Stability of Structure

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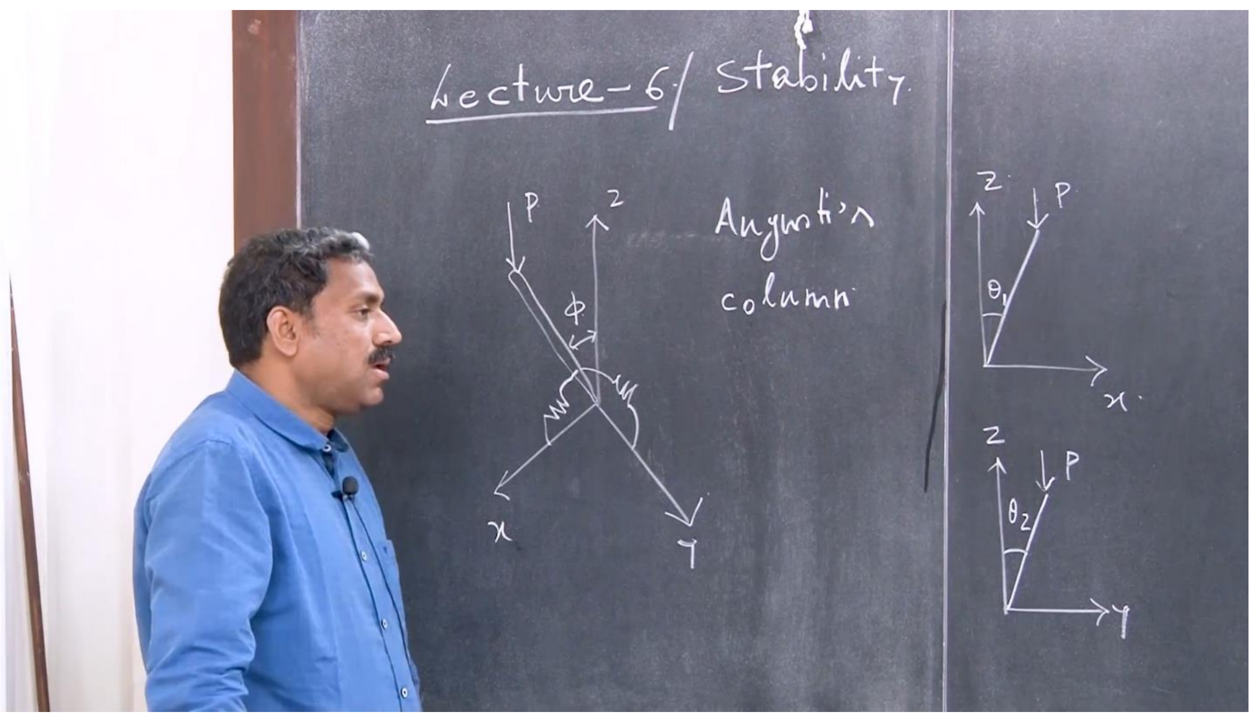
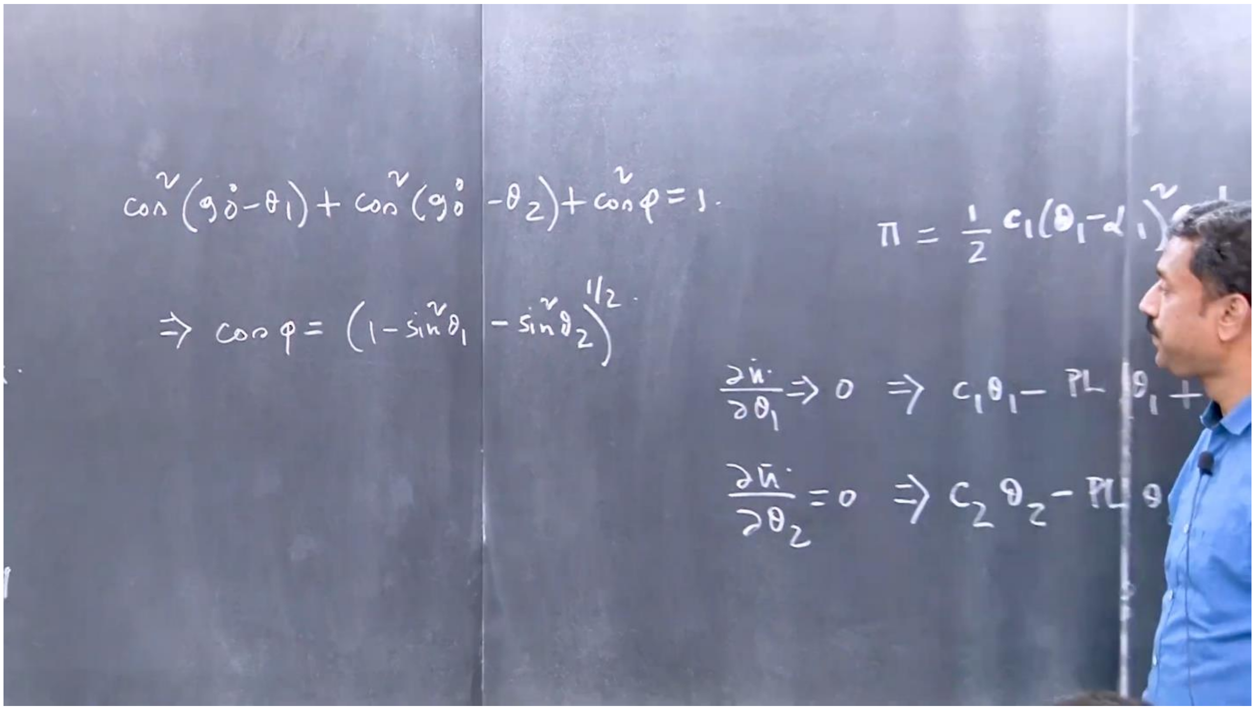
WEEK-03

LECTURE 6: Modal Interaction and Bergan's Truss

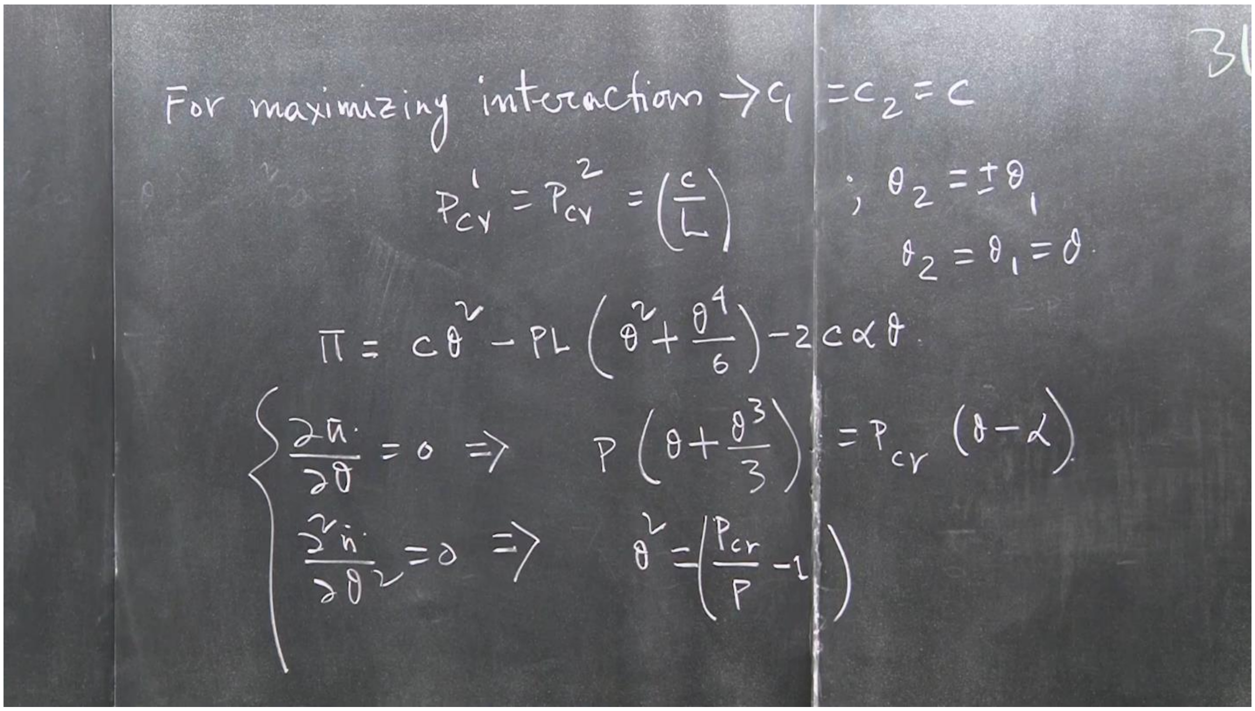
Okay, welcome to the sixth lecture on the stability of structures. So, we are considering August's column. A system is called Augustis's column to demonstrate the influence of modal interaction. We will demonstrate how an apparently imperfection-insensitive mode becomes imperfection sensitive due to modal interaction. And this is demonstrated using a simple example: this is a column. And this is a rigid column that is hinged here and restrained by two springs, C1 and C2. These are both rotational springs. Right. And then we discussed this in the previous class. So, it is projected in the XZ and YZ planes, and then we satisfied the equation for the direction cosine to find the cosine of φ, where φ is the angle with the Z-axis.



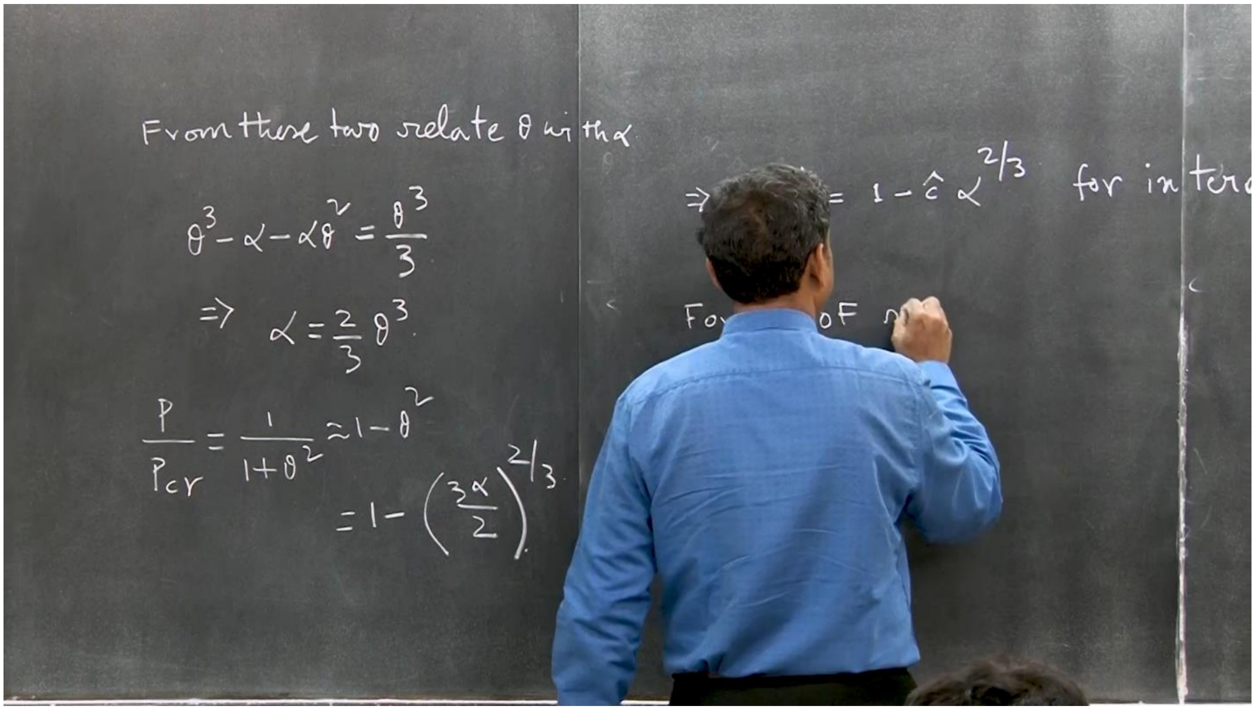
Okay, and then we basically wrote the potential energy functional down. Okay, theta 1 and theta 2 at the two degrees of freedom. Please note that there will be. Two independent coordinates, you know. So, these are two degrees of freedom coordinates. Okay. So, theta 1 and theta 2, because phi can be expressed in terms of theta 1 and theta 2 by virtue of this direction cosine, right? So, then we basically need to simplify this expression. And then we have considered the expansion of $\sin(\theta_1)$ and $\sin(\theta_2)$.



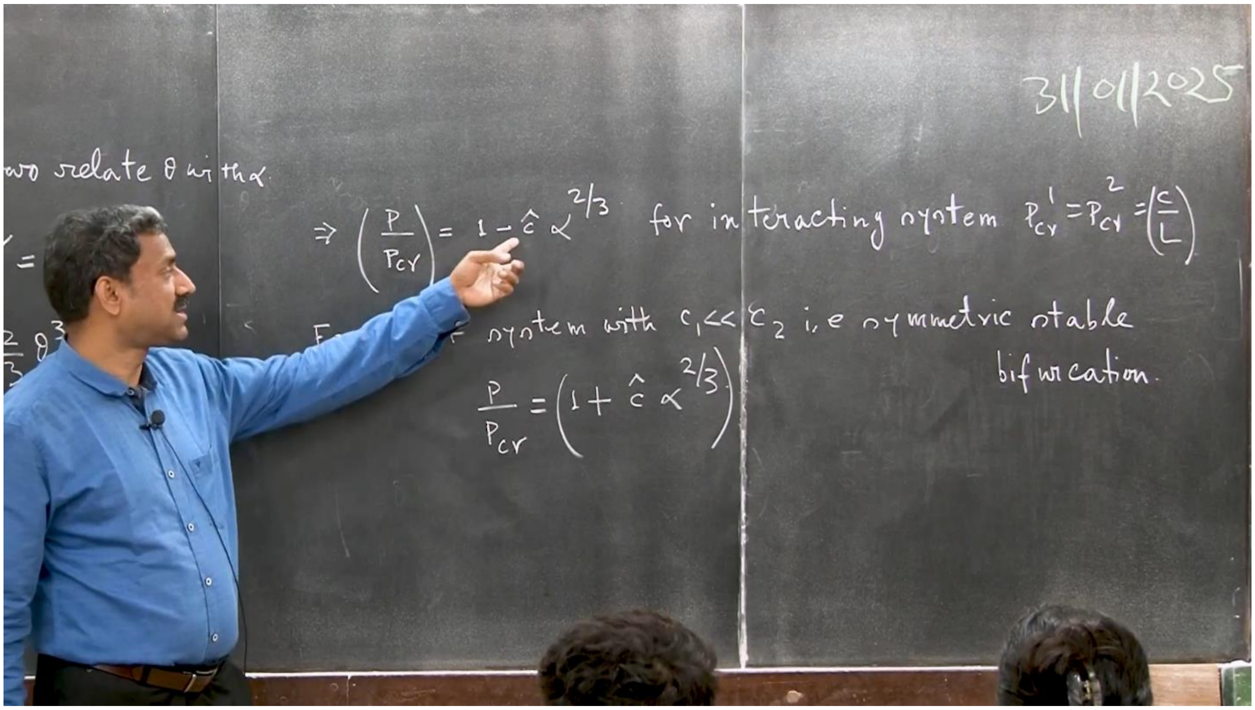
Okay, we included both θ cubed and θ factorial. And then, while squaring, we retain only θ_1^2 and θ_2^2. Then, the cross term between θ_1 and θ_2 and the sixth-order terms are neglected, okay? Then, for the equilibrium path, we have minimized the potential energy functional. Please note that if we want to conduct a linearized analysis, then what will happen? We are going to return only this thing and this thing, right? And all this because θ_1 cubed, θ_1, θ_2^2, θ_1^2, θ_2; these are coupling terms, okay? And then imperfections are only linear. We are assuming that imperfections α_1 and α_2 are the same imperfections. We are given the initial rotation of these springs; okay, C1 is the rotational stiffness and so is C2. So, from here we know linear analysis. From linear analysis, we can find the eigenvalue problem; then there will be two critical loads: critical load C_1/L and another one, critical load C_2/L. Okay, L is the length of the bar. Now I'll simplify and write the final expression. In that case, when C_1 is equal to C_2, the expression will be simplified, and we will still retain some nonlinear terms, and I will write down the expression you will get. Then, of course, the critical load for P_{CR1} and P_{CR2} will be nothing but C/L.Okay, the critical load for that case is... θ_2 is equal to plus or minus θ_1. Let us consider θ to be equal to θ_1, which is equal to θ. If you assume θ_2 is equal to $-\theta_1$, that's also fine. Okay. Then, if you substitute, we will get the expressions for potential energy. The potential energy expression will be $C\theta^2 - PL(\theta^2 + \frac{\theta^4}{6}) - 2C\alpha\theta$. Okay, this will be the expression for the potential energy when this mode multiplicity is occurring. Okay,



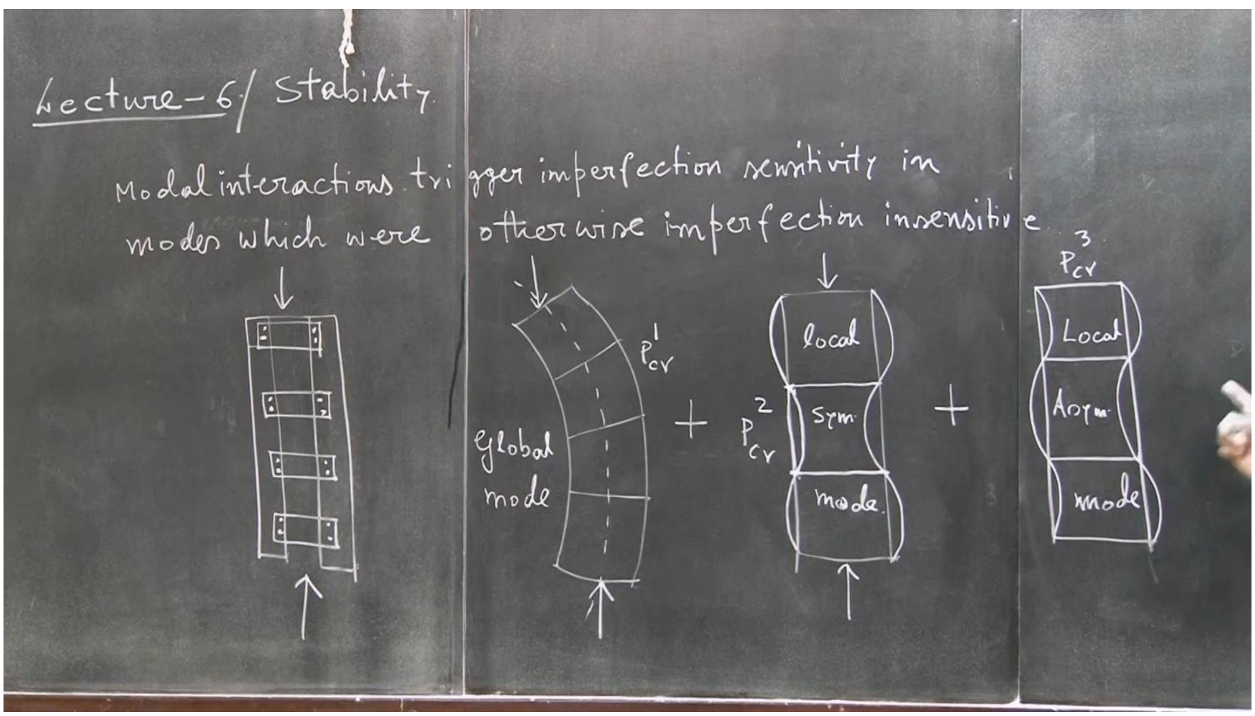
I'm substituting θ_2 equal to θ_1 equal to θ, and both the critical loads are identical. Okay, then this is the expression for π. And from there you will see that, if you differentiate $\partial\pi$ with respect to $\partial\theta$, it is equal to zero for the equilibrium path. Here you will get, you know, $p(\theta + \frac{\theta^3}{3})$ is equal to $P_{cr}(\theta - \alpha)$. And $\partial\pi/\ \partial\theta_2$ equals zero. From where you will get P_{cr} here. This will give $\theta^2 = (P_{cr}/P - 1)$. You can see whether these expressions are correct. Okay. So, from these two, you see that we are going to do for modal interactions what we did in the previous class on imperfect sensitivity. So, we'll just substitute P_{cr} with P. We can write P_{cr} as nothing but $P(1 + \theta^2)$. So, combining these two expressions, let us relate θ to α, okay. So how will you relate it? Please see that $\theta^3 - \alpha - \alpha\theta^2$ is equal to $\theta^3/3$. Okay. And from here you will get $\alpha = \frac{2}{3}\theta^3$. And then I will just write P/P_{CR}, which will be nothing but, you know, $1/(1 + \theta^2)$, which is nothing but $1 - \theta^2$, and then θ^2 will be $1 - (\frac{(3\alpha)}{2})^{\frac{2}{3}}$, okay. So, I'll just remove all these things. From here, I can write P/P_{cr} is nothing but $1 -$ some constant; maybe I can write $c\alpha^{2/3}$. Okay. So, what we see follows a $2/3$ imperfect power law. This is combined for an interacting system.



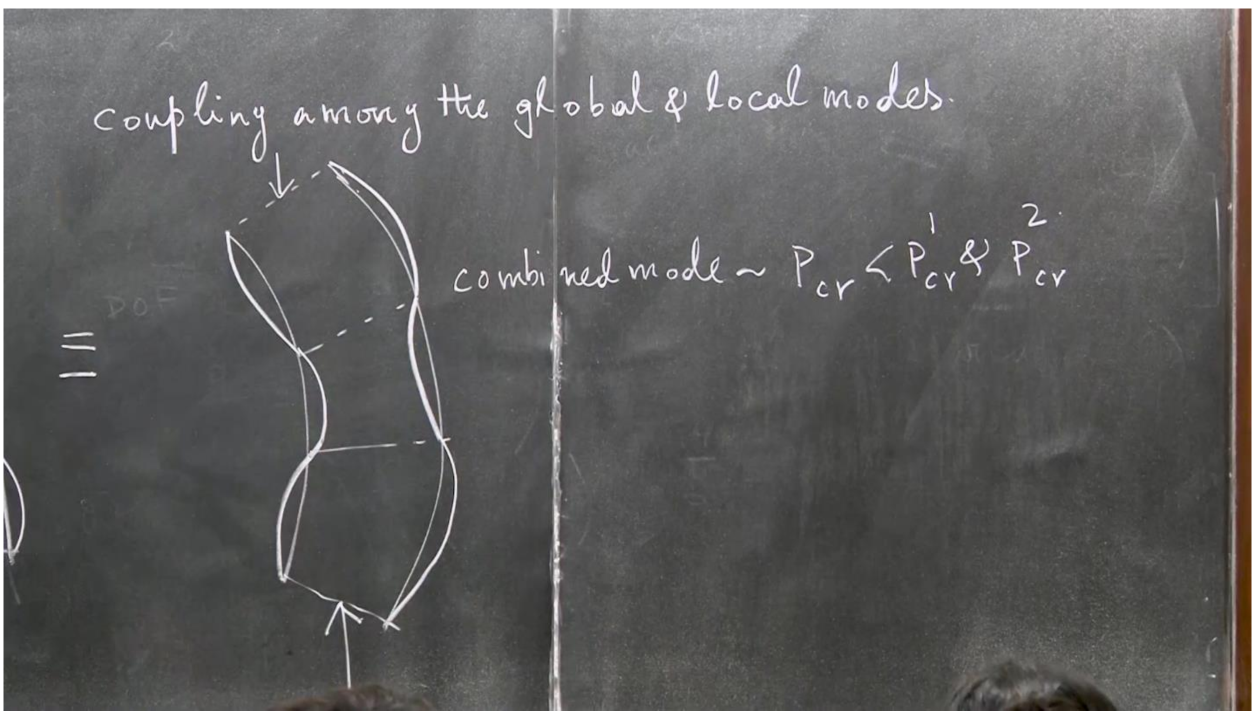
Why is it interacting? Max, because P_{CR1} is equal to P_{CR2}, and it is equal to C/L, right? And we have considered the coupling term between the two modes, right? We have considered the nonlinear term; that's why this cubic and all other terms are coming, right? But now, go back to the individual. So, if I assume that C_2 is much less than C_1. Now, what will the buckling load, $P_{critical}$, be in terms of C and L for a single-degree-of-freedom system? This refers to the model with a rigid bar restrained by a rotational spring of stiffness C. We have already determined this critical load in our first discussion, where the system demonstrated a symmetric, stable bifurcation." So, for an SDOF system with C_1, it is basically much less than C_2. So, what is P in P_{CR}? That is symmetric stable bifurcation. the system which, we'll showing the symmetric stable bifurcation right. the first example we have considered a rigid bar rested by rotational spring. that was showing the symmetric bifurcation and stable post critical path right. for that it was not imperfection sensitive. it was what? P/P_{cr}. It was $1 +$ some constant $\alpha^{2/3}$. That means the load-carrying capacity increases with imperfections. Right? But here, because of the interaction, what is happening? This plus becomes minus. So, it becomes imperfection sensitive. Do you see that? So independent modes, which were imperfection insensitive, become imperfection sensitive. What is the reason for that? Modal interaction. So, modal interaction triggered imperfection sensitivity.



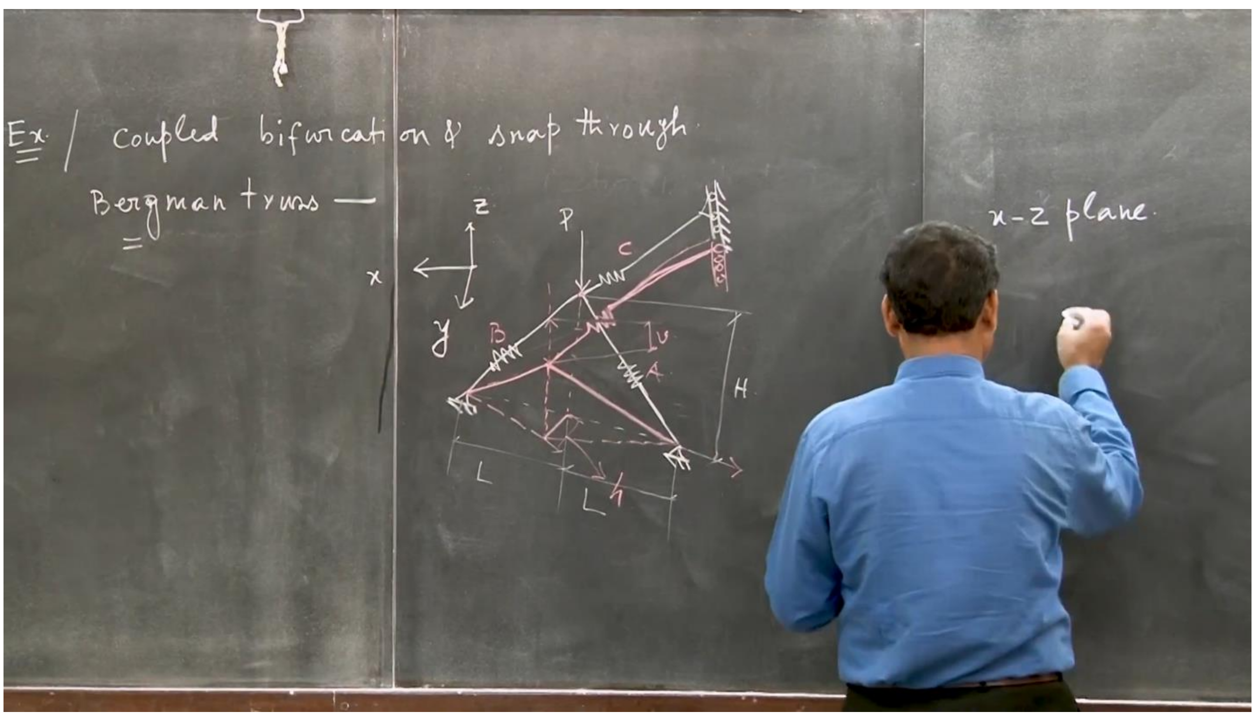
That is the takeaway from this and how it is demonstrated. We have considered August's column, which is a two-degree-of-freedom system resting on two rotational springs. Right? So, you'll see the importance of modal interactions. Apparently, an imperfection-insensitive system becomes imperfection-sensitive, but its imperfect sensitivity is two-thirds sensitivity. What does two-thirds mean? Mildly imperfection-sensitive, right? So, this is the takeaway from the model interaction: we can write that model interactions may trigger imperfection sensitivity in a system. But how will I ensure model interaction? Look, if you do linearized analysis, that means eigenvalues, which lead to the buckling problem; you'll get only uncoupled modes. You know they are actually coupled. If you consider geometric high-order nonlinear effects, these separated buckling modes are actually coupled, and then you perform a linear analysis. This leads to an eigenvalue buckling analysis. For this kind of structure, you will see a multiplicity of modes, and these modes are very close together. That means they tend to interact because they are geometrically coupled by geometric nonlinearity, and that is basically the reason for what happens in cell buckling: why the critical load drops due to all this mode multiplicity or closely spaced modes. Some modes are imperfection sensitive, and some modes are imperfection insensitive. However, most of these modes—whether symmetric or asymmetric—will be imperfection-sensitive. Asymmetric modes are typically more imperfection-sensitive when compared to symmetric modes, though symmetric modes can also be mildly imperfection-sensitive. Asymmetric will be more imperfection-sensitive after half an hour. So, when they interact, it will further enhance the imperfection sensitivity, right? So that's basically what happened during shell buckling, which caused a huge reduction in the critical load. So modal interactions trigger imperfection sensitivity in modes that are otherwise imperfection insensitive. It's clear that this is a big deal and how we have demonstrated it. So, I will give you an example of that. Okay, you must have learned in your steel design class about examples of button columns, right? These columns are connected or articulated columns, and this button system, right, is either welded or bolted, correct? This is the button plate, right? So, these are the two different angles of the channel section that are connected. And why are they buttons? Because you want to use them as a single unit, right? gSo, when it is subjected to a buckling load, what kind of mode will there be? There will be multiple modes, right? For instance, one global mode and one local mode. What will the global mode be? Global mode will be that this column may buckle out of plane due to bending, right? You see that this is the buckling mode, right? Something like this, right? So, this is the central line under axial force and axial compression. So, this is global buckling, right? And there can be local buckling. How?



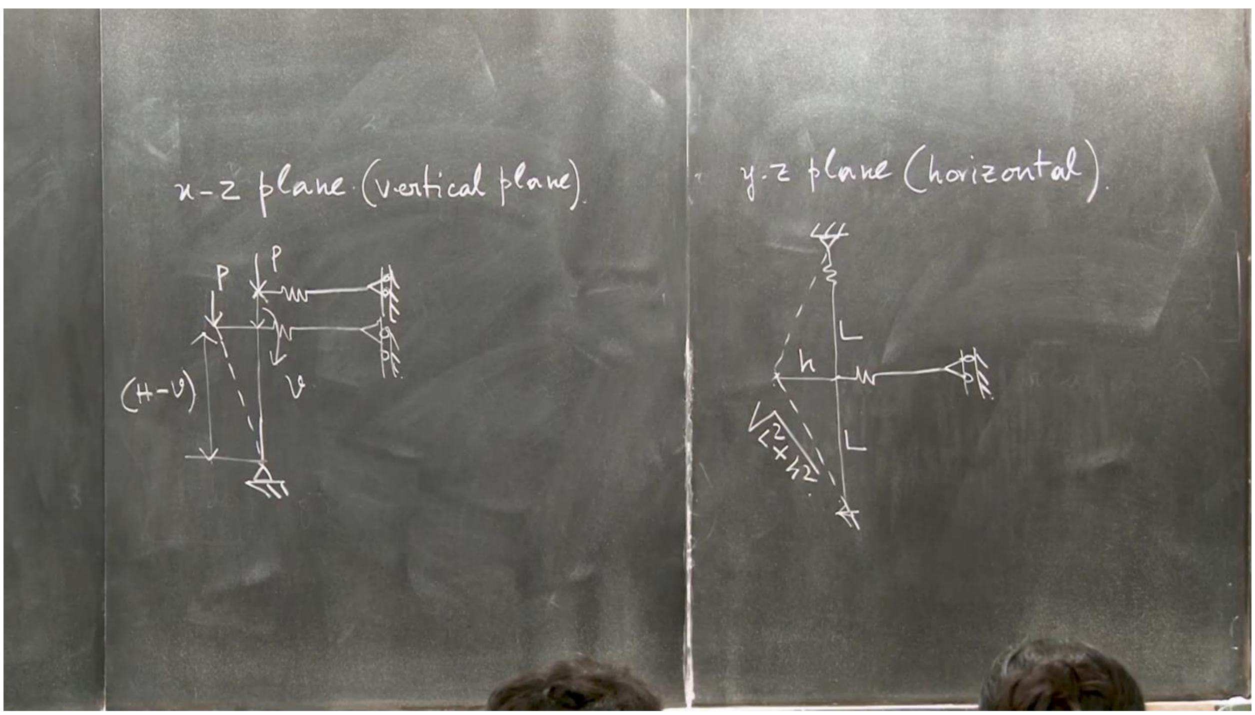
This is a global buckling mode. There can be another local mode. How will it be? because they are button now. So, there can something like this. This is local mode and this is symmetric mode. Why it is. local buckling mode? Because the portion of the column that is between the two button plates is buckling, you see these as points of contraction. At this point, you see that it is buckling. So individually, the intermediate portion of the column is located between the two buttons. Because the button provides some kind of lateral restraint, right? So, these are all local modes, right? Now, this is a local symmetric mode, and there is a local asymmetric mode. How will it look? See if you want to put it in local asymmetric mode; it will look something like this. You see that this is local asymmetric mode, symmetric mode, and buckling mode, right? Now what happened to the global local mode? There can be some critical load, P_{cr1}. Now, for this case, whether it is a local or a global mode, the buckling is ultimately caused by axial compression. Therefore, both of these modes will also have their own specific critical loads. I'm assuming this critical load is also P_{cr2}. Here, the P-critical load is P_{cr3}. You understand what I'm trying to say, right? Now, it may happen that all these modes interact. When and how will they interact? Because when there is little out-of-plane deformation, there will be a significant onset of geometric nonlinearity. Due to geometric nonlinearity, there will be coupling between this mode and the global and local modes.



Local mode can be symmetric and asymmetric right modes, and because of this coupling, it will ultimately look like this: this combined mode will look like something like that. So, you see this is a combined mode and how this combined mode is occurring. This combined mode is a combination of this global mode and this symmetric mode, right? Because this is symmetric. Similarly, there can be a combined mode, a global mode, plus an anti-symmetric mode. So, when this happens, you will see the P critical CF. This P critical will be less than whatever the combined mode PR. This P critical will be less than P critical one and P critical two. Okay. And then it will be imperfection-sensitive. So, this analysis can be done, I mean, analytically. It may be a little challenging, but if you do, even using any finite element method, nowadays there are many commercial software options. You can clearly do this exercise in Abaqus or any other code, and you can clearly see how it affects the modeling. So, this is one example, one instance of modal interaction. Okay. Now I will solve an example. That example shows manifest coupled bifurcation and snap-through, meaning both snap-through and bifurcation. So, it's a combined thing. Okay. So, the system this is called bergman trust von Mises truss. It is the combination of this.so how it looks like something like this. Okay. So, there is a spring, okay? Because of spring stiffness, or you know, axial stiffness, whatever. So, this is nothing but what? This is called a Mises truss, right? So, this axial load is P, okay? But in addition to that, there is another spring that is attached here, okay, and this spring is supported here. So, it's a combination of a von Mises truss resting on an additional spring.

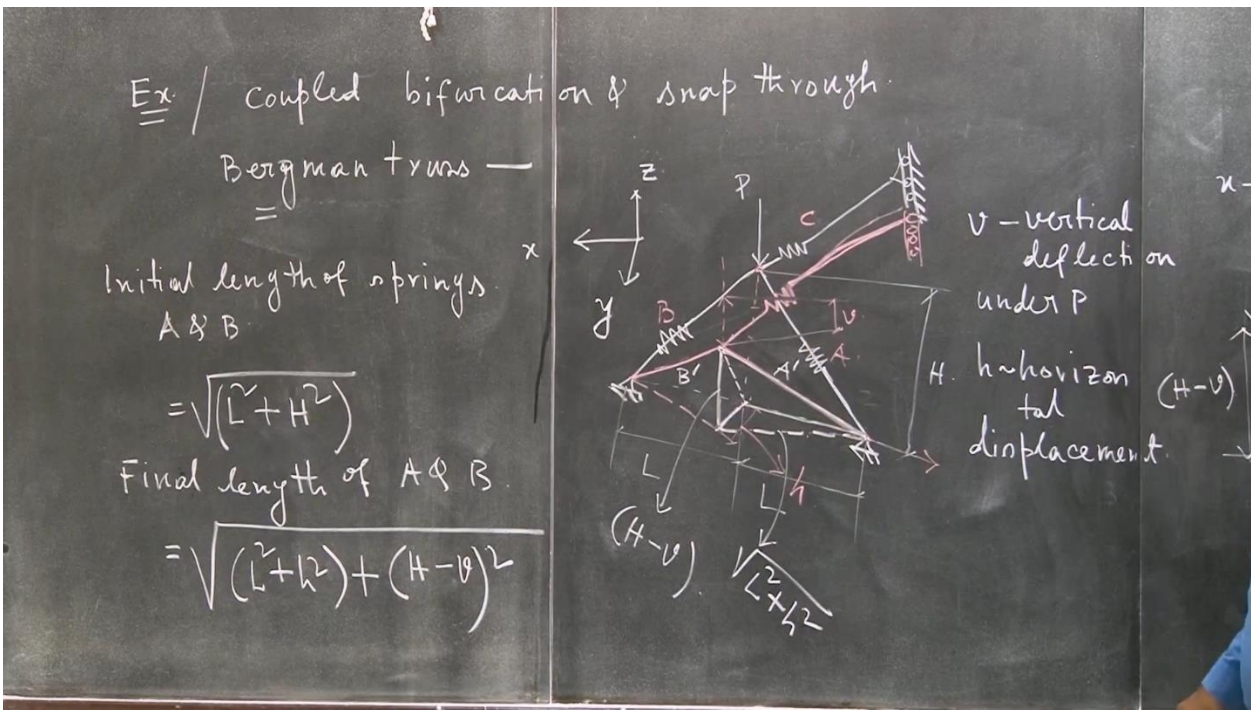


Okay, I am just giving you the dimensions, and this is basically L, and this height is H, okay, like that. So now you tell me this stress is called a Bergman truss, and it will show a couple of bifurcations. Let us analyze that. Okay. So how will you analyze it? See, when you have to consider the deformed right configuration. So, you see that I'm assuming this is spring A, this is spring B, and this is spring C. Okay, so now this spring can move vertically, and this one can move horizontally. The point under the load P can move vertically as well as horizontally, right? So, I'm assuming that when it is moving horizontally, it will come to this position, right? And then, when it is moving horizontally, it will come somewhere here. So, I'm just, you know, drawing it something like this. Here I mean maybe that it will top off that. So, I just put it here and leave it there. So, this is coming horizontally. You see this fellow is basically whatever it is coming down somewhere here. Okay. So, you see that this is the vertical component. I may want to write this component as V, okay? And the horizontal one, I write as small h, okay? All of you got it. What is the ultimate configuration of this? So, this and this, you see, okay? So now, if I'm assuming this is maybe X, okay, and I'm just adding Z, this is X and this is Y. Okay. So, if I show you this view of the X-Z plane, how does it look? The X-Z plane means which plane? The XZ plane means a vertical plane, right?

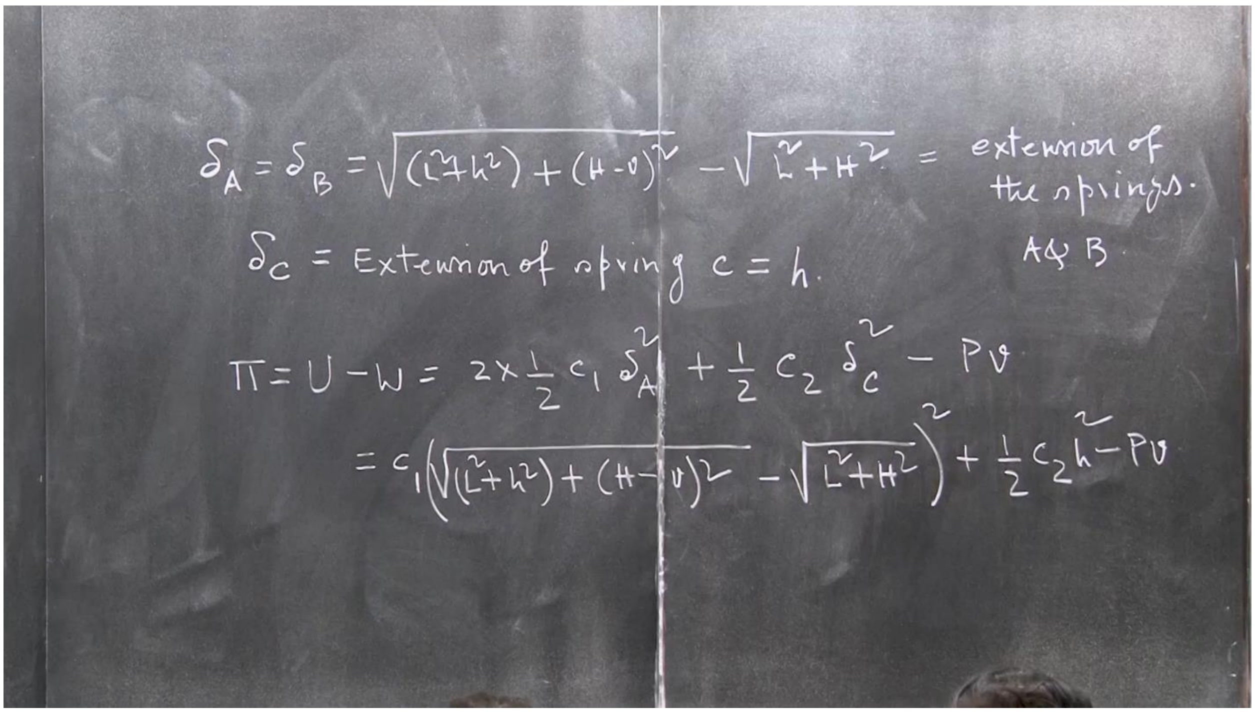


Vertical plane, how will it look? So, what is happening? This is nothing but a vertical plane. Note that, huh? So, P is moving here, and then this vertical movement, this component is nothing but V. Okay. So, this distance of V is how much? $H - V$. because this distance was H, right? so when you visualizing the system correctly okay. Because what we have to find out is ultimately the length, okay? And how will we do that? This will be nothing but this square plus this square, okay?

This length squared plus this length will be a dot, and b is coming to b dot. So, this length is nothing but the square of this plus the square of this. Okay. X-Z plane, and then I will write the YZ plane, which is basically a horizontal plane. Here it is coming to this position. This was basically nothing but small h. This is L. So, you see that this length is projected; this length is nothing but $\sqrt{L^2 + H^2}$, right? So $L^2 + H^2$ is nothing but this length. So, this length is nothing but $L^2 + H^2$ because this is H and this is what L was, right? Okay. So, this length is the same as this length, right? $l^2 + h^2$, okay. So, this is $h - v$, and this is the square root of $l^2 + h^2$. Clear? This is capital $H - v$, and this fellow is nothing but the square root of $l^2 + h^2$ here. Now all of you have the final length, so the initial length of the spring or the stiffness of this spring is given by this Excel stiffness of the truss. I'm assuming that maybe some stiffness springs A and B are identical, okay? It is nothing but $\sqrt{l^2 + h^2}$, right?



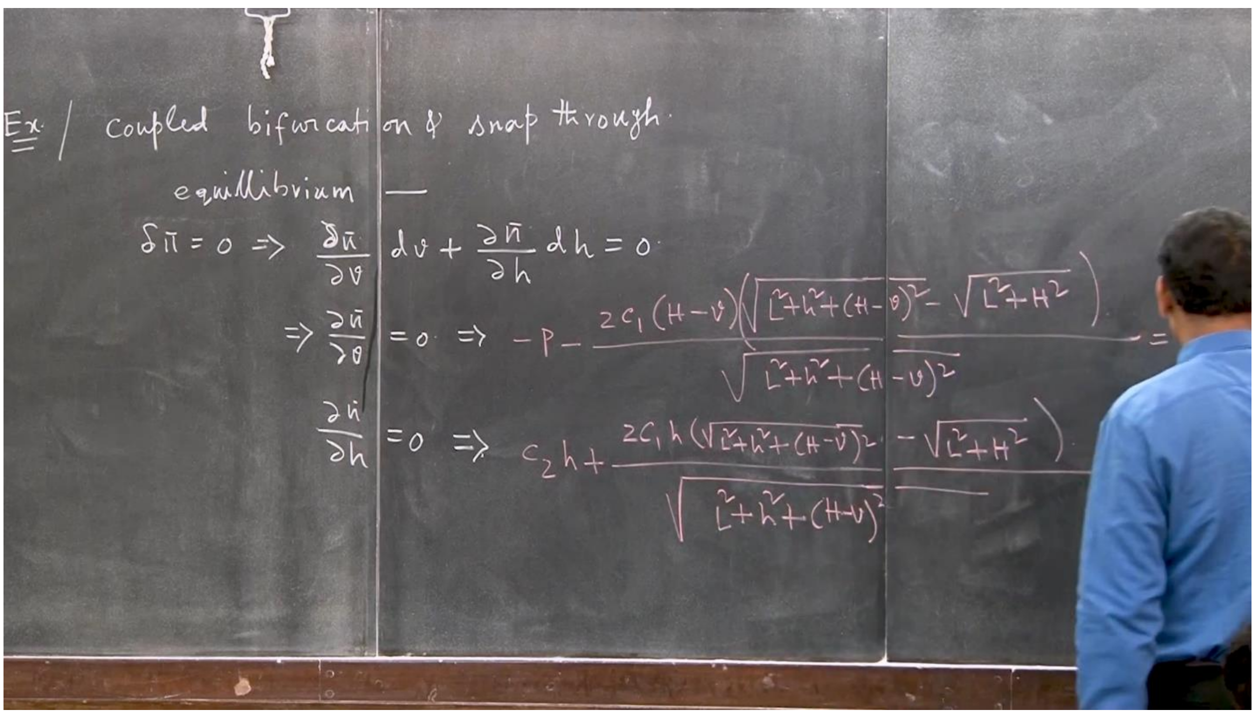
Why $l^2 + h^2$? Because this length was L and this length is H. So, this length was $\sqrt{l^2 + h^2}$, right? The final length of springs a and b is nothing but $\sqrt{l^2 + h^2 + (H - v)^2}$, where v is the vertical deflection and h is the horizontal deflection, right? You can please note that. The vertical deflection under P and H is the horizontal displacement, right? Did all of you understand? So, the inclined springs, B and A, had an initial length of this and a final length of this, right? So, you understand the deformation kinetics, right? Understand the deformation kinetics. What about this? This spring is only going through extension. Extension is nothing but the horizontal "h," right? Okay. This fellow was in one plane, but this is in another plane, right? This is away from the plane. Do you understand now? So, this is the von Mises truss. And this point is restrained by another spring. So, it is like this, and it is supported by another spring in the out-of-plane direction. Clear? So, I'm just removing these two. Okay. So now this, once you understand this deformation thing, then other things become very simple. Okay. So now we'll write down the potential energy.



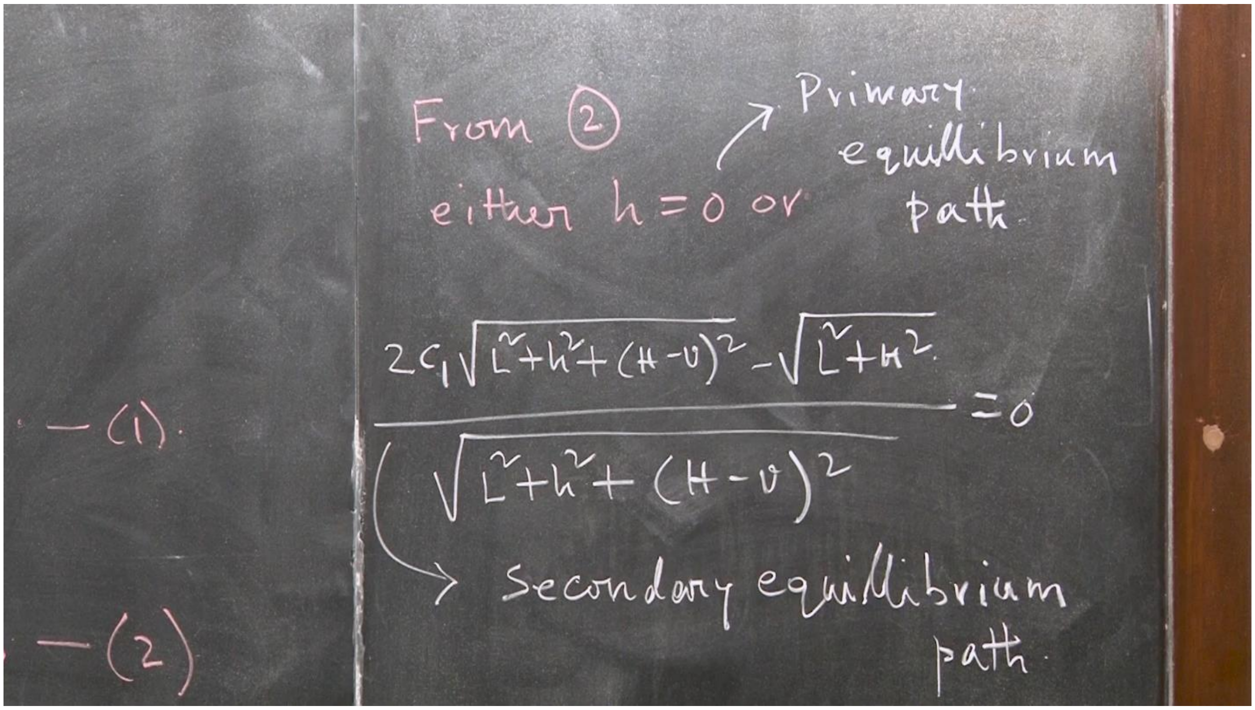
So, δ_a is equal to δ_b, which will be nothing but $\sqrt{(l^2+h^2)+(h-v)^2} - \sqrt{L^2+H^2}$. This is the final deformation. This is the initial length, and this is the final length. Okay. So, this is the extension of springs A and B. Okay. And then the extension of spring C is nothing but the same as the horizontal extension. Only H is moving because the deformation is purely out of plane, which corresponds to H. Thus, point P is displaced by H. So now we are going to write the potential energy function. The potential energy is strain energy minus work done. Strain energy is contributed to by both springs. So, $2 * \frac{1}{2} c_1 * \delta_A^2 + \frac{1}{2} c_2 \delta_c^2 - Pv$, where c_2 is the spring stiffness and δ is basically c. What is the work done by p, the vertical load? So, the only component it will work on is due to vertical. So, it will be minus p times v. So, strain energy minus work done. Strain energy in springs A and B; that's why I multiplied by two, because both springs are working, right? So now I'm just substituting the expressions for δ_A, δ_B, and δ_C over here. So C_1 will be nothing but $\sqrt{(l^2 + h^2) + (h - v)^2} - \sqrt{L^2 + H^2}$, right? This one. Huh? Now, may I remove this plot? This is a Bergman truss. It's a combination of a von Mises truss and a spring. The spring is restraining the top point, the apex of the von Mises truss. Okay, so once we find this out for equilibrium, right? For the equilibrium configuration, we have to set $\delta\pi$ equal to zero because this is a two-degree-of-freedom system with horizontal translation and vertical deflection. Right? So, $\delta\pi/\delta v$ into dv plus $\partial\pi/\,\partial h$ into dh must be equal to zero. Now dv and dh cannot be zero. So, from here, you see that $\partial\pi/\,\partial v$ is zero, of course, and then $\partial\pi/\,\partial h$ is zero, right? So, this will give you two equations. Let us write down these two equations. The partial derivative of π with respect to v will give you

$\frac{-p-2c_1(h-v)\sqrt{(l^2+h^2)+(h-v)^2}-\sqrt{l^2+H^2}}{\sqrt{(l^2+h^2)+(h-v)^2}}$ = 0. And for the other one, basically, you will get

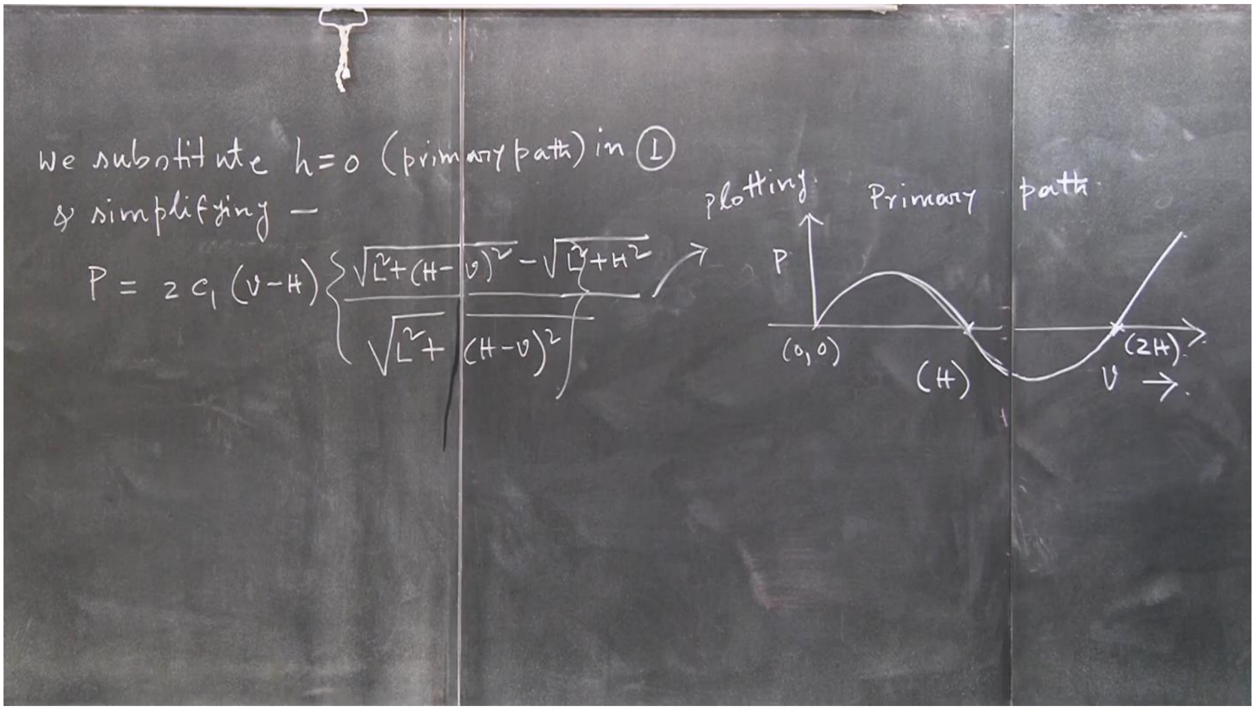
$C_2h + \frac{2C_1h\sqrt{(l^2+h^2)+(h-v)^2}-\sqrt{l^2+H^2}}{\sqrt{(l^2+h^2)+(h-v)^2}}$ = 0. this is (2)



So now we'll see. You can take H outside. I am writing Equation One and I am writing Equation Two, okay? We have two equations. Now let us first write Equation One and Equation Two. These two will give us two equations. But we have to simplify this. There are two h's in both expressions. Either h is zero, or the rest of the expression is zero, right? Or the other three are zero. Okay, the other one is nothing but $\frac{2C_1\sqrt{L^2+H^2+(H-v)^2}-\sqrt{L^2+H^2}}{\sqrt{L^2+H^2+(H-v)^2}}$ = 0. Look, when H = Z, that means it cannot happen. So that is nothing but the fundamental path, right? H = 0 means what? This was non-trivial. That is the fundamental path, right? It does not even mean there is no deflection configuration, right? So, this one is the primary path, which is equal to the primary equilibrium path, right? The primary equilibrium path is given by a = 0, and the other one gives us the secondary equilibrium path.



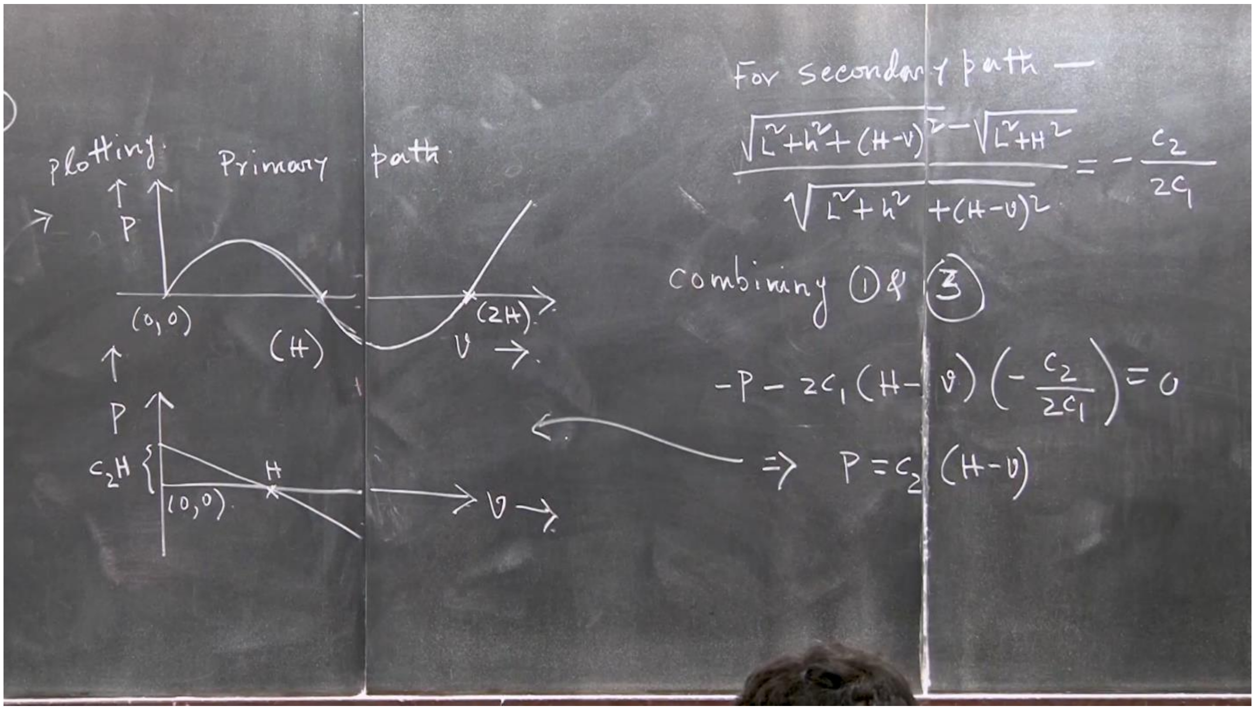
Now please note that there is equation one. Okay, we are yet to do something with equation one. Okay, we'll do it; no problem. So, what are we going to do now? From equation two, we are obtaining the primary equilibrium path for h equal to 0 or the secondary equilibrium path. The other expression is zero, and we must not forget the first equation. So, now what we are going to do is substitute h = 0. That means the primary path h is equal to 0, which means the primary path is one. That is, we are going to substitute h = 0 in this equation. Then we will simplify, and we get $P = 2C_1(V-H)\frac{\sqrt{L^2+(H-V)^2}-\sqrt{L^2+H^2}}{\sqrt{L^2+(H-V)^2}}$. So, P is equal to H. In this expression, you are going to get P right. So, H is equal to zero. You are substituting then in terms of P. Okay, so this one is what?



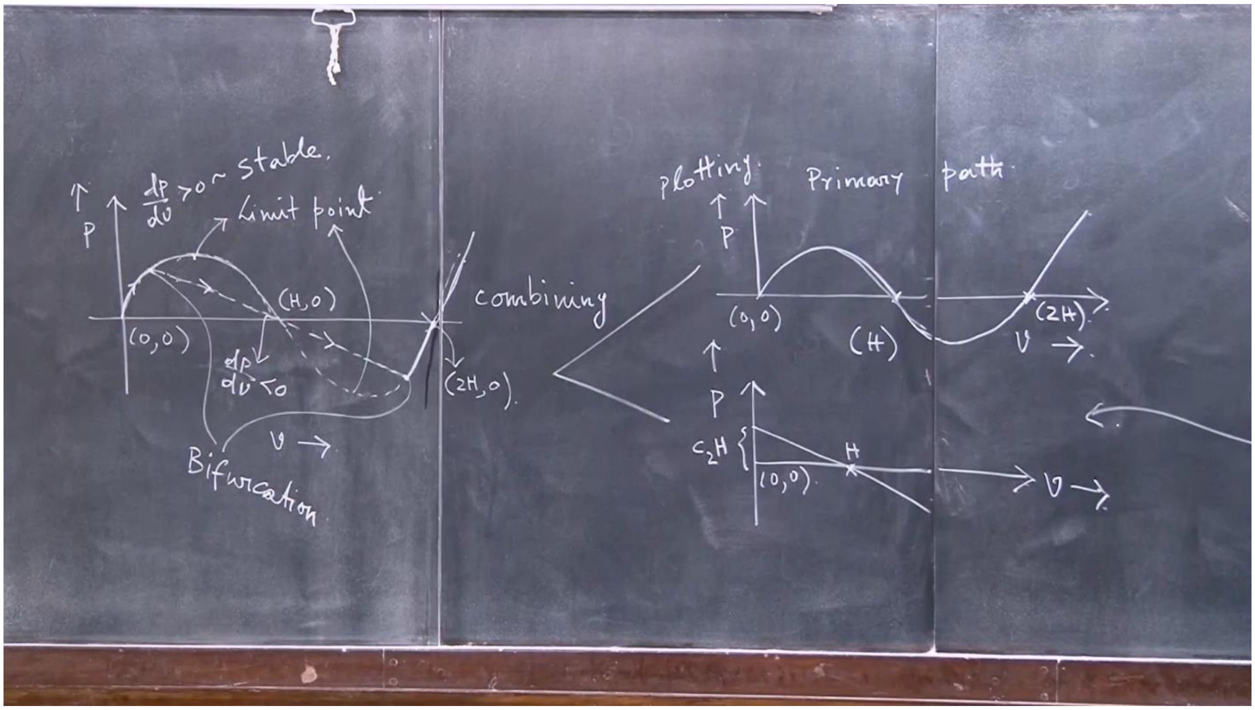
It is the primary equilibrium path, right? So, the primary path that I have written is correct. Let us plot it. If you plot it, it will look like this: P. You can clearly see that for 0, it is equal to zero. So here, this is basically V, okay? So, if we set V equal to 0, then it will be zero, and there will be this situation, something like this. And then this point is nothing but h, and this one is nothing but 2h. So, in three places, it will be zero. Please check whether it is correct. Okay. Oh, h is equal to v. If v is equal to h, then this will cancel out things. Okay, so you see that we get something like this. This is the primary equilibrium path. Now, for the secondary equilibrium path, we have to simplify it a little bit. So, if h not is equal to zero, here, if you see, then I think I just missed C_2 plus there. Please, I inadvertently erased that, okay? So, the secondary equilibrium path from here, I'm simplifying it. Okay, please note that it is equal to $-C_2$. It involves a little manipulation, but do not worry; I will get equation three. I will soon complete

$\frac{\sqrt{L^2+(H-V)^2}-\sqrt{L^2+H^2}}{\sqrt{L^2+(H-V)^2}} = -\frac{C_2}{2C_1}$, this was simplified X. Now we'll combine equation (3) with (1) Okay, this expression. So then if you combine this, what will happen? Of course, this expression will simplify. So, combining one and three means both the secondary and primary parts are substituting equation three into this. Okay. So, then you will see what you are going to get. It will be $-P - 2C_1(H-V) - \frac{C_2}{2C_1} = 0$ Okay, please check it, but I hope that we're doing it correctly. So, from here you will get

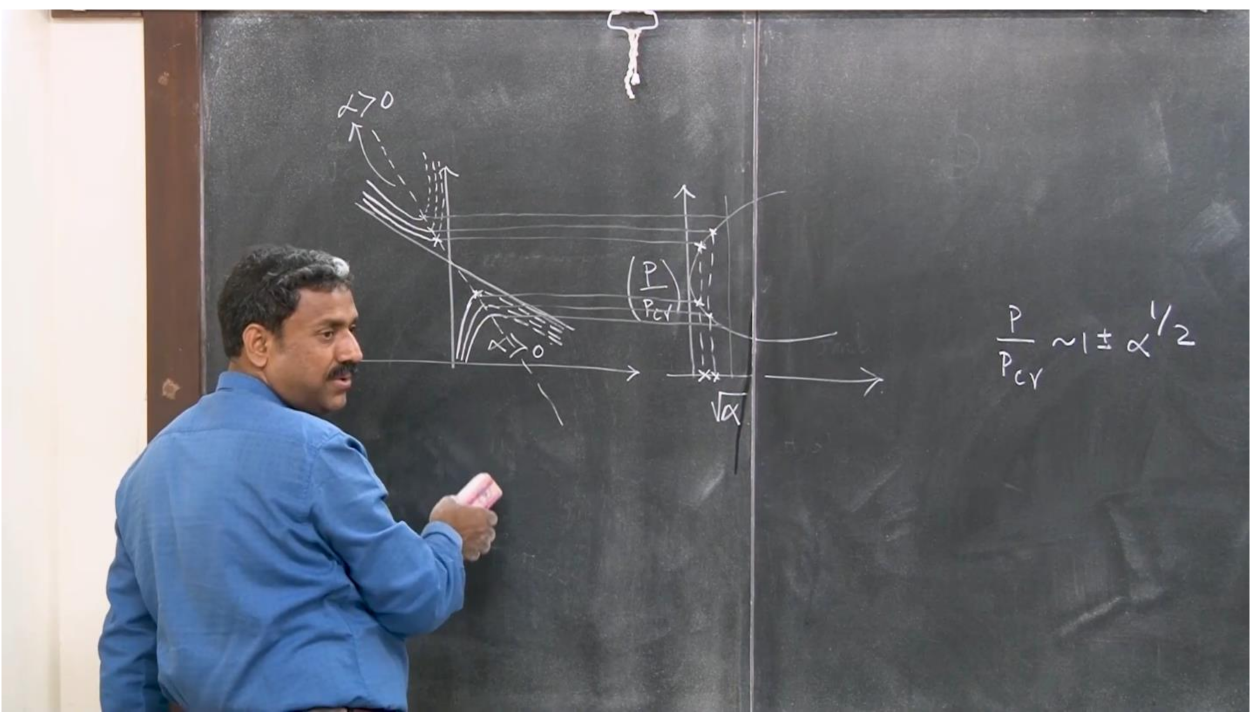
$P = C_2(H - V).$



Do you see that? Now we are going to plot this one. So how does it look when you plot it? P is equal to this, and here it is P, and then here it is V. This is like this. It will go to C_2H, and here it is basically H. This is (0, 0). Now it's a combined system. Now I will add these two. Okay, so you have noted this, right? So now I will combine, and I'm removing this part, and I'm going to combine these two. So, if I combine one, because it is the two paths we are obtaining from two solutions, right? so this system will have combined behavior. if you combine how, it will look like? it will look like this, well you know I'll just draw this Okay. from here C_2 into H. from here I'm going to do this one. Huh. So, this is stable you know stable. Look at what is going to happen. If the system I'm loading is by displacement, I'm going like this. Okay. From here, you see that this path is bifurcating into two. One is following and reaching a limit point. Because this is the maximum root and the point where it is attaining the maximum root, it is stable until this point; then it becomes unstable. Why is it stable on this positive slope? Because here, the derivative of P with respect to V is greater than zero, indicating stability. See, I didn't explicitly obtain the stability because I didn't want to unnecessarily complicate the equation, right? But you can understand that this is a stable path because it has a positive slope.

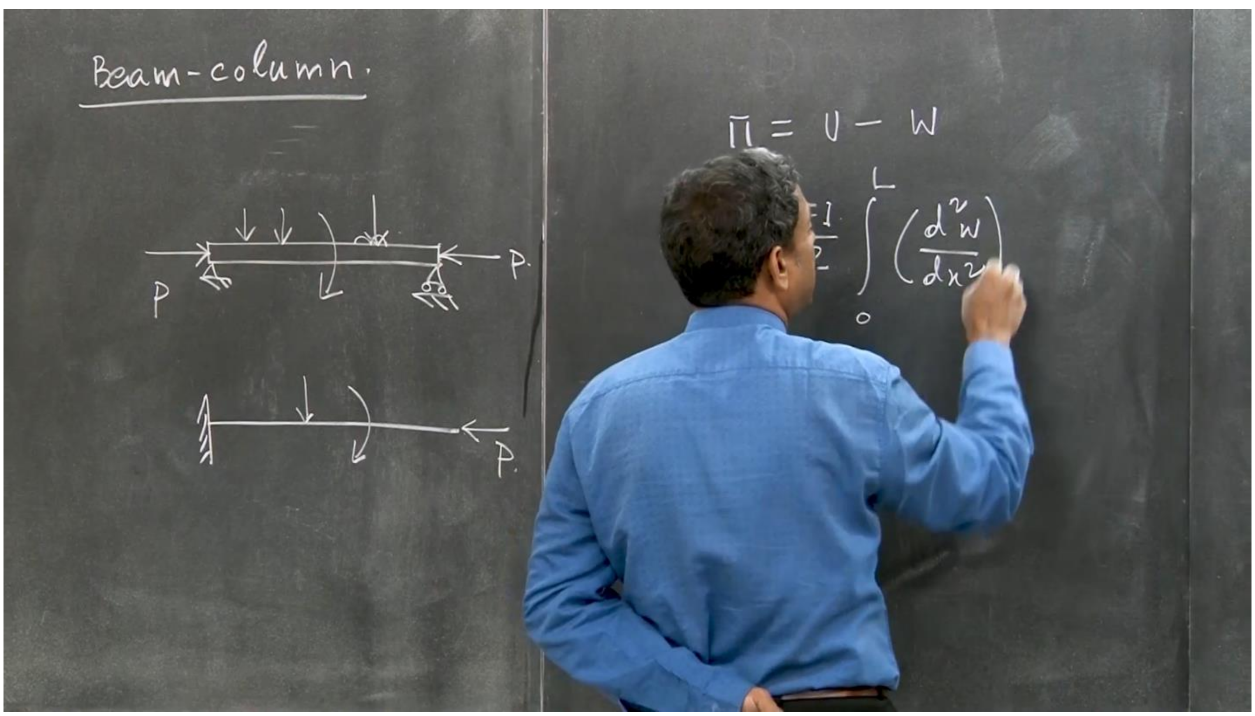


However, when it reaches here, it will either follow this path or that path. So, if it follows this path, it will go. If you make this displacement continuously, it will jump into this by snapping through. But if you do load control, it will follow this path. Clear? Similarly, during unloading as well. Now, if you go this way, and then instead of this, if you follow this path, what will happen? From here, if you go by load control, it will suddenly go into this position, right? You can see that this implies a bifurcation. Even in a bifurcation, it won't suddenly jump from here to here. The transition from this point to that one won't be a snap-through or similar behavior because it's an adjacent path. But it is an unstable path because the slope is negative, right? This path is unstable. Why? Because $\frac{dP}{dV} < 0$ for this slope, right? But if you try to follow this path, at least there will be no snap-through. Snap-through will only occur if you reach the limit point and go beyond it. Do you understand? So, depending on your choice after bifurcation, which path should be followed, right? Of course, it will not follow that path because one will be energetically stable. This is a stable path, and this is an unstable path, right? So, unless it is constrained, it will not follow a particular path. Okay. Whether this is stable or unstable, okay? So, if you reach a limit point, it will suddenly jump and snap through a lock. If you go here, then from this point, it will go there. Okay. So, it shows coupled bifurcation as well as snap-through. Right? Somehow, you have to restrain the system to achieve this so that it doesn't follow this path. Okay, otherwise it will come there, and you know that this path will gradually follow these things. Okay, but it's an unstable path. Note that, okay? So, what we have learned is a system that shows coupled bifurcation as well as snap-through behavior. And what is snap-through? Which one is bifurcation? So, it is not that whatever four fundamental systems we have discussed; there can be a mechanical system that will show coupled behavior. For this, it is a simple system of the Bergman truss. If you consider in-shell buckling, what happens is that you have bifurcation, but a snap-through also occurs. This is due to the coupling between the snap-through and bifurcation behaviors, which are characteristic of shell buckling. When the cylindrical cell is under axial force and when a spherical cell is under uniform pressure, buckling occurs. So we are going to start. One more thing I would like to point out here is that you may recall when we were talking about asymmetric bifurcation. You may recall that part was something like this, right? Okay, then this was the asymmetric bifurcation, and if you recall how the path was, this was the perfect system and these were the imperfect systems, right? Something like this, isn't it? Right? So, this was $\alpha > 0$. And here it was something like this, right? And these are the points. So, this was the stability boundary if you can recall, huh? Right?

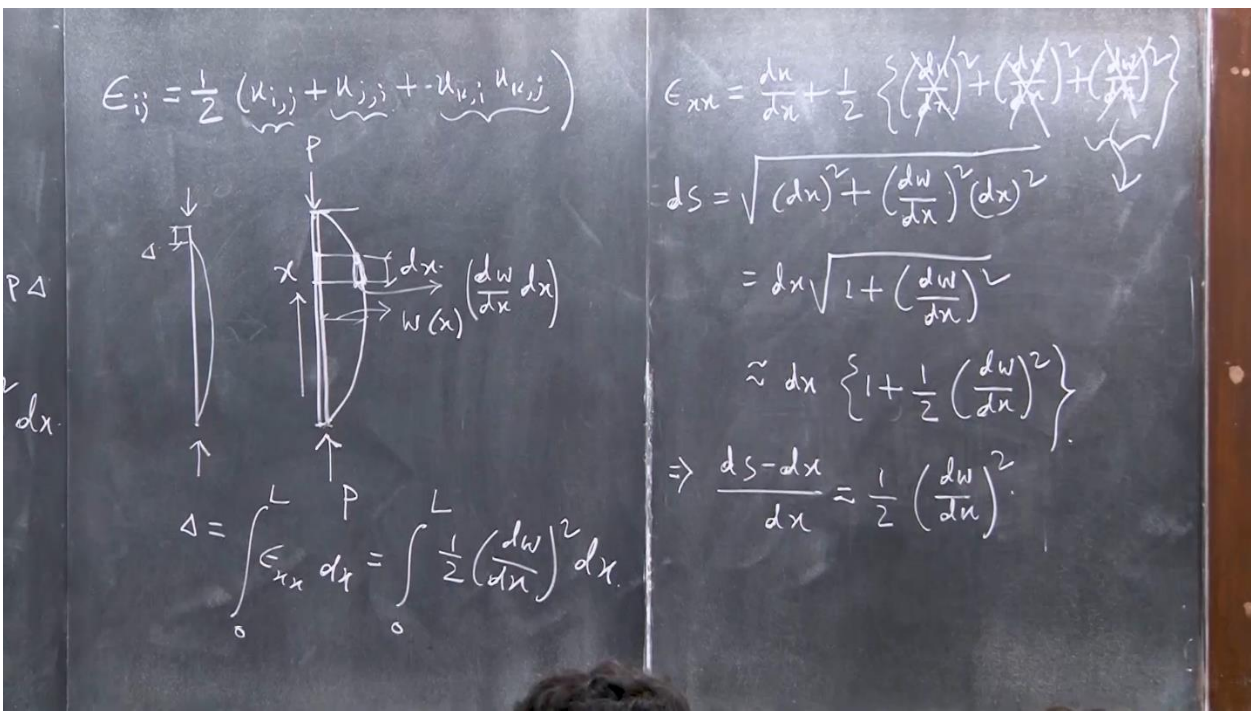


Here, this quadrant was also $\alpha > 0$, right? So here I just saw the imperfection sensitivity for this right. So, $\frac{P}{P_{cr}} \sim 1 \pm \alpha^{1/2}$ was, right? Okay. So $\alpha^{1/2}$ means what? If you take the square root, then it will be plus or minus, okay? And then if you can recall, the way I drew it was something like this, right? So, I was showing you that this part is an unstable part. These are unstable, and these are stable, right? Because it is rising, it is a decreasing path, right? Ascending and descending, this was ascending, right? So, for any α, this is the imperfection sensitivity diagram. So here I was just talking about P/P_{cr}, and this was basically the root α, right? It was something like this. You see these two roots? Notice how they correspond: if you drop here, and then similarly here, you drop again. So, what happens? For any α, there will be two roots—this is one root, and this is another root. So, this root corresponds to this; another root corresponds to this. Okay, one is stable, and the other is unstable. Clear? So, for this, it is an unstable bifurcation. That's why it is sensitive to imperfections. The load is decreasing. For the other one, the load is increasing. Do you understand that? Similarly, for this one, you know it is plus or minus α. In this branch, the load is decreasing because this is the limit point. From this limit point, the path is diminishing, and it is going down and becoming an imperfection-sensitive load. But if you go to the top, this load will increase because it is imperfection insensitive. So that's why it is an upper limb. Similarly, here. What I just wanted to show you is this correspondence, okay?

The correspondence between the stability diagram and the imperfection sensitivity diagram in the case of asymmetric bifurcation with stable-unstable behavior, okay? Now, I think we have covered the basic toy systems. To summarize, we have learned how to formalize the stability problem using an energy-based formalism. Okay, the minimization of the energy gives us the equilibrium configuration that we learned about in the previous courses. But then, in order to find out the nature of the equilibrium—whether it is stable, unstable, or neutral—you have to explore the higher-order derivatives. The Hessian of the potential energy functional depends on whether it is a single-degree-of-freedom or a multi-degree-of-freedom system; it will give you the multi-degree-of-freedom system and an eigenvalue problem. So, you can linearize all the problems. Okay, sometimes linearization will be meaningful; sometimes it will not be meaningful. For example, if there is a significant pre-buckling nonlinearity and there is a huge snap kind of behavior, then the linearization will be meaningless, right? However, if you can ascertain the stability behavior by looking at the functional expression of the potential energy function, it shows that the system undergoes symmetric bifurcation, with stable or unstable post-critical behavior marked by the quadratic term, as well as the quadratic terms of sixth and eighth orders. A system with asymmetric bifurcation and stable or unstable post-critical behavior is marked by the presence of odd-order terms in its potential energy expression, such as first, third, fifth, or seventh-order terms. In contrast to symmetric systems, these asymmetric systems are generally not imperfection sensitive. Other systems do not exhibit bifurcation but instead fail via snap-through. In these cases, the stable-to-unstable transition is a dynamic event. While we can predict the critical point using static analysis, the actual snap-through behavior involves a rapid release of kinetic energy, which is ultimately dissipated as heat and other forms of energy. We have also demonstrated systems that exhibit coupled behavior, combining both snap-through and bifurcation. Furthermore, we have shown the implications of modal interaction: when modes couple, an otherwise imperfection-insensitive mode can become highly sensitive to imperfections. So, our discussion is about the first two chapters. Okay. Now we will start to look into a little real example. First, we'll start with the beam-column. What actually happens and how the governing equations we learned in undergraduate or sub-level classes work, and how the equation for beam bending functions. We'll see that if there is a column and the beam is under compression, then it is called a beam-column, right? We will now see how the governing equations differ for these systems. We'll begin with a very simple system. While there are different analytical approaches, we will start with the most mathematically elegant one: the energy method.



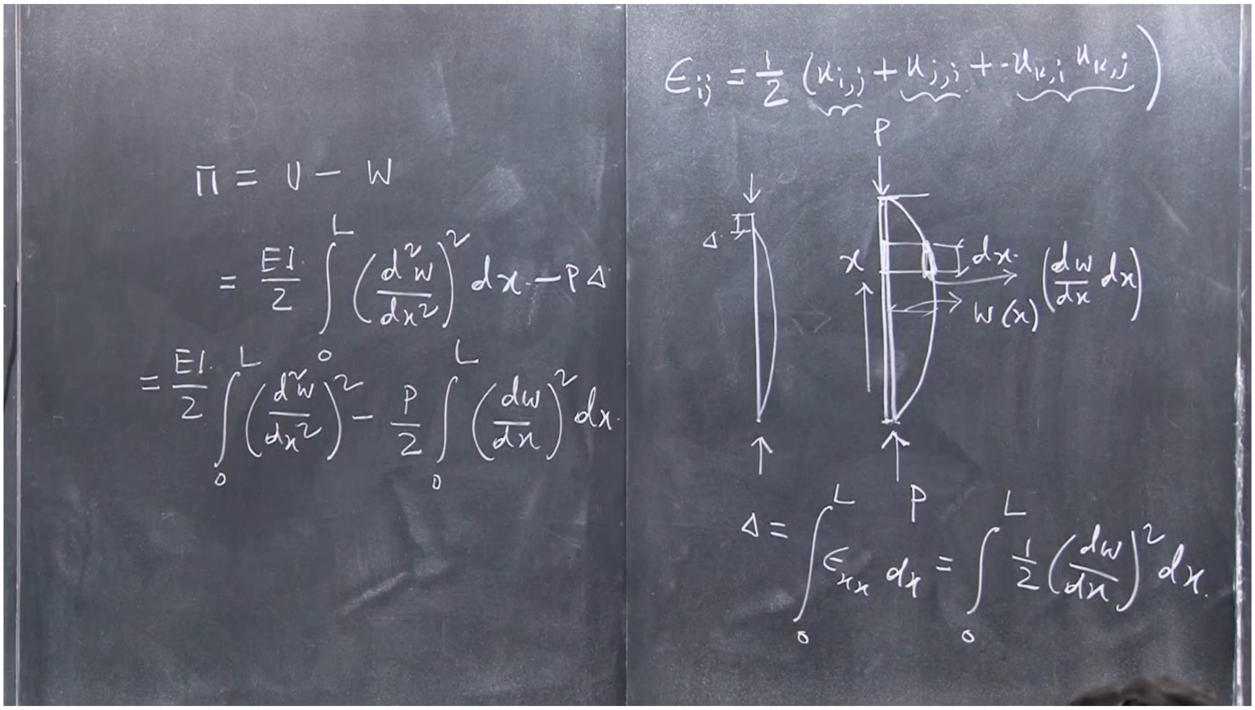
So, why do we use an energy approach? Because it is very intuitive and easy to work with energy, you have quite a bit of discussion on energy functionals in your finite element course. So now, when we are studying beam columns, what is a beam column? A beam is subjected to a transverse load. A beam with transverse loads, such as a moment or a concentrated load, that is also subjected to an axial force P is called a beam-column. It can have various support conditions; it can be simply supported, or it can be a cantilever. Similarly, a cantilever can be subjected to a concentrated load, a moment, and axial compression. So, compression is the one that causes instability, right? The most common instability is buckling, right? So, we are going to derive the equation. Okay, now the potential energy function P. Once again, I am going to consider a beam-column. So, it will be potential energy minus the work done. So, for a beam-column, I am not considering the loading on the beam because the destabilizing force is only P. In our first lesson on stability analysis, we must consider equilibrium in the deformed configuration or the perturbed configuration. So, I am assuming a longitudinal x-axis and a transverse deflection $w(x)$, representing the out-of-plane deformation. You know I'm just changing the axis. Okay, this is a beam-column. Now, what is the strain energy and how does it resist load when it is going through out-of-plane bending? Bending strain energy is expressed in terms of W. We can express it in terms of the bending moment. So, you have learned that $\frac{EI}{2}\int_0^L (d^2w\,/dx^2)^2 dx$, right? How is this coming?



Because the strain energy of bending is nothing but what? Moment multiplied by curvature, right? So, the moment is multiplied by the curvature, and curvature is nothing but d^2w/dx^2. Neglect higher-order curvature, right? So, then the moment is d^2w/dx^2. So half is coming, you know. Why is half coming? For strain energy. Because half integration $\sigma^T \epsilon$, right? Now what is work done?

I'm assuming that, well, it is subjected to some axial force P, right? Now I'm considering a simple case here, of length d. This fellow, when it is deforming, this is dx, and this one is nothing but what? dW/dx into dx. So, this chord length of this square, what? The length of this square is nothing but $\sqrt{dx^2 + (\frac{dw}{dx})^2 (dx)^2}$. You understand what I'm doing, right? So, I have the beam column, and then we are deforming, right? It's so deflected. So, this is this chord. What is the length of this chord? The length of this chord is nothing but this square. I am approximating the chord length of the deformed element, assuming dx is infinitely small. The deformed length is $\sqrt{dx^2 + (\frac{dw}{dx})^2 (dx)^2}$. What is (dw/dx) multiplied by dx, correct? So, this is $1 + (dw/dx)^2$, and then I am writing $dx\sqrt{1 + \frac{1}{2}(dw/dx)^2}$, which is nothing but strain, right? So $\frac{ds-dx}{dx}$ equals what? It is nothing but $\frac{1}{2}(dw/dx)^2$. Which is nothing but strain. Out-of-plane deflection is causing axial strain. And this strain is nothing but what? This is nothing but von Karman nonlinear strain. Because if you can recall from your axial rotation,

$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$. So, this will not be there. Okay. But then this square is going to be there. So, when I'm simplifying ϵ_{xx}, ϵ_{xx} is what? It is nothing but $du/dx + \frac{1}{2}[(du/dx)^2 + (dv/dx)^2 + (dw/dx)^2]$. Now for a beam, the in-plane direction U is very, very small. So, its gradient is r. So, sorry, there is no V, and then only this important term will be there. So, this strain is called von Karman nonlinear strain, and this is valid for moderate rotation. Okay, many people call it large deformation strain. No, this is wrong. Not large deformation; it is moderate deformation. Okay. So, this is nothing; this is another simple way to derive von Karman nonlinear strain. Now, this strain is how this is under this loading. This fellow is what? This fellow will come, and this fellow is deforming. So, there is some deflection. This deflection is nothing but what? The integration of axial force $\int_0^L \epsilon_{xx}$ dx = $\int_0^L \frac{1}{2}(dw/dx)^2$dx. This is Δ. So, $\frac{EI}{2}\int_0^{:L}(\frac{d^2w}{dx^2})^2 dx - P\Delta$. That means the strain energy integration for the beam-column is nothing but $\frac{EI}{2}(\frac{d^2w}{dx^2})^2 - \frac{p}{2}\int_0^L(\frac{dw}{dx})^2 dx$. Do you see how that is coming out right?



For the beam-column, the strain energy, other than bending, will always have a component that is obtained by integrating the work done by the axial deformation using the von Karman nonlinear strain. So, it is basically axial strain due to plane deformation. We know that if there is an extension of this bar, then what is strain? Strain is nothing but change in length divided by initial length. But here, the change in length is along the x direction that you have to consider; deflection is along the W direction. So, in order to capture that, you have to consider higher-order nonlinear terms. That's why strain becomes nonlinear, right? Because du/dx, there is no plain deformation. But still, there is plain strain, and why is it happening? Because plane strain is occurring due to out-of-plane deformation, and that's described by von Kármán nonlinearity, clear? So, this is a functional expression for the strain energy. Now we'll play with it. So, in the next class, we'll do this: we'll derive the governing equation for the beam column. Okay, thank you very much.