**Stability of structure**

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WEEK-12

Lecture 24: Buckling of Cylindrical Shell Under Various Loading Condition

We have a previous class we have derived, so good afternoon, everybody. So today we are going to continue the lecture on cell buckling, and if you can briefly recapitulate what we have done in the previous class. So, we have derived the Donnell equations. And we have stated the assumption clearly. So, if you can briefly recapitulate or recall a little bit of what we are discussing. So, from the beginning, we have presented a simplified formulation for the buckling of cells in which we assume the axisymmetric deformation, right? and then I have derived by using a small strip of the cylindrical cell and we have seen that we have obtained some equation which is Similar to the buckling of beams supported in an elastic foundation. But then we have seen that this kind of solution, what we have done, is hardly, you know, followed except in the initial regime of short and thick cells, okay. So, then we started to solve the equation; nevertheless, it basically gives us the correct dependency that the buckling load under uniform axial compression, uniformly distributed along the circumference, is correct. That the dependency of Eh^2/r is clearly represented by that, okay. But then we started, you know, putting more and more physics into it, and then we derived the Donnell equation, which governs the buckling of shells and the Donnell equation for cylindrical shells, right? and then there was a certain assumption. So, we have presented a simpler, you know, derivation following Burger's equation, and the prime assumption was that several assumptions were made. The first one is that the sale we assumed to be a shallow cell or quasi-shallow shell, okay. This is given by the fact that in circumference, you know that whatever the buckling modes, the wavelength of the buckling mode is much smaller compared to the overall length of the cylinder, and then pure bending is valid, meaning that plane sections remain plane after deformation; there is no shear deformation, right? And along with that, another important thing, a very important thing, is that there is no initial curvature.
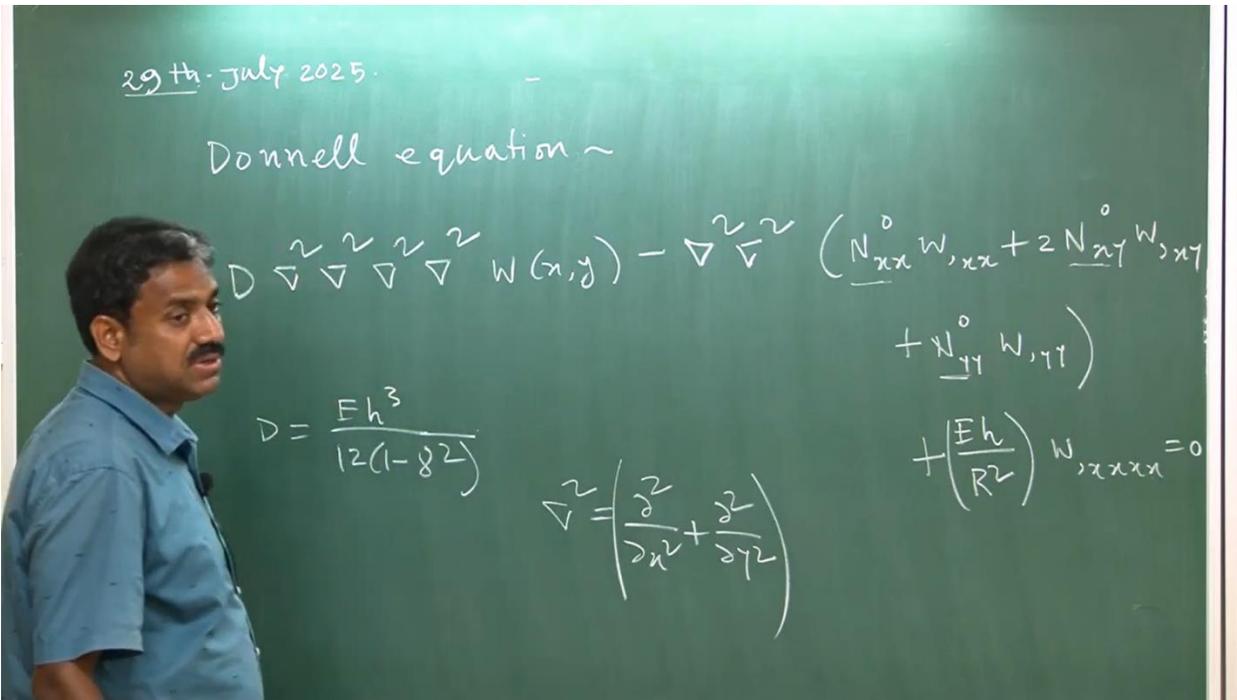
So, there are no initial imperfections in the cylinder, okay. And if you can recall, the governing equation looks like:

$$D\nabla^2\nabla^2 w + \frac{Eh}{r^2}w = 0$$

where D is the flexural rigidity:

$$D = \frac{Eh^3}{12(1 - v^2)}$$

You know the shell section. So, it is nothing but $\frac{Eh^3}{12(1-v^2)}$, the same as whatever was in the plate.

So, the derivation follows very similarly to the cell, except that the additional strain term comes from curvature geometry, and there was also an additional term. But some of these terms were neglected given the fact that r is very large compared to the other terms. So, whenever there was a $1/r$ denominator or a $1/r^2$ term in the denominator, we neglected those terms in the strain expression, right? So, this was the equation, if you can recall. In the book, you will see ∇^8, but essentially, I am writing it in this form:

$$D\nabla^2\nabla^2\nabla^2\nabla^2 w(x, y) - \nabla^2\nabla^2\left(N_{xx}^0 w_{,xx} - 2N_{xy}^0 w_{,xy} - N_{yy}^0 w_{,yy}\right) + \left(\frac{Eh}{r^2}\right)w_{,xxxx} = 0$$

where ∇^2 is the Laplacian operator:

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

See, Donnell's equations can be linear and non-linear, because if we consider this to be a fixed constant, then of course it can be nearly linearized, right? But because of this initial curvature and initial deformation, it is not the case that there will be no N_{xx}^0. There will be all these terms present, and then with the incipient deformation, these terms are going to modify. So, that way, if you consider it, it can be non-linear, okay. So, please make sure that Donnell's equation is mostly used to find out the critical load. So, we will present several solutions. The first one is a cylindrical cell under axial compression. So, here we take C; the length is L and the radius is R. And then it is subjected to N_{xx}^0 because the longitudinal axis is x, okay. And if you see the other two, then y is along this and z is perpendicular to this, okay.
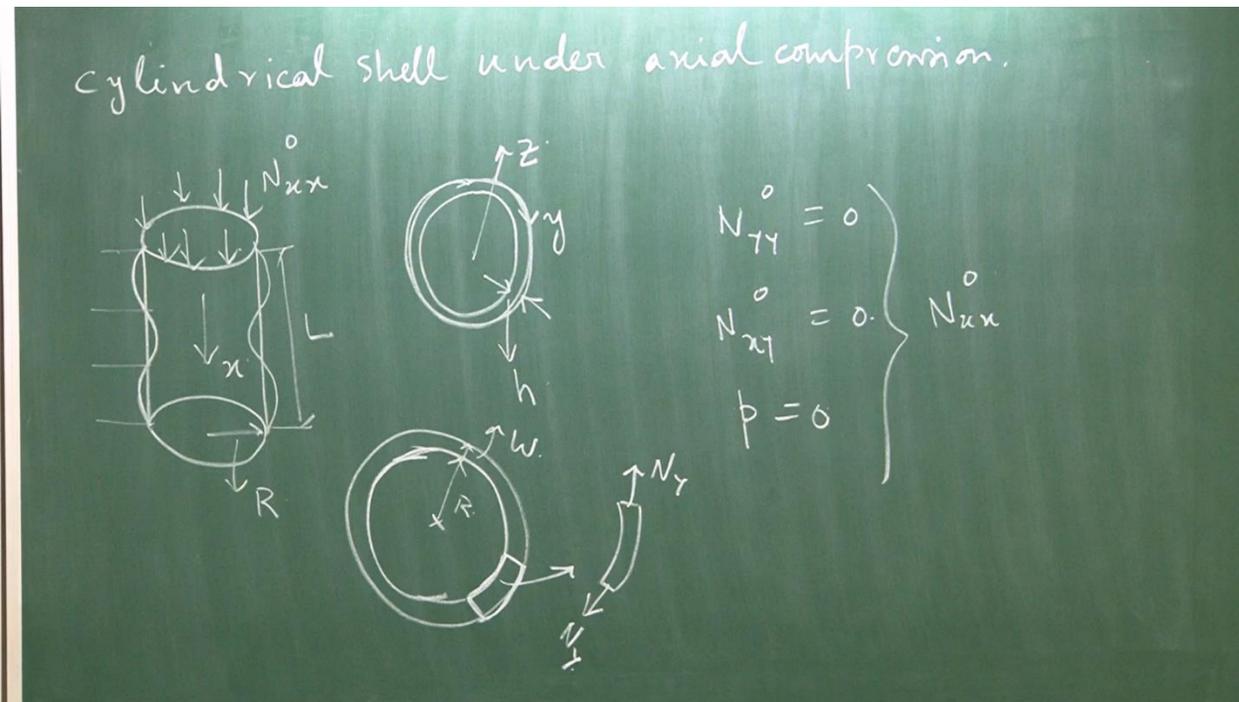
So, this is one cross-section of the cylinder; it is circular, okay? And this is a thickness h, right? h thickness. So, that is the choice of the coordinate system, right? So, of course, from here, the initial stresses show that $N_{yy}^0 = 0$, $N_{xy}^0 = 0$, and then we have only that there is no external $P = 0$, no lateral pressure P, so the only thing that is remaining is N_{xx}^0, right? So, with this, when it is,

now it is going to buckle; now we know that initially, when we consider axisymmetric deformation, we have seen that we will have this buckling pattern that will look something like this. You see that this kind of... The wavelength will be there. So, these are all half wavelengths. So, the solution along the x direction can be expressed as a sinusoidal function, right? And similarly, earlier when we were considering axisymmetric buckling, what we assume here is that the circular section, which has initial curvature, is extending with the amount w, right? So, now the radius increases from r to $(r + w)$, but we did not consider any circumferential change over that, okay. We did not consider any sinusoidal kind, so we did not consider any circumferential wave number, okay? We have considered that because of the axisymmetric assumption, it is just expanding, and that allowed us to take a small section and to write down that governing equation in fourth order, okay. Then we have considered the equilibrium of a small element, and that is what we have solved, right, for the axis symmetry. But here, that is not the case. Here we will also assume that along the circumference here, along this, So, just like in the longitudinal direction, along the circumferential direction also, there can be, you know, if you consider, there can be this kind of, you know, So, we will approximate the lateral circumference and displacement field using a sinusoidal function.
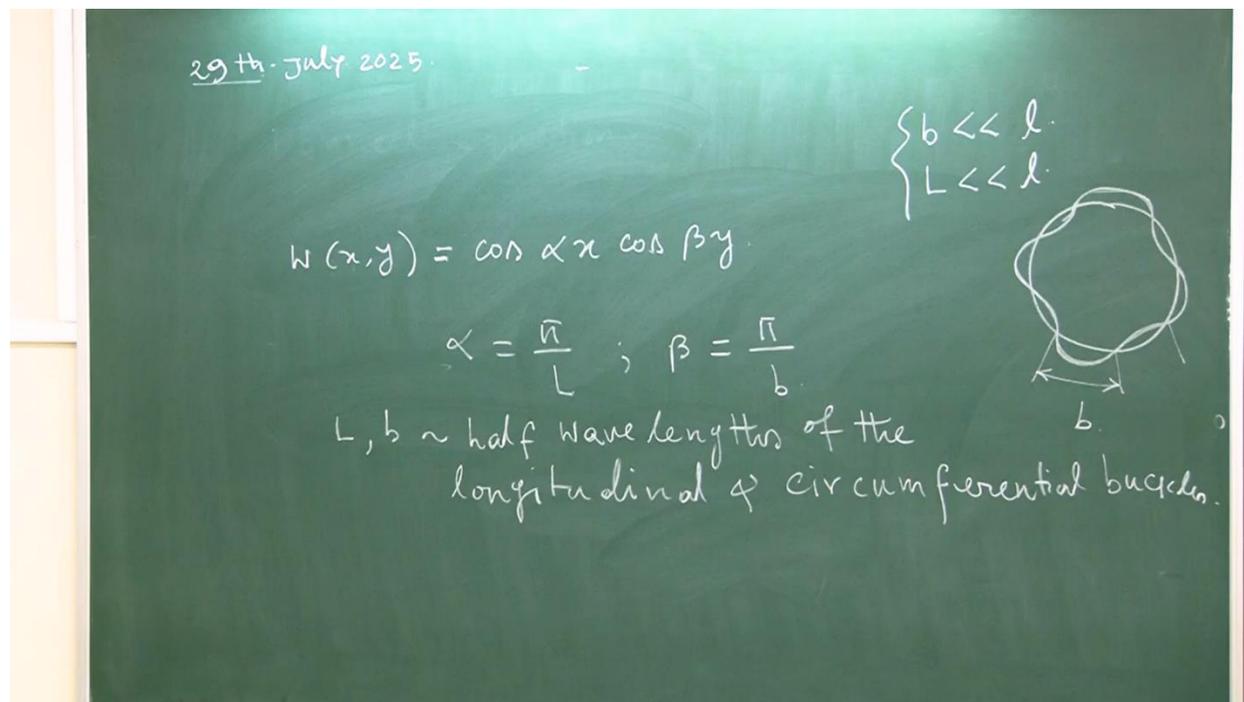
So, these are the half-wavelength numbers, okay? We are going to assume that here this wavelength is L and this wavelength along the circumference is B, right? So, as you see, for the

shallow-shell approximation, we are considering this total length. Is small l right? So, please note that for the shallow-shell assumption, B must be much less than R, and L, or rather R, you know, L and capital L are also much smaller than L. These two are right there. So, we will assume that the solution can be taken as the cosine of, and you see it must be a function. So, αx, cosine of βy, α, and β are the half-wavelength numbers. So, α is basically π/l, and β is basically π/b. These are the half-wavelength numbers, okay, α and β. So, L and B are the half wavelengths of the longitudinal and circumferential buckles. So, with this assumption, we can substitute it into the governing equation. So, the way you see that, the lambda operator is $\frac{\partial^2}{\partial x^2}$, alright, $\frac{\partial^2}{\partial y^2}$, so if you see, it differentiates twice, so it will be what, $(-\alpha^2 - \beta^2)$, right, twice, right, so then if you consider. Governing equation, $\nabla^2 \nabla^2 \nabla^2$, 8 to the power of 8. So, this is going to show if we substitute in Donnell's equation.

 So, what are we going to get? Please note that when we assume this, we are not considering the boundary condition much, but if you want to satisfy the boundary condition, you can also assume sin sin; that is not a problem. Okay, here we are assuming cos cos, but please see whether it satisfies the condition; here we are assuming we have yet to enforce the boundary condition. But please note that if you want to enforce the boundary condition that, you know, it is simply supported, then

it is better to assume sin, sin, right? Okay. So, then, with this, if you substitute, it will look like the differential equation will be substituted for an algebraic equation:

$$D(\alpha^2 + \beta^2)^4 + \frac{Eh}{r^2}\alpha^4 - N_{xx}^0 \alpha^2(\alpha^2 + \beta^2)^2 = 0$$

So, you see that this homogeneous equation is here. So, that is what an eigenvalue problem is, okay. And these are nothing but the eigenfunctions. If you consider this as w_0, you see that w_0. So, w_0 this should, it can take any value w_0. So, these are the deformation patterns. So, these are the eigenmodes and w_0 does not really matter, right. From here you will see that if we simplify, then you will get N_{xx}^0, which is equal to N_{xx}. The critical load is basically:

$$N_{xx}^0 = N_{xx}^{Cr} = \frac{Eh}{\pi^2 R^2 L^2}\left(\frac{1}{L^2} + \frac{1}{B^2}\right)^{-2} + L^2 D\pi^2\left(\frac{1}{L^2} + \frac{1}{B^2}\right)^2$$

Please note that these are all shallow shell theories. So, l and b are very, very less than r. Okay, sorry, it is not very, very less than l with respect to r. Okay. Half wavelengths are much, much smaller than others; that is what the shallow-shell assumption is based on, with Donnell's theories, as you know, derived right. So, what we see is that this solution, of course, you know how many a and b will be there; we have to minimize this. So, we have to minimize with respect to L and b, as we have done in plates and others. So, for certain combinations of a and b, which are nothing but half of the wavelength, we have to minimize this. So, we have to turn this result into a minimization problem. Now, regarding this minimization that people have done, they made various assumptions. For example, the people who have contributed profoundly here are Chajes, Reed-Alexander, Chazes, and then Caladine and others, okay. We are not going to present the minimization here, but we understand that it needs to be minimized, differentiated with respect to L and B, and that there are basically two input variables. to minimize with respect to L as well as B. So, you $\frac{\partial \pi}{\partial L}\frac{\partial \pi}{\partial B} = 0$ and then you have found this. So, if this is an unconstrained minimization, you will see, but we will consider some special cases, okay. If you see the special case or one case, we want to see whether it mimics the one where we started the discussion for an axisymmetric Shell buckling. So, for the axisymmetric case, you know, $B \to \infty$ implies that $\beta \to 0$. As $\beta \to 0$, then $\cos\beta \to 1$, right? So, when $B \to \infty$, what does that mean? The half-wavelength is very, very large. That means along the circumferential, and B is nothing but the web number along the

circumferential direction, right? So, what it means is that there is only one. So, one of the half wavelengths is so large; so large means there will be only one wave number here, you understand? That represents nothing but the axisymmetric mode, right? So, this one is the axisymmetric mode. And this was the axisymmetric mode that we had derived earlier. We will let us see whether it is mimicking that, okay? So then for this case, you will see that:

$$W(x,y) = W_0 \cos(\alpha x)$$

this cosine of αx means that now you have only longitudinal wave numbers, right, wavelength, and then, so these basically correspond to the axisymmetric modes that we have discussed. So, what we see is that Donnell's equation mimics, as a special case, the case of axisymmetric buckling. where the circumferential dependence you are not taking into account, assuming the axisymmetry of deformation and then only one single you know mode is governing ok, 0th mode you can consider and that is why this term drop and essentially that was a differential equation in terms of x, x was only the independent variable right. Isn't it?
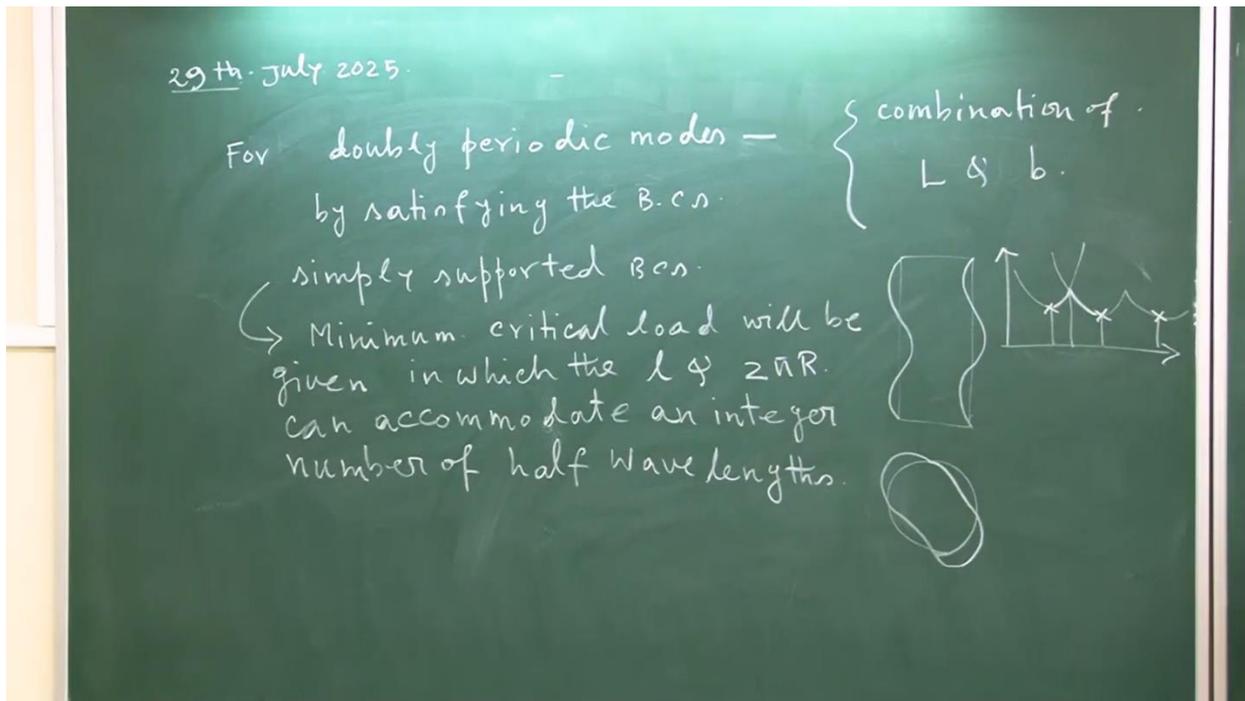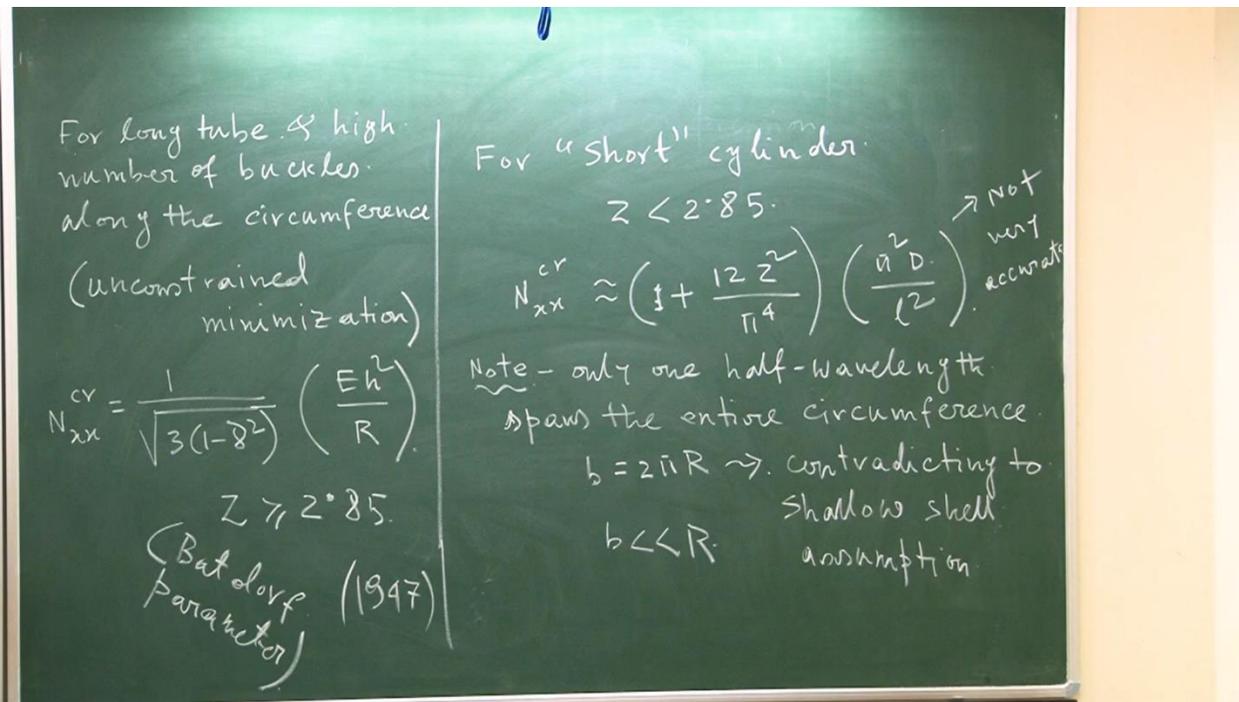

So, that's okay. So, with this, we can mimic that case. Now we will distinguish between two different things using some variable z, okay. Now for the axisymmetric mode that is given. Now for W periodic mode. So, for W periodic mode, what? Where both the longitudinal wave numbers

will be there anyway, right? But, W is periodic in the circumferential; that means it will have a combination. This means a combination of both L and B. Wave numbers are also here, you know, along this direction; wave numbers are present. But here, circumferentially, it is also there. So, it is something like that, okay, or some. So, for doubly periodic mode and by satisfying the boundary condition, for different cases, you have to satisfy the boundary condition. So, that means when you assume this cosine-sine function, that is where you know, when you satisfy the boundary condition, those are the things you have to keep in mind, right? Which functions will you assume, okay? So, for that, for example, a simply supported boundary condition, right? Simply supported boundary conditions. For that, you know the minimum; where will you get the minimum critical load? For a simply supported boundary condition. So, the minimum critical load will be given. In which the length and $2\pi r$ represent the radius length, the radius can accommodate an integer number of half wavelengths, right? So, that is very important. You have seen that right in the plate that wherever our plate dimension B is such that it accommodates an integer number of half wavelengths, that is where it was minimum, right? Can we recall that you know the Arnold term structure, right? It was something like this, something like this, something like this, so what is happening? If it is an integer, you know. If it is an integer, then it is attaining the minimum value, right? Otherwise, in both directions, it is going up because that gives an energetically favorable configuration, right? So, the same thing here as well, okay? So, boundary conditions also have an influence on the critical load, right? So, we will distinguish between the two cases. One case is for a long tube and a high number of buckles along the circumference. A long tube means a long cylindrical shell. So, for that, people have done inconsistent minimization, okay. Inconsistent minimization to obtain the critical load, okay.

To obtain the critical load, the n-axis critical is:

$$N_{xx}^0 = \frac{1}{\sqrt{3(1-v^2)}} \left(\frac{Eh^2}{R} \right)$$

You see that the dependence Eh^2/r, and please note that there is a parameter; this is valid for $z \geq 2.85$. The Z parameter is for the Burt-Dorff parameter, the Batdorf parameter that basically distinguishes between the short and long shell.

For long tube & high number of buckles along the circumference

(unconstrained minimization)

$$N_{xx}^{cr} = \frac{1}{\sqrt{3(1-\vartheta^2)}} \left(\frac{Eh^2}{R}\right).$$

$$Z \geqslant 2.85.$$

(Batdorf Parameter) (1947)

For "short" cylinder.

$$Z < 2.85.$$

$$N_{xx}^{cr} \approx \left(1 + \frac{12 Z^2}{\pi^4}\right) \left(\frac{\pi^2 D}{L^2}\right). \quad \nearrow \text{Not very accurate}$$

Note — only one half-wavelength spans the entire circumference.

$b = 2\pi R \rightarrow$ contradicting to shallow shell assumption

$b < < R$

Let our other parameters, such as ω and other parameters introduced by the European code, be mentioned, but I am not discussing that. But this, so it was done in the Batdorf in 1947, the year we gained independence in India. So, for $Z > 2.85$, that case basically represents where this cylinder is long enough and there are many buckles actually along the circumference and direction. Then, if you do constant minimization, that means minimizing with respect to L and B; then you will see this kind of critical load. So that you get a closed-form expression by doing the minimization with respect to L and B half wavelength. $\frac{1}{\sqrt{3(1-v^2)}}\left(\frac{Eh^2}{R}\right)$. Do you see the dependency on $\left(\frac{Eh^2}{r}\right)$ is universal? Okay, but This is for the so since $z > 2.85$, this is the long shell, right? Similarly, you know that for a short cylinder, what is considered a short cylinder by Z is greater than 2.85:

$$N_{xx}^{Cr} = \left(\frac{\pi^2 D}{L^2}\right)\left(1 + \frac{12 z^2}{\pi^4}\right)$$

this formulation was given by Chazas, okay. So, this is for a short cylinder, okay. Please note that the parameter that distinguishes between the two, A long tube with a high number of buckles along the circumference of the short cylinder is, of course, the Batdorf parameter, and there is a critical value for that: 2.85, okay. However, there is a note; there is a point here. Please note that here. The

note is that, you know, only one-half wavelength spans the entire circumference here. Here. That means $d = 2\pi r$. See, for this, you know, whatever was obtained by Chazes around 1967, you see that. Unlike this one, a large number of buckles occur along the circumference here, right? But here for this case, only one half-wavelength spans the entire circumference. What does it mean? That means one means it is very similar to this axisymmetric one, right? $B = 2\pi r$, right? Half a wavelength is nothing but this. This is contradictory to the shallow shell assumption, right? Why is it contradicting the shallow shell assumption? Because in the shallow shell assumption, we have assumed that D is much smaller than R. So, this cannot be correct. So, this one, this expression is not very correct. It is not very accurate, you know; not very accurate. And you understand why it is so because, although we have solved the mathematics, and you have obtained that, but the thing is that since $B = 2\pi r$, it is not very, very small; only one wave number is spanning over the held circumference. So, this is contradicting the fact based on which the Donald's equation has been derived, which assumes that the shallow is shallow shell assumption right, that we, okay. To other extreme cases, we must consider here: if the shell is very, very long—very, very long means the shell is very high—then what happens? Then bugling cannot be governed by the Donnell equation. Why? It will behave like a long one, so for a very long shell it is right; it behaves like a column. Like a long column, not governed by, you know, it is not by the Donnell equation, please note that, okay. Another thing, if the cell is very, you know, very short, it means you know $z \to 0$. So please note that you know whether, when we are discussing that $z > 2.85$ and $z < 2.85$, it is not encompassing the very long and very short, very short cell $z \to 0$. Then the shell will behave like, you know, it will behave like a plate, behave like a plate. because very short shell that means it is so, so individually this one you know it is so short, that means the shell action will not be there. So, the thing is, it will behave like a plate; you see that, and then its buckling will be governed by the blade wheel, right? Other than these two cases, there are also two extreme cases, in which one case is for a very long shell that behaves like a long column. And for a very short case, $z \to 0$, it behaves like this, right? Good. Another important thing is that there is one very, very important fact, which is the most striking aspect of this solution. So, you will see that, in case there are multiple solutions involving doubly periodic modes. What is the doubly periodic mode? Doubly periodic mode means that you know along the longitudinal direction, there are modes, and then along the circumferential direction, there will also be modes. So, the doubly periodic mode and the axially symmetric mode: if we consider the first one, it is the axially symmetric mode, right?

You know which one is the axially symmetric mode because one half wavelength spans over the entire circumference.
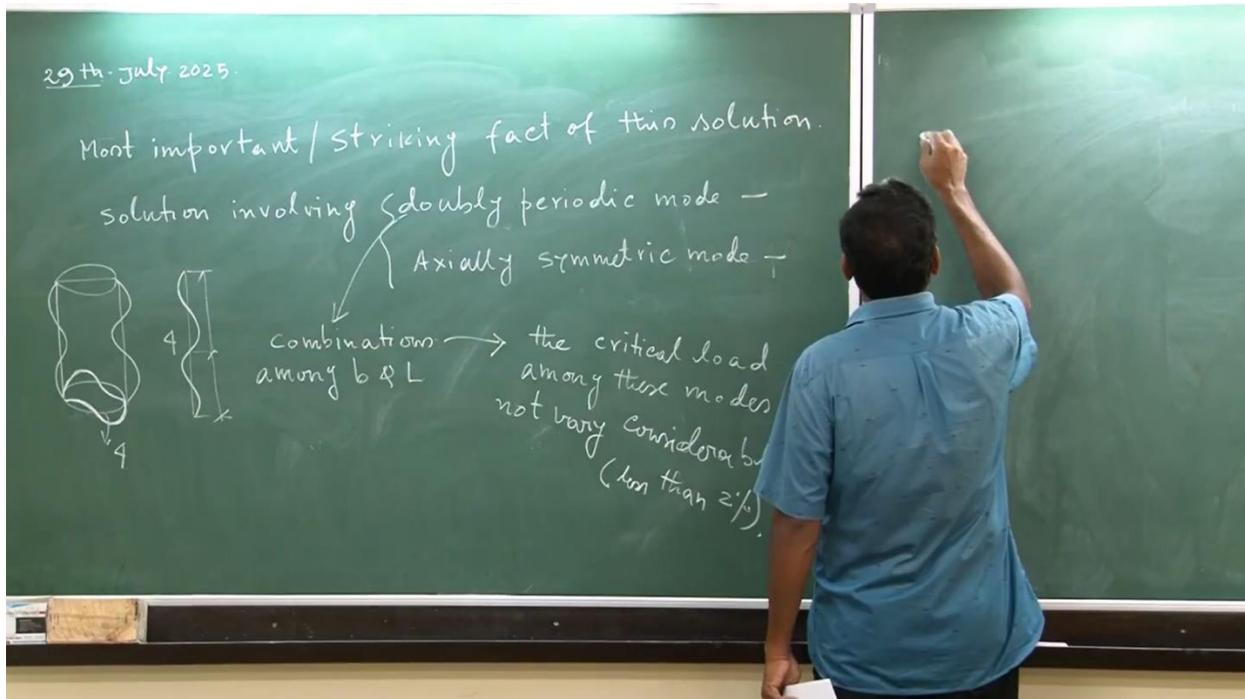

So, you see, these are the two modes: the initial mode if you consider the fundamental mode and things. So, initially, what we consider an axially symmetric mode, in the case where there is only a single web number spanning the work, and then a doubly periodic mode means there are, I mean, several, two numbers along the whole circumference. Now, here also in doubly periodic mode, there can be combinations of B and L, correct? You know, combinations among B and L. Why combinations among B and L? That means, you know, along the circumference, around the, see, you can have, see, you like having like this, right? And then you can have, so what happens is that, of course, here I am having more modes, okay. More wave numbers, but you can consider then two wave numbers. So, there can be a combination between the longitudinal as well as the circumferential half wavelength, right? So, when I am calling the doubly periodic mode, if we have, you know, doubly periodic means, say consider that this is 1, this is 1, so this is one complete period, right? And this is one complete period, isn't it? Along the longitude. Now in the circumferential, this is one complete period, and this is one complete period, right? So, the wave number here is basically half; the wave numbers are 1, 2, 3, and 4. Here are 1, 2, 3, 4, right. So, there can be an n number of combinations between these two, right? These and this can be outward,

inward, or something like that, right? There can be various combinations of them. You know, there can be 4×4 types of combinations, right? It has been seen that if you consider either an axially symmetric mode or a doubly-periodic mode with any combinations of B and L, Then a very interesting and striking fact is that the critical load, you know, among this mode, do not vary considerably. That means it is less than 2 percent, approximately 2-3 percent, okay. So, what I am trying to say is that the critical load among these modes, you know, whatever solution you obtain by solving Donnell's equation does not vary. What does it mean? That means they, you know, cylindrical shells under axial compression have closely spaced modes, right? Or a multiplicity of modes. And when it has a multiplicity of modes, that means they will interact; there will be modaling and all this mode. See, in the linearized regime, there will be no coupling, but please note that. They are coupled anyway by geometric nonlinearity, right? I emphasize that we can linearize it, but Donnell's equation is non-linear as well because N_{xx}^0, N_{yy}^0, and N_{xy}^0 are always with incipient deformation. They will be changing, and there will be coupling. So, there will be this. All these modes are geometrically coupled. all these modes, okay. All these modes are coupled modes. So, they interact, right? Coupled. And this led to modal interaction. Do you see this? Modal interactions. And when this modal interaction is present, it basically reduces the critical load. From the very beginning of the classes, I have shown that when there are two modes interacting. Then the combined mode will have a much-reduced load because it will become imperfection-sensitive, right? Two coupled modes with the same or closely spaced, closely similar, or very close critical load interact if they are coupled correctly, and they will further reduce the load. We have demonstrated this phenomenon using Auguste's column, right? So, these will significantly reduce the critical load, and that is why you will see there is a significant difference. This shows a significant difference, a significant disparity between theoretical and experimental critical load, as you can see. So, the way it will. So, it is Z, and here I will just show you that. So, this is the most interesting, or you know, fact out of all these things, because the solution of Donnell's equation gives closely spaced modes. And then, if you plot it for the varying of z, and then if you normalize it, you know the N_{xx} critical; you normalize it with $(\pi^2 Dz/L^2)$. It is logarithmic, and this here is, you know.
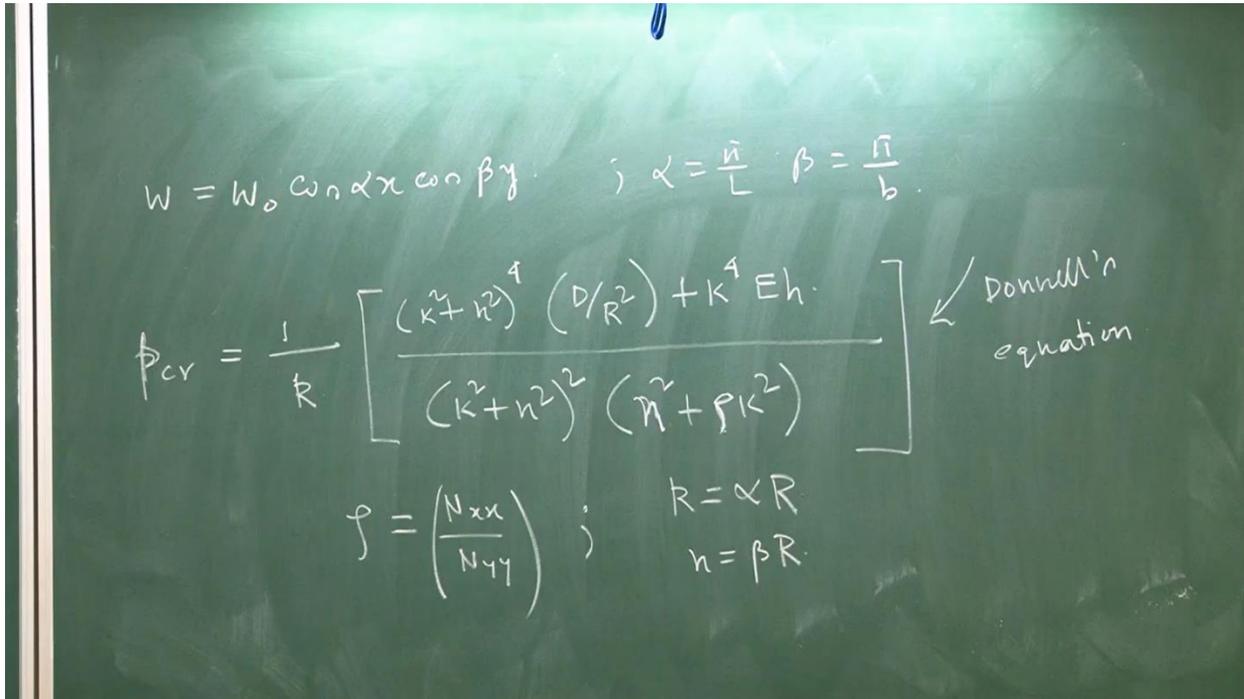
So, you see from here, there will be a huge drop, you see that. So, whatever this critical load is, it is forming some kind of upper bound, but there is a significant drop in this load. Why is this drop happening? Because all this closely exposed mode conspires through modal interactions and then reduces the load, which of course is necessary in order to study that load drop. You have to solve it once by accounting for the non-linearity, and that is, we will use a slightly different version, but essentially the elements are the same to show this drop, okay? So, this is the most interesting fact here, okay. Please note that for the shell under axial force, this kind of drop in critical load is occurring. But this is not the case. Please do not be misled by the fact that this will be true in all cases. No, this is not true. If the cylinder is subjected to a transverse load along with an axial force. You know lateral pressure; this is hydrostatic pressure, or instead of hydrostatic pressure, if you can simulate this situation by two ways. Once you take a cylinder and then compress it, you can hydrostatically apply pressure the way you do in this triaxial test in solid mechanics. Either you can apply pressure by pumping up the liquid, or another easier way is to use a suction pump to remove the air. So, atmospheric pressure will push it, okay. If you do this, then you will see a slightly different kind of thing. So, what we will see is what I will do now. So, what we have understood from this is. that we distinguish between long and short shells. Of course, there can be super long ones, and there can be super short ones, okay. And then the Batdorf parameter is the one that distinguishes between the two. Of course, for very short shells, the things $Z < 2.85$ are not quite valid, because that violates the shallow shell assumptions. To summarize, and then, most

importantly, it gives you huge deductions in the critical load. Because of the multiplicity of closely spaced modes which are coupled with geometric non-linearity.
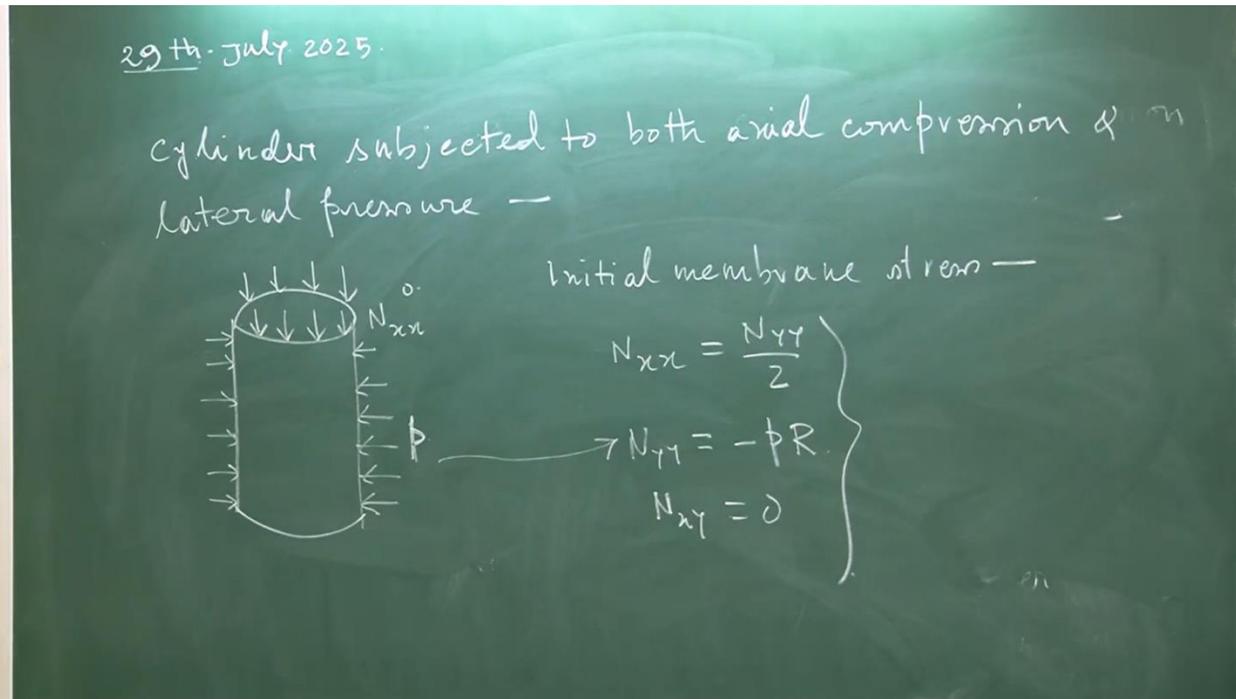

Through geometric non-linearity, there is an interaction between the modes that also reduces modal interactions, which is a very important aspect here. So, please note that with all this reduction in load flow, there is a huge discrepancy between the theoretical and practically observed load. Theoretically, we will see how the drop can be obtained, but at least this analysis will not be able to predict that. But it can definitely predict the closeness of the mode and the critical load between different modes. So now, we will consider a case, you know, of the cylinder. A cylinder subjected to both axial compression and lateral pressure. So, a cylinder, which is both axial compression and lateral pressure, if you consider this one, right? So, here it is, axially compressed, N_{xx}^0, and this pressure. this is P. So, for this case, initial membrane stress. You see, N_{xx} will be, of course, $N_{yy}/2$; you just apply the hoop stress and meridional stress from that theory in undergraduate mechanics. You know $N_{yy} = -PR$ and $N_{xy} = 0$, right? Membrane, this is the membrane state subjected to a P load, right? So, we are basically converting how P can be given as input in the Donnell equation because the Donnell equation can only take care of the membrane stresses. So, we can take care of that right, by substituting. Of course, N_{xx} already has N_{xx}^0 over there. So, if you do take this and then substitute it into the governing equation. So, assume $w =,$

once again, the similar solution to $\sin(\alpha x)\cos(\beta y)$, right? And α and β, as you know, are half wavelengths, right? $\alpha = \pi/L$ and $\beta = \pi/b$, right?

$$W = W_o \cos \alpha x \cos \beta y \quad ; \quad \alpha = \frac{\bar{n}}{L} \quad \beta = \frac{\bar{n}}{b}.$$

$$P_{cr} = \frac{1}{R} \left[\frac{(k^2 + n^2)^4 (D/R^2) + k^4 Eh}{(k^2 + n^2)^2 (n^2 + \rho k^2)} \right] \quad \leftarrow \text{Donnell's equation}$$

$$\rho = \left(\frac{N_{xx}}{N_{yy}} \right) \quad ; \quad \begin{array}{l} R = \alpha R \\ n = \beta R \end{array}$$

Then I am not doing it, but you can easily do it; it is not a problem. You know donnell's equation, I mean if they assume solution and then assume this N_{xx} of value, we can substitute, of course here will be two unknown on although one equation, So, we cannot solve for it; we can only solve for it by assuming a certain ratio between N_{xx}^0 and p because there are two unknowns. So, we will define some ratio $\rho = (N_{xx}/N_{yy})$. This ratio we are going to define for the critical one is right. Then you will see the way we will obtain that P critical is $1/k$. Here, ρ is the ratio of N_{xx} to N_{yy}, and this parameter $k = \alpha r$, and then n is nothing but βr; r is the radius of the cylinder, right? So, as you see, you are obtaining P critical, but as a function of this ratio, okay. Ratio is N_{xx}, and this is ρ, right? Because you have only one equation, but what I would like to tell you is that now you have obtained this from the Donnell equation, right? So, we will see Donnell's equation. So, M and all the other parameters we have defined; we have defined k, and we define N and M, okay. So, here, then I will explain what happened; there is an interesting fact I am going to share. Dot log z, and here I am going to put $\frac{P_{cr}}{\left(\frac{\pi^2 dz}{L^2 r}\right)}$; $\rho = 1/2$, okay. Here you see that this solution is given by

Donnells, right? All this data point is coming from an experiment; these are from experiments, and

these experiments were conducted by Gerard. Gerard has made many significant contributions to inelastic buckling of shells, and this is also the solution she has proposed. So, that is what Gérard has proposed.
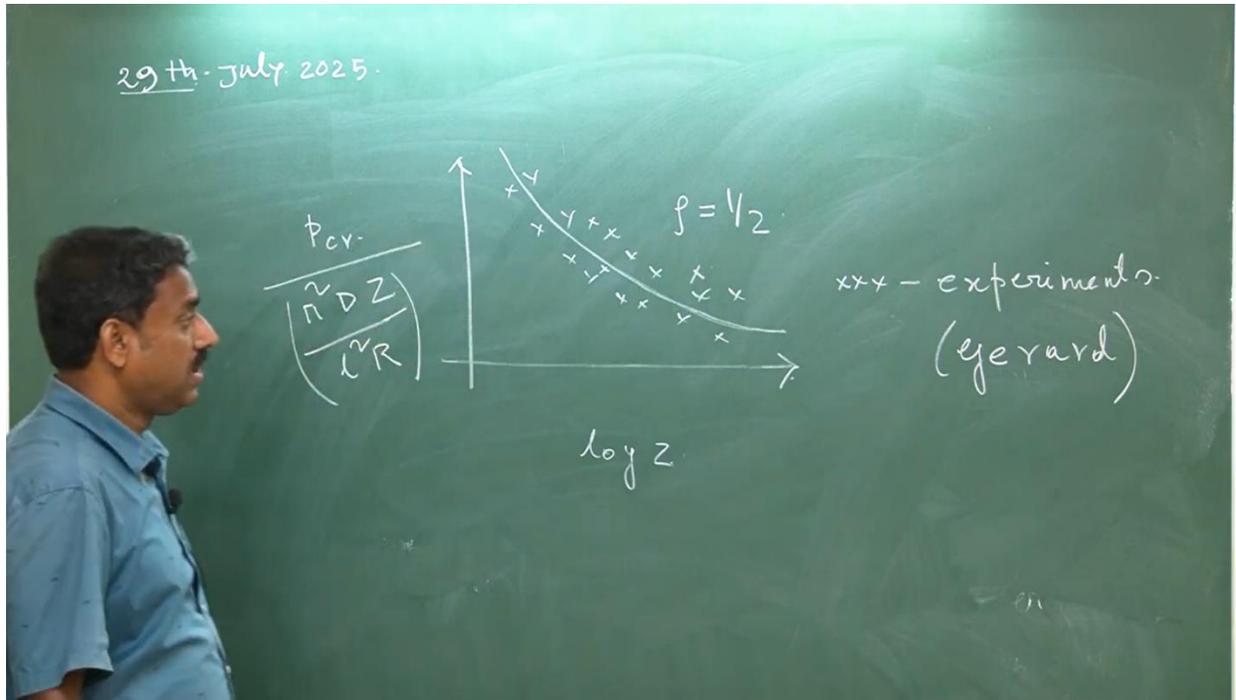
So, you see that there is a reasonable match. So, what is really happening over here? The thing that is happening here is that if you see the P critical for different combinations of k and an, you will see that ok or $\alpha\beta$, rather than, are not well spaced. Here, the modes are well separated. The modes are well separated, which means there is no multiplicity. No modal multiplication, okay. So, they do not interact, since there are no interactions among these modes which are coupled; there is no decrease, and the post-critical load drop is not there. So, there is a good match between the experiment and theory. So, what do we see? This is an interesting fact that a cylindrical cell, if it is only axially loaded under axial compression, shows imperfection sensitivity because of the closely spaced modes that interact, right? And there is a huge difference between the experimentally observed load and the theoretically predicted load. If the same shell is subjected to axial compression accompanied by lateral pressure, okay. Then the imperfection sensitivity is essentially arrested because the critical loads are no longer closely spaced, and there are well-separated critical mode shapes, you know, well-separated for buckling, and there is no interaction. So, an interaction will not be significant; you see that, right? So, the lesson from this is that whether

it will be imperfection sensitive or there will be multiplicity modes also depends not only on the geometry of the cell and its parameters, you know. but also depends on the mode of loading and what kind of loading it is subjected to. One more solution I am going to present is based on Donnell's equation, and then we will move to the next. So, here is the same cylindrical shell, a cylindrical shell subjected to torsion. We are coinciding, you know, and here it is. So, this will be all you know. You see all twisted, right? You see. So, there will be all this helical; this structure is a helical structure, right? So, you see that the wave will assume $w = w_0 \sin(\beta y - \alpha x)$. Because this helix can be represented using this instead of the multiplication of the two cosine and sine functions satisfying the boundary, here we do not care about the boundary condition. Please note that there is no inclusion of boundary conditions. It will be extremely difficult to satisfy the boundary; it is not possible, okay? So, $w = w_0 \sin$; this basically mimics the helix, and then you substitute in Donnell's equation, right? And if you do see substitute, then of course here, as you see where you will get N_{xy}, only the critical load you will get here; there is no N_x, there is no N_y, there is only nxy, so shear stresses, right? It will be expressed as:

$$N_{xy}^{Cr} = \frac{Dn^2}{2R^2 k}(1 + k^2)^2 + \frac{Ehk^3}{2h^2(1 + k^2)^2}$$

$n = \beta R$, and $k = \alpha/\beta$; circumferential wave numbers are ok β, and k is the slope of the helix. Slope means this one, ok. So, this one, when you take the ratio of β to α, then this slope will be there, right, with the longitudinal axis. This is the slope, okay? The slope of the helix is relative to the generating circle of the cylindrical surface. Now, we are not considering boundary conditions because it is somehow valid for a long cylinder in which, you know, we are assuming that this solution is away from the boundary. So, whatever for far field solutions does not really influence it. What happened here, see, here in torsional, you know buckling, there will be a much smaller number of wave numbers here, but if you go there, there will be a very small number of wave numbers here, okay. Many waves will be there, but here relatively fewer, 1 or 2. So, from here, if you do minimization, it's okay. So, by minimization, that is what we want to minimize, we will see that there is a condition to be satisfied: this slope of the helix $k^2 = 1$. And then you write:

$$N_{xy}^{Cr} = \frac{1}{2}\left\{\frac{Dn^2}{R^2 k} + \frac{Ehk^3}{h^2}\right\}$$

if you do minimization, okay. That can further be simplified. See, k is very, very small; $1 + k^2$ can be taken as 1, okay. Because in this expression, $1 + k^2$ is 1, that is what we have substituted. So ultimately, you know, we get this expression. can further be simplified to be equal to:

$$N_{xy}^{Cr} = 0.1366nEh \left(\frac{h}{r}\right)^{3/2} (1 - v^2)^{-3/4}$$

Here, you can clearly see there is an "n" here.
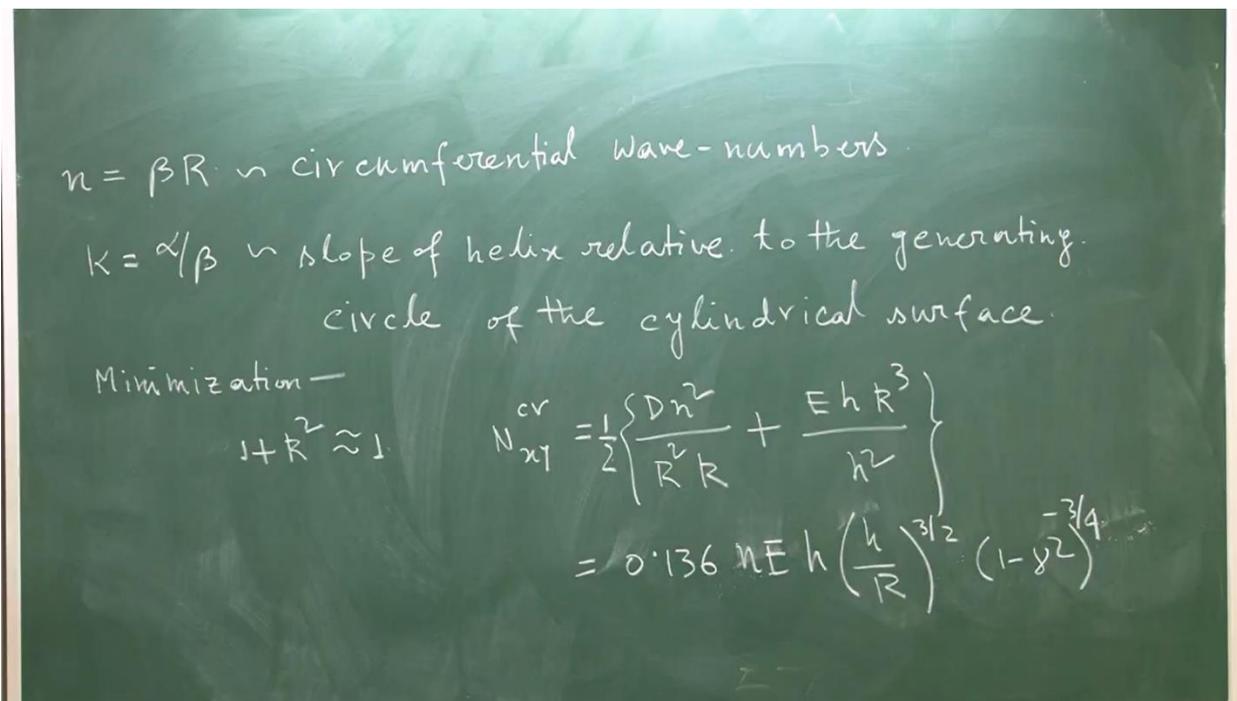
So, the minimum load will occur at 1 when n is 1. What does n being 1 mean? There is only one wave, number here. But one wave number it cannot; it is not a physical thing, right? What does $n = 1$ mean? A single buckle along the circumference, right? So, let me write it down, you know? I will just write N_x; k is basically the slope of the helix, and h is actually very small. So, that is how this simplification can be made, and then ultimately. So, what we see is that:

$$N_{xy}^0 = 0.136nEh \left(\frac{h}{r}\right)^{3/2} (1 - v^2)^{-3/4}$$

So, here you see that $n = 1$; then only N_{xy}^0 attends to the minimum. But when $n = 1$, the problem is that there will be only a single wave number here, and if there is a single wave number, it will

not be able to satisfy the in-plane equilibrium equations. In-plane equation, see, Donnell's equation is basically out of plane, right?
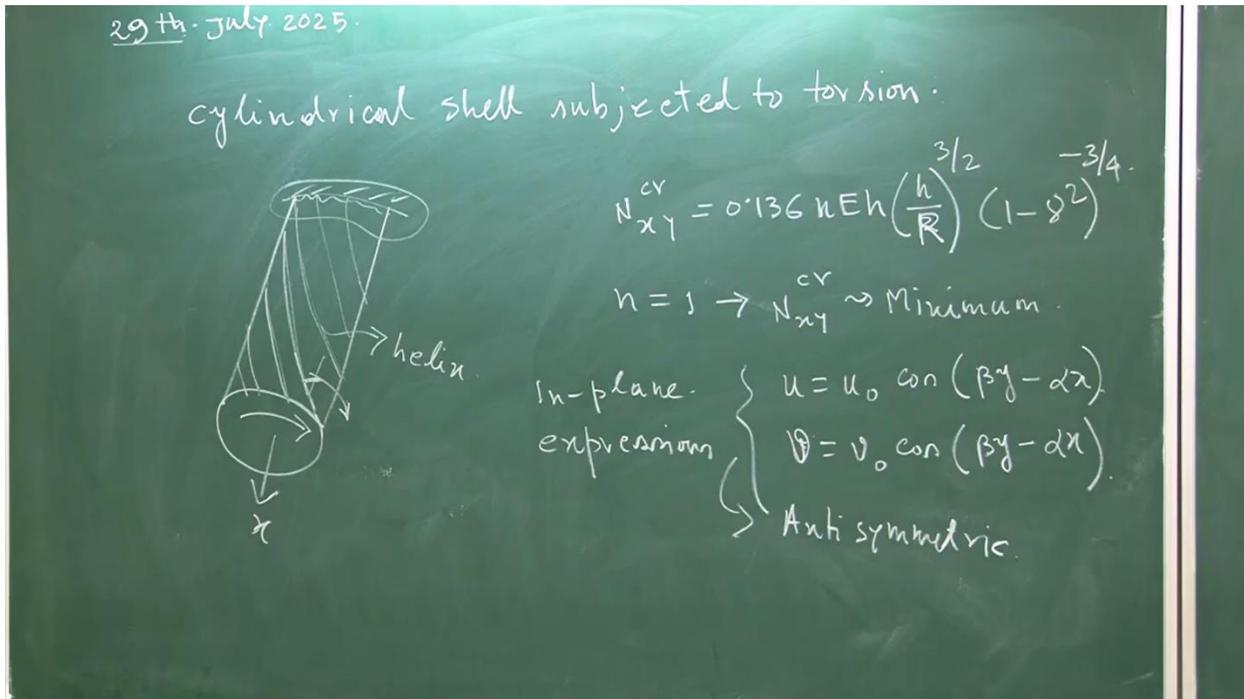
So, the in-plane one is automatically satisfied because we have introduced the stress function, which we substituted. But if you see the in-plane equation, if you want to substitute the strain displacement, okay. And then if you derive, there will be u and v, of course, in plane displacement. The problem is that in plane equations I am not writing here, but it is not very complicated, okay? It is, in fact, much lower in order when comparing with. So, in plain expressions, I am writing okay. In plain expression, if you want to satisfy $u = u_0\cos(\beta y - \alpha x)$ and $v = v_0\cos(\beta y - \alpha x)$. So, if you substitute this in the plane equations, the governing equation that will satisfy. The thing is that here, if you see, these are anti-symmetric. That means if you twist it here, then you see the deformation pattern must be anti-symmetric, right? So, a single wavenumber will not be able to accommodate this thing. So, that is why $n = 1$ is not a viable solution. Now for a viable solution, this must be anti-symmetric, okay. Antisymmetric distribution of axial displacements. The antisymmetric of n, u, and v is basically axial displacement. Axial displacement means, you know, that it is x, y, z, and so on. These are all Ws, right? And x, y, right? So that is what all this cross-section is permitted to do with respect to the longitudinal axis. That is what you have. So, the feasible one is basically for $n = 2$. And $n = 2$ is a viable solution. So, when $n = 2$ and you substitute, you will see that this is a viable solution for it. The only thing is that we never talk about

the boundary condition; see, it is difficult to satisfy the boundary condition. Because that will violate whatever simplification that is, what happened actually is that $n = 2$, which means there are 2 wave numbers in terms of the helix, whatever you have assumed, right? Because that is also conducive to satisfying the axial. The in-plane forces mean in terms of u and v axial displacement, and that is why we require an anti-symmetric field, right? But what actually happened is that at the end boundary, the assumption of the two buckles is not valid. There can be a number of buckles over there and that at the boundary, okay. That is not basically captured by the tsunami that does not, you know, cause a lot of problems that can be accepted as an engineering solution, okay. Another interesting fact here is that when the sail is subjected to torsion loading, it is not, however, the critical load, as you can clearly see from here. The critical ones, $N1$, 2, and 3, are not closely spaced; they are well separated. So, once again, there is neither any mode multiplicity nor closeness of mode. So, there is no modal interaction and subsequent degradation of the critical load. As far as torsional loading is concerned, the cylindrical shell is not imperfection sensitive. So, just as you know lateral pressure, torsional load is also not imperfection sensitive. A cylindrical shell under torsional load experiences torsional buckling, which is not imperfection sensitive. So with this, I will stop here; you see that we have used the Donnell equation to solve the critical load for different cases. Please note that none of the solutes are cases in which it is not imperfection sensitive; that means there is no multiplicity, closeness, or mode. The Donnell's equation gives very accurate results because it matches the experiment. But the case in which there are huge modal interactions and multiplicity or closeness modes, then whatever we have obtained is not valid. Because experimental loads are way lower than the theoretical load, we need to consider the additional formulation. So, that is what we are going to do now, okay. So, we could have directly used this thing, but we will use a different formulation for that. So, now what we will use, until now, we have concentrated on the critical, finding out the critical load under different kinds of loading.

29th. July 2025.

Post-critical analysis of cylindrical shell
(Axial compression)

$W_0 \sim$ imperfection.

initial imperfection —

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right).$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w_0}{\partial y}\right)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) + \left(\frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y}\right)$$

Now we will see the post-critical analysis of the cylindrical field, okay. Post-critical analysis means beyond the buckling of cylindrical cells. And when we are considering post-critical analysis, we know what kind of loading under axial conditions; of course, we will consider axial compression. Why axial compression? Because we know that the buckling modes are closely spaced, they are liable to interact. So, here are the governing equations: So, all these things will be very similar, except that you have to incorporate the initial imperfection. Initial imperfection we have to incorporate, so:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)$$

See, I am not sure how this is happening, but it is not very difficult; it is very simple. See, all these; this was the in-plane displacement, and this is the von Karman non-linear term. These are the additional terms: w is the out-of-plane deflection, and w_0 is the imperfections. This is the imperfection amplitude, okay? So, there are additional terms coming in the strain expression. Please note that $\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w_0}{\partial x}\right)$ This is the imperfect thing. This is the imperfection. This is the imperfection, okay? It is not difficult. It can be, you know, if you consider the initial deformed geometry, and if we just follow the way we derive the strain distribution, you will get it. So, if you substitute that thing into the stress function expression and then the way you have derived the

equilibrium equation, I will write down only the final expressions, okay? So, please note that these are very similar to Donnell's equation, but we are not going to convert it into Donnell's terms, okay? We will have two separate equations, one for the out-of-plane equilibrium in the out-of-plane directions, which means the gate direction, and then another one is the compatibility equation that is expressed in terms of the stress function. So, we are going to write this. So, the one we are going to write is:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

$$-\frac{h}{D}\left[\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2}\right) - 2\frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 w_0}{\partial x\,\partial y}\right) + \frac{\partial^2 f}{\partial x^2}\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w_0}{\partial y^2}\right)\right]$$

$$= 0$$

This one is the out-of-plane equilibrium equation, out of plane. As you can clearly understand, in terms of W ok, this is the equilibrium equation, and then we substitute the strain displacement equation because of imperfection. Please note the effects of imperfection.
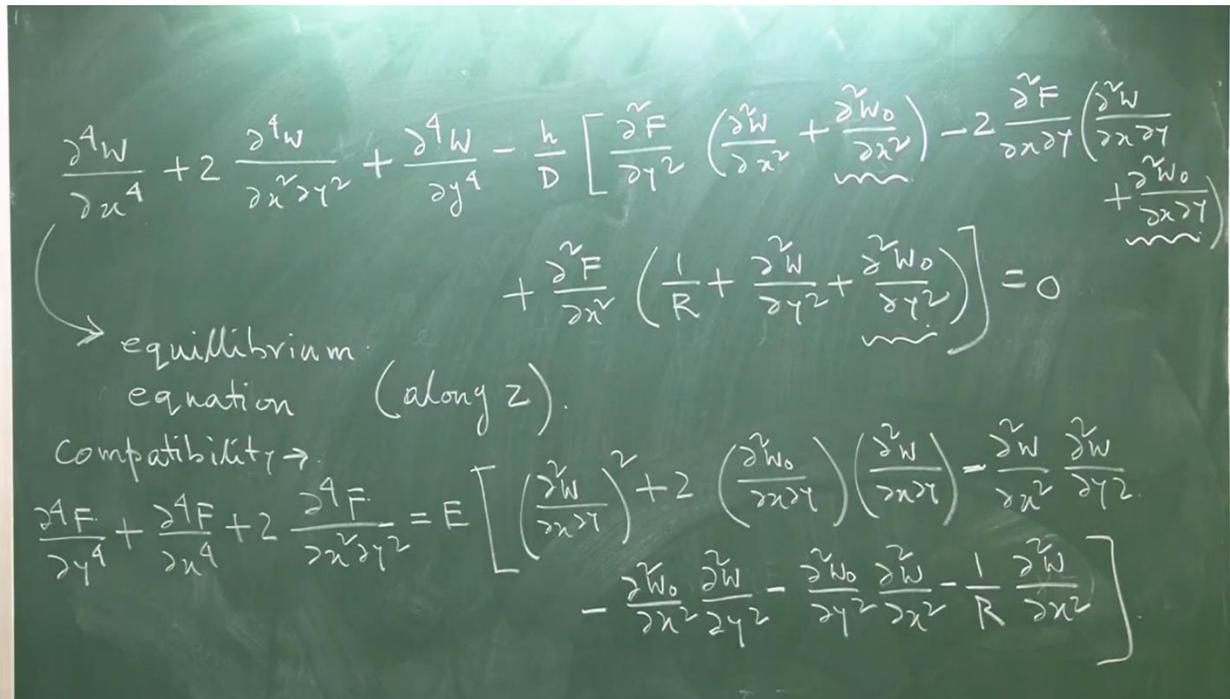

So, this is the effect of imperfection; this is the effect of imperfection; this is the effect of imperfections, okay. So, this is nothing but the equilibrium equation. Of course, it is expressed in terms of displacement. But this is essentially out-of-plane equilibrium. Equilibrium equation, out-

of-plane direction expressed in terms of displacement, okay. Out of plane, along the z out of plane direction, okay. Now, the other one is compatibility; that is what I am also writing, okay. So, the compatibility equation when we are writing, I will write this one something like this:

$$\frac{\partial^4 f}{\partial y^4} + \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2\,\partial y^2}$$

$$= E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + 2\left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right) - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right.$$

$$\left. - \frac{1}{R}\frac{\partial^2 w}{\partial x^2}\right]$$

This is the compatibility, okay? Now you can clearly see that the nonlinearity, of course, other than von Karman nonlinearity, we are not considering all the terms in the, you know, large strain, but the von Karman nonlinear term, okay. But they are, of course, so these two equations are a little big, complicated, and nonlinear equations, okay? And it allows the redistribution of the inflamed forces, and of course, N_{xx} is nothing but what can be derived from the second derivative of the stress function, right? So, we have to solve it correctly.
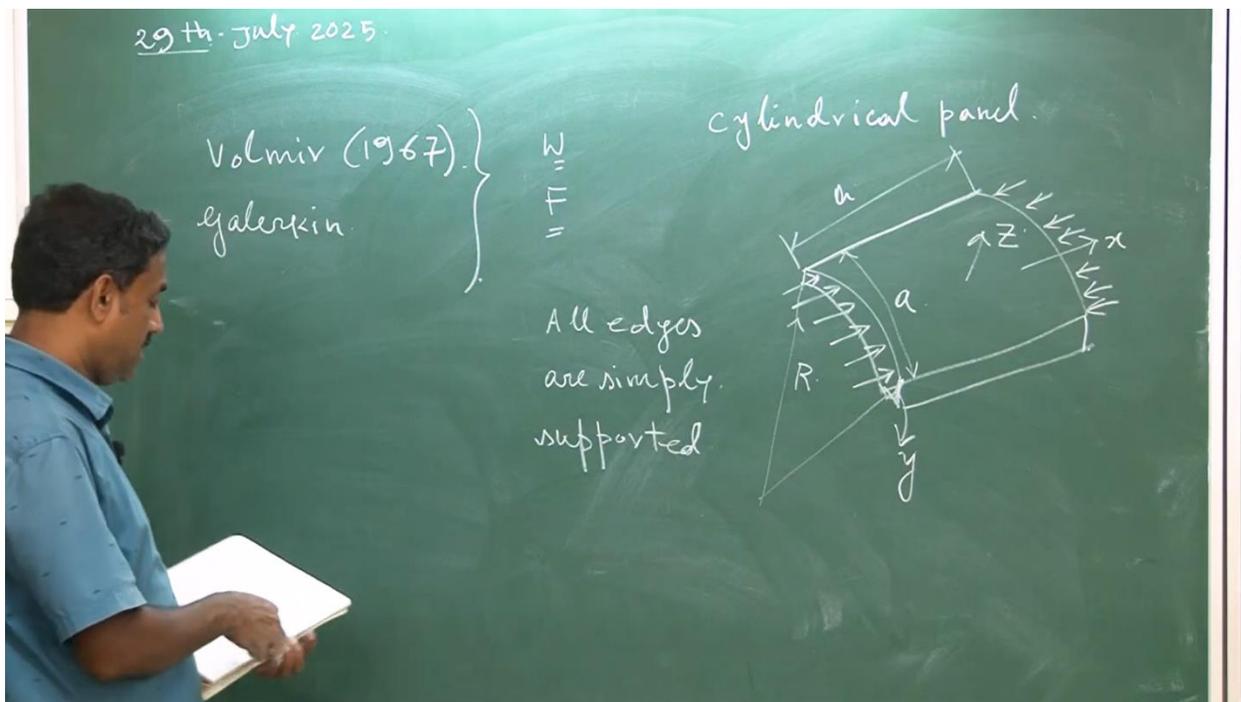
So, in the next class, we are going to solve it, and of course, we will use this Galerkin method, which is given by Volmir. So, Volmir's solution around 1967 is a very little ad hoc solution, but

this is a solution that can be discussed in the classroom because otherwise, this is a complicated matter. These equations can be solved to demonstrate the post-critical load drop, which otherwise is not captured by this solution, using the Bubnov-Galerkin method. So, this is based on an assumption. We assume some w, we have some f, and then based on that, we do. So, we will consider the cylindrical panel; we cannot solve a whole cylinder, but we will consider the case of a cylindrical panel. So, a cylindrical panel and then for simplification, only one curvature is something like this: a cylindrical panel, something like this, and then you know. This one is R, you know this one is Y, this one is X, and you know this one is Z, okay? And we are going to consider that this side is length a and this side is also length a, okay, both sides length a for simplification. So we will assume that, of course, it is subjected to some axial forces in this. Axial force, compression force from there, and all other factors; these two ends are simply supported, and all the edges are simply supported because that will simplify the assumptions of some solution, right? Simply supported, okay. And you will see how we will be able to solve it. So, this solution using Galerkin and Bubnov-Galerkin will be done in the next class. Thank you very much for today's class.