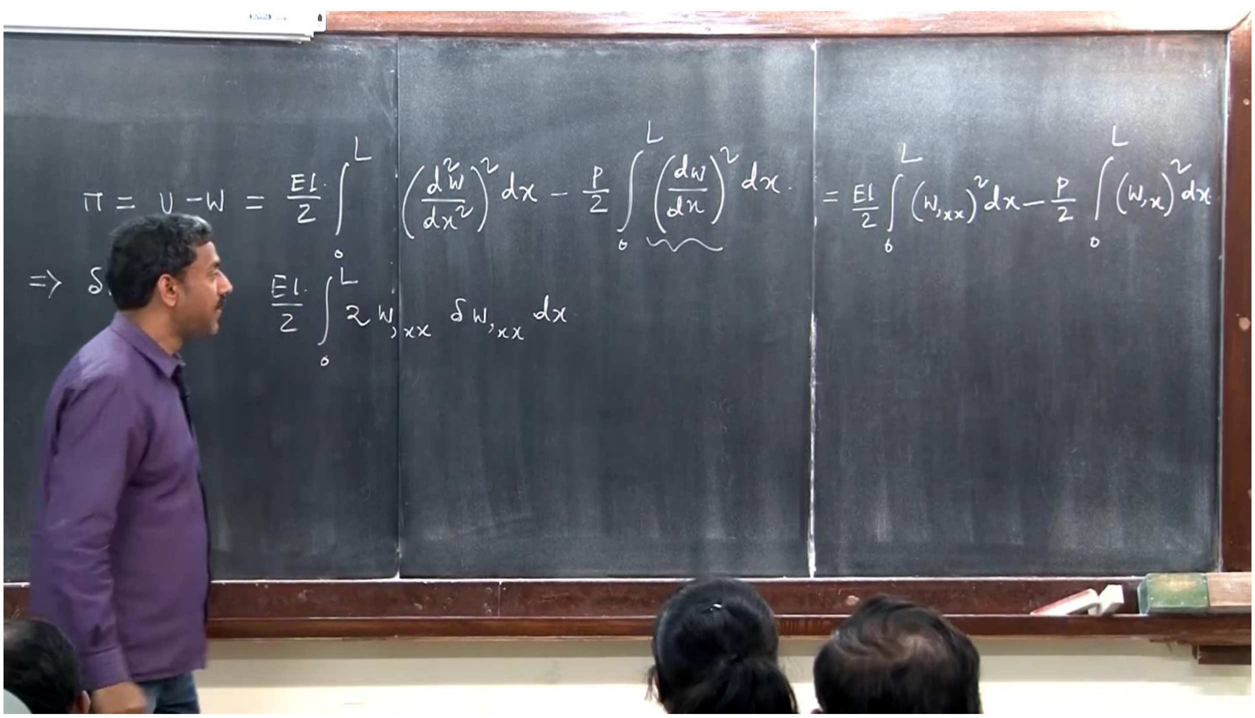
WEEK-04

LECTURE 7: Beam-column system and Response amplification

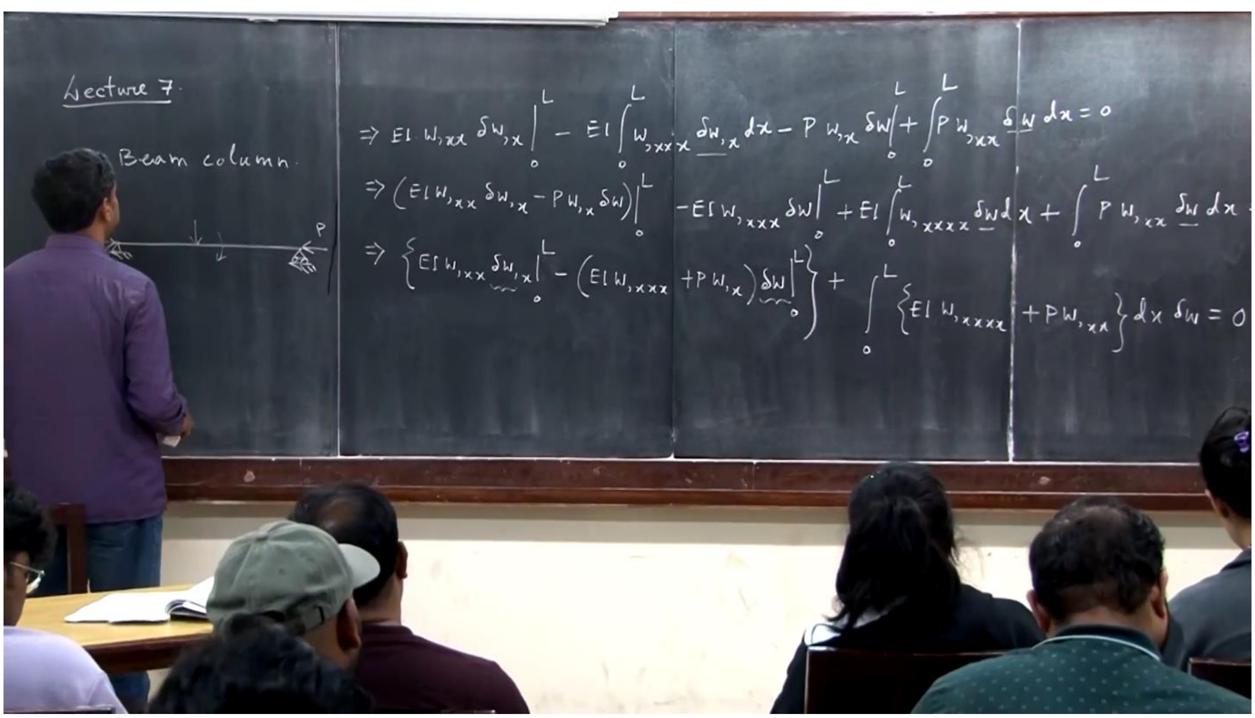
So, welcome to lecture seven on this class's stability of structure. Let us briefly recapitulate what we were doing in the previous class. So, we have just completed the discussion on a simplified system, which consists of a rigid bar and has elasticity imparted by a concentrated translational spring and a rotational spring. And then we have classified the instability into several classes. We have demonstrated the equilibrium path. We have demonstrated how the simple equation system leads to an eigenvalue problem through linearization that provides us with the critical load. And we have to explore the higher-order derivative of potential energy. The first-order derivative of potential energy gives us the equilibrium. Now, in order to ascertain the nature of equilibrium, whether it is stable, unstable, or neutral, we have to explore the higher-order derivative of a potential energy function.



Thereafter, we have also, discussed the imperfection sensitivity of modal interactions of mode right, with those simple examples, and then what we started in the previous class that we briefly touched on beam-column. Okay. So, you all are aware of the buckling of columns that we were taught in our undergraduate classes, right? In strength of materials or mechanics of materials, right? So, here we're just trying to extend this concept, okay? So, you're discussing the beam-column, right? So, a beam column is a structural system. A beam is a one-dimensional structural component that is loaded in the transverse direction. A column, on the other hand, is a member that is loaded in axial compression. Therefore, a beam column means it is both a beam and a column. So, it is axially compressed, actually like this P, right? While there are many approaches for deriving the different governing differential equations, the approach that we are going to follow is based on energy. Once again, we have seen the beauty of the energy approach, and precisely we have used the energy approach throughout all the discussions for stability analysis. There are various formalisms of formalism for stability analysis. The most convenient approach is the energy formulation, which uses potential energy and its various derivatives. In the previous class, we derived the potential energy for a beam-column.

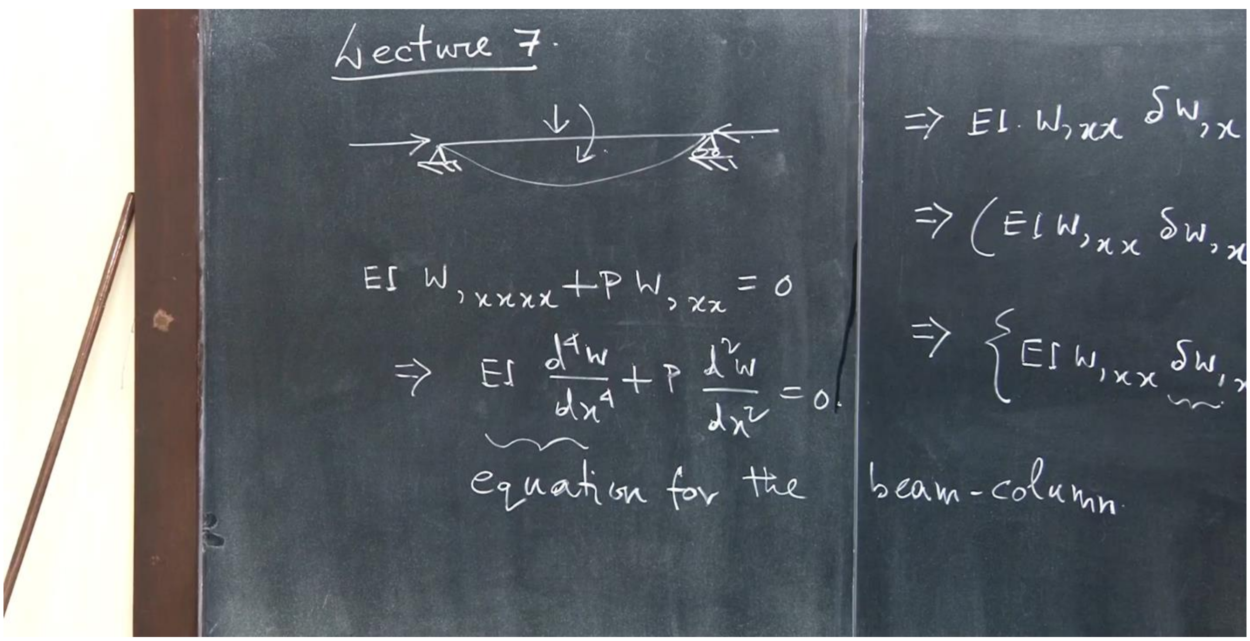


Strain energy minus work done, where the strain energy of bending is nothing but $\frac{EL}{2}\int_0^L(\frac{d^2w}{dx^2})^2\,dx$, right? And then in this beam column, what we have assumed is that we have not considered the governing differential equation for beam bending, right? So, here in the beam column, this is basically the governing term that describes the strain energy for bending, right? And then the work done by the axial load is nothing but $\frac{P}{2}\int_0^L(\frac{dw}{dx})^2$dx . So, if you can recall, this term is nothing but a term that comes from the von Karman nonlinearity, which is a very essential term for buckling, even for linearized buckling analysis. We'll make use of the von Karman nonlinear term, okay? So, then from here we'll just use a simplified little EI by 2 integration 0 to l this I'm going to derive $(W_{,xx})^2$ dx. So, xx means double derivative. Okay, So, we are just simplifying the expression. Now, the first derivative is this one. So, $\delta\pi$ needs to be zero. Okay. First variation of the potential: it is a function and a potential energy function. first variation of this potential energy function to be zero. So, if you do then you just do that. So, Ei by 2. So, differentiation is allowed under the integral sign you know Leibniz rule is valid, So, because a certain function has to satisfy that, and w is the function, because it's an elastic displacement continuous differentiable function, right $\int_0^L 2W_{,xx}$ dx. Okay, you see we are differentiating minus $\frac{P}{2}\int_0^L 2W_{,x}$ delta x, and that must be equal to zero, right? So, we will just simplify this further: the integral $\int_0^L 2\, W_{,xx}\delta W_{,xx}$dx minus the integral from 0 to l of $W_{,x}\delta W_{,x}$ dx equals zero. Okay, So, this is the equation we are getting by taking the first derivative of the potential energy function, right?



So, I'm removing that part, and then I will do further. Okay, removing this one. We'll simplify this term. How are we simplifying it? First, let me simplify, and then our aim is to convert $\delta W_{,xx}$

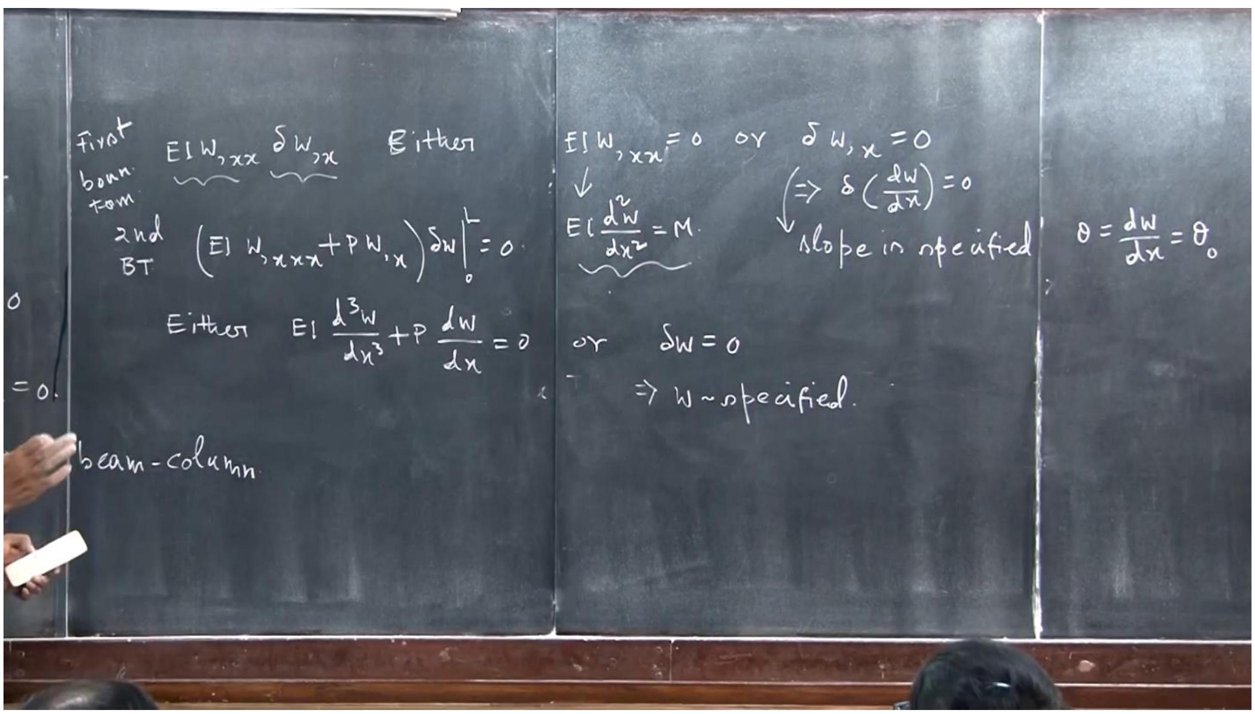
into δx. Okay. So, this we are following is nothing but variational calculus, a little bit of infinite element. We have learned a little bit of variational calculus, right? So, our aim from simplification is to convert it through bifurcation integration W and all this. We want to convert it into δW; okay, this is our aim for simplification. So, now what we're going to do is let this be outside, So, first function, okay. First function integration of the second minus integration derivative of the first integration of the second whole integration minus then. Here P second first function integration of the second minus whole integration - a will be plus derivative of the first integration of the second integration equal to zero. Now the boundary term, whatever we are defining, we'll take it okay. It takes it together EI $W_{,xx}\delta W_{,x}$, and this one I'm bringing here minus $PW_{,x}$ from 0 to l, okay. now what I will do look here it is δW right but here it is still $\delta W_{,x}$ is there. So, you have to first once again integrate it further okay. So, you have this term. So, this one I am just once again integrating So, first, the function integration of the second minus the whole integration plus the derivative of the first integration of the second, this term right, and then the last term is from $\int_0^L PW_{,xx}\delta$wdx. So, the simplification I'm doing tries to follow that; essentially, I want to convert the derivative term.



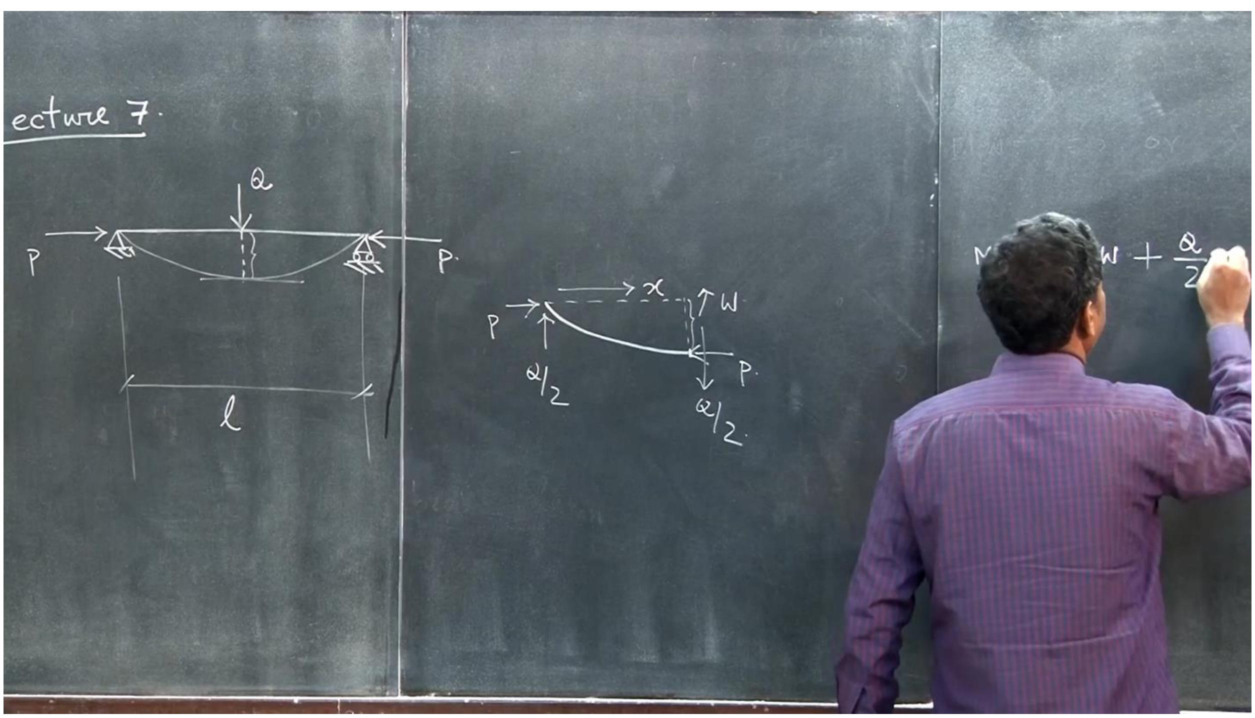
Now I will take it inside. So, what I'm going to do is integrate EI $W_{,xx}\delta W_{,x}$ into l, and this thing, this and this minus here, I'm going to do EI $W_{,xx} + pW_{,x}\delta W$ integration from 0 to l, right?

So, the boundary term I am basically combining, okay? Please note that $\delta w_{,x}$ and δw, okay? Plus, here, what I'm going to do here is integrate zero. Please note that here it's d δw δw, So, I'll take it together Ei $w_{,xxx}$ + d $W_{,xx}$, you know dx δw is equal to zero. Clear? So, this is the boundary term, and this is the integral term, right? Okay. Now, what I'm going to do is that all of you have noted this down. Please note it down. Now you see that δw is not equal to zero, right? Because δw is not equal to zero. Well, I mean not necessarily. Well, first let us see that integration from 0 to L of this one dx δw is equal to zero. Okay. So, δw is not equal to zero. Right? And so, this one, when I'm making it integration from 0 to L, we are integrating over the domain, but these two limits can be arbitrary. So, if δw is not equal to zero, then this must be zero, okay, if these two limits are arbitrary. So, not all the cases can you write that right δW. This basically gives us the governing equation. From here, what I can write is that

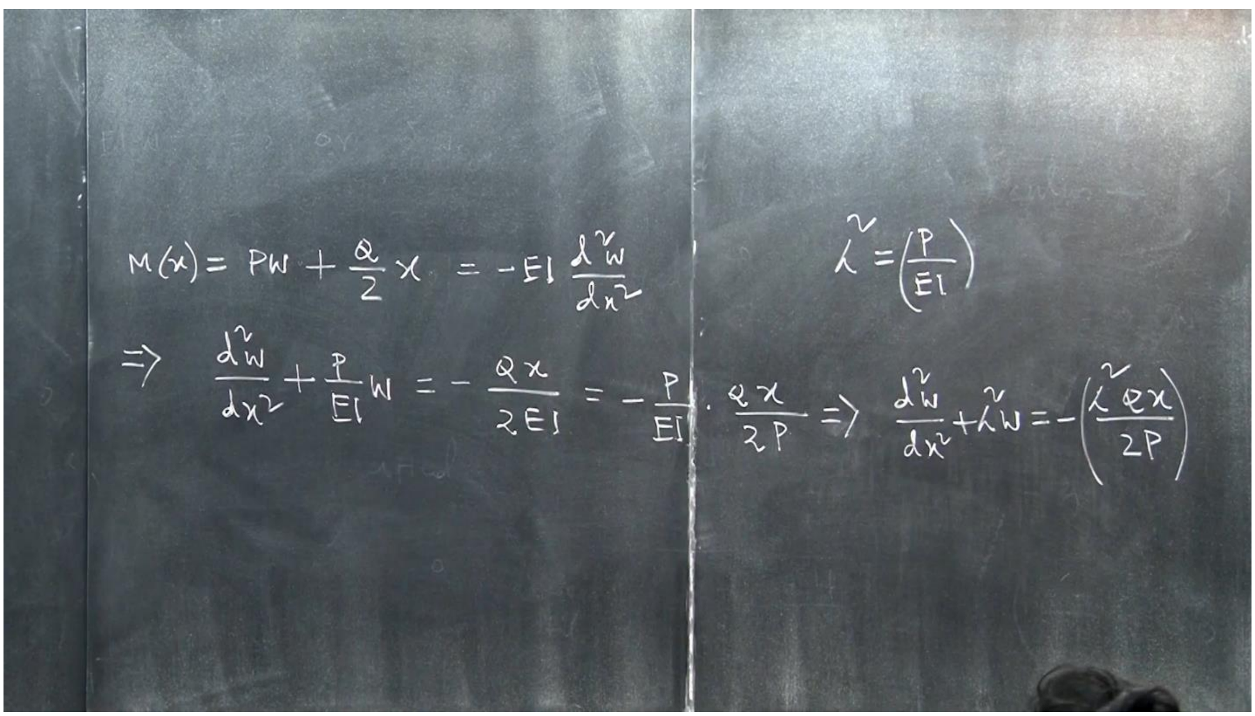
$EIW_{,xxxx} + PW_{,xx} = 0$. δW is equal to Z because A and B are integral; we are writing from 0 to L. You know, between any A to B, we can write. So, this is nothing but the governing differential equation: $EI\frac{d^4w}{dx^4} + p\frac{d^2w}{dx^2}$ is equal to zero.



This is the equation for a beam column. This was nothing but the beam, and when there were no column and no axial compression, this term was not there. So, because it is a beam-column, the beam is under axial compression, and an additional term is coming. Do you see that? So, this is the equation for the beam column and δ. Now, this gives us the governing equation. Now, what about the boundary terms? The boundary terms will give us the boundary condition, okay? That I will, I'm going to show; that's the beauty of this variational principle. So, from the energy function we started with, okay, this will give us not only the governing differential equation but also, give us the boundary condition, okay. So, if we look at the boundary term first, this is the one boundary term. So, bound when we had discussed when we were talking about $EIW_{,xxx}\delta W_{,x}$ when either they need to be evaluated at zero or l, right? Any a and b, right? Wherever the boundary is right now, either this needs to be zero or this needs to be zero if it has to vanish, right? So, either $EiW_{,xx}$ needs to be zero or $\delta W_{,x}$ needs to be zero, right? If the boundary needs to vanish, right? So, what does $\delta W_{,x}$ mean? This is nothing but dW/dx. What does it mean? It means the slope is specified, and it can be zero. If the slope is constant, then the variation of that must be zero. The variation of a constant is zero, right? So, the slope is specified. That means $\theta = \frac{dw}{dx}$, which is equal to θ_0, understand? If the slope is specified, then you don't... and what is this expression? Now EId^2w/dx^2 is what? This is nothing but the bending moment. So, if the bending moment is specified, then the slope is unknown. If the slope is specified, then the bending moment is unknown. Do you see that complementarity of the relationship, right? Either respective displacement is known. If you know the displacement, you don't know the force at a support condition. Then you know what the displacement will be, right? You don't know the force; if you apply some force, you know the force, but you don't know the displacement, right? That's what essentially happens. So, either the slope is known or the bending moment is known. If the bending moment is known, that is the force boundary condition. So, for example, when the slope is specified, you consider what will happen if it is simply supported. Simply supported means what? The bending moment is zero. We know the bending moment; we don't know the slope, right? Now, at the free end, we don't know the slope, but we know the bending moment will be what? Zero. Right? Okay. So, we are done with the first boundary term. What about the second boundary term? Now the second boundary is $EIW_{,xxx} + PW_{,x}$. This is multiplied by δw 0 to l, which needs to be zero, right? Okay, that means either $EI\frac{d^3w}{dx^3} + P\frac{dw}{dx} = 0$ or $\delta w = 0$. What does it mean? If δw is zero, it means that the deflection is specified. If the deflection is specified, then you don't know the shear force. If you know the shear force, then you don't know W specific. For example, here W is zero. If there is sinking support, W is specified such that del W is zero, right? Or if you are at the free end, you don't know the deflection, but you know the shear force is zero, right? So, this expression is what? It's nothing but shear force. Shear force is $EI\frac{d^3w}{dx^3} + P\frac{dw}{dx}$, and if you can recall, this is nothing but $\frac{dm}{dx} + P\frac{dw}{dx}$. So, that means for a beam-column, the bending moment of shear force is nothing but the derivative of the bending moment. But in addition, the term $P\frac{dw}{dx}$ is alSo, present. So, in the beam-column, there is an additional term due to shear force. The additional term of this component is P. Axial force is contributing to shear. So, the shear force is different. So, what we have seen is that you see the beauty of the energy function. We didn't consider any physics, just mathematics. We have considered the energy functional, and from the energy functional, by doing this little manipulation and whatever we are doing, it is basically called variational calculus. It gives, through some simplification, not only the governing differential equation but also, the boundary condition. So, if you want to Solve this differential equation, you need to have this boundary condition; otherwise, it is also, true. That means if a governing differential equation is given and augmented by sets of boundary conditions. You can derive a function. by minimization of which you will get the Solution of the problem.

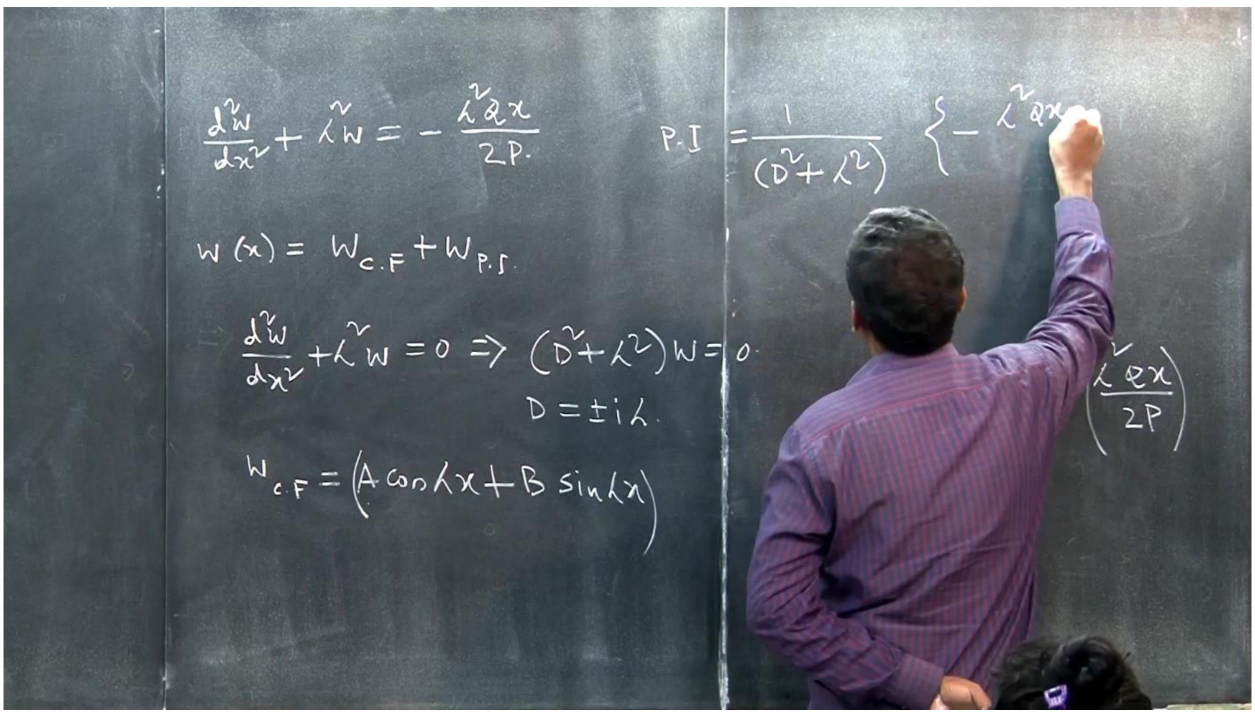


That's what essentially you do in finite element. Infinite element what you do? you are given a differentiate Bubnov-Galerkin method or Petrov-Galerkin method. what you do? You are given an equation. See, when you start with the truss element, you start directly from the energy, right? But if you are given any differential equation for heat conduction, fluid flow, and so, on, when you derive the finite element, what do you do? From the equation and sets of boundary conditions, you try to derive some functional, and you minimize that function to get the Solution, right? So, the way to get it is called variation. This is the subject of variational calculus. Okay? Knowingly or unknowingly that little bit. Okay? Of course, there are many more elements. These are nothing but very simple lessons. Right? Okay. So, what we want to derive is the governing equation from the beam column. This is the governing equation, and these are the boundary conditions. Right? So, it's an extension of the beam bending equation, right? But there is an additional term. So, the beam bending equation $EI\frac{d^4w}{dx^4} = 0$, but this additional term is due to the column; this p is coming, and then the shear force has some change; otherwise, the rest of the things are the same. Okay.

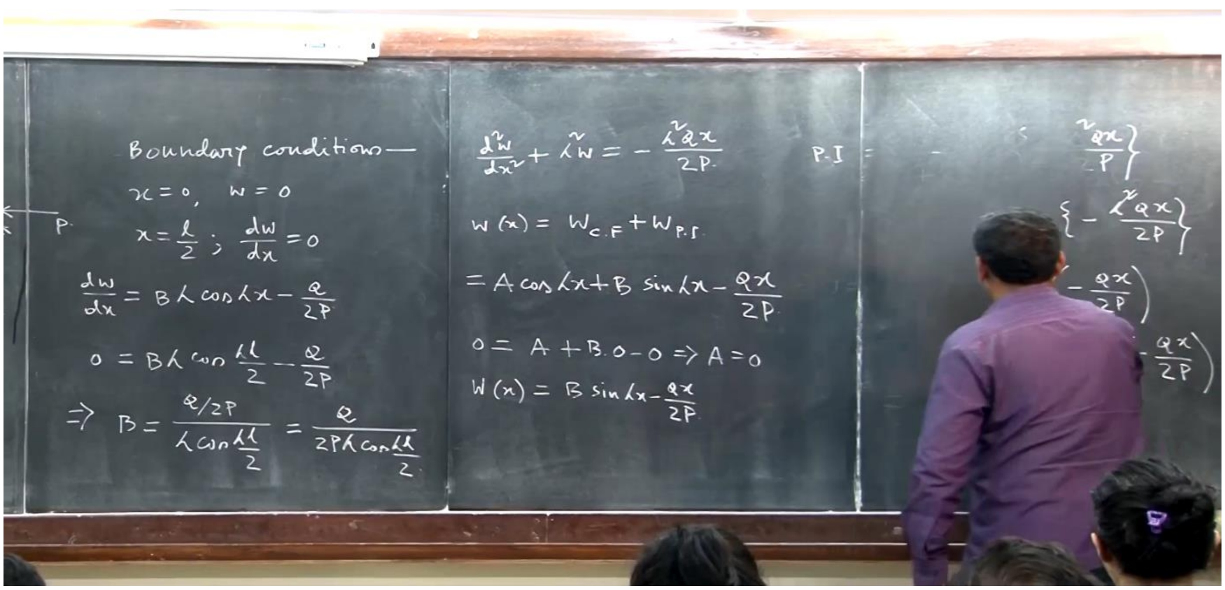


Now we are going to discuss various characteristics of the beam column that we are considering. Here it is subjected to P. And we're assuming that, well, I mean, there might be some forces, etc. So, now you know it is maybe deflecting Something like that and we assume that it is subjected to some load Q here. Okay. And there is a mean spin deflection; Something is happening, δ_0, whatever. Okay, this deflection, δ_0. Okay. So, this is of length L. Okay. So, now we can directly start with the governing equation, and you can directly start with the governing equation we can

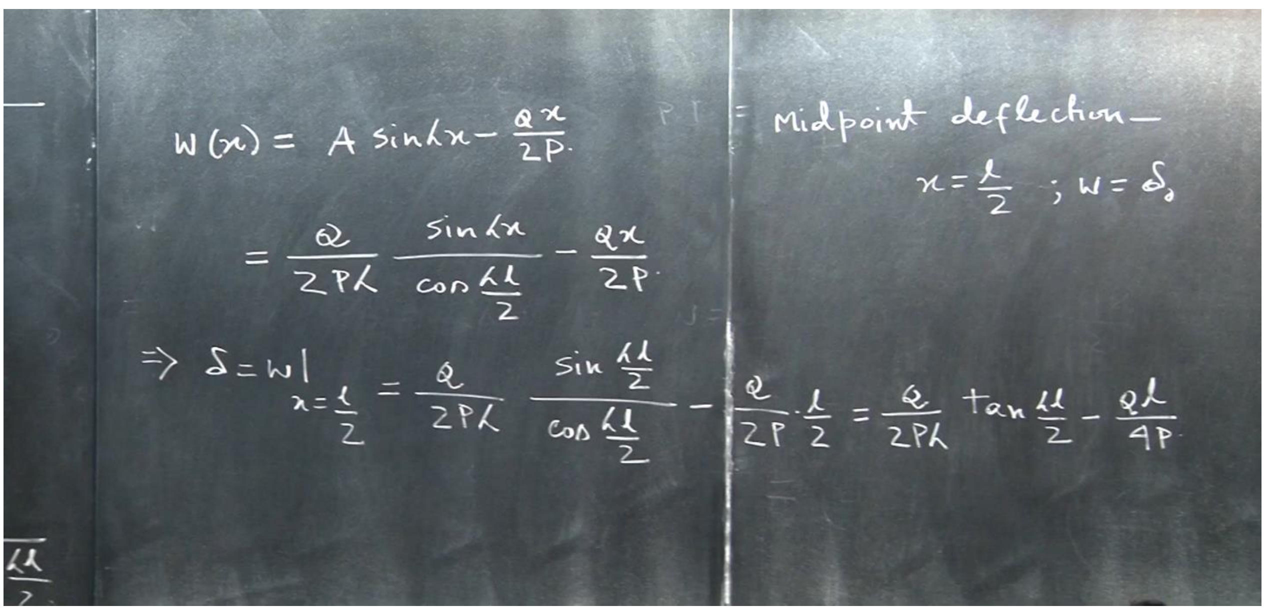
Solve, but here I will do a little differently, okay? So, here, this is symmetric, right? So, if I consider the deformed configuration, okay, this part is P, and here it should also, be P. And if this is Q, then there is a reaction force Q by 2, right? Then there will be some downward reaction force Q/2, right? And I'm assuming that this is maybe W, and here from a distance X. Okay. This is the beam column in the deflector configuration. Right? So, what is the bending moment here? The bending moment as a function of X. What is the bending moment here at this section, right? At this section, the bending moment will be nothing but see this P into W, right? And then these two Ps are forming a clockwise couple, right? So, this one Q by 2, plus Q/2 into X, right? And this is nothing but $-EI\frac{d^2w}{dx^2}$, right? So, let us quickly Solve it. So, now what we are going to do is $\frac{d^2w}{dx^2}$ here; it will come out as $+\frac{P}{EI}$ into w. This is equal to $-\frac{Qx}{2EI}$, right? This means $-\frac{p}{EI}$ into x. So, we are going to define Something: $\lambda^2 = \frac{P}{EI}$, okay? So, then here what we are going to get is $\frac{d^2w}{dx^2} + \lambda^2 w = -\frac{\lambda^2 qx}{2p}$, right? So, from the bending moment, there is a governing differential equation $EI\frac{d^4w}{dx^4} + p$;



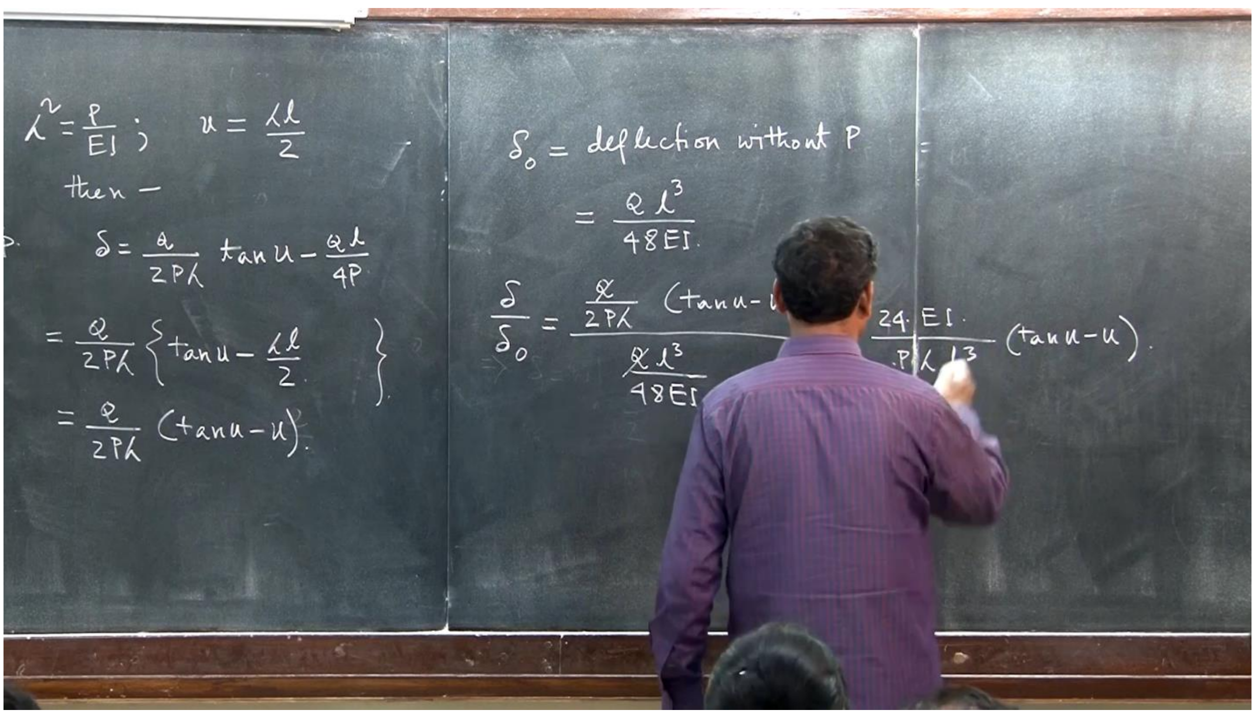
however, why we are trying to follow this is that we want to Solve that differential equation. It's a fourth-order differential equation. If you want to Solve the fourth-order differential equation, then you need to have a boundary condition that allows the third-order derivative. So, you have to consider both the force and displacement boundary conditions, as well as the force boundary condition, because we need to have four boundary conditions, right? Because I want to keep the constant difference order of the in differential equation low. So, that's what I directly work with: the bending moment. You see that we have a second-order equation; it is easier to Solve. That's the reason, right? Nothing else. We could have also, solved it using energy; that is also, another way. Okay, good. So, now what are we going to do if this is the equation? Let us Solve it. Huh? And keep that $\lambda^2 = \frac{p}{ei}$. Ei is the flexural rigidity, and I is the second moment of area. Okay. So, you see that $\frac{d^2w}{dx^2} + \lambda^2 w = -\frac{\lambda^2 Qx}{2p}$. So, it's a non-homogeneous differential equation. So, $w(x)$ will have two parts. One is the complementary function w, and the other one is the particular integral or particular Solution w. The complimentary function is nothing but the Solution of the homogeneous equation. The homogeneous equation is $\frac{d^2w}{dx^2} + \lambda^2 w = 0$. Right? So, if I give this d, the differential operator $(d^2 + \lambda^2)w = 0$. This is a differential operator, and the roots will be $\pm i\lambda$. So, if you just substitute, you get the complementary function Solution to be $A\cos(\lambda x) + B\sin(\lambda x)$; that is the complementary function, right?



This is why. Because exponential $i\lambda + e^{-i\lambda}$ by Euler's Theorem is nothing but $\cos\lambda x + i\sin\lambda x$, right? So, then a and b are the constants, right? Now we are going to do the particular integral. So, the particular Solution will be nothing but, once again, I'm going to write that format $d^2 + \lambda^2$, and here I'm going to write $-\frac{\lambda^2 qx}{2p}$. Right? Here I want to do this: λ^2 I will take out $(1 + \frac{d^2}{\lambda^2})$, then $\frac{\lambda^2 qx}{2p}$. So, then λ^2 will go, and then $(1 + \frac{d^2}{\lambda^2})^{-1}(-\frac{qx}{2p})$. And if you just do this binomial $\frac{d^2}{\lambda^2} + \frac{2^4}{\lambda^4} +$ whatever minus operate it on d, which is the differential operator, we see that the double derivative on x will be zero, right? So, this is the only thing: the particular integral is nothing but $-\frac{qx}{2p}$, right? So, the total Solution I'm going to write is $A\cos(\lambda x) + B\sin(\lambda x) - \frac{qx}{2p}$, right? That's the total Solution $w(x)$, right? Now we are going to enforce the boundary condition. Okay. So, the boundary conditions for this are: at $x = 0, w = 0$, and the second boundary condition we are going to apply is symmetric. Therefore, under the load at the midpoint, the slope must be zero because of the symmetric boundary. So, $x = \frac{l}{2}$. We are going to do that $\frac{dw}{dx} = 0$. These two boundary conditions are required because there are two unknowns. So, when I'm doing to do when $x = 0$ and $w = 0$, means $0 = A + B \times 0 - 0$. So, basically, a is nothing but zero, right? So, a is zero. So, now simplified as $w(x) = B\sin(\lambda x) - \frac{qx}{2p}$. Right? Now I'm going to the second derivatives, which is nothing but $\frac{dw}{dx} = b\lambda\cos(\lambda x) - \frac{q}{2p}$, and zero, and $b\lambda\cos(\lambda\frac{l}{2}) - \frac{q}{2p}$. So, B is nothing but $\frac{Q}{2P}$ divided by $\lambda\cos\lambda\frac{L}{2}$. That means here it is $\frac{Q}{2P\lambda\cos(\lambda\frac{l}{2})}$. Okay, B. So, I'm going to remove all other terms, right? You see.

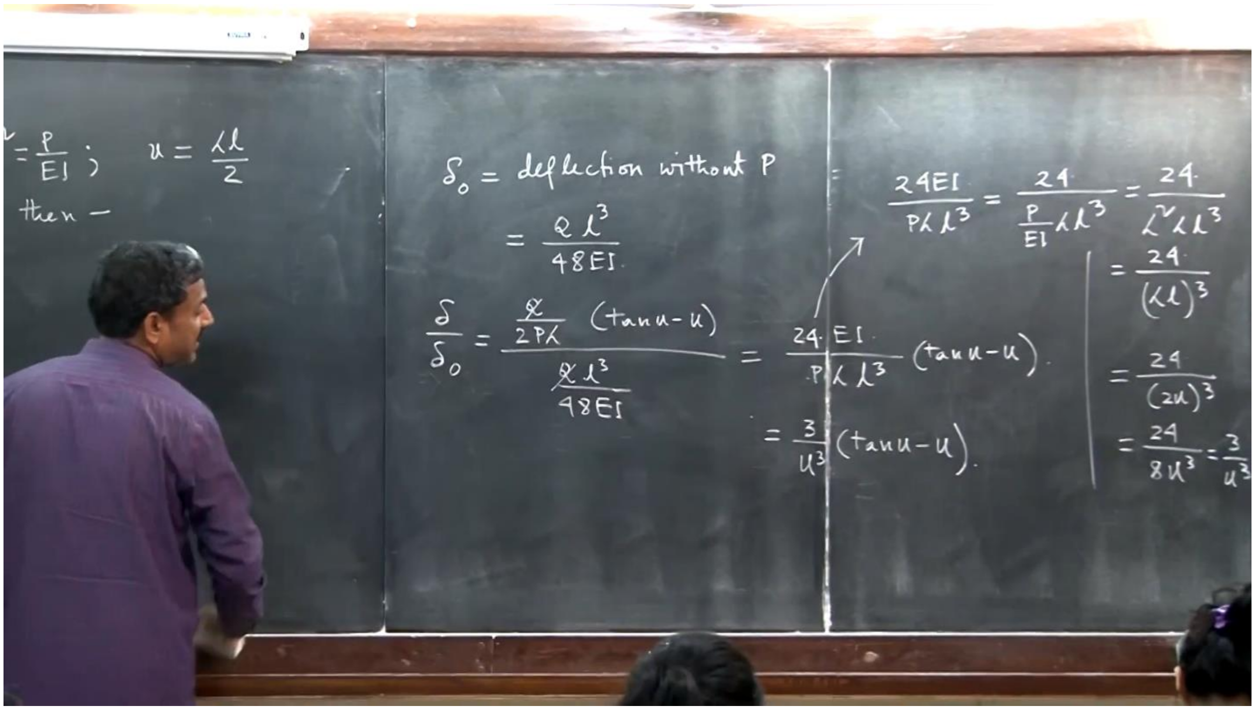


So, $w(x)$ becomes $A\sin(\lambda x) - \frac{qx}{2p}$, right? And then A becomes, you know, if we just simplify, A becomes $\frac{Q}{2P\lambda}$, and here it is $\frac{\sin(\lambda X)}{\cos(\lambda L/2)} - \frac{QX}{2P}$, right? That is the Solution we are interested in; we want to find out the deflection at the midpoint, right? So, if you want to find the midpoint deflection, that means at $x = l/2$. I'm assuming that w is equal to δ_0, okay? So, here I'm going to write δ as $w(x) = l/2$, and then if you substitute q with $2p\lambda\sin(\lambda l/2)$. If here is $\cos(\lambda l/2) - \frac{q}{2p} l/2$, then here you will see what we are going to get. $\frac{q}{2p\lambda}\tan(\lambda\frac{l}{2}) - \frac{ql}{4P}$ right. We'll do a little more simplification to get them in one deflection, that we have assumed $\lambda^2 = \frac{p}{i}$ right. Here we are going to define another parameter, u, which is equal to $\frac{\lambda l}{2}$. That's what we assume: $\frac{\lambda l}{2}$. Then, what we obtain is that δ is nothing but zero. Okay, δ_0 midpoint deflects, or rather than δ, is nothing but $\frac{q}{2p\lambda}$. So, well, you know, and then what we can do is $\frac{Q}{2p}$, then $\frac{l}{2}$, by λ, $\frac{l}{2}$. Okay. So, $\frac{Ql}{4}$, p, and u. So, wait, let me see, okay. This $2\lambda\tan u$ is minus $\frac{ql}{4p}$. Okay, So, then what are we going to do? So, $\frac{Q}{2P\lambda}$, if you take outside, and then $\tan U -$ what will this be? $\frac{Q}{2P\lambda}$. Q you are taking out, P you are taking, 2λ. So, here it will be inside. You will see that λl is by 2, So, q is by $2p\lambda\tan u - u$, and it is nothing but $\frac{\lambda l}{2}$. Okay.



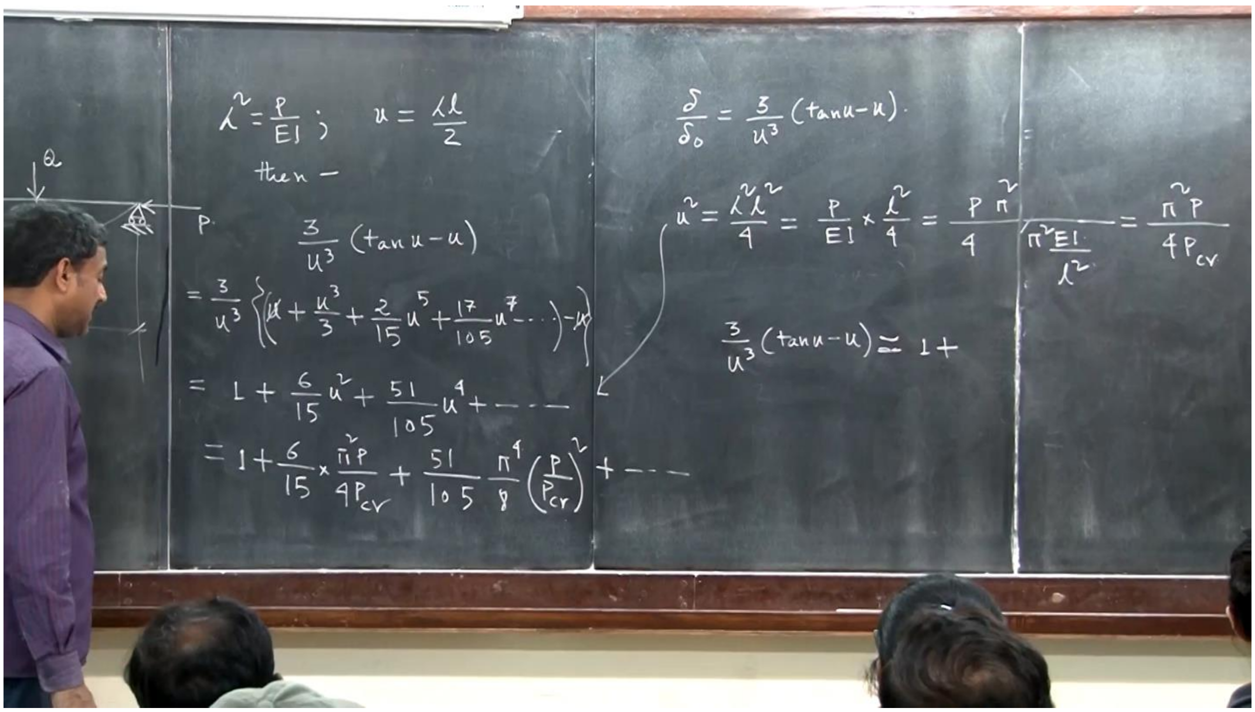
We'll do a little more simplification. Okay, So, now δ_0 means deflection without P. What is the deflection? The deflection is the concentrated load of $\frac{QL^3}{48EI}$, right? That was the deflection, okay.

So, what we can do $\frac{\delta}{\delta_0}$ is going to be, where δ is what? $\frac{\frac{Q}{2P\lambda}(\tan U - U)}{\frac{QL^3}{48EI}}$, right? So, this Q will go. Okay. And then here what is happening is that $\frac{48EI}{2P\lambda L^3}(\tan U - U)$. Huh. Now this one let's simplify though. So, it is further. So, this is nothing but basically $\frac{24EI}{P\lambda L^3}$. Okay.



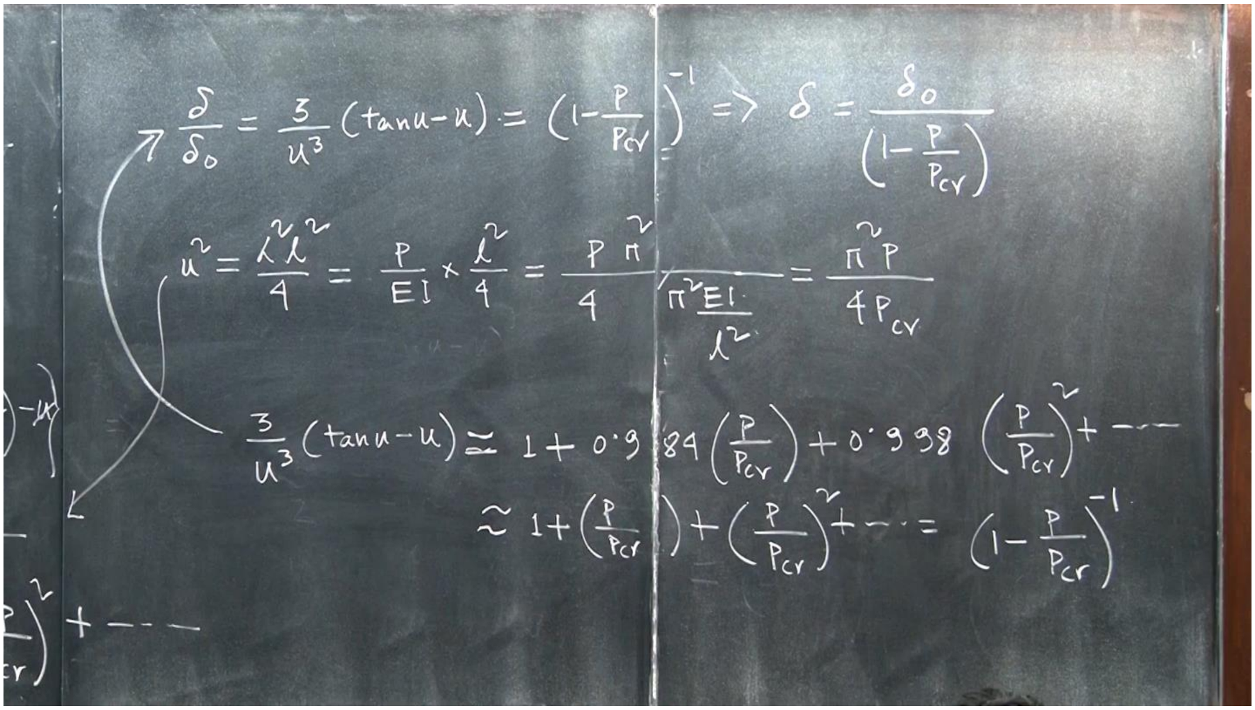
So, this term is coming up as $\frac{24EI}{P\lambda L^3}$. Here we'll do one thing: 24 we will write $\frac{P}{EI}$ into λL^3, So, $\frac{24}{\lambda^2 L^3}$. Then $\frac{24}{\lambda L^3}$, okay. And $\lambda L1$ means $\frac{24}{8u^3}$, then $\frac{3}{u^3}$, right? So, this term is nothing but $\frac{3}{u^3}(\tan u - u)$, okay. we're doing some substitution just to simplify it please note that okay. So, we'll further simplify it okay. So, you see that $\frac{3}{u^3}(\tan u - u)$. Okay it will be $\frac{3}{u^3}\left\{u + \frac{u^3}{3} + \frac{2}{15}u^5 + \frac{17}{105}U^7\right\}$ Something like this. Okay. If you evaluate the factorial, then you will get this, and you see that we are writing minus u, and then just this and this will go. So, what is essentially remaining is that u^3 will go. So, there will be one, and then what will remain is $1 + \frac{6}{15}u^2 + \frac{51}{105}u^4 + \cdots$ and things like that, okay? So, all of you are following the derivation, right? So, we obtain $\frac{\delta}{\delta_0}$, which is nothing but $\frac{3}{u^3}(\tan u - u)$. This is how we are getting it, okay? So, now what will we do? So, u^2 is nothing but $\frac{\lambda^2 l^2}{4}$, right? And λ^2 is what $\frac{p}{Ei}$, right? λ^2, this is $\frac{L^2}{4}$. Okay. So, P and this four I will take here.

So, $\frac{EI}{L^2}$. Okay. So, here I am going to write π^2, and on the top, I'm alSo, going to write π^2. So, here I'm going to write $\frac{\pi^2 P}{4P_{cr}}$. If it is simply supported at both ends, then $\frac{\pi^2 EI}{L^2}$ is nothing but the critical load P_{cr}. So, $\frac{\pi^2}{4}$ here and $\frac{\pi^2}{4}$ is nothing but so, what I'm going to write is this: please note that from here I'm going to substitute it.



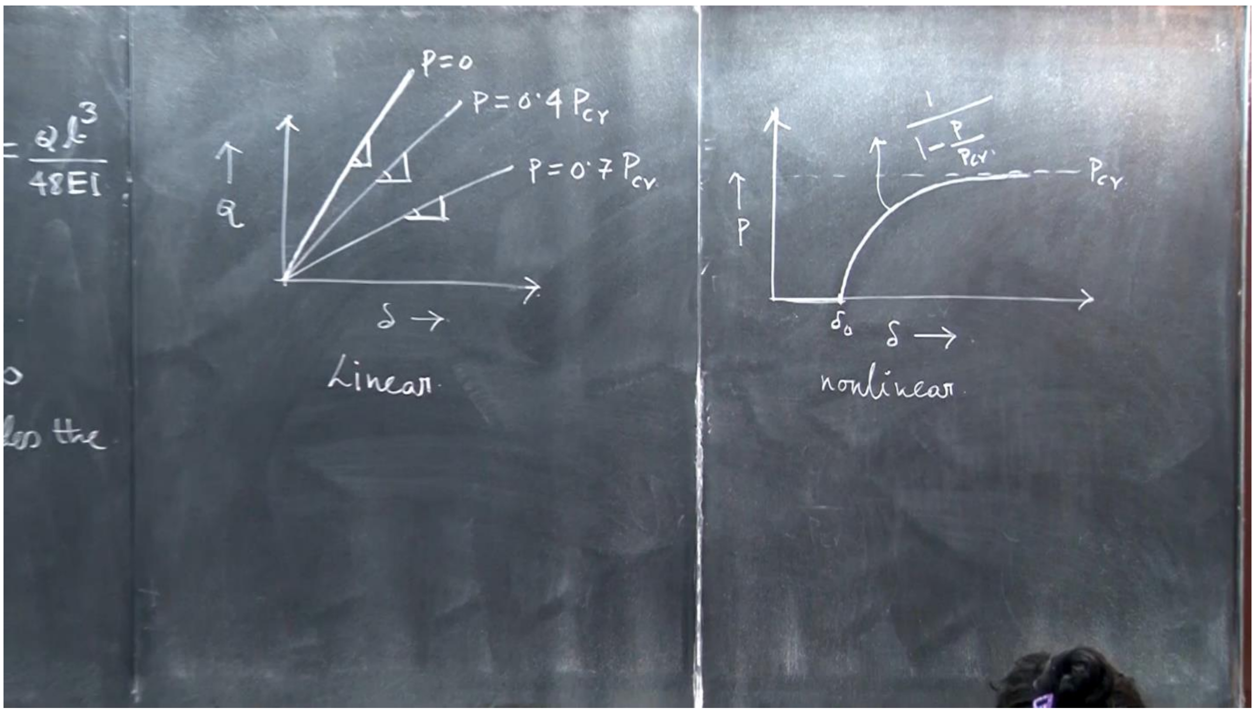
Here you see that $1 + \frac{6}{15}u^2 * \frac{\pi^2 p}{4P_{cr}} + \frac{51}{105}\frac{\pi^2}{8}(\frac{P}{P_{cr}})^2 + \ldots$ whatever.

And then you will see this will further be simplified to, So, $\frac{3}{U^3}(\tan u - u)$ further be simplified to, you see that this $1 +$ you know π^2 is nothing but 0.984 Something by 10. So, it will be very close to one. Okay, that's why I am going to write, it will be $1 + 0.984(\frac{P}{P_{cr}}) + 0.998(\frac{P}{P_{cr}})^2 +$ whatever. Okay. What does it mean? So, may I write it like this? $1 + \frac{P}{P_{cr}} + (\frac{P}{P_{cr}})^2 +$ whatever, may I write it like this: $(1 - \frac{P}{P_{cr}})^{-1}$? Say it's a binomial series, right? May I write it like So, Ultimately, this becomes. $\frac{\delta}{\delta_0}$ is nothing but $(1 - \frac{P}{P_{cr}})^{-1}$, or I can write δ is nothing but $\frac{\delta_0}{1 - \frac{P}{P_{cr}}}$. So, this will give us the interesting thing.

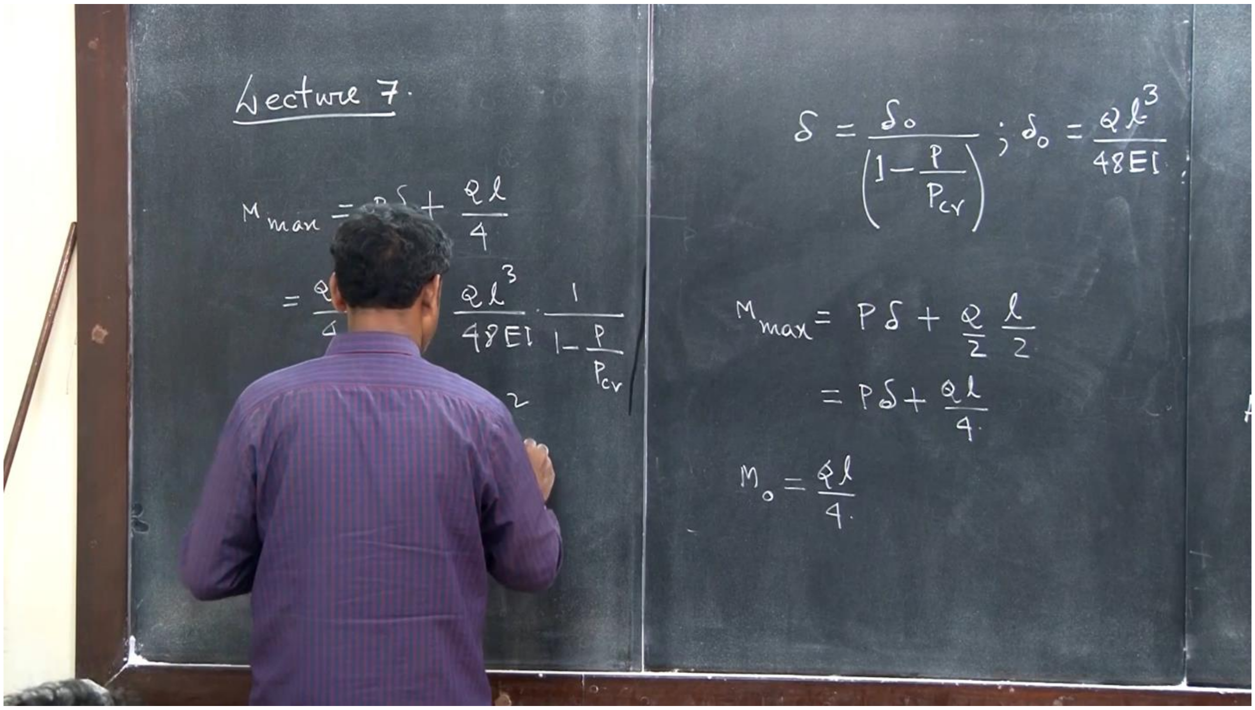


Why? Because you know I'm removing all the other things. Okay. So, ultimately what I obtain is δ, which is nothing but $\frac{\delta_0}{1-\frac{p}{P_{cr}}}$. What does it mean? That means if P is zero, then δ is nothing but δ_0; if P tends to P_{cr}, then δ tends to infinity, right? So, out-of-plane deflection enhances with increasing P. What does it mean? The flexural stiffness reduces. So, axial force reduces the flexural stiffness. Right?

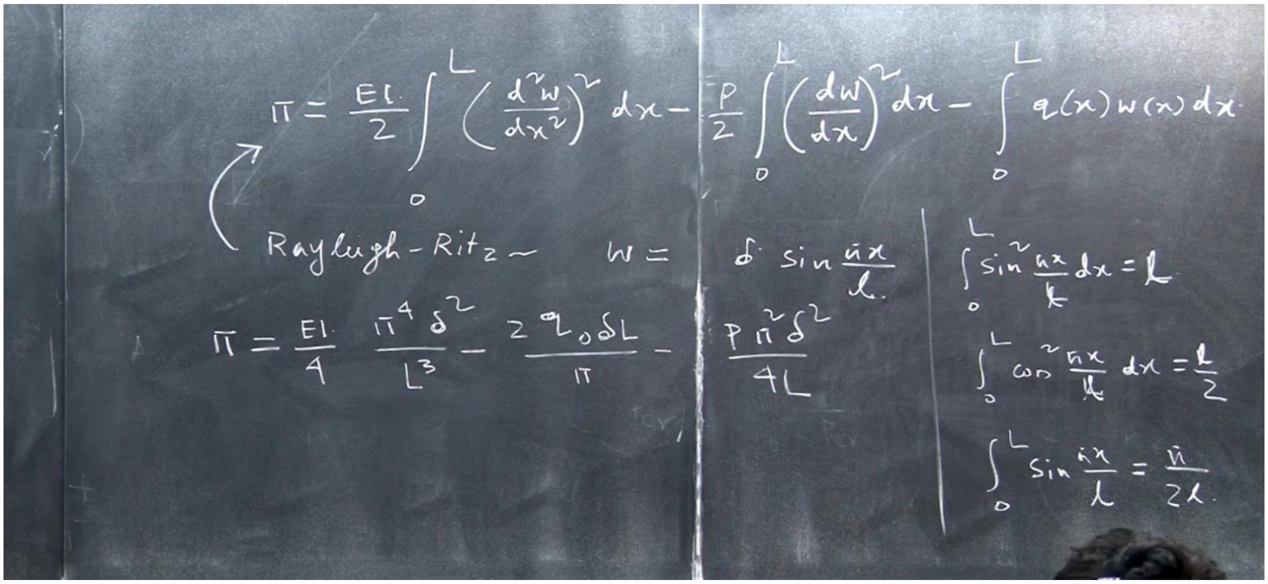
So, here you see that axial compression erodes the flexural stiffness. And during buckling, what happens is that the flexural stiffness is less than that of a column, which will try to deform in flexure. Any elastic system tries to attain the minimum configuration of energy in any system, right? The elastic system is not an exception, right? So, you see that I am a column and Somebody is compressing me, right? Initially, I will try to deform elastically by axial compression. I will have $\epsilon_x \frac{du}{dx}$; I'm going to compress a strain, but that strain energy will be stored inside me, right? And that will control potential energy, but then as P increases, you will see that because it erodes the flexural stiffness, it will be energetically easier to deflect out like this. That's what? Basically, buckling, right? So, buckling is essentially what happens energetically? So, beyond a certain threshold value of axial compression, the out-of-plane deflection occurs. The strain energy demanded by out-of-plane deflection becomes less than the inflame strain energy due to inflame deformation. And then the column tries to attain an energy that is energetically favorable by out-of-plane deflection. Because axial forces tend to erode the axial stiffness. This will be even more clear; you'll understand it when we do finite matrix formulation or finite element formulation, including geometric stiffness. So, here, I am going to plot it. Here you see that if you plot δ versus Q, then there will be three scenarios. You can see that for three different cases: when P is zero, when P is equal to maybe $0.4P_{cr}$ (40% of P), and when P is maybe $0.7P_{cr}$.



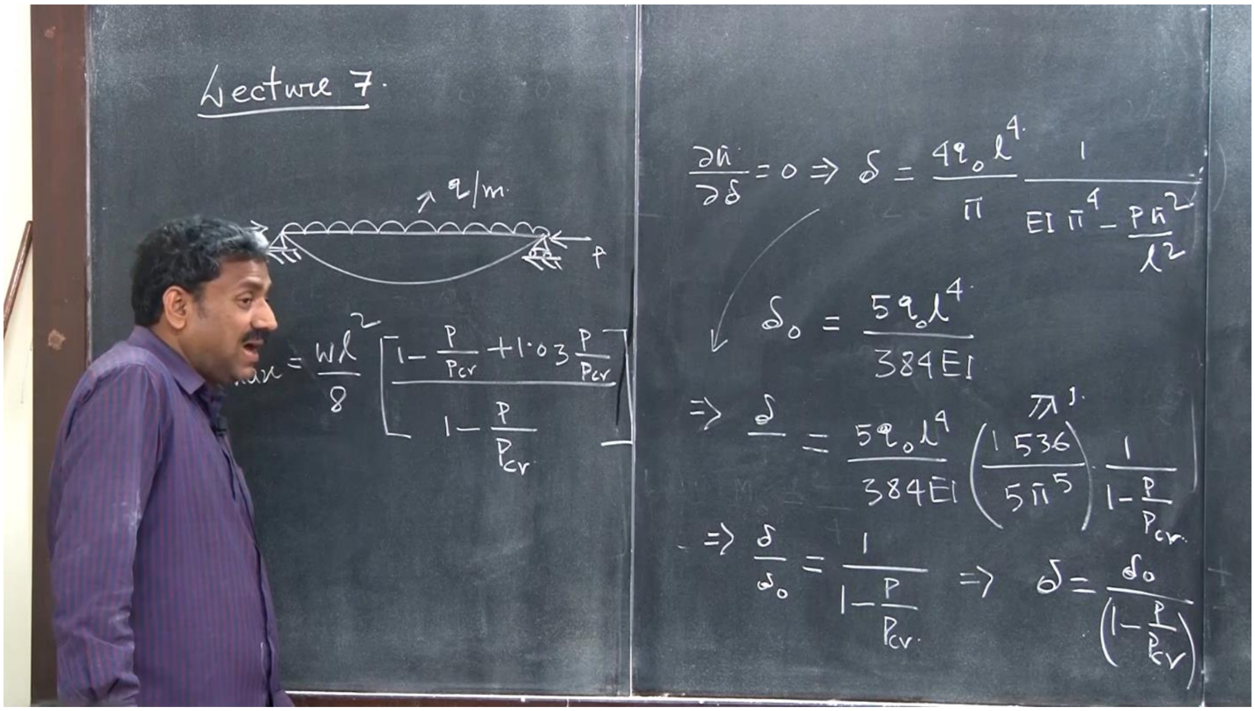
So, that means here you see that the relationship is linear. δ_0 is nothing but what? δ_0 is nothing but $\frac{QL^3}{48EI}$. But then $\frac{QL^3}{48}$ is being eroded. So, here, the axial slope of this line reduces, and the slope of this line is nothing but the flexural stiffness. So, you see, this inclination is reducing, and this is nothing but stiffness. So, that relationship between Q and δ remains linear. You see that there are three distinct values of P axial force, and they are in different fractions of the critical load. As the critical load increases, the axial force becomes a higher fraction of the critical load, and you see that the flexural stiffness reduces. Now, what is the deflection? What is the relationship between P? Here I'm going to write δ, and I'm going to give P. Okay, you see the and here $P_{critical}$. Okay. So, here you will see that at P0 there is Some threshold deflection δ_0, and after that it will be asymptotically right. So, this is nothing but, you know, $\frac{1}{1-\frac{p}{p_{cr}}}$. Okay. So, you say that if there is no axial force, then Some deflection will happen after that as it is tending to p_{cr}. The deflection will nonlinearly increase because this is a nonlinear function; $\frac{1}{1-\frac{p}{p_{cr}}}$ is a nonlinear function.



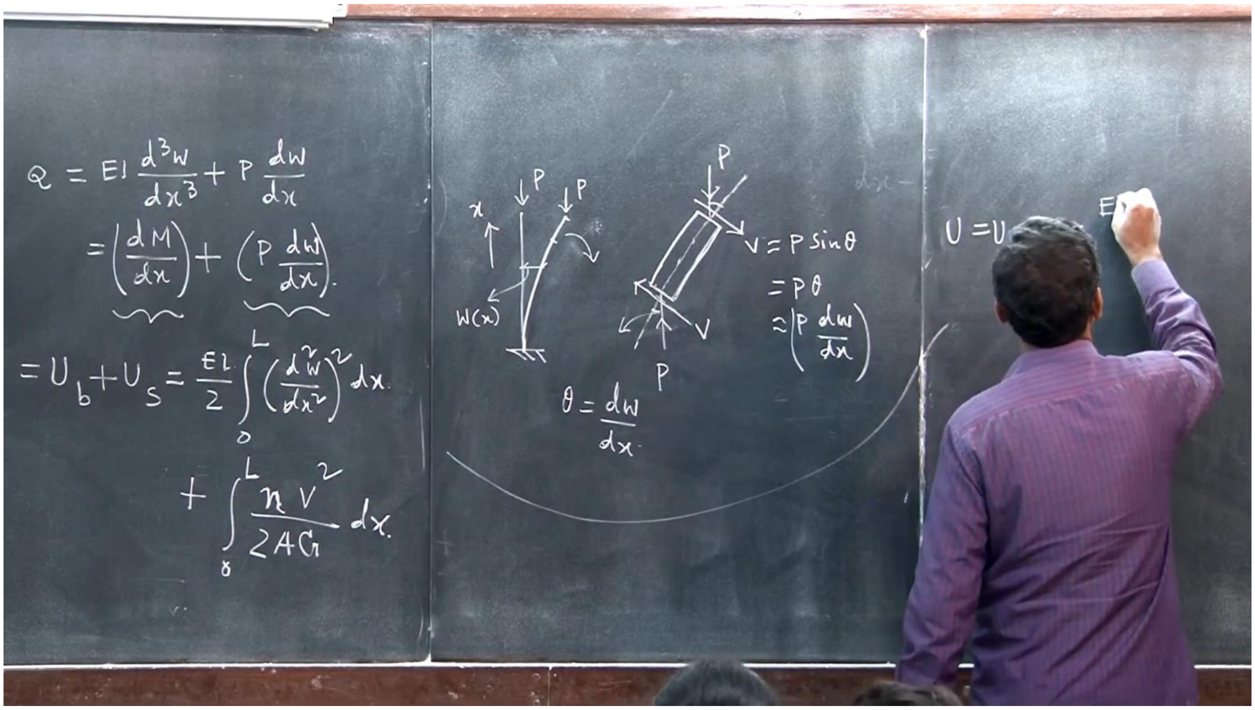
So, the relationship between q and transverse force versus deflection is linear. But axial force versus transverse displacement is nonlinear. This linear relationship is perfectly linear. So, it allows the superposition to be valid. But when the relationship between transverse deflection and axial force is concerned, it is nonlinear. Therefore, you cannot apply superposition for axial force. To find out the influence of the two axial forces on out-of-plane deflection. Okay. So, you have to be careful there. You see that. Now, what I'm trying to say about the bending moment is that it is basically a factor. So, what we see is that an application of axial force amplifies the deflection. We can define this factor as the amplification factor, and this amplification factor depends on $\frac{P}{P_{cr}}$, right? This is the kind of displacement amplification factor. The amplification factor is what? $\frac{1}{1-\frac{P}{P_{cr}}}$, right? So, this amplification depends on the $\frac{P}{P_{cr}}$ ratio. Now, the same amplification has been seen in dynamics. Dynamics also, provide amplification, but the mechanism is different. That dynamic amplification occurs; it's a function of what? Frequency ratio: frequency of loading ratio divided by natural frequency and the damping also, but here it is static. There is no inertia force involved, right? So, there is some displacement amplification factor for axial load, right? Now, for the bending moment, what is going to happen? The bending moment will be maximum at the midpoint. There will be nothing but $P\delta + \frac{q}{2}\frac{l}{2}$, right? So, $p\delta$ of $\frac{Ql}{4}$. Then you just substitute it there, and this is M_{max}, but then M_0. If there is no p, it is nothing but $\frac{QL}{4}$, right? So, there will be amplification in the bending moment also, So, this amplification will be M_{max} is e $\delta + \frac{ql}{4}$, and then you substitute that. So, here it will be $\frac{ql}{4}$ and P into $\frac{Ql^3}{48EI}$ into $\frac{1}{1-\frac{P}{P_{cr}}}$ here, and then if you simplify it. You simplify it by multiplying 4 $\left[1 + \frac{PL^2}{12EI}\frac{1}{1-\frac{P}{P_{cr}}}\right]$, and then you'll substitute this $\frac{PL^2}{12EI}$. So, everything we express as $\frac{P}{P_{cr}}$, then you will get P into $\frac{\pi^2}{\pi^2}$ into $\frac{EL}{L^2} = 0.82\frac{P}{P_{cr}}$. Okay. And then, if you substitute this, you will get M_{max}, which is nothing but $\frac{QL}{4}$, and here it will be $\frac{1-0.18(\frac{P}{P_{cr}})}{1-(\frac{P}{P_{cr}})}$. So, what we have seen is that this is the bending moment without the actual force, right? So, this is M_{max}, this is M, and this is M_0. Okay. So, amplification is one minus because it is negative, and it is slightly lesser, at least for this case, but it is not necessarily. So, there is amplification in the bending moment as well. So, amplification in the out-of-plane deflection and amplification in the bending moment both. And you see that amplification for the bending moment depends on this predominant term $1 - \frac{p}{P_{cr}}$, where the denominator is slight because the erosion of this term is little, 0.18 around 20%. Right fine. So, the same exercise can also, be repeated for a beam, which is distributed. Okay. Here you see that you take a beam. Okay. And it is actually a force P. Okay. And maybe it is a uniformly distributed load of intensity small Q per unit length. And here, you can also, find out what amplification is and things you do not essentially require; you do not always need to use whatever approach you are using, but you can also, use the energy approach.



The energy approach is a little significant. You try to do this exercise yourself; I'm just giving you the hint on how to do that, okay? So, you see that when you write the potential energy function π as $\frac{EI}{2}\int_0^L(\frac{d^2w}{dx^2})^2dx - \frac{P}{2}\int_0^L(\frac{dw}{dx})^2dx$. And then, because there is a transverse note q here, it will be $\int_0^l q\,(x)w(x)dx$, right? And then you can use the Rayleigh-Ritz method. Okay, what is it, right? You have done region dynamics. Find the natural frequency, right? You can assume w is equal to $W_0 = \delta\sin(\frac{\pi x}{L})$, assuming that it is simply supported. Simply supported means this basically satisfies the essential boundary force conditions. So,, the essential boundary condition means here w = 0 when $\sin(x) = 0$ and x = l when $\sin(\pi) = 0$, right? So, then you basically substitute this expression in, right? Then, if you substitute this expression and then integrate and simplify it, you will get it, okay? So, I am just writing down the intermediate steps, but you can follow along yourself, okay? So, here, if you do then π, you will. If you substitute it here, then π will be $\frac{ei}{4}\frac{\pi^4\delta^2}{l^3} - 2q_0\delta\frac{l}{\pi} - \frac{P\pi^2\delta^2}{4l}$. Please note the integral. Okay, these integrals are standard integrals, right? $\int_0^L \sin^2(\frac{\pi x}{L})dx = \frac{L}{2}$. Okay, please recall this integral. Okay. $\int_0^L \cos^2(\frac{\pi x}{L})dx = \frac{L}{2}$, and for all this identity, you have to use $\sin(\frac{\pi x}{L})$. Okay, these are all small l; it is nothing but $\frac{\pi}{2l}$. So, all these things you use, and then what we get, you will see that. So, this is the potential energy function, and then you simplify $\frac{\partial\pi}{\partial\delta}$, which must be equal to zero. When you do, you will get that δ is nothing but, you know, $\frac{4q_0L^4}{\pi}$. Into $\frac{1}{EI\pi^4 - \frac{p\pi^2}{l^2}}$. Okay fine, and if you want to substitute δ_0, it is nothing but Eq L^4. You know what the deflection is; if it is a uniformly distributed load, it is $\frac{5Q_0 l^4}{384EI}$, right? Isn't it? So, then you will see that $\frac{\delta}{\delta_0}$ if you substitute, and you will see once again that $\frac{5q_0 l^4}{384EL}\left(\frac{536}{5\pi^5}\right)\frac{1}{1-\frac{p}{p_{cr}}}$.



You know this term is nothing but it will be one, okay. So, here, the $\frac{\delta}{\delta_0}$ is nothing but. Well, this is Δ, actually. Okay. And this is Δ. Δ_0 is nothing but this. This is nothing but $\frac{1}{1-\frac{P}{P_{cr}}}$. So, do you see how the same thing is coming? $\delta = \frac{\delta_0}{1-\frac{p}{p_{cr}}}$. Even for this, the same relationship is what we are getting. So, the amplification factor is nothing but $\frac{1}{1-\frac{P}{P_{cr}}}$ for the uniformly distributed. However, the expression for bending moment amplification will be a little different. Bending moment expression I'm just writing, this will be a M_{max} is nothing but like without P the bending maximum bending $\frac{WL^2}{8}$ Here it will be, this expression will be a little different; you work it out yourself, but I'm just writing: $\frac{P}{P_{cr}}$, $1-\frac{P}{P_{cr}}$, and plus $1.03\frac{P}{P_{cr}}$. So, what we see is that the amplification factor for this uniformly distributed load is the same, but in the bending moment, the amplification will be a little different here. See, earlier it was $1-0.18$ Something, but here it is $1-\frac{P}{P_{cr}}+1.03$. So, here, amplification is a little higher than the previous one, right? Because earlier in the denominator it was $1 - 0.18$, but here the resultant term, if you see, there is a 3% increase in the numerator; the denominator remains the same. Okay



So, as far as splitting moment amplification is concerned for this case, it is slightly higher than the previous case. Okay. So, taken together, we can conclude that the presence of axial force in the beam-column basically enhances the deflection; there is some amplification of displacement, which depends on the $\frac{P}{P_{cr}}$ ratio. And also, subsequently, there is amplification in the bending moment as well. Okay. But bending moment amplification varies from case to case. Okay. And we have demonstrated both the differential equation approach as well as the energy approach. Clear? Now before the completion of today's lecture, we are going to demonstrate one very important step: the effect of shear deformation on buckling load. This is extremely important because failure occurred by neglecting shear deformation. There was a large truss that failed because the effect of shear was not accounted for in the design; at the time, our knowledge base was not sufficient. It was not extended, and the effect of shear deformation on the buckling load was not known. And that leads to a failure and caused a significant number of casualties and loss of life. Okay, So, that's what we are going to demonstrate. So, here, what we will do is consider a case; we can consider a simply supported beam, okay? TP okay, So, what we know about shear force is nothing: $EI\frac{D^3W}{dx^3} + P\frac{DW}{DX}$, and this is nothing but the expression that comes from the bending moment, and the other component comes from the axial force, right? This is the total H right contribution from the axial force contribution of the bending moment. Now you see the strain energy of bending. So, the strain energy will be two types: one is bending and the other one is shear. So, the component of actual force that contributes to the shear is nothing but $p\frac{dw}{dx}$, right? Okay. And we can alSo, see it, I mean, if you consider the axial p, okay, and then this is deforming here. So, you just consider this P, and you consider a small thing here, P, and here it is P. Okay. Now you see this angle θ. What is this angle θ? θ is $\frac{dw}{dx}$. If we consider transverse deflection defined in terms of w(x), I am considering x to be aligned along the axis of the column. Right? So, now you see that some of these components will act here and then this will act here. So, I'm considering this shear here, okay, this component because of P. What is this component? This angle is what? θ, So, $P\cos\theta$, and this is nothing but $P\sin\theta$, right? So, $P\sin\theta$ means $P\theta$, and $P\theta$ means $P\frac{DW}{Dx}$. So, you understand the origin of this term $P\frac{Dw}{dx}$. It's a very ad hoc way, So, I thought I would just show it to you. Otherwise, you'll wonder what the hell you know physically about this term. You see that? What I get, of course, is that the moment contribution is there because of this. So, now, if I want to write down the bending energy, I want to use the Rayleigh-Ritz energy method, right? So, you understand the origin of the $p\frac{dw}{dx}$ term. See what we have obtained, $PW\frac{DW}{DX}$, previously from the variational approach, right? But it is alSo, physically meaningful; you understand from here, right? Because of p, there is an additional term that is coming from the bending right. That is $\frac{dm}{dx}$, which is nothing but $ei\frac{dw}{dx}$. Okay, So, now if I want to write the bending energy. What I have written is that $\frac{EI}{2}$, the bending energy from 0 to l, is $(\frac{d^2w}{dx^2})^2$ into DX; that was the bending energy right $\frac{EI}{2}$, this one. Plus, if I now want to discuss the contribution of shear, may I say that you all know what shear is? All of you have studied the strain energy chapter in your undergraduate courses, right? What is the strain energy for shear? V is the shear force, I'm assuming. What is the respective shear deformation $\frac{V}{A_g}$? A_g is the shear stiffness, A is the area, and G is the shear modulus, but there is a shear correction factor, right? What is the shear correction factor N? The shear correction factor takes into account that traditionally, in beam bending theory, you don't care about shear deformation, but rather the first-order shear deformation, as proposed by Timoshenko's deformation theory. First-order shear deformation in beams is Timoshenko, in plates it is Reglíñear and Milnes' method, and in shells it is Naghdi and others. So, first-order shear deformation assumes that it is nothing but one additional rotation of the deformed section of the beam, right? But here, stress distribution is never uniform; it is always parabolic. So, there is some incompatibility in order to achieve the consistency you need to have another factor. Okay, So, that factor is nothing but the shear correction factor, and what is that factor? This factor depends on, I mean, properties of the sex and material, etc. But if you take 5 x 6, it can vary from $\frac{2}{3}$ to $\frac{5}{6}$, typically you say now. Take So, it is integration 0 to L $\frac{nv^2}{2AG}$ into dx, where n is the shear correction factor, v is the shear force, and AG is the shear stiffness, right? So, now you see this expression we are going to simplify a little bit. From here, I'm going to write total strain energy. What is the total strain energy? I will still write bending plus shear, but what will I do? I will not use this term directly; I will rather use a simply supported beam column. What is the other way to write the bending energy in terms of bending moment? Please recall the other $\frac{m^2 dx}{2ei}$; that was the expression for the bending moment. This is another way to write this in terms of displacement, but this is in terms of bending moment plus integration from 0 to L. I'm writing $\frac{n}{2AG}$, and v is nothing but $p(\frac{dw}{dx})^2$ into dx. Please note that $\frac{DM}{DX}$ has been taken from the bending energy, So, I'm only considering the term $P\frac{DW}{DX}$. Okay, So, this way here. Okay, now what is the bending moment here? Here, the bending moment m(x) is nothing but p times w here. Because it's an out-of-plane deflection with respect to w, for w(x) at any section, you know p times w, right? That is the bending moment, So, it is pw, and the rest of this is in terms of w and p. I'll do this in the next class, okay? Thank you for today's lecture.