Stability of Structure

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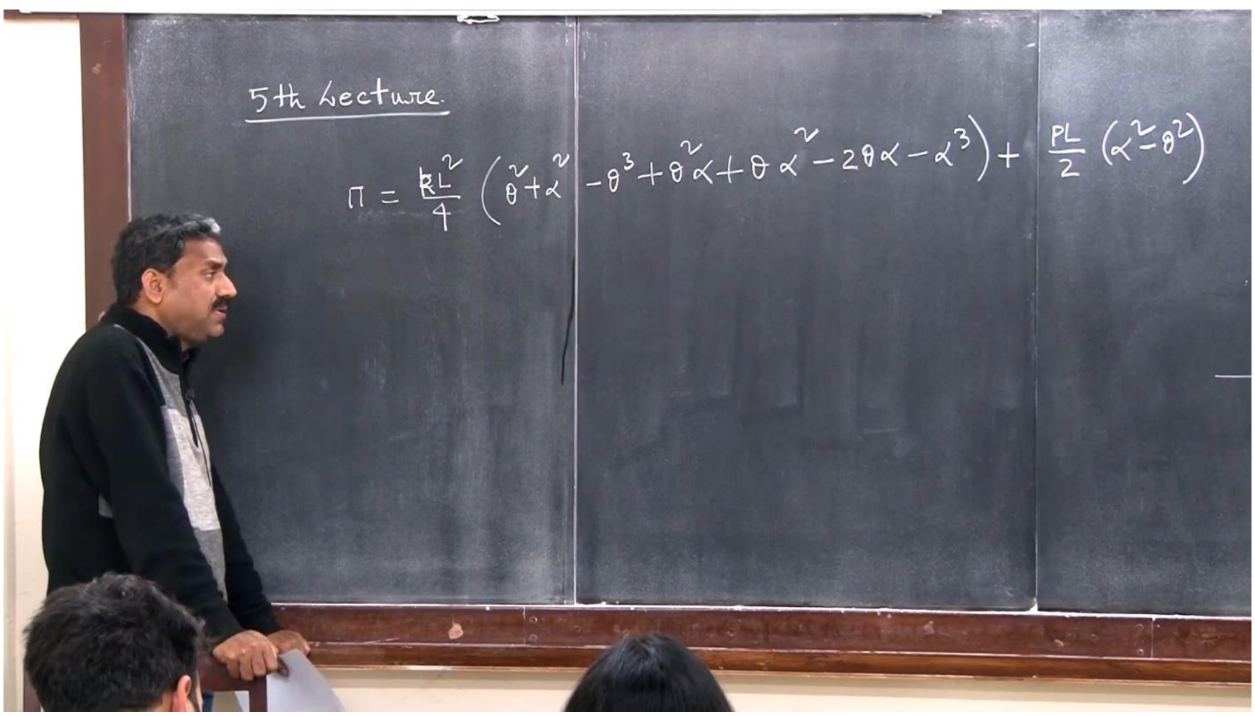
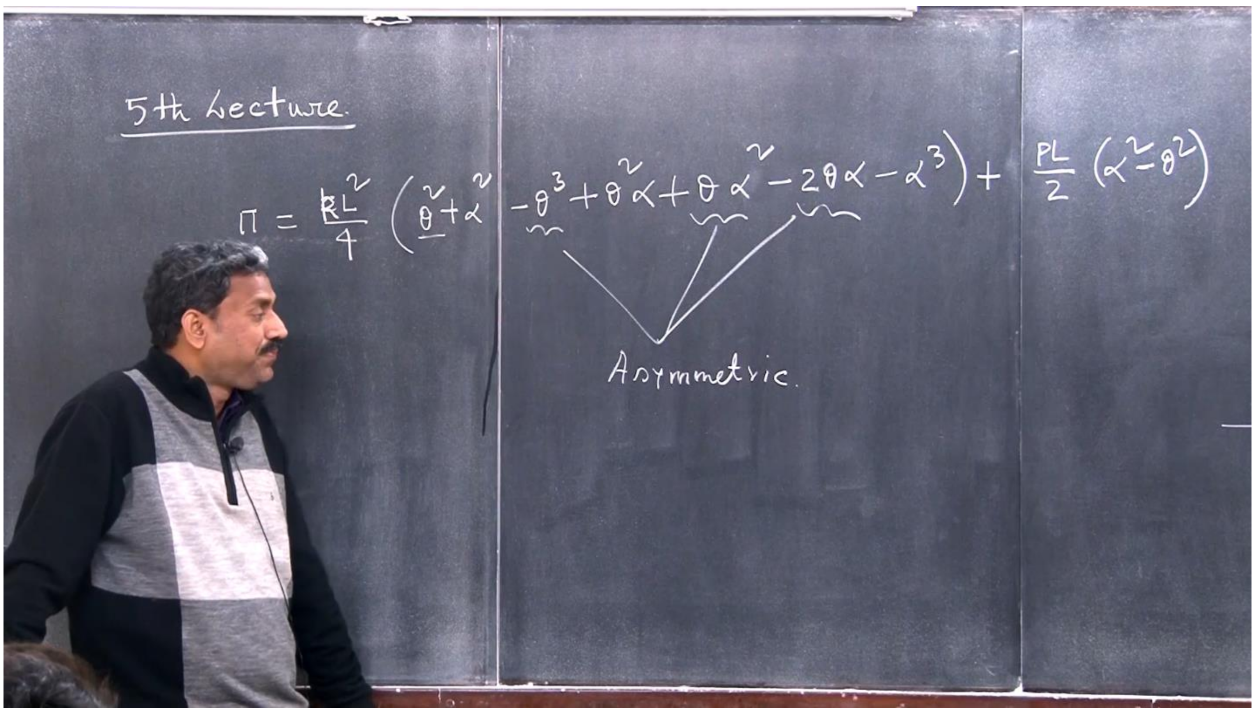
WEEK-03

LECTURE 5: Stability Behavior of the Von Mises Truss

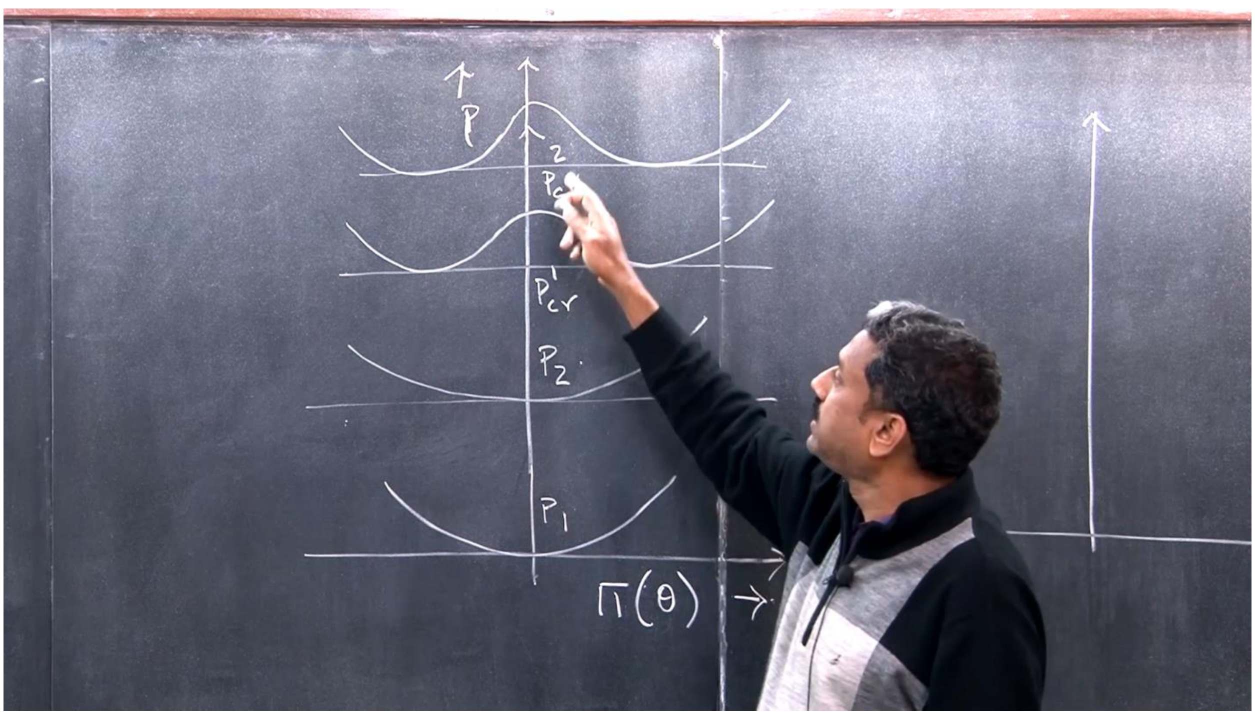
Okay, welcome to the fifth lecture on the stability of structures. So let us briefly recapitulate what we were discussing. So, you're discussing asymmetric bifurcation with stable or unstable postcritical behavior, and we have demonstrated. So, this behavior that is portrayed by a system in which a rigid bar is restrained by inclined spring. Okay. So basically, the restoring force varies with positive theta and negative theta differently, because the lever arm, you know, when you are taking the restoring moment. So, the magnitude of this will depend on the value of θ. For positive θ (that is, clockwise rotation), it will be less, and for anticlockwise θ, it will be more. So, depending on the value of θ, the system will exhibit either stable or unstable behavior.



So, a stable fundamental path will bifurcate into an asymmetric post-critical path. The right path, which is the descending path, will be unstable, while the left path will be stable. So, in the bifurcation stability diagram we demonstrated, the equilibrium path of the imperfect system asymptotically approaches that of the perfect system upon incorporating imperfections. The system shows notorious imperfect sensitivity, as we have demonstrated. So, it follows the half power law regarding Koiter's imperfection sensitivity. That means if there is a slight imperfection, there is a significant drop in the critical load. And then, if we look into the potential energy function, you know these are the expressions that I have written. So, theta is basically, you know, the degrees of freedom, right? And then alpha is the imperfection. So, what we assume is that this is a symmetric term, this is an asymmetric term, right? This is an asymmetric term, and this is also an asymmetric term. So, this asymmetric term, right?

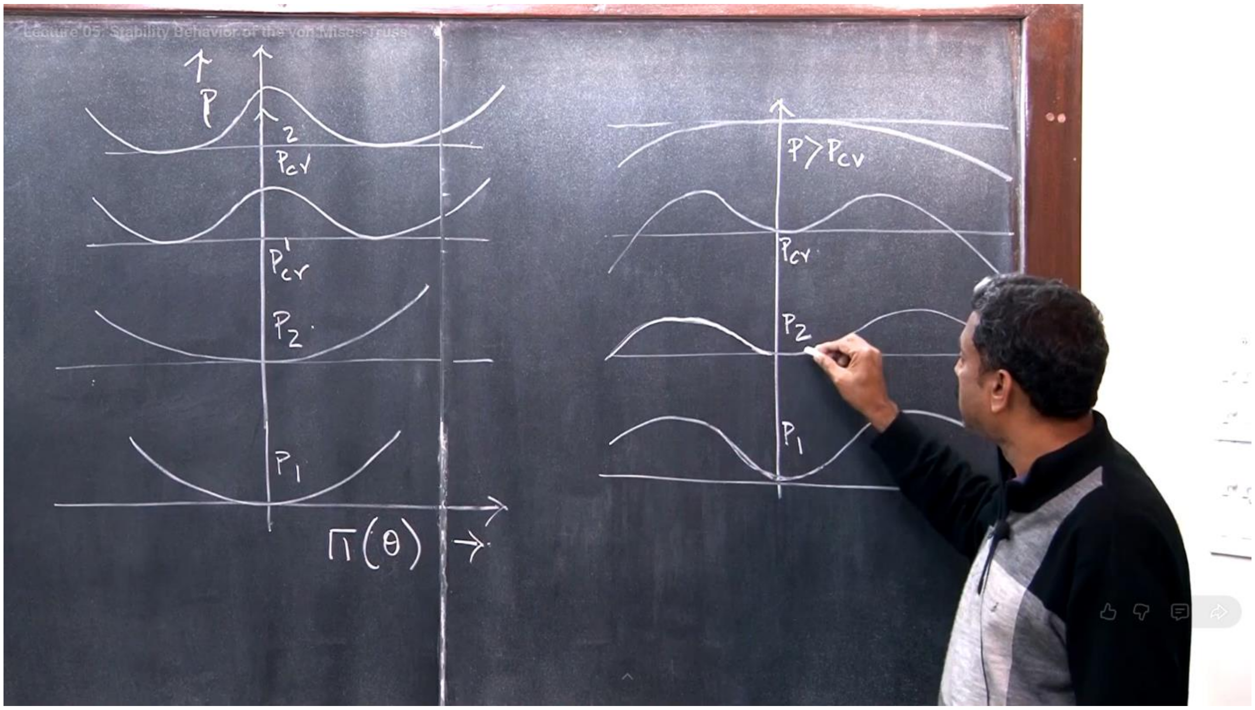


This basically demonstrates the dependency of the system's behavior on the stable and unstable on theta. So, for positive θ, the system will follow an unstable path, whereas for negative θ, it will follow a stable path. This observation also correlates with the slope of the equilibrium path. So, this kind of asymmetric term in the potential energy expression will not appear in a symmetric bifurcation, regardless of whether the post-critical behavior is stable or unstable. In that case, the expression contains only even-order terms, such as quadratic, sixth, or eighth terms. Okay, if you consider higher-order terms, in the symmetric case you get only even-order terms. But here, you can see the presence of a linear (first-order) and a cubic term, both dependent on θ—that's the point I wanted to highlight. So, we have covered all three behaviors. The first one is a symmetric bifurcation with stable post-critical behavior, which is imperfection insensitive. The second one is also a symmetric bifurcation but with unstable post-critical behavior, which is mildly imperfection sensitive. The third one is asymmetric postcritical behavior with both stable and unstable types of postcritical behavior, and it is notoriously imperfection-sensitive. Now, the fourth system we are going to consider. Okay. But before that, for these three systems, we are going to see the topology of the potential energy function. Okay. So, for the first kind of system, I'm just plotting, you know, π here, the potential energy function π, as a function of, you know, θ, right? And this is for different p. So, for the stable bifurcation, it will look like this. I am increasing the value of P. Okay, so this is for P_1; maybe this is for P_2. This is for P_3. You know P_3 means the critical one, rather. Okay. And after that, I will draw it. So, what you see here corresponds to the first type of system, where the bifurcation is symmetric with stable post-critical behavior. You can observe that the potential energy function forms a concave surface, right?

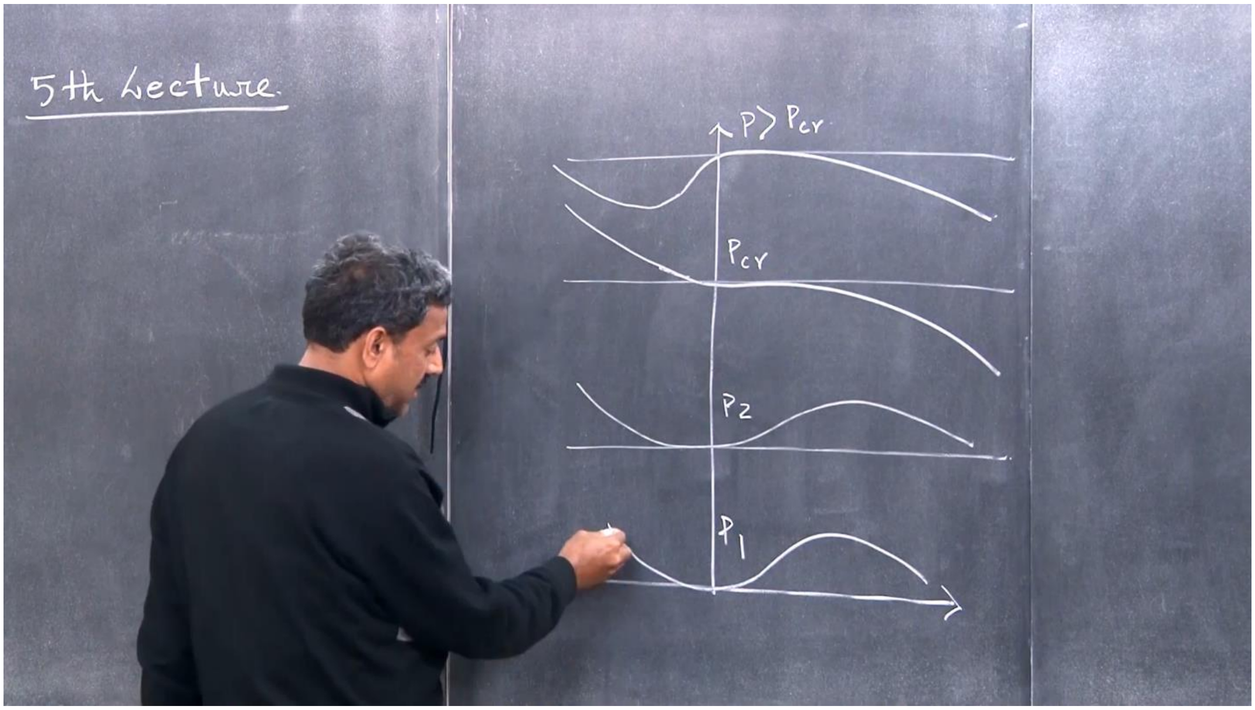


Then, as soon as P_1 and P_2, are less than the critical load P_{cr}, the stable configuration at $\theta = 0$ is not a stable configuration; this corresponds to the stable configurations of the fundamental path. Right, but then you see here it is convex, here it is unstable, right? So, θ is equal to 0; it is no longer stable. Okay, and then with increasing value, the increasing value of P critical, you know P_{cr2} means a slightly higher value. Then this potential will be a little expanded and things like that. Clear? So, this is for the first kind of system. Okay. Then second kind of system it looks like this.

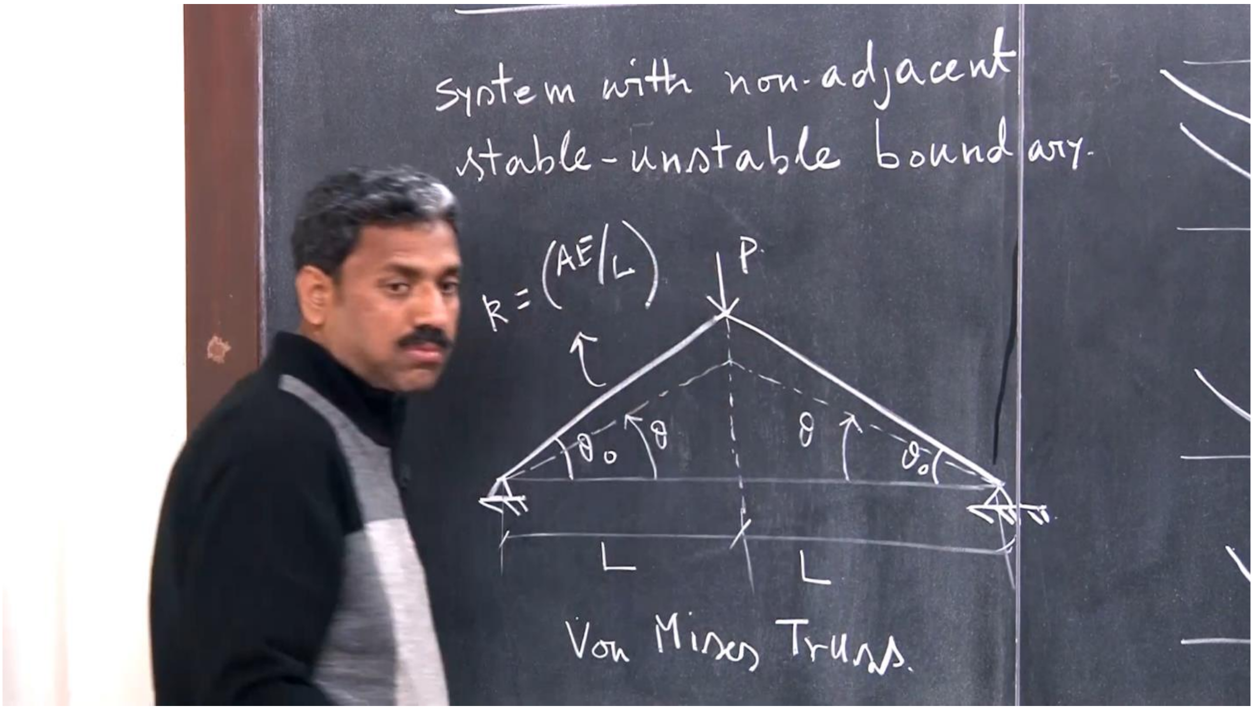
You see that for the symmetry bifurcation, but unstable postcritical behavior; you see that the fundamental path is stable here, right? This concave surface, after that, is unstable, right?



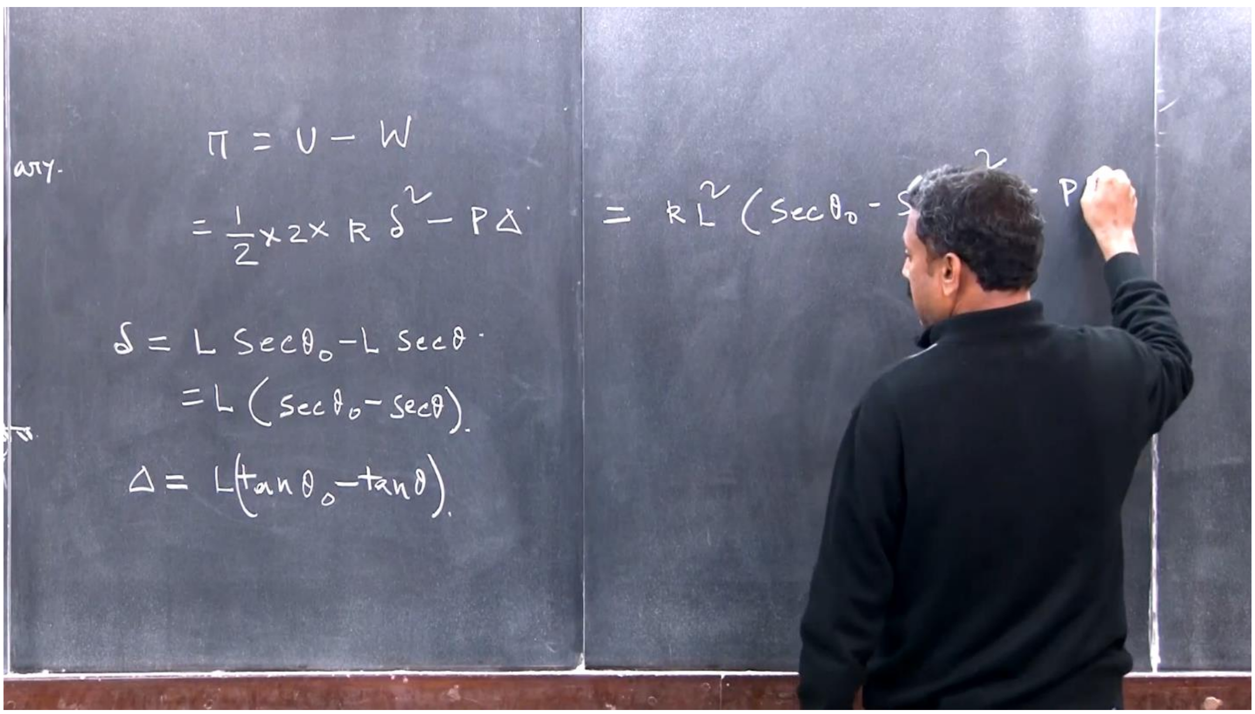
And then slightly higher load. Okay, these are all, you know, smooth. Please note that. Okay, still, you know, slightly higher P_2, but still, this P_2 is less than big critical. It may be that you know potential will expand. Okay, but still, it remains the same. As soon as it reaches a critical value, you see this become unstable. But here, you know it still remains like this. Okay. But as it is much higher than critical, you see what happened. You see that these are unstable. You see that this becomes convex. Okay. That means the postcritical part is unstable. Understand? This is for the symmetric but unstable post-critical behavior. Right. Now for the third thing which we have discussed just now how it will look like. Let me say so that thing to you here. So here, do you notice that in the previous two cases, the potential energy function was symmetric with respect to the fundamental configuration $\theta = 0$? In both cases, the energy surfaces were symmetric. Here it is asymmetric and stable when p is less than p_1, which is less than the critical right. But here, it is unstable. Do you see that? As I mentioned, for θ in the anticlockwise direction, it is stable, but for θ in the clockwise direction, it is unstable.



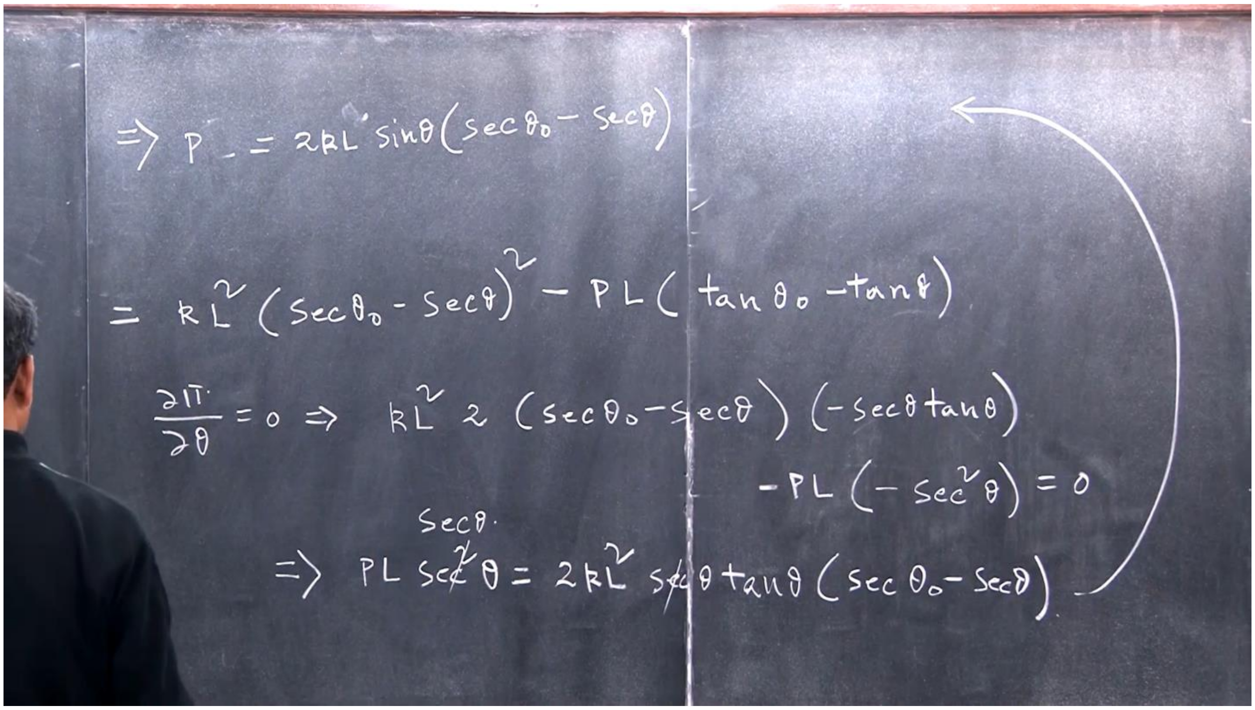
Then slightly above, you know, slightly higher, but still it is this potential well expanded a little bit. But still, it is unstable. As soon as it reaches P critical, it is not unstable. This is also unstable, right? And this is kind of flat. So, okay, then what happens when P is greater than P critical? That means the left-hand side is stable, but the right-hand side is unstable. You see that here it is stable because it is concave, but here it is unstable. On the right-hand side, it is unstable; on the left-hand side, it is stable. Clear? So, this is due to asymmetry, clear. So, do you understand the topology of the potential energy functional, right? And in all these cases, the terms in the potential energy function appear as even powers — squared, fourth, sixth, eighth, and more and more term include — which result from expanding the trigonometric functions in the potential. But in this case, it will include the first term. Okay. This distinguishes between the positive or negative cubic term, the fifth-order term, the seventh-order term, and so on. Whenever these odd terms are present, it will be, as you know, this notorious imperfection sensitivity; in all these cases, it is half power sensitive. In these two cases, it is insensitive; it is two-thirds power sensitive, right? Clear? Okay. So now we are going to consider the case. So, let us focus on the third type of system, which exhibits asymmetric bifurcation with unstable post-critical behavior and strong imperfection sensitivity. This kind of behavior is typically manifested in an asymmetric buckling mode in a shell.



you know both the cylindrical shell as well as spherical shell Under pressure. cylindrical shell under axial compression and spherical shell under uniform pressure, they will show both symmetric as well as asymmetric buckling mode. Okay. And this asymmetric buckling mode will show this kind of behavior. Okay, that's why. So now we are going to consider all the previous examples we have seen, where the transition from stable to unstable was adjacent, meaning the gradual transition you can see. Okay. So, for the perfect system, that was bifurcating from the fundamental part to the secondary path or post-critical path, right? That bifurcation was adjacent to the stable and unstable transition, or adjacent to alternate equilibrium configuration, right? But now we are going to consider a system, and that's the last system. You know we are encompassing essentially all the types of behavior that are portrayed by elastic structural systems. So now we are going to consider a class of systems in which we are trying to demonstrate using a very simple example, namely what is called the von Mises truss. which is simply supported, and then it is basically two members with excel stiffness AE by l, okay. And then subjected to concentrated would be the top, and these are length L, initial inclination was θ_0, and then after perturbation, that is taking a configuration θ, which is defined by θ. Okay, intermediate configuration. So, this is the system in which stable to unstable transitions are not adjusted. Okay, so the equilibrium quantification for stable to another stable equilibrium configuration is far apart. I will show you how this is possible. Now, for this kind of system, the first step in any stability analysis is to consider the system in its initial configuration, with $\theta = 0$. So, theta, what is the potential energy functional $\pi =$ strain energy(w) minus work done(w), right? What is the strain energy? The strain energy for the system $U = \frac{1}{2} \cdot k \cdot \delta^2 - P \cdot \delta$. So, k is the stiffness of an individual, and this stiffness is nothing but AE divided by l. A is the area and E is the modulus of elasticity. You know it will not be L, but you know if this is L, then it will be $\sec\theta$ or whatever, right? But I'm assuming this is K, right? So, two into $\delta^2 - P\Delta$, this capital Δ is nothing but this one, right? And this δ will be nothing but this. So, this δ will be square initially: this $(L\sec\theta_0 - L\sec\theta)$, right? Then $L(\sec\theta_0 - \sec\theta)$, and capital $\Delta = (L\tan\theta_0 - L\tan\theta)$, right? So let us see $kL^2(\sec\theta_0 - \sec\theta)^2 - PL(L\tan\theta_0 - L\tan\theta) = 0$.



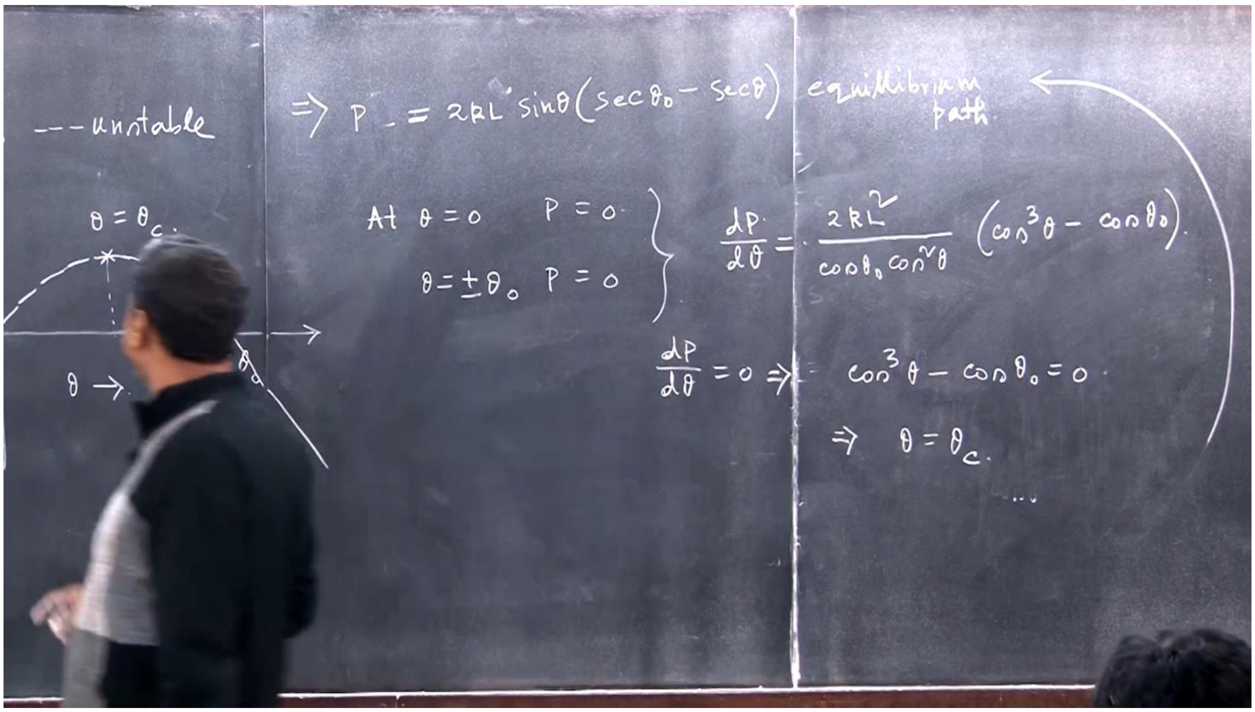
so now we'll see equilibrium configuration $\frac{\partial\pi}{\partial\theta} = 0$ or equilibrium path let us differentiate. so $kL^2(\sec\theta_0 - \sec\theta)2(-\sec\theta\tan\theta) - PL(-\sec^2\theta) = 0$. And then if you see, you can write it like this: $PL\sec^2\theta = 2kL^2(\sec\theta_0 - \sec\theta)\sec\theta\tan\theta$, or from here you know if I further simplify it. Then we will get $PL = 2KL\sin\theta(\sec\theta_0 - \sec\theta)$. So, this is basically giving the equilibrium path. Now I'm removing this part, and I'll come back further. So here it will look something like this: then at $\theta = 0$, $P = 0$, and at $\theta = \pm\theta_0$, $P = 0$, right? Fine. But at some point, if you plot it, it will look something like this. So, you see that $\theta = 0$. This is P, and this is theta. You know this is basically $\theta = -\theta_0$, and this is $\theta = \theta_0$, and this one where this takes value is $\theta = -\theta_c$, and this is $\theta = \theta_c$.



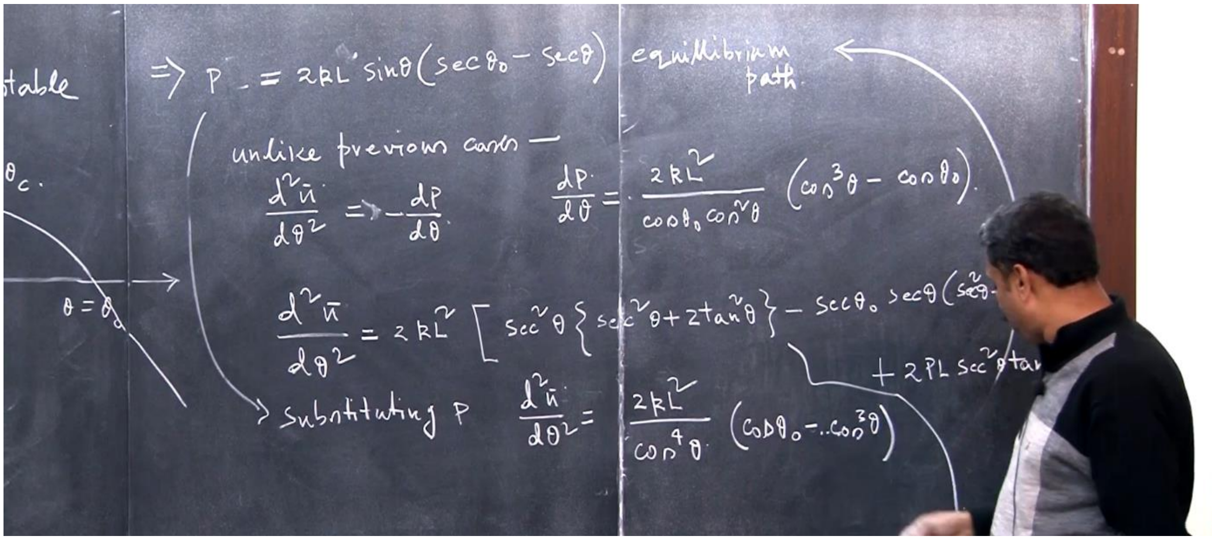
I will just explain to you why it is dotted. You know the solid line represents the stable path; the dotted line represents the unstable path. But this plot is for the equilibrium path. Okay. This is the whole equilibrium path. Okay. Now if you go further, it will still continue. Okay. Something like this, it will continue.

Huh? So now at what value you know it is being maximum. So, for maximum P; $dP/d\theta =$ zero So how is $dpdtheta$ coming? You know, you just differentiate it. So, $\cos\theta(\sec\theta_0 - \sec\theta) + \sin\theta(-\sec^2\theta - \tan\theta)$, okay? So then, simplifying, we will get: this is $\cos\theta\sec\theta_0 - 1 - \sin\theta(1/\cos^2\theta)(\sin\theta/\cos\theta) = 0$. And this one will be $(\sec\theta_0/\sec\theta) - 1 - (\sin^2\theta/\cos^2\theta)(1/\cos\theta) = 0$. Okay.

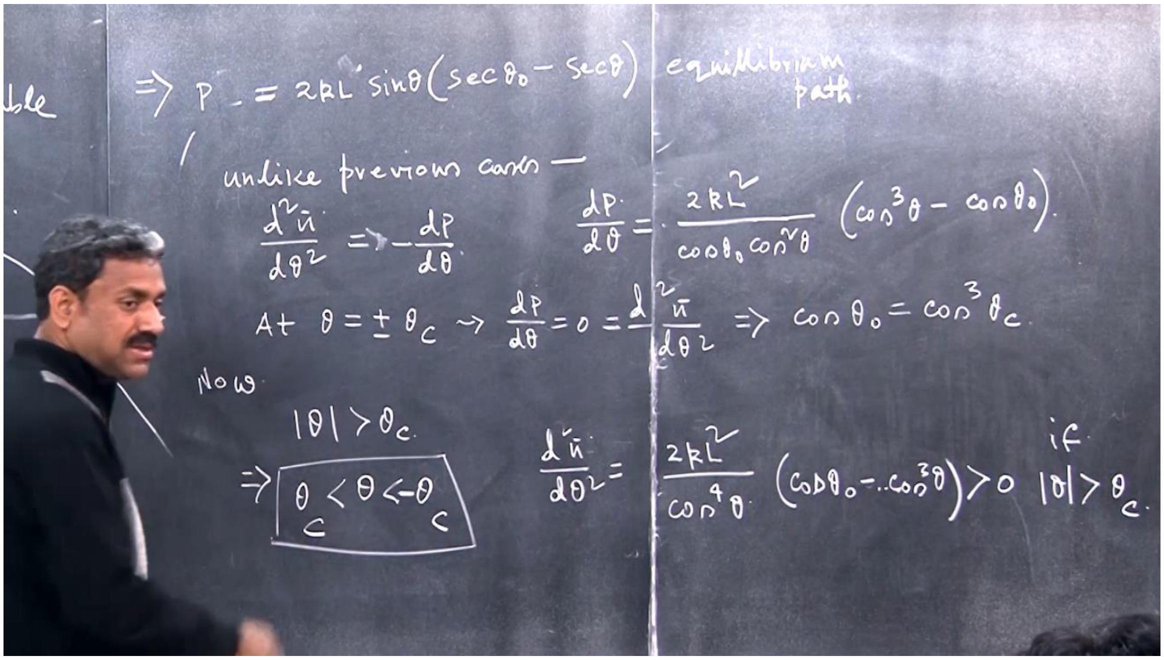
So here, what you are going to get is simplifying it further. If you simplify further, then you will see it's going to be, you know, $(\sec\theta_0/\sec\theta) - 1 - (\tan^2\theta)\cdot(\sec\theta) = 0$, okay? And then $(\sec\theta_0 - \sec\theta - \tan^2\theta \cdot \sec^\theta = 0)$. So ultimately, you will get an expression $\cos\theta \cdot \sec\theta_0 - \sec^2\theta = 0$, something like this you will get, okay. Final expression will come as $dP/d\theta = (2kL^2/\cos^2\theta\cos\theta_0)(\cos^3\theta - \cos\theta_0)$. Okay, this is the expression you will get. So now this will be the final expression. Okay, I was trying to derive it; please check it. Ultimately, you will get this expression. Okay, if you differentiate this. Then $dP/d\theta = 0$, which is for maximum, and $(\cos^3\theta - \cos\theta_0) = 0$. So this equation needs to be solved. And then by solving this, θ is equal to θ_c, where θ is equal to θ_c, $dp/d\theta$ is zero. That means these two points, where P is maximum, are indicated as θ is equal to θ_c, and then there will be two roots, okay. One θ is equal to θ_C, you know, because cos of minus θ equals θ, right? So, at θ equal to θ_C, the maximum value of P is occurring, okay? So, that's where we are going to get that thing by solving this equation $(\cos^3\theta - \cos\theta_0) = 0$, which is known; so, from here, you know θ is nothing but $1/3\cos\theta$.



That's okay. So, θ_c is obtained. Now, you may recall that $dP/d\theta$ is nothing but $d^2\pi/d\theta^2$. The expression will remain the same, except for a change in sign. Okay, that's what I just wanted to mention — please note it down: $dP/d\theta$. And if you try to find $\delta^2\pi/\delta\theta^2$, I'm just writing the expression directly to save time. "So, this expression will come from that. Whatever the expression for π was, it will be: $2kL^2[\sec^2\theta\{\sec^2\theta + 2\tan^2\theta\} - \sec\theta_0 \cdot \sec\theta(\sec^2\theta + \tan^2\theta) + 2PL\sec^2\theta\tan\theta]$. It's a little long expression. Now, what is P for the equilibrium path? What was P?" P was this.

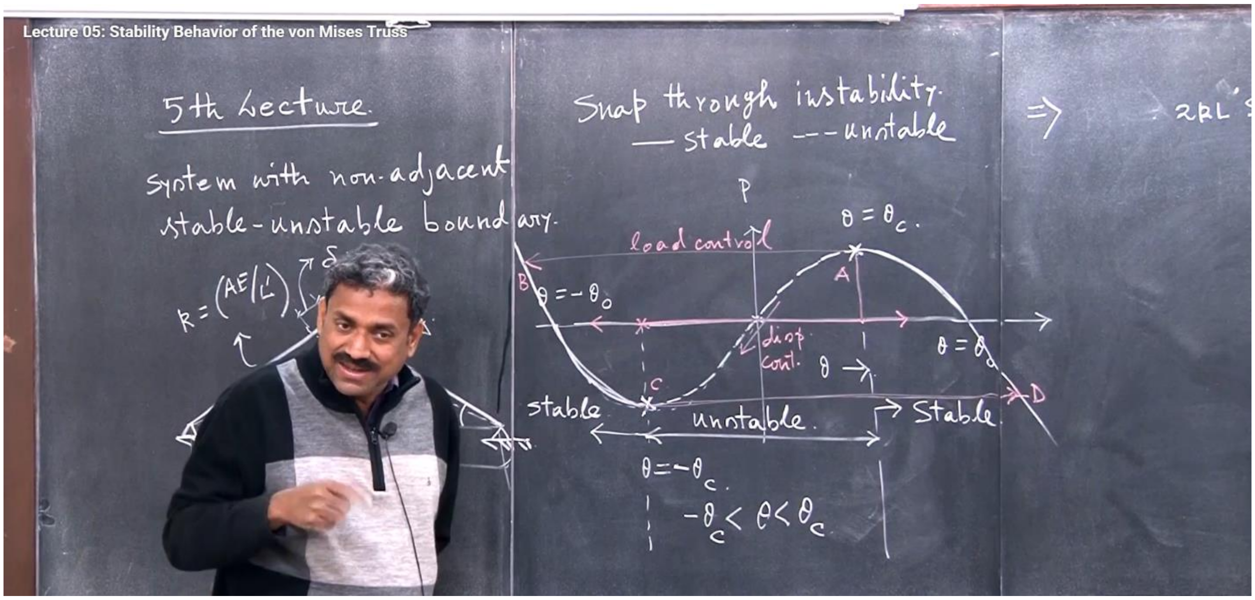


So, if you substitute P in $d^2\pi/d\theta^2$, then you will get $(2KL^2/\cos^4\theta)$, which will be a relatively simpler expression, but you have to do quite a lot of manipulation. Ultimately, you'll get this expression $(\cos^3\theta - \cos\theta_0)$. So, what we did to the potential energy function was differentiate it once, then differentiate it a second time, and then substitute the equilibrium path into that. So $d^2\pi/d\theta^2$ is in terms of θ_0. "And now, what is $d\theta/dP$? Do you see the difference? The difference is that this expression is essentially the same in both cases. Here also, $\cos^4\theta$ is always positive, and there as well it remains positive. The only difference is that this expression is exactly the negative of the previous one. " So, that means, unlike all the previous cases, here $d\theta/dP$ is nothing but $-d^2\pi/d\theta^2$. This is very important to note. So, what do you see here that unlike all previous cases, but in all these cases here, they are qualitatively the same except for the difference in sign, this is nothing but minus $DPd\theta$ in both cases, nothing but tangent stiffness, right? "So now, let me explain why this happens. A system will be stable when $d^2\pi/d\theta^2 > 0$. That is the condition for stability. Now, recall that we solved the equation $d\theta/dP = 0$ to find the value of $\theta = \theta_c$, which corresponds to the maximum value of P. So, if $\theta > \theta_c$, then $d^2\pi/d\theta^2$ becomes positive, indicating a stable equilibrium." "When it is positive, you can see the stability. So, from these two observations, we know that $d^2\pi/d\theta^2 = dP/d\theta$. Now, I'll remove that for simplicity. What I want you to focus on is this important point — please note it carefully and look at it." So, at $\theta = \pm\theta_c$, we have seen that both $dP/d\theta = 0$ and $d^2\pi/d\theta^2 = 0$. This occurs because $\cos\theta_0 = \cos^3\theta_c$, which is how we obtained θ_c.



Now, if this second derivative is greater than zero, stability occurs if and only if $|\theta| > \theta_c$.When the magnitude of θ exceeds θ_c, the expression is positive. But what does $|\theta| > \theta_c$ mean? It means $\theta_c < \theta < -\theta_c$. So, this is nothing but a stability boundary. So, which one is stable? This one is stable. This one is stable. That's what I have explained to you. That's why I have drawn this solid line as stable. This is stable and in between θ, between $-\theta_c$ and θ_c; that is, when $-\theta_c < \theta < \theta_c$. This is an unstable boundary. This is stable; please note that. So, from here to here, this is unstable. This is unstable here. It is stable. Beyond this, it is stable. Beyond this, it is unstable. The stable, that's what this is: a solid line. Now, what will happen? I will explain to you. I have taken θ, some value. Please be very careful: some θ where θ is greater than θ_0. Okay. So, what I am going to do is conduct an experimental load control test.

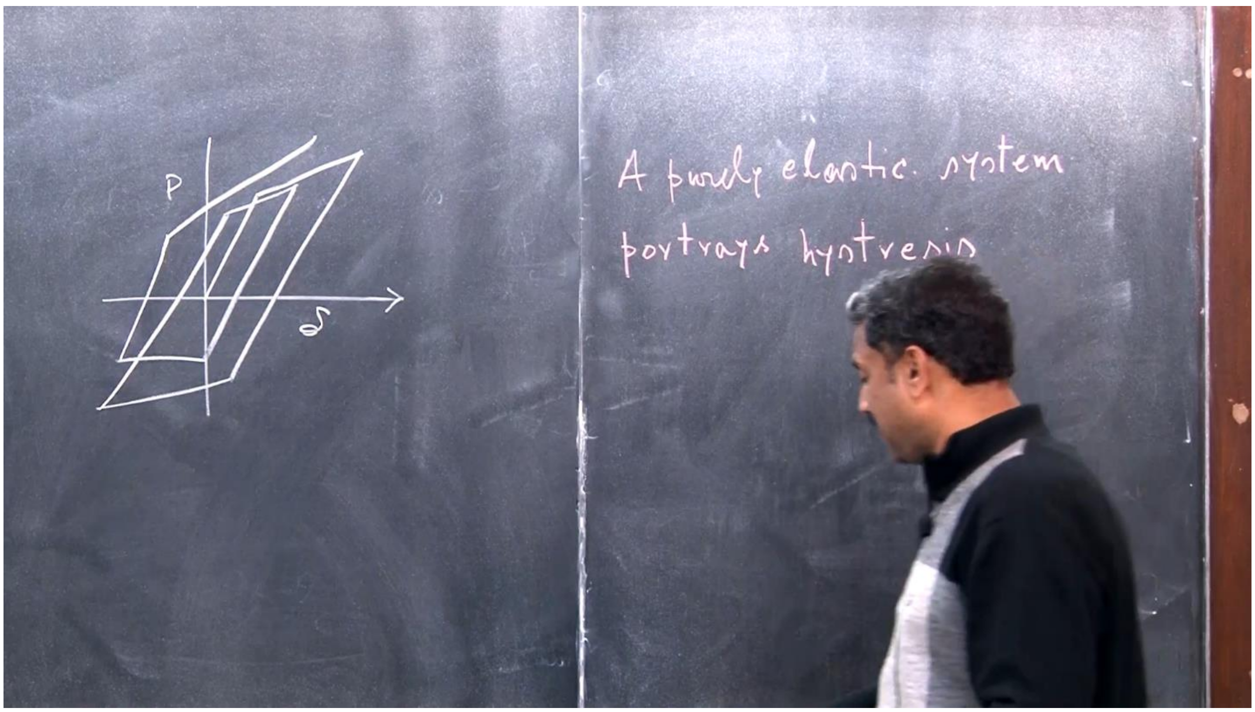
I will gradually increase the load, following this path, until I reach the point where $\theta = \theta_c$. That means I am continuously increasing the load until θ becomes equal to θ_c. If I am coming to this point through load control—you all know what load control and displacement control mean since you perform experimental methods—then this is the approach I'm taking. If I'm doing load control and gradually increasing the load, the value of θ decreases as the load increases. Now, as I keep increasing the load and reach a certain point, if I continue under load control, what will happen is that the system will not follow the stable path—it will suddenly jump. That means, instead of following the curve smoothly, the response will suddenly jump from point A to point B if I am going under load control. What is going to happen then? This point will suddenly jump to the other side, and θ will become equal to $-\theta_0$. That means that, from here, it will suddenly snap through to $\theta = -\theta_0$. In other words, the system will completely come down and it will be something like that $\theta = -\theta_0$, because that position is stable. Why is it not possible to follow this path under load control? Because this is an unstable path. Under load control, you cannot follow an unstable path. So, it will directly jump to a stable configuration.



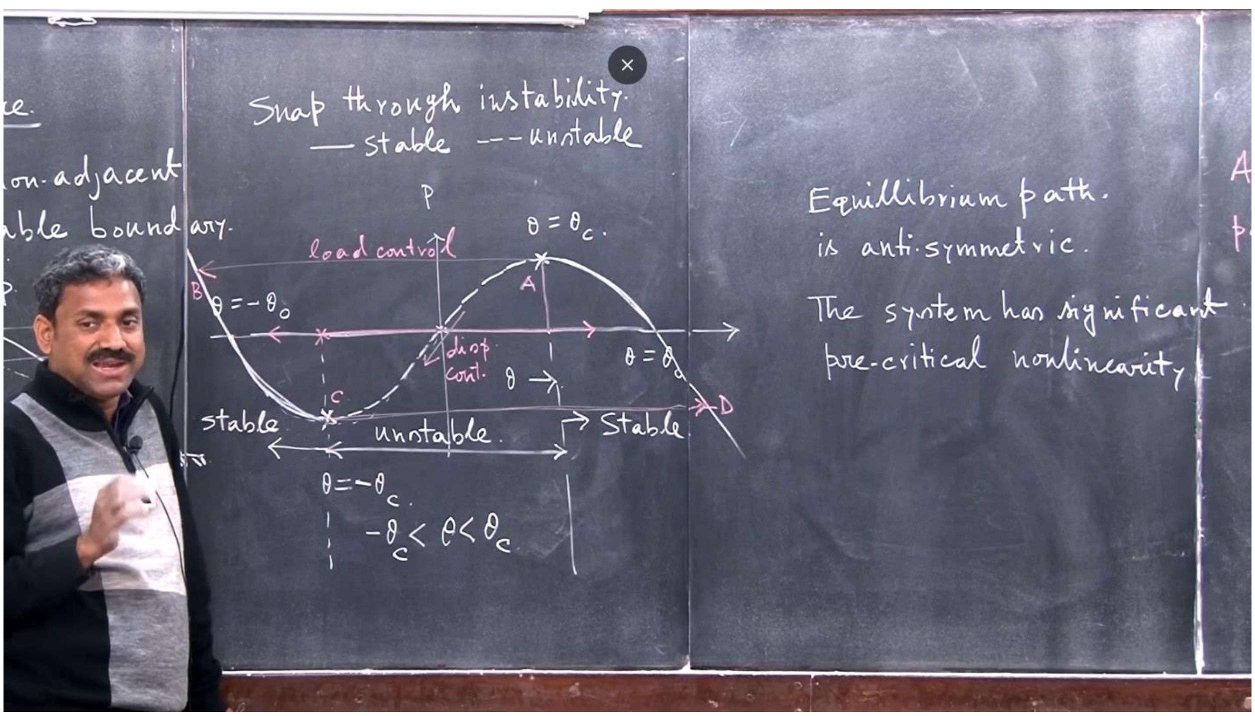
This was a stable configuration. From there, it will come to a stable configuration. But what is happening? So, a stable configuration to adjacent stable configurations is not adjacent. They're a finite distance apart. Do you see the difference between bifurcation instability and this type of instability? This one is called snap-through instability. What is happening here is snap-through instability. Now, at this stage, you are decreasing the load. That means shear when it comes down here; the load is still acting vertically. From this point onward, I start decreasing the load—going in the negative direction. Negative P means that the direction of the load is reversed; instead of acting downward, it is now acting upward. Now the load is acting upward — I'm pushing it. As I continue to push, I'm still under load control. But if I keep increasing the load further, the system will not follow this path smoothly. Instead, it will suddenly jump from here to there. Once again, another snap-through will occur. That means, as I keep pushing, the structure will suddenly jump — it will go upward or downward abruptly — and reach the next stable position. So, I am writing from A, B, C, and D. From point C, the system will suddenly snap through to configuration D —

from one stable configuration to another stable configuration. Once again, notice that this stable-to-stable jump is not between adjacent points; it occurs over a finite distance, right? Now, does that mean it can never follow this path? Of course, not — it can follow this path under certain conditions. How? If you make the experiment displacement-controlled instead of load-controlled, and if you allow it to be displacement-controlled, then this path can be, you know, traversed, right? So, this can only be traversed through displacement control. Okay? Do you understand that? Now, another point to note: look at why $dP/d\theta$ has a negative sign in the expression for $(d^2\theta/dt^2)$. Here, if the slope $dP/d\theta$ is positive, the system is unstable. That means $dP/d\theta$ appears with a negative sign in $(d^2\pi/dt^\theta)$ because it is a positive slope that can correspond to an unstable path. Normally, a positive slope would indicate stability, but here it does not. So, what is happening here is that this kind of system mises truss, It doesn't lead to a traditional bifurcation, It lead to a snap-through Of course, in literature, you will see a conflicting little statement that there are, I have seen that people are referring to snap-through as bifurcation; the only thing is that the bifurcation is occurring from one stable configuration to another stable configuration, but it is not a distant effort; it is not adjacent. Okay, non-adjacent. So, another thing what you please see that for when it is going from A to B. this transition, we are doing a static analysis is none of the cases we are doing a dynamic analysis inertia we have not considered static analysis. We are just simply minimizing the potential energy, right? But do we think that when the system is transitioning from A to B, it will happen statically? It will not happen statically; it will always happen dynamically. From here when it will jump there it will happen momentarily. That means there will be significant inertial force and there will be kinetic energy and it will oscillate. Okay. And eventually, if there is no damping, it will perpetually oscillate, but there is damping. So eventually, it will come to rest. Thus, the kinetic energy will be dissipated in the form of heat. And what about the amount of energy involved? The energy is represented by the area of the loop. See, when the system jumps from one point to another and then back again, the area enclosed by this loop corresponds to the energy dissipated due to the snap-through behavior. In other words, as the system moves from here to there, the energy dissipated is equal to the area of this loop. When the system moves from here to here, if you consider the whole loop, you see that even a purely elastic system shows hysteresis.

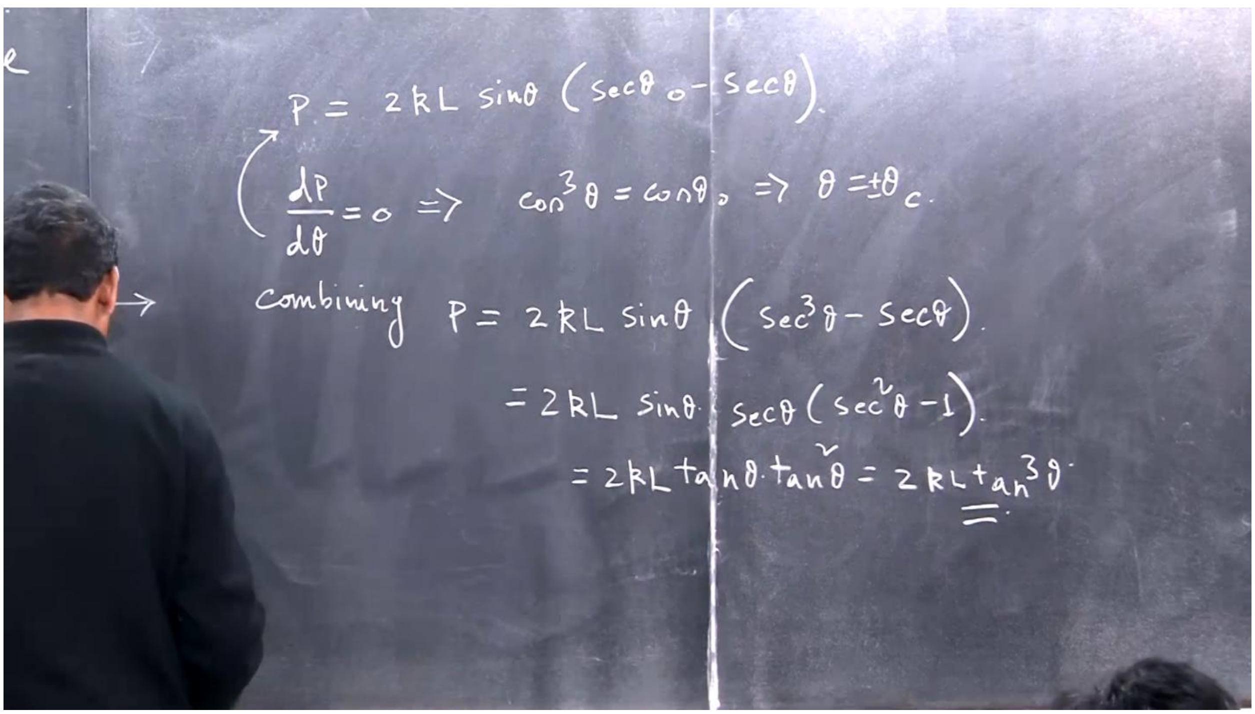
Why do we observe hysteresis in structural dynamics or earthquake engineering? It usually happens when the system exhibits material nonlinearity, meaning the material is yielding. So, you see, in structural dynamics, when a concrete or steel structure is subjected to loading, it may start yielding. As the load is applied and removed repeatedly, the structure moves back and forth — going, coming back, and then going again. This results in cyclic loading and unloading, which produces the hysteresis behavior we observe. So, this is the hysteresis that you have seen, but this hysterisis was what? due to elastoplastic hysterisis in your structure any reinforced concrete column, of course that hysteresis will be little more complicated or steel structure, when it is going right, but here it's a purely this is called elastoplastic hysteresis right Load displacement: here it is load, here it is displacement, but this is a purely elastic system, yet the system is showing hysteresis.



A purely elastic system portrays hysteresis, which is somewhat uncommon. Elastoplastic systems can show hysteresis. It's a geometrically nonlinear system. Okay. So, what should we comment here? I'm just writing this, but please note that there is no bifurcation in this case. The transition from one stable configuration to another happens over a finite distance and occurs momentarily. Also, please note that imperfections in the system don't play a significant role here; we are not studying imperfection sensitivity in this case. The systems that demonstrate snap-through and hysteresis don't show much influence from imperfections, which is why we don't focus on imperfection sensitivity here. But another very important point is that, unlike the previous systems we studied, this system is influenced from the very beginning by geometric nonlinearity. And it is this geometric nonlinearity that is responsible for the snap-through and hysteresis behavior we are observing. So, this is called a Mises truss. Now, let me show you something: in this system, we never approximated the sine, tangent, or secant terms that appear in the potential energy function or in the load-displacement equilibrium path. We never linearized anything. Do you see that? In all previous cases, we were basically — or preferably — linearizing the system. We were expanding terms like sine and cosine using the Maclaurin series. So, a trigonometric expression, we were preferring to express it in terms of you know polomial, because potential energy function when expressed in terms of polomial, will reveal important clue about the instability behavior both the stability, stable path unstable path It also provides insight into imperfection sensitivity and the nature of bifurcation, whether it is symmetric or asymmetric. Please note that earlier we were talking about symmetric bifurcation, asymmetric bifurcation, and equilibrium paths that could be either symmetric or asymmetric. Here, however, the equilibrium path is antisymmetric. Here, the system is exactly mirrored, so the equilibrium path is antisymmetric. It is neither symmetric nor asymmetric — it is antisymmetric. Imperfections don't play much of a role here.



Another important point is that the system has significant pre-critical nonlinearity, and I will show you why pre-critical nonlinearity is important. Please note that in all the other cases we have studied, when we expanded the potential energy functional, we typically linearized the system by dropping higher-order polynomial terms. Okay, when we linearized the system, what did that lead to? It led to finding the critical load, which in turn resulted in an eigenvalue problem. Initially, it was trigonometric because it came from setting $\partial\Pi/\partial\theta = 0$. So, for all the previous systems, everything was expressed in terms of trigonometric functions. But ultimately, we expanded them into polynomials, and by neglecting higher-order terms, the problem was converted into an algebraic eigenvalue problem. By solving this eigenvalue problem, we were able to obtain the critical load as well as the buckling mode shapes. These modes then served as a legitimate basis to further simplify the potential energy function, ensuring that there were no cross terms and that all terms were uncoupled. This also helped in tracking the topology of the potential energy function, and by examining just the coefficients, we could comment on stability. But here, in this system, if we arbitrarily linearize, it will be very misleading. I'm going to show you what happens if you linearize a system that displays snap-through behavior.



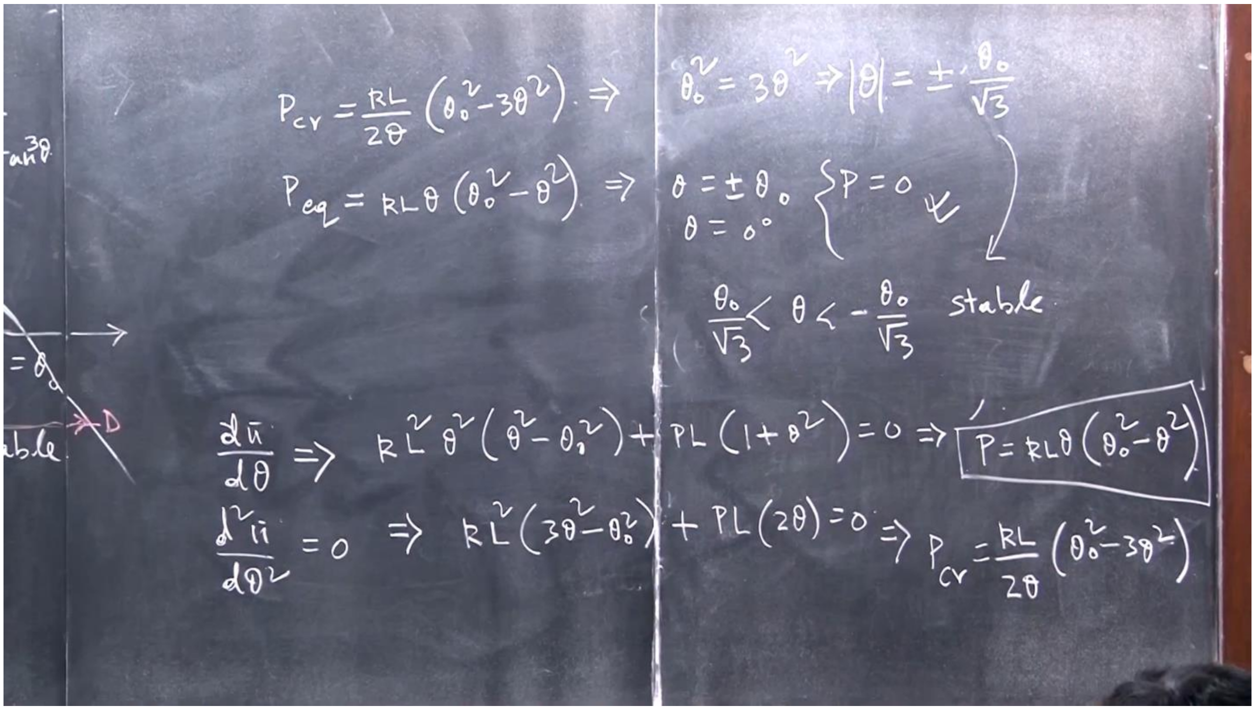
You cannot linearize it arbitrarily. You cannot just express this in terms of polynomial functions, and then if you analyze based on this kind of simplification, it will be very, very misleading. That's what I'm going to show you. Now, another thing I would like to point out is the equilibrium path. We are expressing the load as $P = 2KL\sin\theta(\sec\theta_0 - \sec\theta)$ and then we calculate $dP/d\theta$. $dP/d\theta$ that was making zero was giving you the equation that cos cubed θ must be equal to $\cos\theta$ times $\sin\theta$. So, from there, by solving, you are getting θ is equal to θ_c. Okay. Now, P is basically okay.

So, if you substitute this in, after combining and simplifying, we get: $P = 2KL\sin\theta(\sec^3\theta - \sec\theta)$. Then P is $2KL\sin\theta\sec\theta(\sec^2\theta - 1)$, then $2KL\tan\theta\tan^2\theta$, then $2KL\tan^3\theta$. This, basically, you know, is all; if you just draw it here, you know that will show you tan cubed θ. Okay, so here please show it.

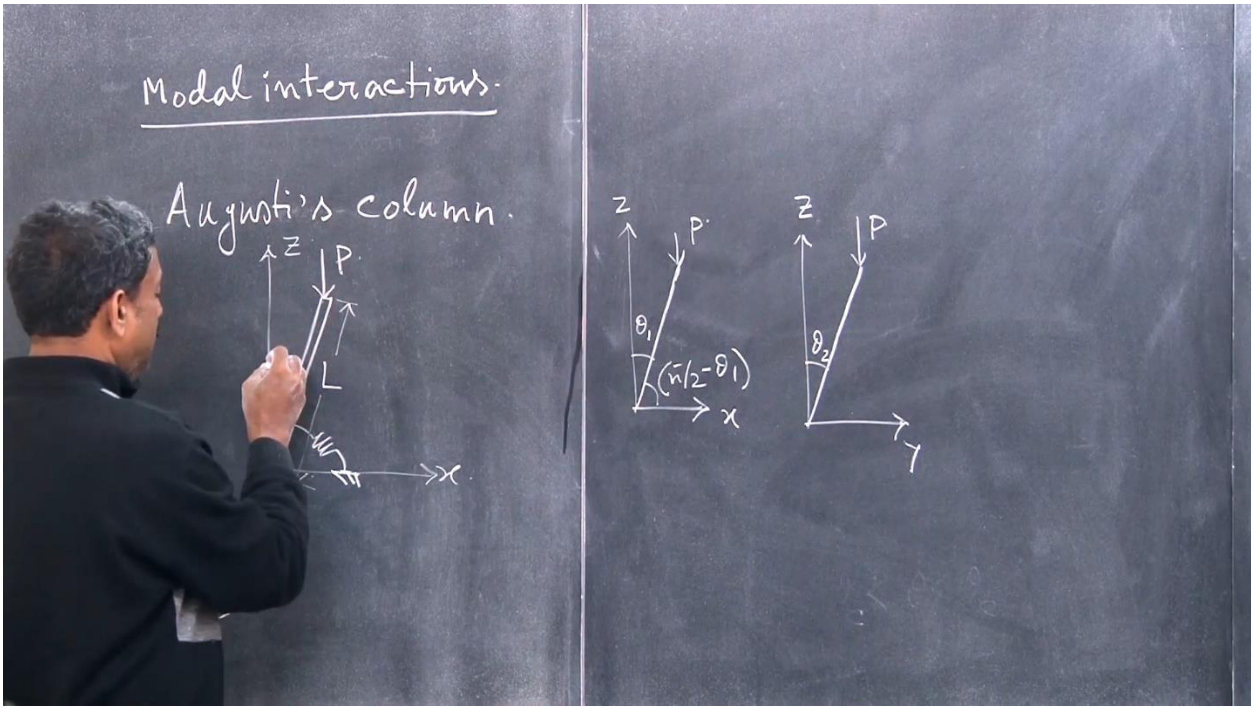
I am just drawing tan cube d θ. Okay. So, this is basically showing the stable–unstable transition. Please note how the tangent function looks, especially when it is raised to the third power — it has a cubic shape. So here, we have: $P = 2KL\tan^3\theta$. This expression represents the stable–unstable boundary of the system. I'm just imposing this simple form to illustrate the concept. Please make a note of it. So, this is nothing, but this is the stable unstable boundary. Huh? So essentially, what happens is that $dPd\theta$, from whatever expression we have, if you substitute it back, the equilibrium path becomes simplified. That's what I'm sketching here. This simplified path will intersect the two-peak curve, illustrating the stable and unstable regions. In one way, you can say: now, I'll show you — please note this down — what happens if we linearize the system. So, the expression for the potential energy functional was $\Pi = KL^2(\sec\theta_0 - \sec\theta) - PL(\tan\theta_0 - \tan\theta)$, which was the expression for the potential energy function. Now what I'm going to do is try to linearize it, meaning I am trying to approximate this trigonometric function in terms of its expansion. And let us see what happens; this will give $\approx KL^2\{1 + (\theta_0^2/2) - 1 - (\theta^2/2)\}^2 - PL\{\theta_0 + (\theta_0^3/3 - \theta - (\theta^3/3)\}$. Please note that I am neglecting the quartic term and only considering the quadratic term, okay? So, it is approximate. For the tangent function, I'm only including the cubic term, okay? So then when I'm including. I will just, you know, simplify it further $\Pi \approx \frac{KL^2}{4}(\theta_0^2 - \theta^2)^2 - PL[(\theta_0 - \theta) + \frac{(\theta_0-\theta)^3}{3}]$, okay. So, you see, we are expressing everything in terms of polynomials and neglecting higher-order terms. This is the first step toward linearization. Next, when we take: $d\Pi/d\theta = 0$.we get a simplified expression for the equilibrium path. This allows us to analyze the system more easily, while still capturing the essential behavior. I am writing the simplified equation $KL^2\theta^2(\theta^2 - \theta_0^2) + PL(1 + \theta^2) = 0$, and this will give the equilibrium path. So, the P equilibrium path is nothing but $P = KL\theta(\theta_0^2 - \theta^2)$.

Okay, so you see, we have obtained the equilibrium path from the linearized minimization of the potential energy. Next, we can perform the second derivative of the potential energy: $d^2\Pi/d\theta^2 =$

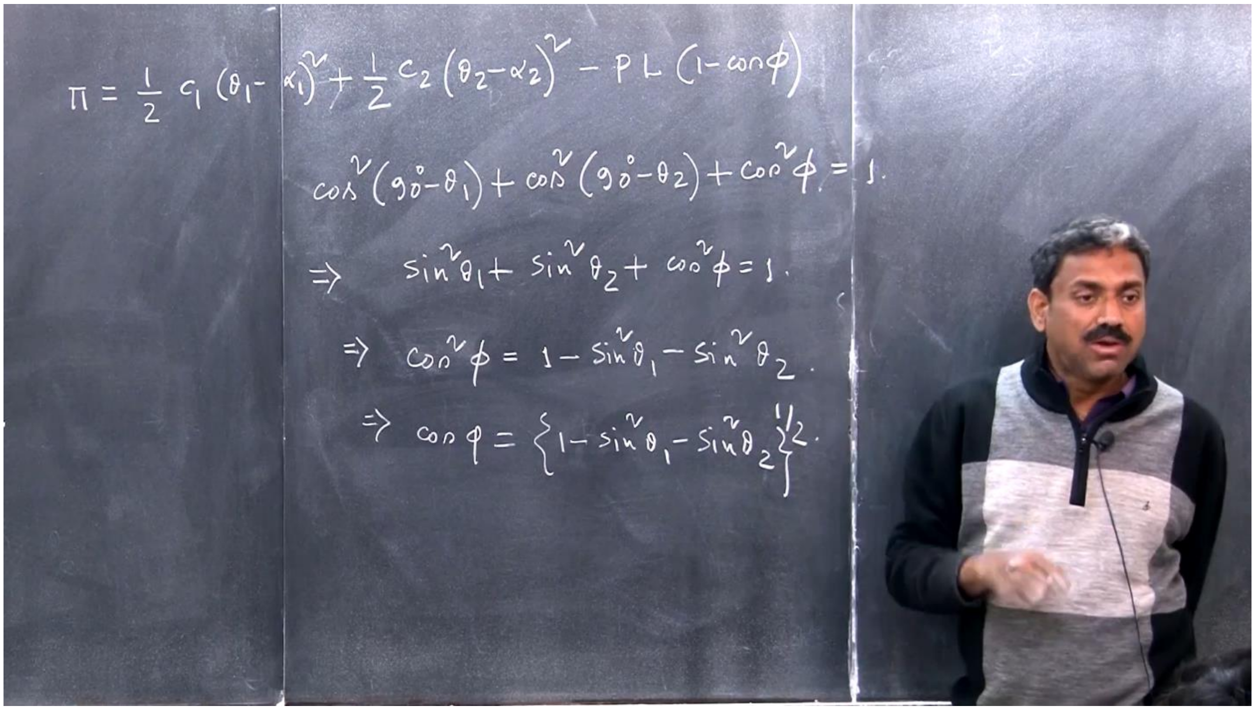
$KL^2(3\theta^2 - \theta_0^2) + PL(2\theta) = 0$. From this, we can find the critical load: $P_{critical} = \frac{KL}{2\theta}(\theta_0^2 - 3\theta^2)$.



So here, I'm just keeping the essential terms for clarity. Please make sure you note this down. What we see is that $P_{critical} = \frac{KL}{2\theta}(\theta_0^2 - 3\theta^2)$, $P_{equilibrium} = KL\theta(\theta^2 - \theta^2)$.From this, we understand the equilibrium path. When P = 0, we can solve for θ: $\theta = \pm\theta_0$. Is that correct? So yes, that's correct. The equilibrium positions for zero load are: $\theta = \theta_0$, $\theta = -\theta_0$. And of course, $\theta = 0$ is another trivial solution. Even when P=0, at $\theta = 0$, the load is also zero. That's correct. So, through linearization, we are correctly capturing this information. Now, what about the critical point? The critical point defines the stability boundary. From the second derivative, we get: $\theta_0^2 = 3\theta^2 \Rightarrow |\theta| = \pm(\theta_0/\sqrt{3})$. This gives the critical angle. It's important to note that the stable region is not simply $(\theta_0/\sqrt{3}) < \theta < -(\theta_0/\sqrt{3})$; rather, the correct boundary comes from $\theta_c = \pm\theta_0/\sqrt{3}$. So $\theta = \theta_c$ does not correspond to $\theta_0/2$. Actually, θ_c is not equal to $\theta_0/3$. The correct θ_c is obtained by solving the trigonometric equation: $\cos\theta_0 = \cos^3\theta_c$. By solving this equation, we get the true value of θ_c. By solving this equation, we get the true value of θ_c. However, if we use the linearized approximation, it gives a misleading estimate of the stability boundary. The linearization suggests $\theta_c = \theta_0/\sqrt{3}$, which is incorrect. Nevertheless, the linearized approach still correctly indicates that the transition from stable to unstable — or from one stable configuration to another non-adjacent stable configuration — is non-adjacent, which is consistent with the snap-through behavior. The estimate is just numerically inaccurate. So, this basically implies that you cannot arbitrarily linearize every system. A system that shows snap-through behavior — a system that demonstrates or displays snap-through instability — cannot be accurately represented through linearization. If you try to linearize such a system, it will give you inconsistent or misleading results. If you linearize a system that exhibits snap-through behavior, it will give you an inconsistent and misleading result. Why does that happen? Because this system, rom the very beginning, possesses significant pre-buckling nonlinearity. Of course, other systems also show some degree of nonlinearity before buckling, but in this case, the nonlinearity is much more significant, and it cannot be meaningfully linearized. So, you should not be misguided by the idea that every system can be linearized just because linearization leads to a neat eigenvalue problem from which you can easily compute the critical load. That approach can be very misleading for systems that display snap-through instability. In fact, several structures — such as cylindrical shells or spherical shells under external pressure — also exhibit snap-through behavior, and their analysis requires considering geometric nonlinearity from the very beginning. Even there are other examples as well. Okay. So, these kinds of systems shouldn't be linearized. In fact, this mistake has been made by many scientists in the past. In fact, when people were basically trying to solve various systems. One such common example is the shallow arch—or the flat arch—an arch in which the rise is relatively small compared to the span. You know all about arches, right? You have studied the mechanics of arches? But when you study the buckling of arches or post-buckling behavior. If you linearize it, it will not show the snap-through. It will not reveal the snap-through behavior. The same goes for arches. So, in fact we have written one of our papers. I mean when we were writing them, the misleading information was even published in the literature. We have referred to their arbitrary and indiscriminate linearization, which leads to misleading and apparently conflicting results and wrong information. Because people didn't realize until that time that this system could not be meaningfully linearized, they simply thought, "Okay, let it be linear, and we'll solve it that way. Okay. Understand that. So, we have essentially covered all four basic systems we call toy problems. But essentially, with these four toy systems, we are able to demonstrate all the physical behaviors that we encounter in the elastic stability of structures. Right now, we are going to demonstrate one important thing about imperfection sensitivity, and that is through modal interaction. Okay, for that we are going to consider a system called Augusti's column.



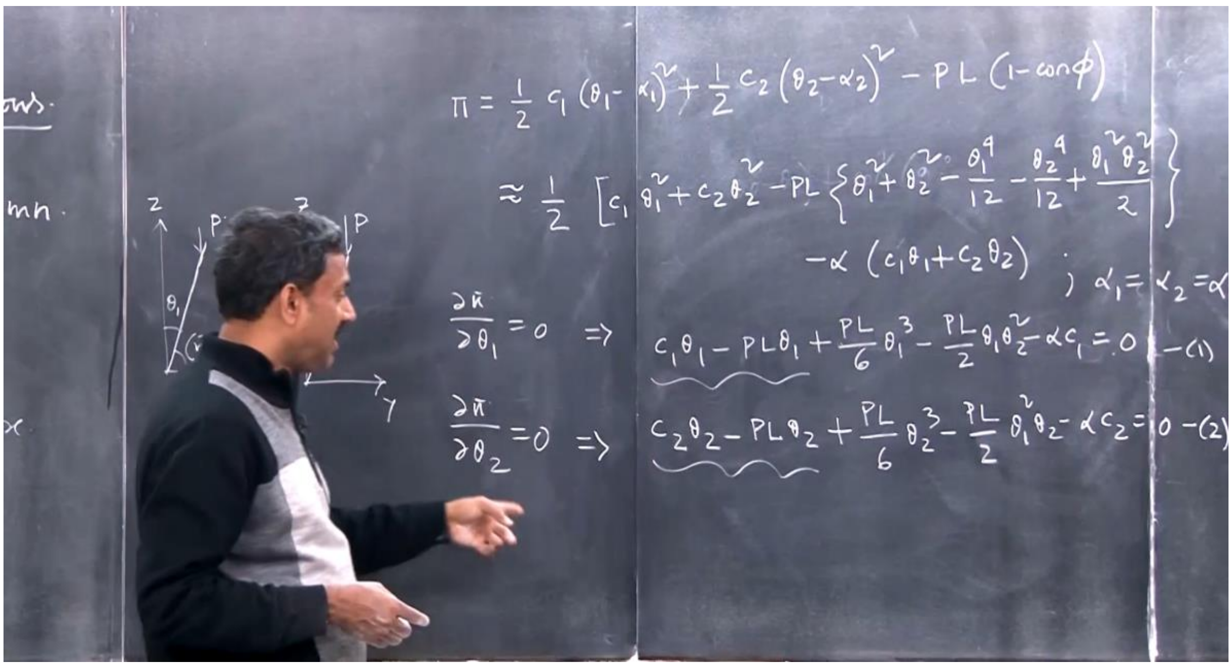
What are we going to demonstrate? We are going to demonstrate the implications of modal interaction. What happened when you did modal analysis and dynamics? We have seen that in a linear system; you have well-defined modal sets. Similarly, for buckling, you also have distinct buckling modes. But there are systems in which this buckling mode interacts and why it will interact because in the presence of nonlinearity, what will happen? So what we are going to demonstrate is that it is an apparently simple system with two uncoupled nice modes that will be imperfection insensitive, but as soon as they interact through nonlinearity, that coupled behavior becomes imperfection sensitive. That's what we are going to demonstrate using a very simple example. So, the example we are going to consider is called Augusti's column. So, this is named after scientists. So, in this column, you can consider it like this: maybe this is X, this is Y, and this is Z. There is a system, and this is a rigid bar. And this rigid bar is connected to the system with this spring, huh? And this is P, and this is length L. Okay. They're assuming that maybe with the Z axis, if we just projected with X, this is P, this is X, Y, and Z. Assume this is the angle θ_1; maybe this angle θ_2. The angle complementary to θ_1 is $(90° - \theta_1)$, and similarly for θ_2. So, the angle that makes you know the z-axis. I'm assuming this is ϕ, right? So, from the properties of the direction cosines, we can write that $\cos^2(90° - \theta_1)$ plus $\cos^2(90° - \theta_2)$ plus $\cos^2\phi = 1$. ϕ is the angle with the z-axis. So then this is $\sin^2\theta_1 + \sin^2\theta_2 + \cos^2\phi = 1$, then $\cos^2\phi = 1 - \sin^2\theta_1 - \sin^2\theta_2$, then $\cos\phi = \{1 - \sin^2\theta_1 - \sin^2\theta_2\}^{1/2}$. Right?



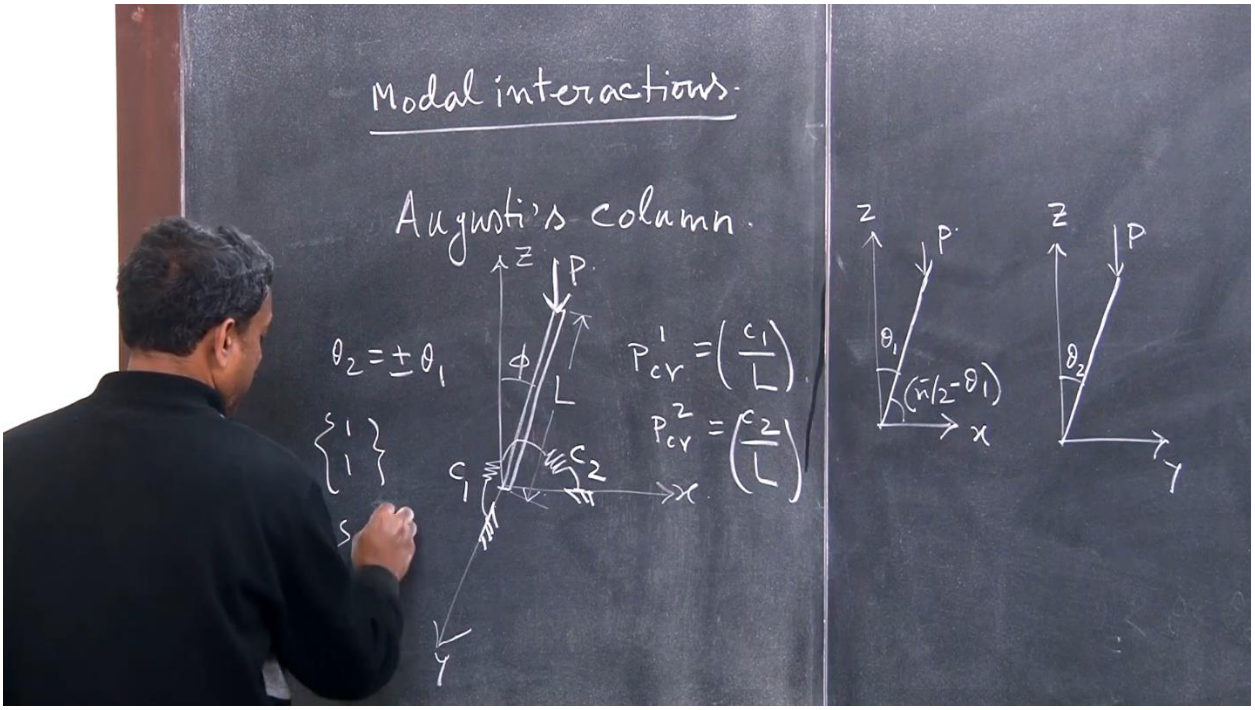
You understand what I'm trying to do? Right? This angle is ϕ, so I'm assuming this is C_1. This is two rotational springs. Why are we calling it a rotational spring? Because we know that a rigid bar is supported by a rotational spring. So, the system is stable by bifurcation, and therefore it is imperfection insensitive. So, what is the potential energy function? The potential energy function is nothing but $\Pi = \frac{1}{2}[C_1(\theta_1 - \alpha_1)^2 + C_2(\theta_2 - \alpha_2)^2] - PL(1 - \cos\phi)$ So you may wonder why I'm not including $\cos\alpha_1$ and $\cos\alpha_2$, because if you see here, it is appearing directly in α_1 and α_2, right? So, cos will inherently involve 1 minus, you know, α_1^2 by 2. The square or quadratic terms α_1, α_2 involving are neglected because they are very small compared to the linear terms. Therefore, the influence of imperfections in these higher-order terms can be ignored. Do you see what I'm trying to explain? So ϕ, I obtain this expression. And where does this come from? It comes from any point in the system; the direction cosines must satisfy this rule, and it can be expressed in terms of θ_1 and θ_2. So essentially, this system, Augusti's column, has two degrees of freedom. This is a 3D plane. Okay. It is inclined at an angle ϕ in the z-axis and some angle θ_1 in the x-axis and θ_2 in the y-axis. Okay. So, I will simplify $1 - \sin^2\theta_1 - \sin^2\theta_2$. So $1 - \{\theta_1 - (\theta_2^3/6)\}^2 - \{\theta_2 -$

$(\theta_2^3/6)^2\}$ means $1-(\theta_1)^2-(\theta_2)^2/6\}^2+(\theta_1)^4/3$. So this $(1-\sin^2\theta_1-\sin^2\theta_2)^{1/2}$ then approximates to $1-1-\frac{1}{2}\left\{\theta_1^2+\theta_2^2+\frac{\theta_1^4}{3}+\frac{\theta_2^4}{3}\right\}-\frac{\theta_1^2\theta_2^2}{4}$.

Okay. So here we are, and then substituting, if you substitute, the expression for y will look like this. Okay. So approximately it will be half you know $\Pi=\frac{1}{2}[C_1\theta_1^2+C_2\theta_2^2]-PL\{\theta_1^2+\theta_2^2-\frac{(\theta_1)^4}{12}-\frac{(\theta_2)^4}{12}+\frac{(\theta_1^2\cdot\theta_2^2)}{2}\}-\alpha(C_1\theta_1+C_2\theta_2)$. Okay. So, we are assuming that $\alpha_1=\alpha_2=\alpha$. Okay. So then if you differentiate to find the equilibrium configuration,



we differentiate the potential energy: $\partial\Pi/\partial\theta_1=0$ then $C_1\theta_1-PL\theta_1+\frac{PL}{6}\theta_1^3-\frac{PL}{2}(\theta_1\theta_2^2)-\alpha C_1=0$ and $\partial\Pi/\partial\theta_2=0$ then $C_2\theta_2-PL\theta_2+\frac{PL}{6}\theta_2^3-\frac{PL}{2}(\theta_1^2\theta_2)-\alpha C_2=0$. So, this is the equilibrium configuration. Okay. So, what you see here is that the interesting part is to look until this is linear. So, you can easily form an eigenvalue problem, right? And this nonlinear term, if you linearize it, means the nonlinear term we neglect, right? And neglecting imperfections as well will then give you the eigenvalue problem. And from the eigenvalue problem, you can find out p critical. So $P_{cr,1}=(C_1/L)$, $P_{cr,2}=(C_2/L)$, right? Where are you getting it from? If you linearize, just consider some linear value problem, right? We are linearizing the trigonometric problem into an algebraic eigenvalue problem.



From this, the mode shapes we obtain are: $\theta_2 = \pm\theta_1$. That means one mode has $\theta_2 = +\theta_1$ and the other has $\theta_2 = -\theta_1$. These modes are symmetric and linear. Now, if you include nonlinear terms and others, we'll cover them in the next lecture. Thank you very much for today.