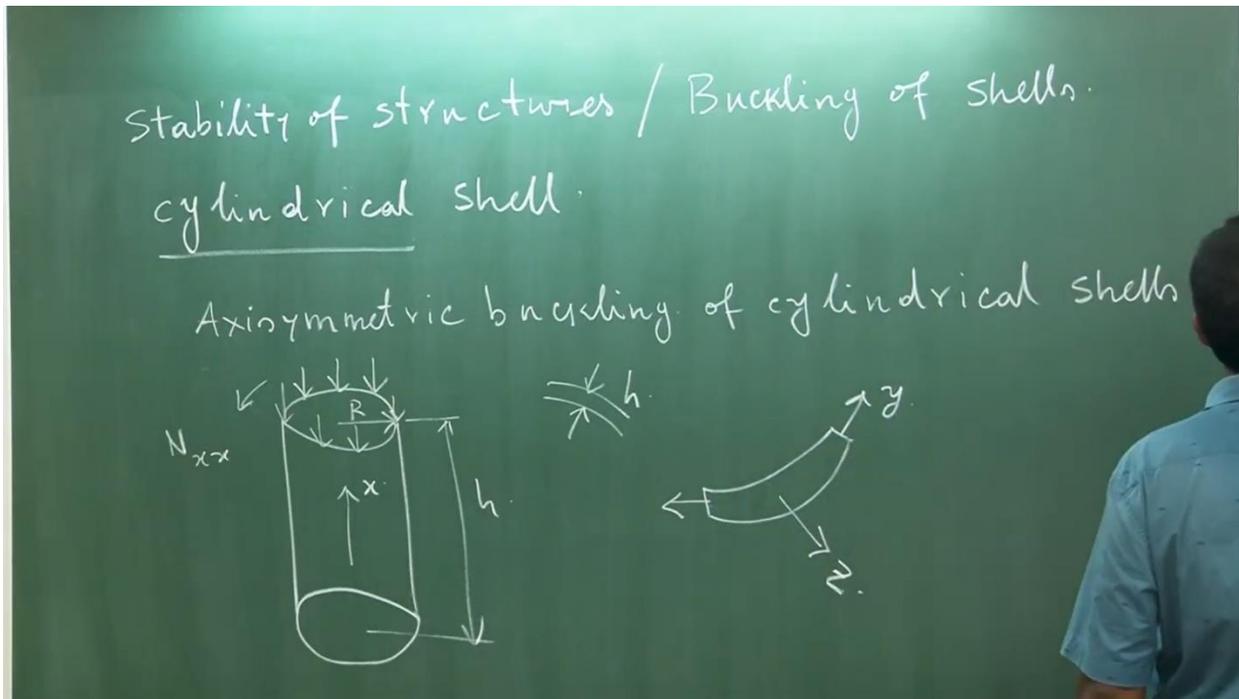


Stability of structure
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WEEK-12

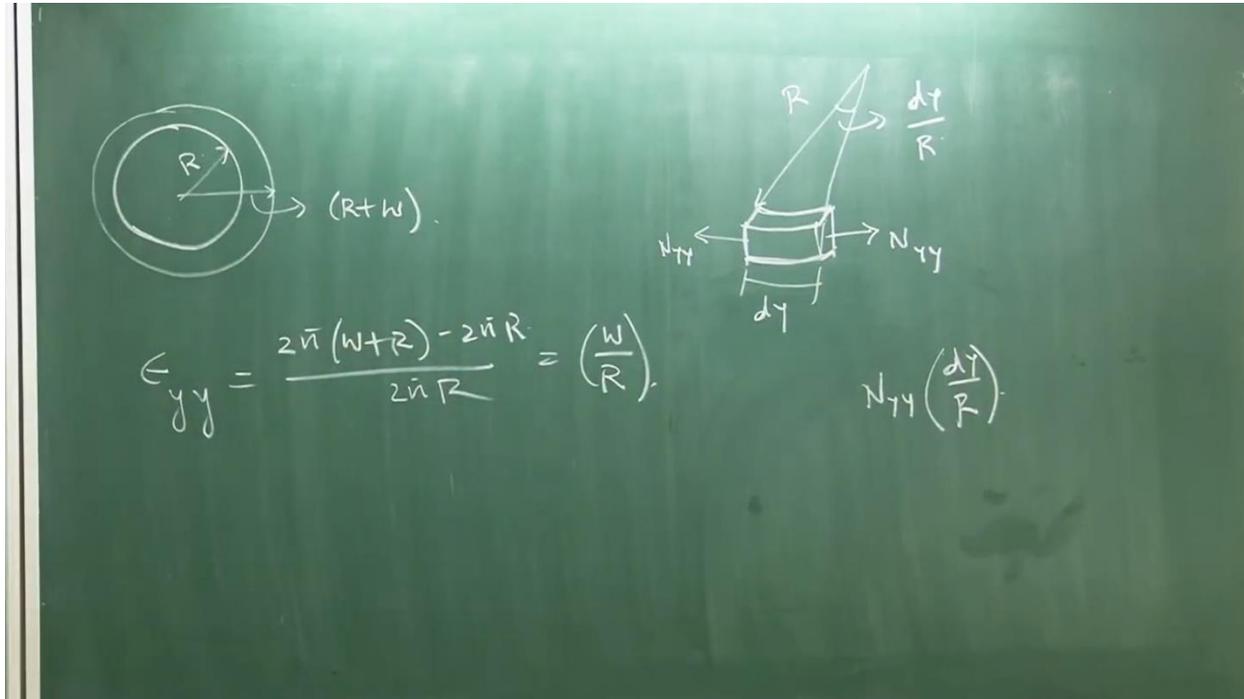
Lecture 23: Donnell Equation for Cylindrical Shell

Okay, good afternoon, everybody. So, here in the courses on the stability of structures, including today's lecture, there are three more lectures remaining, right? So, you are discussing the buckling of the shell. So, first we have considered the buckling of a cylindrical shell. So, if you can recall, we have started the buckling of cylindrical shells and have considered the simplest cases, where we consider axisymmetric buckling of cylindrical shells. An asymmetric buckling of a cylindrical shell occurs.



Let us briefly recapitulate what we covered in the previous class. So, you have considered a cylindrical shell, and you know the radius r , right? And this is the length L , correct? And it has a thickness, shell thickness h , or you can define it in some books as t . And then it is subjected to axial compression, you know, load along the periphery; that is, the intensity we are defining is

N_{xx} , where x is defined as x , and y is defined as z , which is perpendicular to the surface. So, this is this, and this Y -axis is something defined like that, and Z is perpendicular to the surface right along the thickness direction.

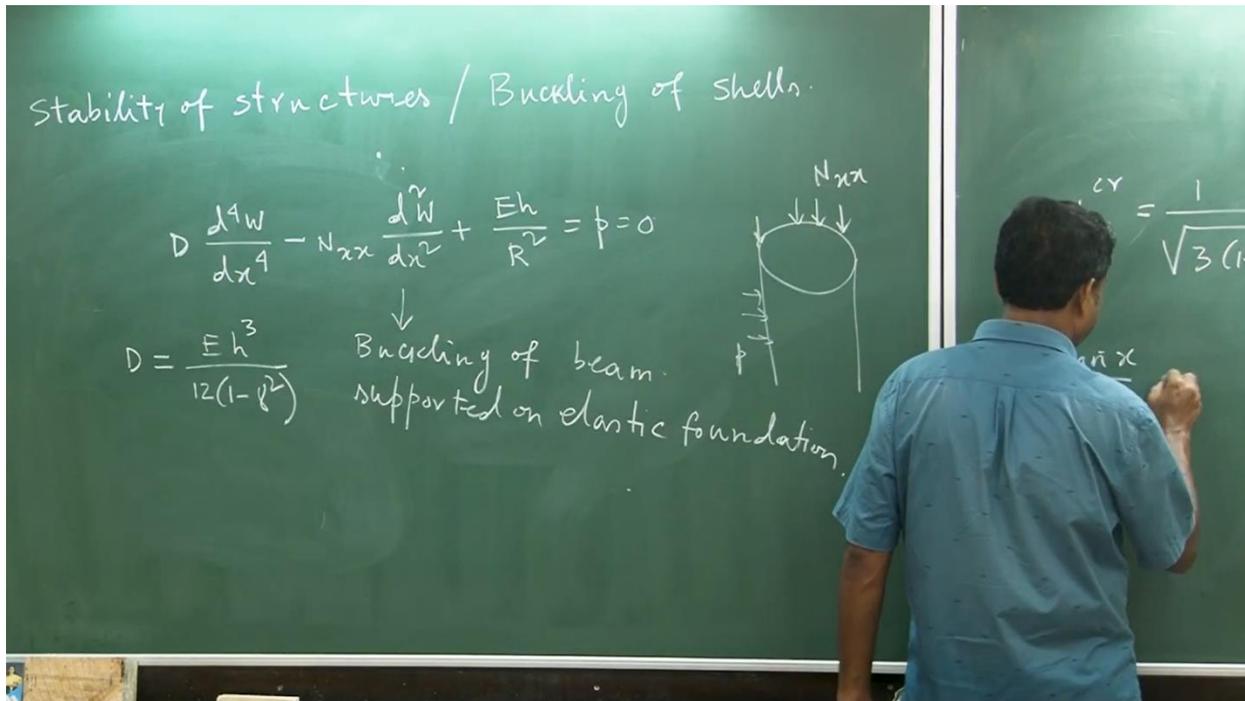


So, the only assumption in the axisymmetric buckling of the shell was the fact that we are considering the strain ϵ_y because it is deforming axisymmetrically, right? So, r coming, you know the radius is $(R + w)$. So, ϵ_{rr} or ϵ_{yy} was

$$\epsilon_{yy} = \frac{2\pi(w + r) - 2\pi r}{2\pi r} = (w/r).$$

So, essentially (w/r) is the radius of the cylindrical cylinder, right? So, with this, we have considered the equilibrium of a small strip, you know, right? We have considered the equilibrium of a strip like this. We have taken it like this, and then it was considered unit length. So, N_{yy} is here, and then it is making some angle here. This here is dy , and this is radius r . So, this angle is (dy/r) . Therefore, N_{yy} will have some component along the radial direction, right? So, the membrane stress N_{yy} will have some component along the radial direction, which is N_{yy} essentially $\sin(dy/r)$. So, (dy/r) is known, and then we consider the equilibrium along this direction; then we obtain the governing equation that we discussed in the previous class. So, I am

just briefly recapitulating it. So, the governing equation that we have derived for this is something like this.



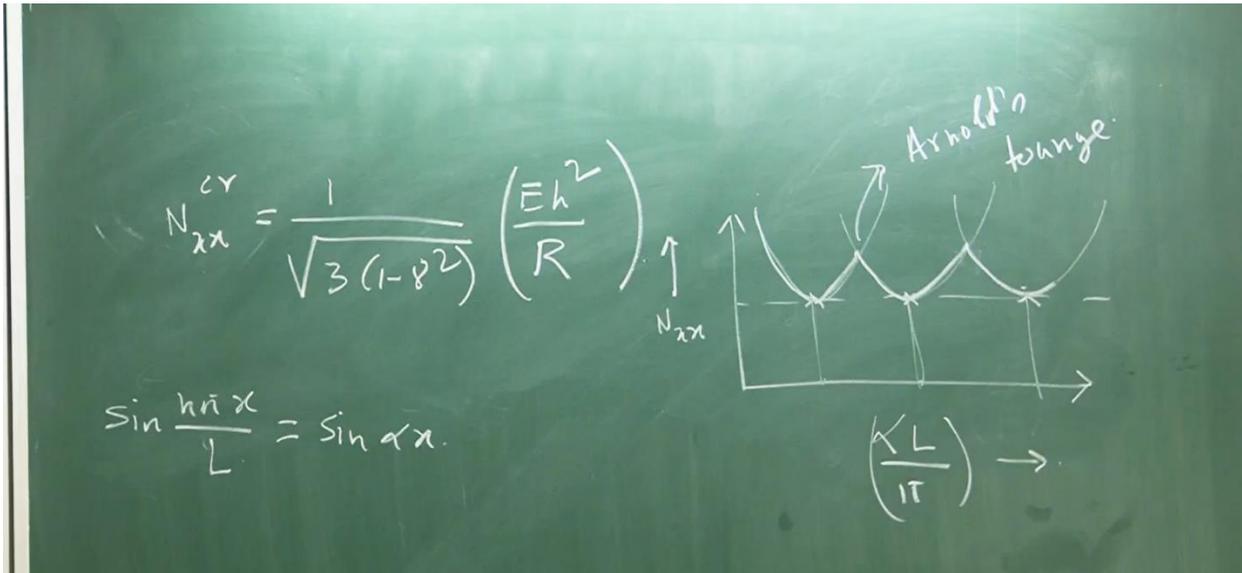
So, we have considered the equilibrium of this strip in the vertical direction. So, N_{yy} will have a component, right? Then we consider the flexural rigidity of the strip, and ultimately, the governing equation is $d^4 w/dx^4$. So, there is only one variable because it is axisymmetric, which permits it to be a one-dimensional equation in terms of space.

$$D \frac{d^4 w}{dx^4} - N_{xx} \frac{d^2 w}{dx^2} + \frac{Eh}{R^2} = P = 0.$$

P is the lateral load. If this cylinder is subjected to some lateral load, you know, I mean, circumferential pressure P , then this P , of course, is 0 for buckling. So, you can clearly see that the axisymmetric assumptions basically reduce the problem to one dimension. So, it is expressed in terms of D . Essentially, what is happening is that we are considering the bending of this strip, okay? Essentially, we are idealizing this as a plate and then writing down the equations, okay? D is the flexural rigidity of this strip. What is the flexural rigidity of this strip? If its thickness is T , then, if you can recall, it is

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Now is the ratio of the question right, and then r is the, so what we can see is this equation similar to what? The equation is similar to the equation for the buckling of a column.

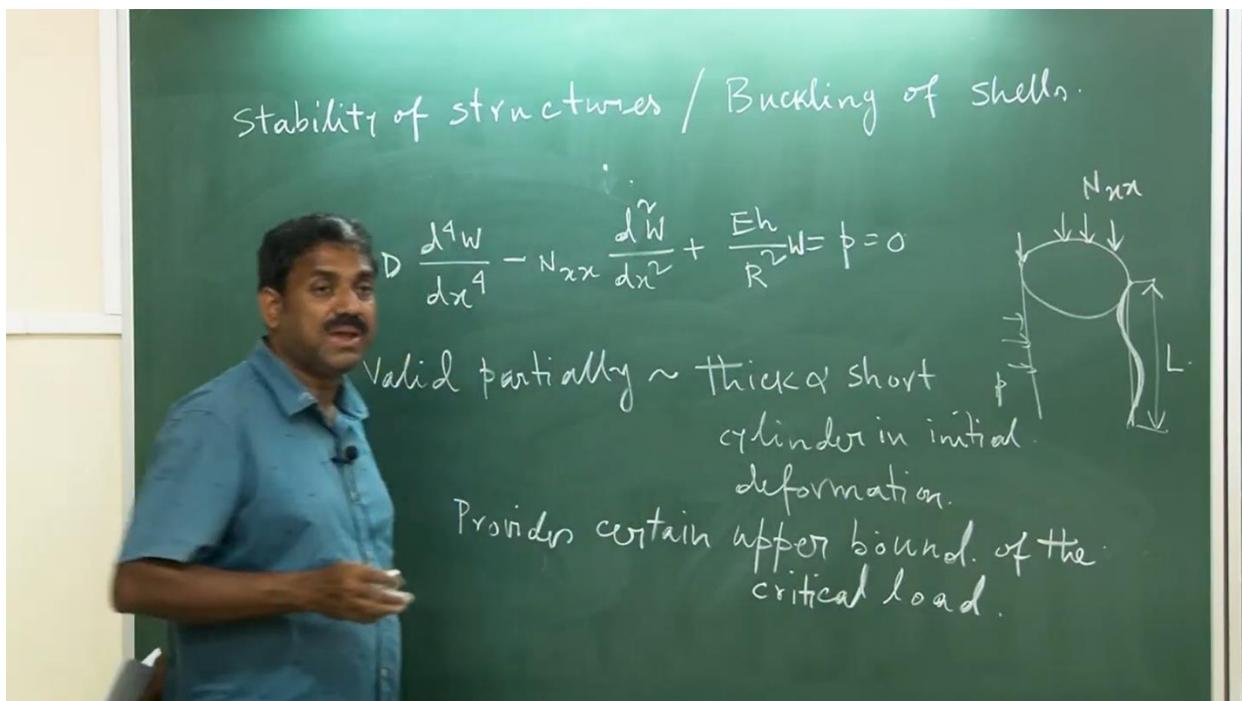


Of a column or beam, buckling occurs in a beam supported on an elastic foundation. So many things can be idealized like that; for example, a pile or a soil-structure interaction problem. So, you see, this is so you can solve these equations, and then you have solved them in the previous class to obtain this, which is of course the eigenvalue problem. Because it is homogeneous, it leads to a trigonometric eigenvalue problem. So, the solution N_{xx} critical value is N_{xx} the critical load, whatever the critical value of the load N_{xx} , ok. It is critical that

$$N_{xx}^{crit} = \frac{1}{\sqrt{3(1-\nu^2)}} \cdot \left(\frac{Eh^2}{r} \right)$$

So, you see this is the equation, right? And then, while getting out the solution, what we have assumed is that we assumed the boundary condition to be simply supported. So, that means the out-of-plane displacement is restrained, and the bending moment is zero. So, that will convert it into $\partial^2 w / \partial x^2 = 0$, at $x = 0$ and $x = L$. Right and w are equal to 0. Of course, the shell has to allow for axial deformation. Because only then will it be buckled, and then I assume in terms of $\sin(n\pi x/L)$, something like that, or basically $\sin(\alpha x)$. So, of course, what we have seen is that

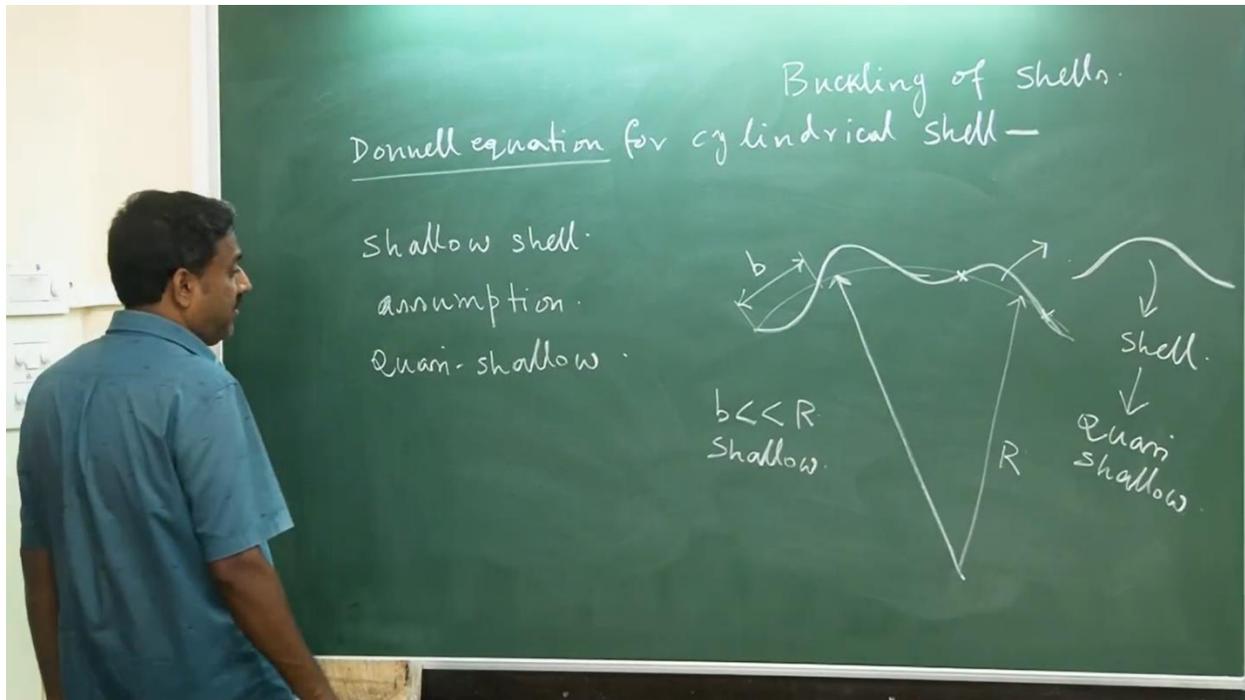
$(n\pi/L)$. You have to show the buckling mode we have assumed to be something like this; you know, something like this, sinusoidal in pattern. Now the maximum, so it is similar to the solution that you assume from the plate buckling. In the plate, we have seen that in one direction, if you make it very long, then what happens is that the buckling load attains a minimum if the number of half wavelengths, the length, and the boundary condition are such that. If it can accommodate, say, an integer number of half-wavelengths, then it attains a minimum buckling load, right? So, the boundary condition should be such that this length can accommodate an integer number of half wavelengths, right? That was the condition to have a minimum, so this corresponds to it. So, otherwise, if you can recall that, just like a plate, we can have this, you know? $(\alpha L/\pi)$, and then here it was N_{xx} . So, otherwise, you will see this kind of thing. So, you can see that if this is an integer $(\alpha L/\pi)$, then if it is an integer, you know the critical load is at a minimum, right? Otherwise, if it is not an integer, then it is increasing, you know. So, if you mark the envelope, it looks like this kind of tongue-like structure, and this is called Arnold's tongue, right? This is similar to the plate equation, right? The origin of these things lies in considering the out-of-plane equilibrium equation for the strip, you know.



So, this is highly idealized. So, in this diagram, you can see that if it is not able to accommodate the sales geometry boundary condition, the sales length is not conducive to accommodating an

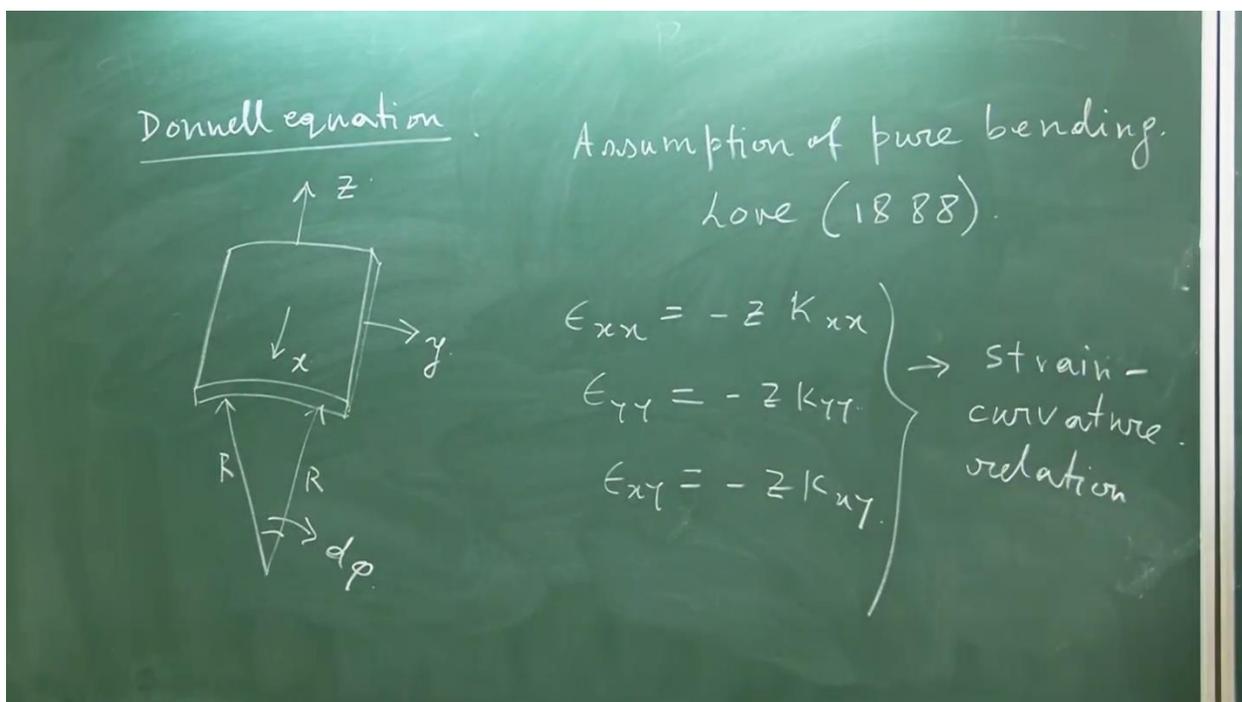
integer number of half-wavelengths. Then the critical load will increase; otherwise, if it is an integer, it will minimize, and that value is as follows: So, this is a very idealized situation given that we assume it is axisymmetric. But please note that none of the shells, cylindrical shells, truly follow this kind of assumption. Well, if the cylindrical shell is like this, it is partially valid for thick and short cylinders in the initial deformation range. So, initially, when it tries to buckle, that means it is trying to lose its stability; then, if the cylinder is thick, if it is short, then it can follow this kind of axisymmetric deformation pattern; otherwise, it cannot. Most of the time, it deviates immediately, even if it initially follows this pattern of being axisymmetric; however, it goes to a non-axisymmetric pattern immediately. And hardly is this kind of load reached experimentally, okay, because of this fact; but that is not a very big problem. The problem lies elsewhere; the issue is that, because of nonlinearity and the interaction among modes, you know the buckling load of this kind of cylinder reduces significantly; it can be as high as 60 to 80 percent, okay? And that is what this is about. You know, this has only academic value for discussion, but it does not have much practical value. However, while it still captures all these gross kinds of assumptions, it captures a very important fact: the dependency of Eh^2 on R . This is a very interesting thing. It shows the dependence on both the thickness and the radius of the shell. This dependency is correct, whatever be the thing. So, this qualitative aspect is captured here, okay? And what it also says is that what you see is a coupling whenever you consider this shell because of the initial curvature. You know R ; because of the curvature in the geometry, there is coupling. In the component of in-plane forces, in-plane deformation and out-of-plane deformations are coupled because the in-plane component is involved. Forces affect the out-of-plane deformation. Okay, why is it happening? You see that if you consider this, there is one W . What is happening? See, whenever it is deforming axisymmetrically, W is the out-of-plane deflection, right? This is okay, and because of this W , some in-plane strain is being developed. So, you know, W/R , that strain ϵ_{RR} , right? Or ϵ_{YY} , which is coming and is in plane strain. So, that will have some in-plane stresses developed, right? That is N_{yy} , isn't it? So, you see that this coupling and that coupling is in terms of W . So, it is first-order coupling; therefore, you see that it reveals the first-order coupling. First order means it is the order of W or coupling between in-plane and out-of-plane deformations. Of plane forces on deformation forces, okay. So, in-plane deformation affects the out-of-plane forces, and in-plane forces are influenced by out-of-plane deformation. In-plane forces and out-of-plane deformations. This is very important; this is not so in a plate. If you consider a plate, in a plate the in-plane forces

and the coupling between in-plane forces and out-of-plane deformation, what is the coupling in a plate? In plate buckling, if you can recall, these terms are due to von Kármán nonlinearity. So, $\left(\frac{\partial w^2}{\partial x}\right)$, you see that $\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$ is in the curvature, right? and $\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$. This product right, all you see these systems appear in square. What is happening? This is the von Kármán non-linear term. This is in-plane strain due to out-of-plane deformation, right? So, the coupling between out-of-plane deformation and in-plane strain or forces, whatever they may be, will result in membrane forces. It is second order, but here it is in first order; that is a very important aspect that has been captured. Secondly, it provides the deformation you know as dependency; it captures the dependency (Eh^2/r) , okay. And another thing is that it gives some kind of upper bound; you can see that. So, this is also important; it gives the upper bound. You know, this formulation provides a certain upper bound of the buckling load. Upper bound of the critical load. So, in spite of all this deficiency in this axisymmetric buckling shell formulation, which is a pretty simple first-order differential equation, second-order differential equation, and fourth-order differential equation in one dimension, right? It is only valid in the initial regime of thick and short cylinders; nevertheless, it deviates as soon as it goes a little further in deformation and becomes axisymmetric. So, none of the cylinders really buckle in symmetrically; you know, except that if they are thick and short, they will be asymmetric. But nevertheless, it captures several important features. One reason is that it provides a reasonable upper bound and reveals the first-order coupling between the in-plane forces and the out-of-plane deformations. Unlike the plate, you know, where the coupling is of second order, it captures this dependency. Clear? Okay. So, there is a refined theory, and then there are those things to capture. So, what we are going to derive is based on the concept of plate theory, you know, the buckling of plates that we have covered. We are going to derive the second thing now, you know? It is a little lap, you know, the very important equation that is Donnell's equation. Donnell's equation for cylindrical shells is specialized for cylindrical shells, but there is a generalized equation called the Donnell-Vlasov-Mustar equation that applies to any shell geometry. So, when you want to generalize the shell geometry, of course, you have to know a little. Differential geometry is where you need to know a little bit about covariant basis vectors, contravariant basis vectors, unit matrix tensors, and things like that. So, those are the things I am trying to consciously avoid here because this is not a close, you know, course on shell theory.



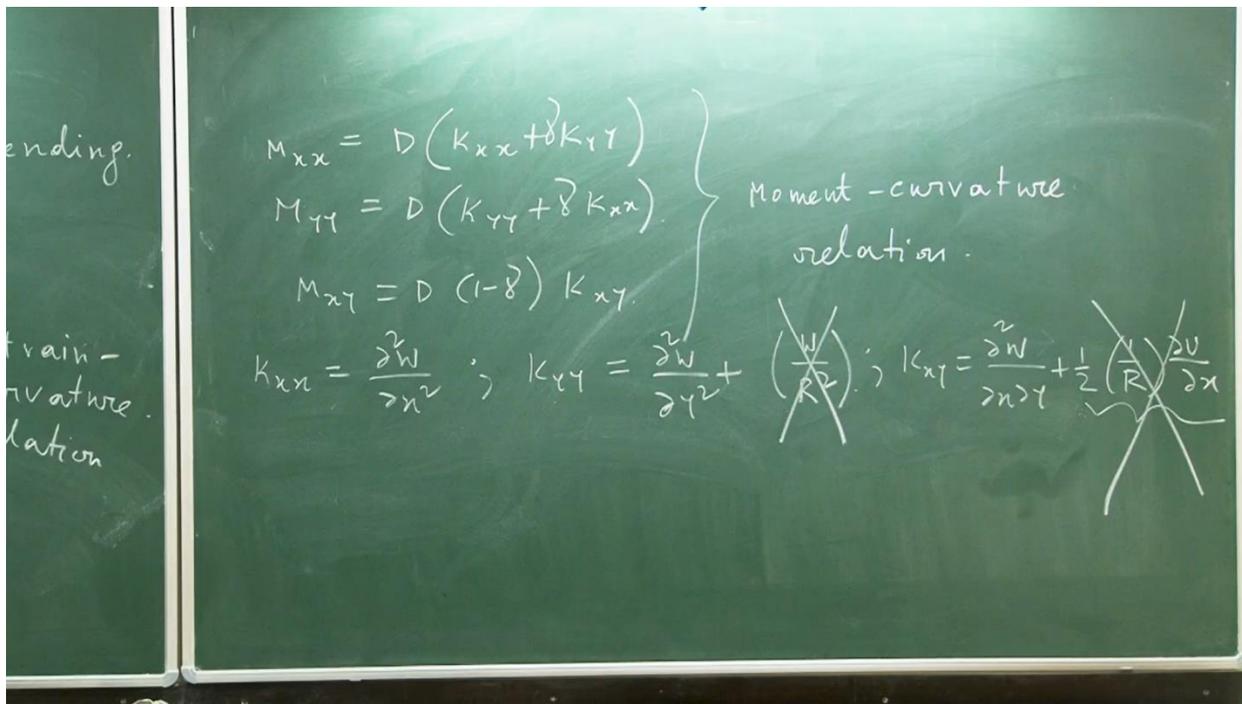
We are only considering shell buckling, essential physics, and a little bit of formalism that we will cover. I will write it down, but please note that the Donnell equation is a specialization of cylindrical shells for the generalized Donnell-Mushati-Vlasov equation for the generalized shell. Donnell's equation for cylindrical shells is okay. We will derive that cylindrical shell, and one more important thing I would like to mention is that previously, the axisymmetric buckling theory we are presenting is similar to that of a beam and the buckling of a beam on an elastic foundation. That was derived by Lorentz in 1910, by Timoshenko around 1910, and then by Lorentz, Timoshenko, and Southwell. All these people contributed to the development of axisymmetric shell-buckling theory. Which does not really represent actual facts; nevertheless, it captures several important aspects that we have discussed along with this limitation. And then comes the well-celebrated day. Also, it is very useful that Donnell's equation is applicable to the cylindrical shell. Here is what we are going to do: we are going to make an assumption called the shallow-shell assumption. A quasi-shallow shell is a shallow shell. So, what happened is that we have some shell geometry; you see that, and then it has a radius of curvature, you know, r . You know, then it is buckling. You know, there is buckling; because of buckling, there will be some undulations right here, right? So, if this is half a wavelength, you know, these are characterized by their half wavelength; okay, B is nothing but $(\lambda/2)$. If B is much less than R , then it is called a shallow shell. B is much less than R , so B/R tends to be 0. Another thing is that if you see all these half

wavelengths, if we consider one such half wavelength, then you see that this half here and here looks like this kind of thing; you see that. So, this can be considered a shell; you see that? So, this representation is called a quasi-shallow approximation of the shell; you see that. So, what is a shell? So, if we say that R is, you know, in all our theory, we will assume R to be the radius of curvature for that shell geometry, which is reasonably high. So, to ensure that the assumption of a shallow shell is valid, understand that, okay? So, the half wavelength of the buckle, you see that half wavelength; please note that the half wavelength of these buckles, right? Buckling deformation, right? The wavelength of buckles, you know, represents a shallow shell, doesn't it? which is much smaller than our radius of curvature or the radius of the shell. So, Donnell's equations give the generalized non-linear equation and its linearized version, okay. So, we are going to derive it using a different approach, okay? The original derivation was a bit complicated. So, you understand and can differentiate between a shallow shell and a deep shell, right? For the deep shell, R will be a little less okay, and B/R is not very small.



So, we are going to consider a shallow shell, okay? So, you see the way we will; you have to recall the little bit you know about plate bending theory. So, the way I am considering a portion of that is okay, and I am assuming this is, you know, the same thing we did earlier. This is Z , and this is along this thickness, you know. This thickness, you know, maybe this along this; this is y , right?

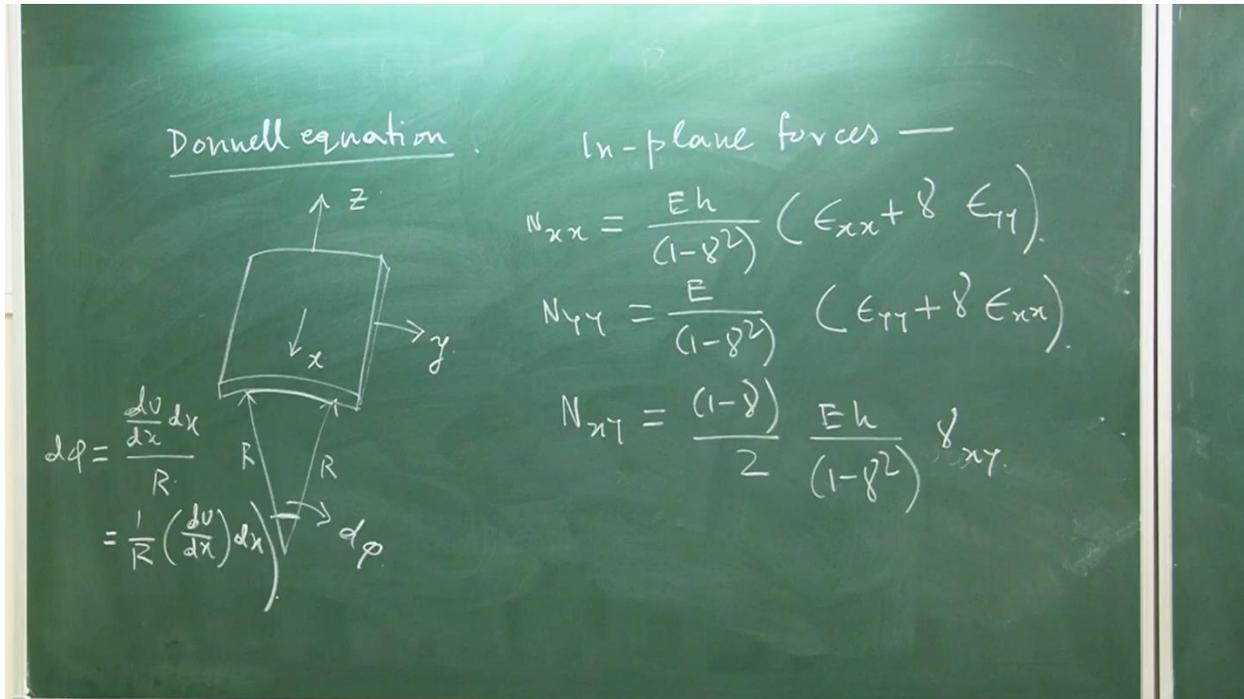
z, and then here is x, okay. So, this is r, and this angle is $d\phi$, okay? So, we are still making the assumption of pure bending of the shell, and this assumption was made by Love in 1888, right? Here you can clearly see that we can write ϵ_{xx} or plate bending; things just recapitulate a little bit of shell bending, okay? ϵ_{yy} and ϵ_{xy} ; so, if you can recall, it was $-z$, you know, κ_{xx} . $-z\kappa_{yy}$ and $-z\kappa_{xy}$, $\kappa_x\kappa$; these are all curvature terms: direct curvature, and this is cross curvature, so the plane section remains planar, you know, for deformation, right? And then, the moment-curvature relationship, if we recall, M_{xx} was $D(\kappa_{xx} + \nu\kappa_{yy})$, you know, M_{yy} was $D(\kappa_{yy} + \nu\kappa_{xx})$; this is two-way bending, that is why two-way bending is coupled by the Poisson ratio, and M_{xy} is $D(1 - \nu)\kappa_{xy}$.



This is a moment-curvature relationship, isn't it? Moment-curvature relationships. We are not deriving because we have derived that thing for the plate, right? And in plate and shell theory, in plate bending, you have learned Navier's plate equations, correct? And then you know this. So, this was Kirchhoff's plate theory, okay? There is no shear deformation, and this was the strain-curvature relationship or curvature-strain distance relation, correct? So, moment curvature, and if you can recall that, curvature can be expressed like this. $\kappa_{xx} = (\partial^2 w / \partial x^2)$; you know κ_{yy} is $(\partial^2 w / \partial y^2)$ because of this. There is a certain difference that I will just mention to you, you know,

and κ_{xy} is $\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2} \left(\frac{1}{r} \right) \frac{\partial v}{\partial x}$. So, look, because of this initial, you know? A curvature, you know, because of the geometry, unlike a plate. See, unlike a plate, which is a flat surface, the first cylindrical shell has this curvature; its radius is r . So κ_{xx} , there is nothing because κ_{xx} , if you see that κ_{xx} , it is a curvature that is infinite; therefore, it will behave like a plate. That is why it is $(d^2 w/dx^2)$, right? But κ_{yy} , that is curvature along y ; this, along here, is where $(\frac{d^2 w}{dx^2})$ is, because of deflection it is there, but there will be an additional (w/r^2) term. Why? Because if you can recall. That w/R^2 strain, and then you have to do $(1/R)$. So, there is this strain. Similarly, if you want to cross curvature that is κ_{xy} because of 2A bending, then you will see that the additional term will also be half of $(1/R)$. It is not very difficult to obtain those terms. So, I am not going into details, but you can understand where that is coming from. Now, we assume the shell is shallow, right? So, $(1/r)$ tends to infinity. So, because they have a $(1/r)$ score, we can neglect this term; $(1/r)$ is present, and we can neglect this term. Essentially, it is very similar to the plate equations, isn't it? So, it can be neglected, and the reason it is happening is that $d\phi$, $d\phi$ was how this term is coming, you see. I have explained why the (w/r^2) term came about because the (w/r) was what? Was the strain correct? So, that was divided by $(1/r)$, okay? But how this half $(\frac{1}{r}) (\frac{\partial w}{\partial x})$ is coming, I will explain. So, it is $d\phi$. Do you see that? $d\phi$ is nothing but what? $d\phi$ is nothing but (dv/dx) times (dx/r) , right? E is right, and this was nothing but $\frac{1}{R} (dv/dx) dx$. So, now this, basically, thermodynamics is going to come, okay. So, $d\phi/dx$ is nothing but the curvature. So, whatever C is, what is the curvature? Change in this angle, right? Change in the slope. So, change in slope, so C , when we are considering cross curvature, what is happening? The $d\phi$, right? What is $d\phi$? $d\phi = \frac{dv dx}{R}$. This is 1 divided by this, and then $(d\phi/dx)$ will be 1 divided by $(d\phi/dx)$, right? So, that is what this term will come across as: curvature, okay? Anyway, that is not very important, so I am neglecting it. Now, we have this strain-curvature relationship, and we have this moment-curvature relationship, right? So what else is remaining? My in-plane forces, right? Because that is very important for buckling. So, for that, I am now going to write down the influencing forces. which will be nothing but N_{xx} , N_{yy} , N_{xy} . So, N_{xx} is

$$N_{xx} = \frac{Eh}{(1 - \nu^2)} (\epsilon_{xx} + \nu \epsilon_{yy});$$



N_{yy} is $\frac{E}{(1-\nu^2)}(\epsilon_{yy} + \nu\epsilon_{xx})$; . These are the implants: shear N_{xy} , N_{xx} , and normal stresses, and shear. You can understand where these things are coming from, right? Done in the mechanics course, so moments and curvature; all these things are there. So, now we are going to substitute into the equilibrium equation; all of you know the equilibrium equation. In plane stress, these are all components, and we just define them in plane strain. And at the middle surface, okay; in plane strain and at the middle surface, okay. So, ϵ_{xx} is

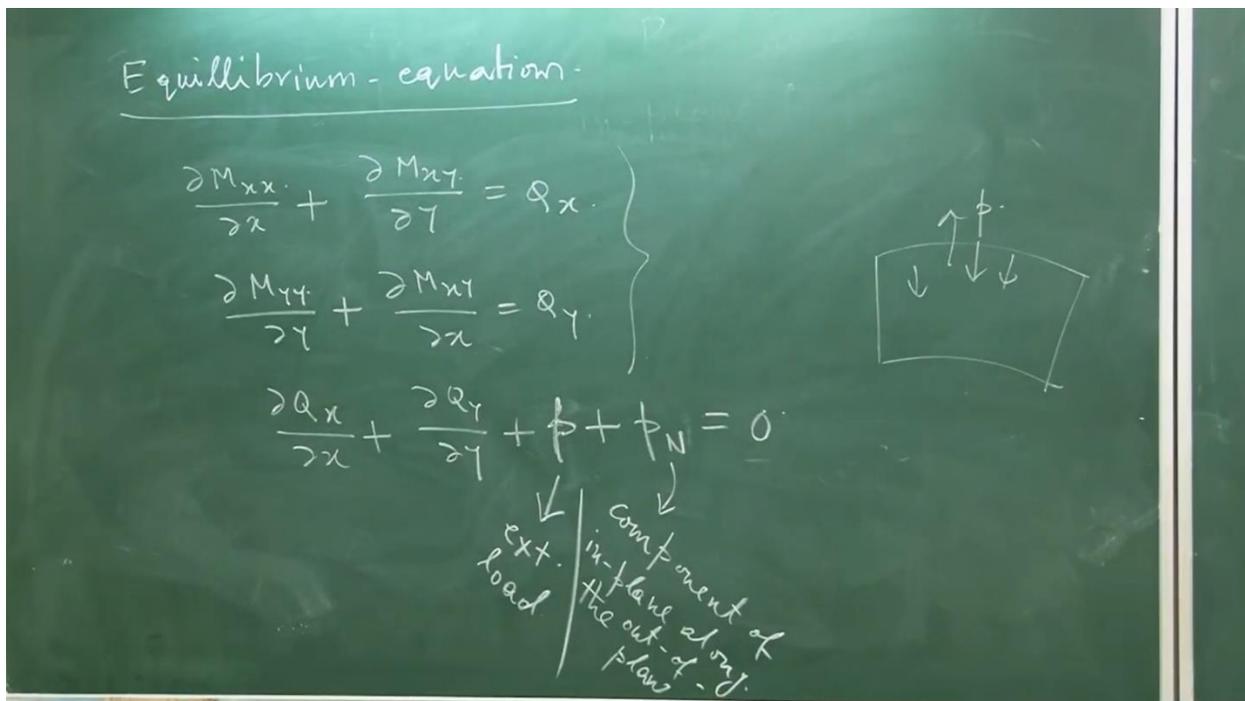
$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2.$$

Please note that when we did axis symmetry, only the out-of-plane deflection was considered, right? But here we have considered u , v , and w , all terms, correct? Ultimately, you will see that I will come there and explain it. So, ϵ_{yy} is

$$\frac{\partial v}{\partial y} + \frac{W}{r} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2.$$

So, these are mid, you know, in-plane strain and the medial surface, okay. See why we have to consider that these are von Karman non-linear square terms. Why? Because we have to express

the potential, see, for any buckling problem, our potential energy must be correct up to second order, right? So, you see that our potential energy expression must be correct up to the second order, right? All these terms are correct, aren't they? Then we will assume that now we all have everything we need to have an equilibrium equation. So, all things are more or less the same except for the fact that there are some changes in the strain-displacement terms. But then all these terms you know because of this shallow-shell assumption; some of these terms got canceled, okay. So here, when we did this, you said that at that moment, we were considering the bending moment ∂M_x ; all these equations we have derived, right? For the plate similar to the shell, Q_x was our shear, but then there is a difference.



This shear is not what we call V_x because we have considered the shear. Is the shear in the bed due to our consideration of equilibrium in the deformed configuration? So, this shear component is not strictly vertical; it is rotating to the right, and that is why it will have a component along the in-plane direction as well, right? So, it will affect that ok. And then the other equation $\frac{\partial M_{xx}}{\partial y} + \frac{\partial M_{yy}}{\partial x} = Q_y$; this and this one, how shear is related to the bending moment, right? And then the equilibrium of the vertical equilibrium, then you know $\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = P$. P means you know it is the transverse load P ; for buckling, P is not important. P is, if you know this is the cylindrical shell, you see that,

and then P is the load in the out-of-plane direction, okay? But for buckling, P is irrelevant, right? But please note that along with P , I will put $P + P_n$, two things: P . So, what is P ? P is the externally applied load in the direction perpendicular to the surface. And this is the component of in-plane forces along the out-of-plane direction, along with the out-of-plane forces. P_n is the component of in-plane forces in the out-of-plane that is P_n . So, P_n is important here because there will be a component of in-plane forces in the out-of-plane direction, correct? That is why, because we have considered the deformed configuration as equilibrium in the deformed configuration, we have perturbed this shell. So, this is one equation, and then if you substitute this equation here, you combine all these elements. This is pretty much similar to that of, say, a plate. Okay, combine these equations. If you combine all these equations, then the governing equation becomes

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = P + P_n.$$

Now, P is taken as 0; that is not extra, but what is P_n ? P_n is the, what is P_n ? P_n is nothing but a component of in-plane forces. In the out-of-plane direction. So, component, what were the components in the out-of-plane direction? One is, of course, a component of N_{yy} because, if you can recall this, you know that N_{yy} is acting like this, right? This angle is ϕ , right? This angle is ϕ . So, this angle will be ϕ . So, they are here $N_{yy} \sin \phi$, and this angle is nothing but ϕ , which is y , and this is r , right? So, then P_n will be $-\frac{1}{r} N_{yy}$, and this is one; one is coming from the intrinsic geometry of the shell. Where is this component coming from? Component of in-plane forces in the out-of-plane direction per unit length; is that correct? But then this is too long. So, the unit length will be divided by dy ; that is why, right? But then, because of deflection, the out-of-plane deflection will have a component. What do those components look like? you have done this $N_{xx} \frac{\partial^2 w}{\partial x^2}$, If you can recall that. there was this N_{xx} you know and $\partial w / \partial x$ okay Then $N_{xx} \frac{\partial^2 w}{\partial x^2} +$, so when you subtract these two, the sum components remain the same way we did in the plate. So, this one is new because of intrinsic geometry, and this one is due to deformation; it is similar to plate bending. Plate buckling is okay, and then $N_{yy} \frac{\partial^2 w}{\partial y^2}$; this is due to deflection, additional deflection, curvature due to deflection, and this is geometry, okay. Please note that this is a linear term; this is very important. You see that this is linear because r is constant, but is it linear? It is not linear. Why is it not linear? Now you may think that in the plate it was linear when considering

plate buckling; if so, it is linear. So, we can consider it to be linear. But for sale, the problem is that from the very beginning, when you apply forces, isn't it? Because of N_{xx} , you know, there will be little deflection developed, correct? And so, from the very beginning, N_{xx} , N_{xy} , and N_{yy} will, you know, be changing. So, that is why it is non-linear; this is non-linear, okay? So, from the plate, there is a difference, you know, because of its intrinsic geometry.

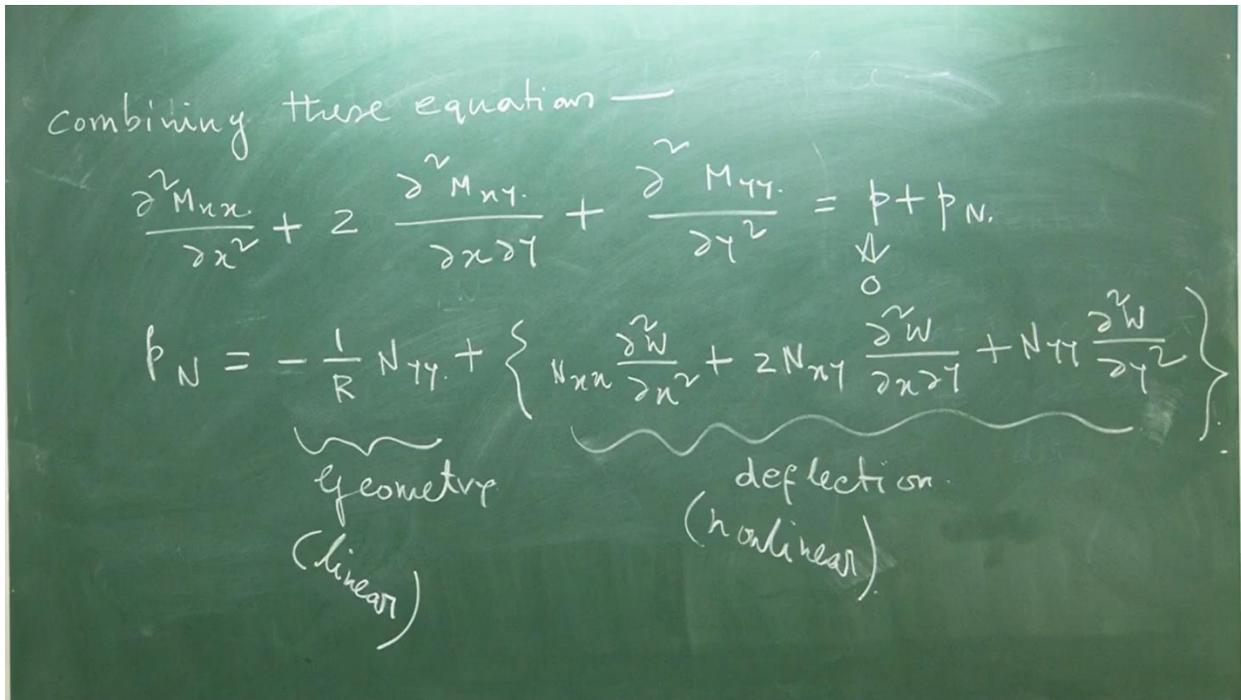
Equilibrium - equation.

equilibrium in the out-of-plane (z) direction.

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = -\frac{1}{R} N_{xy} + \left\{ N_{xx} \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_{yy} \frac{\partial^2 w}{\partial y^2} \right\}$$

$$\left. \begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= 0 \end{aligned} \right\} \text{- In-plane equilibrium.}$$

That will always be, you know, why it is non-linear, you see. Unlike plates, what happened is that in shells, there are initial conditions; you know, the geometry itself has the curvature, right? And this curvature will have some kind of initial curvature due to imperfections. So, when there are initial curvature imperfections, all of these imperfections, you know, there will be certain, $\frac{\partial^2 w}{\partial x^2}$ and because of that, N_{xx} , N_{yy} , and N_{xy} will develop, right? That will modify this term, okay? So, from the very beginning, there will be non-linearities. So, whatever comes from the equilibrium equation, these are out of plane, but then there are in-plane forces and the in-plane equation. So, we can write it down here if you want, you know.



So, this P_{NY} , you can put it here, okay? So, what will we do? So, this is deflection, equilibrium in the out-of-plane direction. Okay, out of plane means along the z-axis here, right? direction, so if you write what we are going to get here is

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = P + P_n$$

$$P_n = -\frac{1}{r} N_{yy} + \left\{ N_{xx} \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_{yy} \frac{\partial^2 w}{\partial y^2} \right\}$$

And from the 2w and 2, you see that this is the equation of the plane. If you can recall, the plane will not be affected by the curvature term, right? So,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0;$$

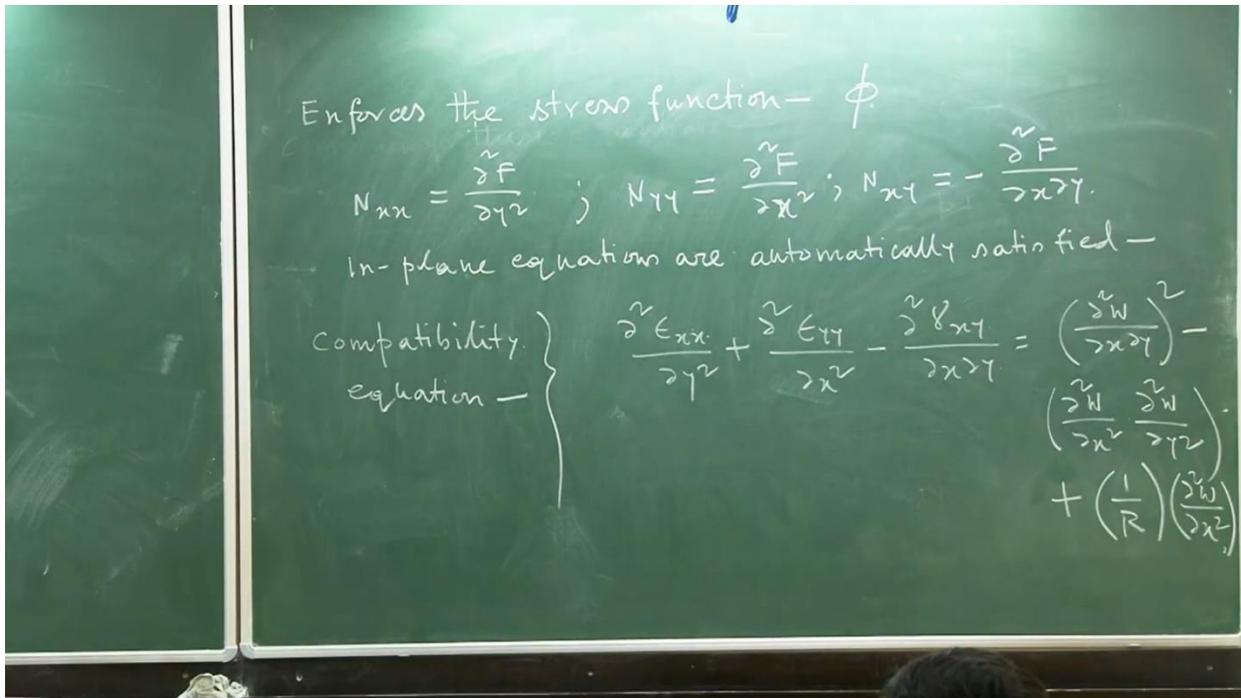
this is the same as in plate

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0;$$

right? So, this is out-of-plane deformation. This is an out-of-plane equilibrium. This is the equilibrium in the plane. In plane equilibrium. Right? In plain equilibrium. So now what will I do? I am just removing things. Now you can recall how we approached this problem with the plates. We approach this problem; there are three unknowns here, but we have employed this stress function formulation. Where the in-plane stresses were derived from some potential function, right? So, we will enforce the stress function as we did on the plate. So, stress the function ϕ when we introduced it, if you can recall. So, one thing I would like to let you know is that there can be one term Q/R . Because of the initial curvature, there may be an additional term, but this can be neglected because of the shallow-shell assumption. So, do you see that because of the shallow-shell assumption, many terms can be neglected, and those are very important for the development of Donnell's theory? So, this is for the shallow-shell assumption; because of the appearance of this R term in the denominator, and since R is very, very large, this term can be neglected. So, these are not exactly the same, okay? This function, if we put then $N_{xx} = \partial^2 F / \partial y^2$, of course, there can be a thickness h multiplied by it. But there, because it is stress, these in-plane forces per unit length are stresses per area, so they must be of unit thickness; but later we will adjust that. Let us assume it like this, okay? But $\partial^2 F / \partial x^2$ and $N_{xy} = -\partial^2 F / \partial x \partial y$. So, if you do that, then these two equations are automatically satisfied, and in-plane equilibrium is satisfied automatically. So, in-plane equations are automatic, which basically reduces the dimension of the problem. Instead of dealing with three, you know, stress components, we can deal with the one potential energy function for which the in-plane equations are automatically satisfied, right? So, we make use of the compatibility equation because it is important. So, equilibrium equations will be satisfied, but you have to consider the compatibility equations. So, the compatibility equation for the stress function formulation is, of course,

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) + \left(\frac{1}{r} \right) \left(\frac{\partial^2 w}{\partial x^2} \right)$$

.So, if you substitute this strain term, we will get something on the right-hand side like that. Now you substitute ϵ_{xx} in terms of σ_{xx} , then substitute it, and then you get both things, okay. I am removing this part, but please keep this in mind, okay?



So, the same thing, this term right-hand side from left-hand side will be the same thing, but the right-hand side, instead of W , will express it in terms of stress. ϵ_{xx} in terms of σ_{xx} , and then that in terms of this, and then we will equate both parts, okay.

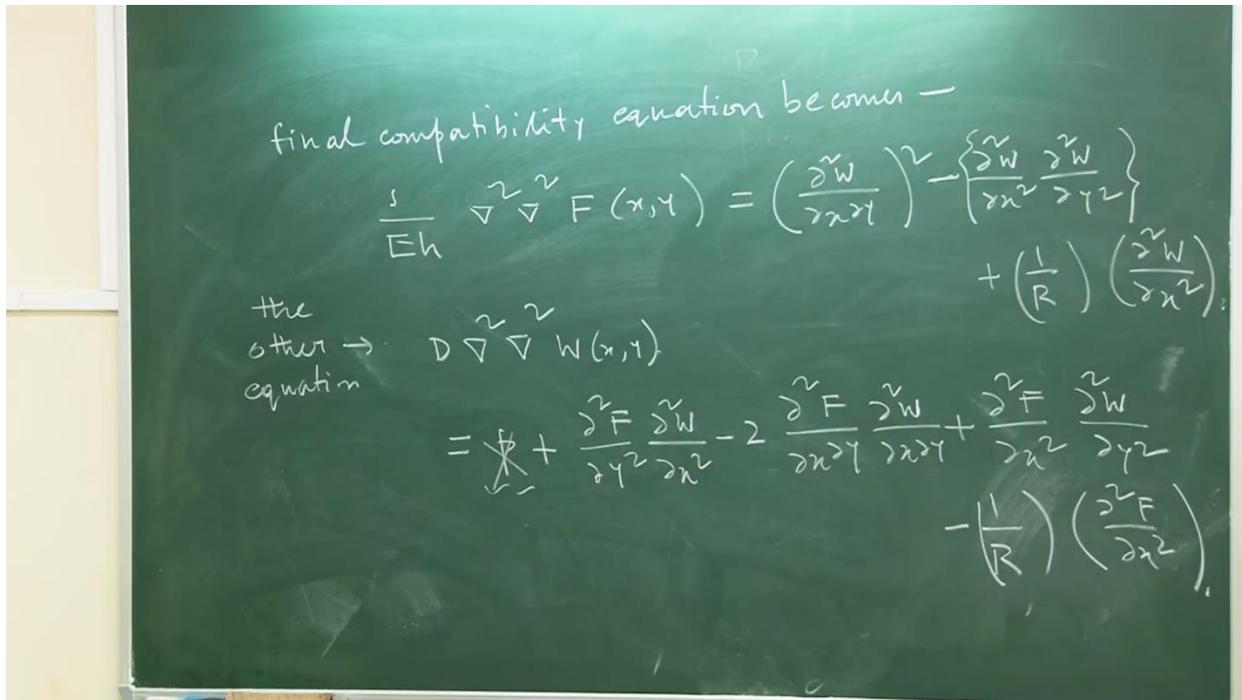
You know, if you do so, and if you express ϵ_{xx} in terms of σ_{xx} , then this compatibility equation becomes. from here. so, final compatibility equation becomes you know you know

$$\frac{1}{Eh} \nabla^2 \nabla^2 F(x, y) = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} + \left(\frac{1}{r} \right) \left(\frac{\partial^2 w}{\partial x^2} \right).$$

So, we know this, but where is it coming from? This is coming as ϵ_{xx} in terms of σ_{xx} , and then ultimately here. So, this is the compatibility, and if we substitute the N_{xx} term into the out-of-plane equilibrium equation, So, the other equation becomes, which I ask you to recall, so that I can set $\nabla^2 \nabla^2 w(x, y) = p$; we will make it 0 anyway. But I am just writing

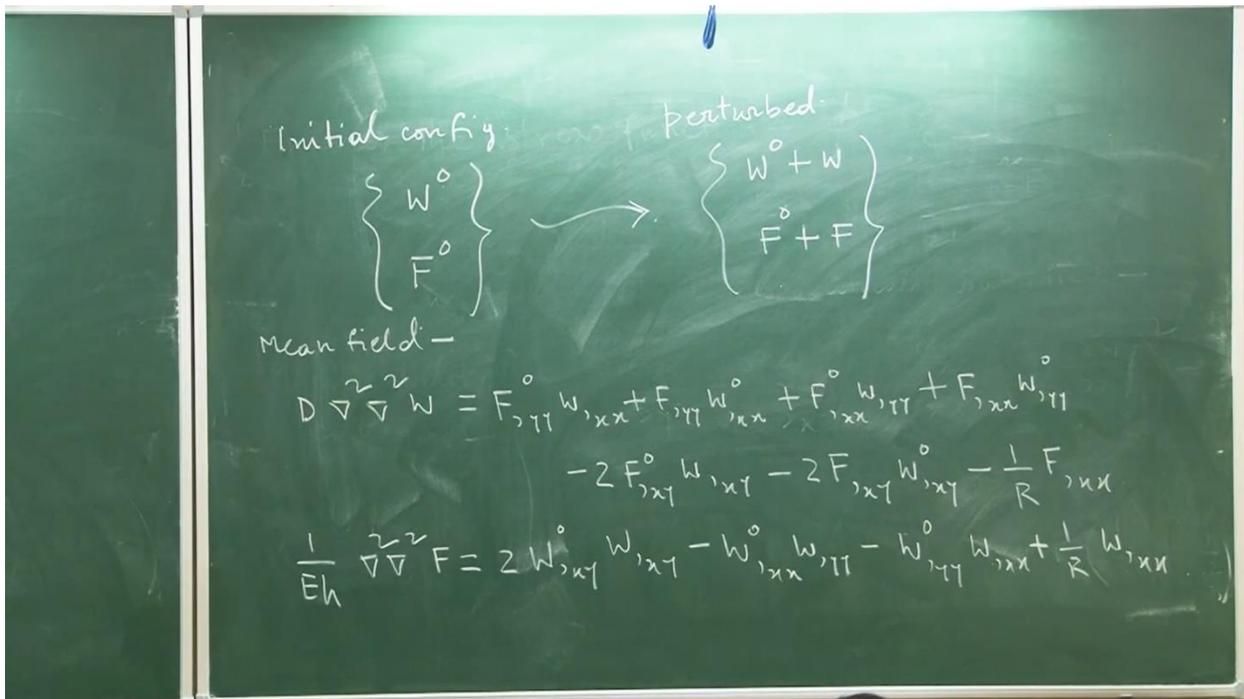
$$D \nabla^2 \nabla^2 w(x, y) = \left\{ \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right\} - 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right) \left(\frac{\partial^2 w}{\partial x \partial y} \right) + \left\{ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} - \left(\frac{1}{r} \right) \left(\frac{\partial^2 f}{\partial x^2} \right)$$

see that the same equations, where N_{xx} and N_{yy} are essentially substituted for n_{xy} , are substituted in. Now, these two equations are essentially the governing equations.



We will combine these two finals. So, we are paving the way for the development of Donnell's equation, okay? Please note that. So, now we see that if you want to combine what we can do, you should proceed accordingly. We can operate from here; see, there is λ^2 and $\lambda^2 f$, correct? So, here we can substitute, okay? By having a little more than I do, let me explain: we operate it further, differentiate one equation, and then substitute it into the other so that we can combine both, okay? So let me see. For the time being, let us keep these two equations separate, okay? Now, one thing let me tell you is that if r tends to infinity, that means if the curvature is infinite, then it can be converted to the shell equation, right? So, shell can be reclaimed from the as a special case. Go another way, okay? So, this is not Donnell's equation, but I have to do a little more stuff, okay? But these are the main things we are approaching; we are paving the way. So, now I will directly derive Donnell's equations. So, let us assume the initial configuration of the plate, okay? The initial configuration I am assuming is $\left\{ \frac{w_0}{H_0} \right\}$, okay? Out of plane deflection, this is then perturbed; now I will discuss the perturbed configuration. What am I doing in a perturbed configuration? $\left\{ \frac{w + w_0}{f_0 + f} \right\}$, please note that. This is the initial one, and this is the perturbed one. Then what am I going to do? I will substitute w_0 and f_0 into these two equations, okay? What will I do? I will substitute these into the two equations, okay? Then I will subtract one from another. So, this is another approach to derive the equilibrium. Sometimes, when you see a complicated structure, it is really difficult

to interpret it, and then you note down the equations, right? If the development becomes too complicated, Taking the components of different forces in deformation can become difficult. That is what I told you about the plate, right? In fact, there is another approach where you can directly start to see how we have derived this equation for plate buckling; you know, I mean for the plate bending equation, Navier's equation, right? You can directly do it from the theory of elasticity starting from $\sigma_{ij,j} = 0$, then integrate over the thickness, multiply it by z , and integrate once again, right? Similarly, if you do this and want to find out the plate buckling equation, then you do not have to start with the Cauchy stress tensor.

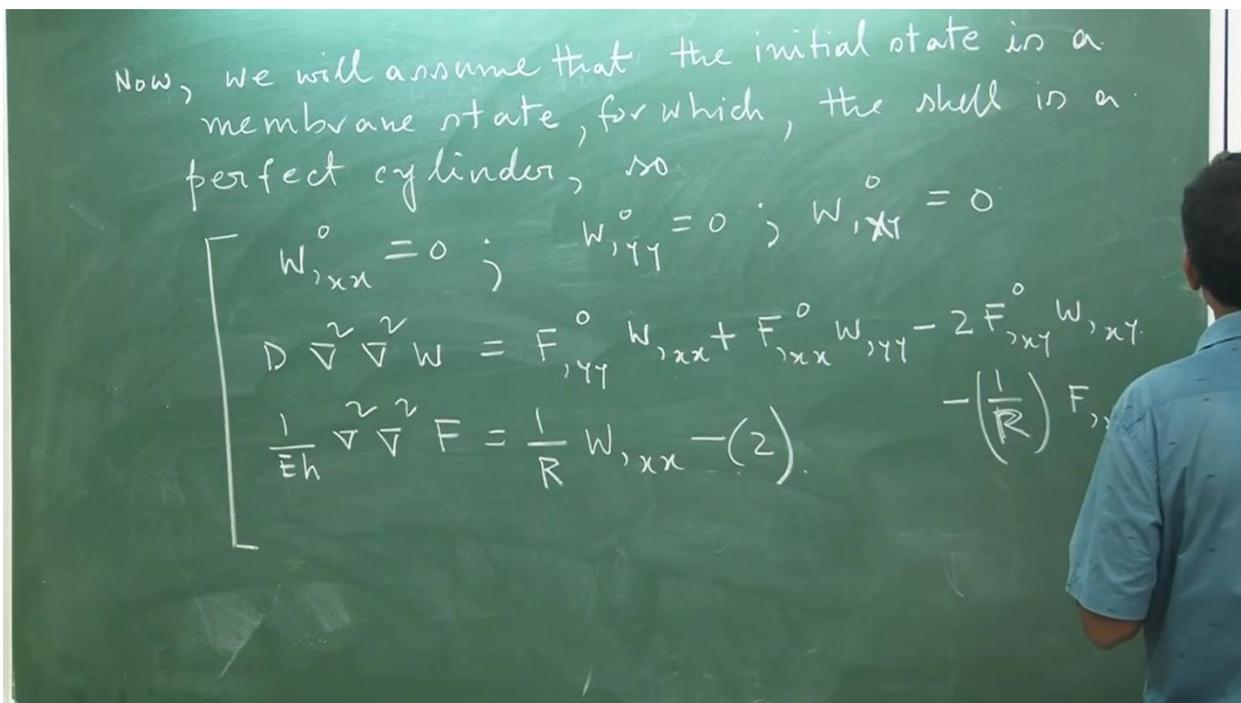


You have to start with the second Piola–Kirchhoff stress tensor. Because you have to distinguish between the deformed and undeformed configurations, right? And the way you can distinguish between the deformed and undeformed configurations at the stress tensor level is by using the second Piola-Kirchhoff stress tensor. So, do the same thing with that, and then you will get it. Okay? So, here is another way to derive this: essentially, what we are doing is getting the equilibrium equations correct. Then what we will do is assume that this is the initial configuration and that these are the perturbed configurations. So, we will substitute these two values here, and then in the perturbed case, we will also substitute, and then we will subtract the substituted values from the mean equation. And whatever incremental equation we get, that equation is essentially

the equation for plate bending and the equation for buckling. So, that is what we will do. So, I will just show you what the mean equation, the mean field equation, is, okay? So, the mean field equation: what is the mean field? Mean field; by substituting this one, I will use comma xx for the derivative. Could you please see how I'm doing?

$$D\nabla^2\nabla^2 w_0 = f_{0,xx}w_{0,xx} + f_{0,yy}w_{0,yy} - 2f_{0,xy}w_{0,xy} - \frac{1}{r}f_{0,xx}$$

And similarly, for compatibility, I am going to write $\nabla^4 f_0 = \dots$, and basically, xx means the second derivative, okay. So, these are the two mean-field equations. Now, you can similarly write the perturbed equation by substituting w_0 with $w_0 + w$ and f with $f_0 + f$, correct? And then, in the perturbed equation, you subtract the mean field equation, okay? We will make one simplification: a simplified assumption.



Now we will assume that the initial state is a membrane state. There is, you know, a way to predict the membrane state for which the shell is a perfect cylinder. And then, if it is a perfect cylinder, what will happen? What we can see, $w_{0,xx}$ will be 0 because w_0 is the mean field initial, right? If there is nothing, then $w_{0,xx}$ and $w_{0,yy}$ are 0; $w_{0,xy}$ is 0, right? So, if you make it 0, this will cancel out; this will cancel out; this will cancel out; this will cancel out, right? So, what will we get? The

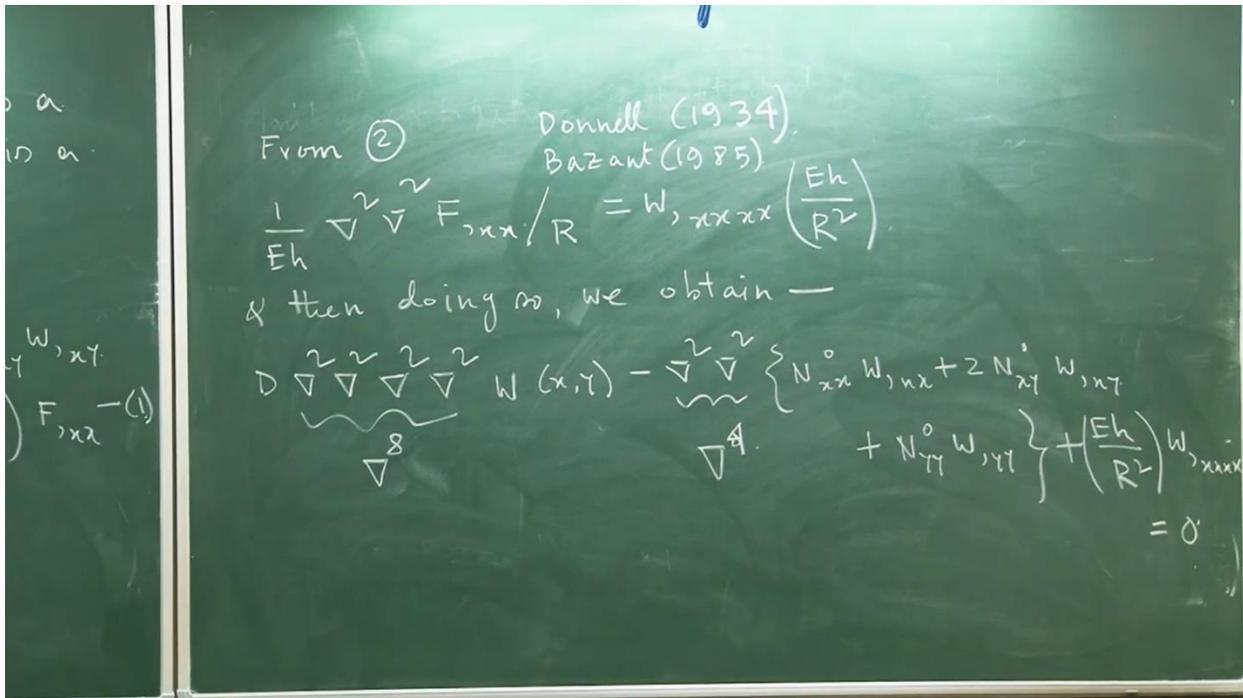
equation will be simplified to $D\nabla^2\nabla^2w = F_{,yy}^0$. Please note that although there is no initial reflection, in-plane stresses can be present; that is why the respective stress function cannot be zero. So, you know air stress function okay $W_{,xx} + F_{,xx}^0W_{,yy} - 2F_{,xy}^0W_{,xy} - \left(\frac{1}{r}\right)f_{,xx}$, okay. This is one equation; what will the second equation be? This will be 0; this will be 0; this will be 0. So, it is very simple; thus, this equation becomes

$$\frac{1}{Eh}\nabla^2\nabla^2F = \left(\frac{1}{R}\right)W_{,xx}$$

You see that all these assumptions are important. This derivation has been given in Bajan's book; please note that you should know it; otherwise, the main derivation in the original paper is very complicated and somewhat intractable. So, now we will combine these two. So, what will we do? The mean field we have subtracted is correct, so what should we do? We, instead of W , can substitute this for XX . Similarly, we prefer to know what we will be doing. We will operate it once twice ok. So that it comes in fourth place, okay? you know these are all $W_{,xx}$ and then we will do, so here we will operate here on ∇^4 , another ∇^4 , ∇^4 and here also we will operate with xx . So, then we will combine; we will substitute these places: this with this, this also with this, and things like that, okay. So, what we will do, then you know from these two equations is I will just write. So, $\nabla^2\nabla^2F_{,xx}/R$, you see that. This is $\lambda f_{,x}\left(\frac{1}{Eh}\right)$. Let us write it like this: Eh is equal to divided by r , which is equal to $W_{,xxxx}\left(\frac{Eh}{R^2}\right)$, right? Do you see that? How are we doing? Do you see that? So, if we differentiate twice, we have to do this as well, right? So, we can substitute it here. If we do, then what will we do? similarly we will not only, similarly we will also operate $F_{,yy}$, something like that, $F_{,xy}$, something like that and then you will substitute here, right? So, for that, we will operate it further here, another ∇^4 ; then only it will come in fourth, xx , xx it will come, xx , y something; that combination permutation we will do. So, by combining and doing so, we obtain.

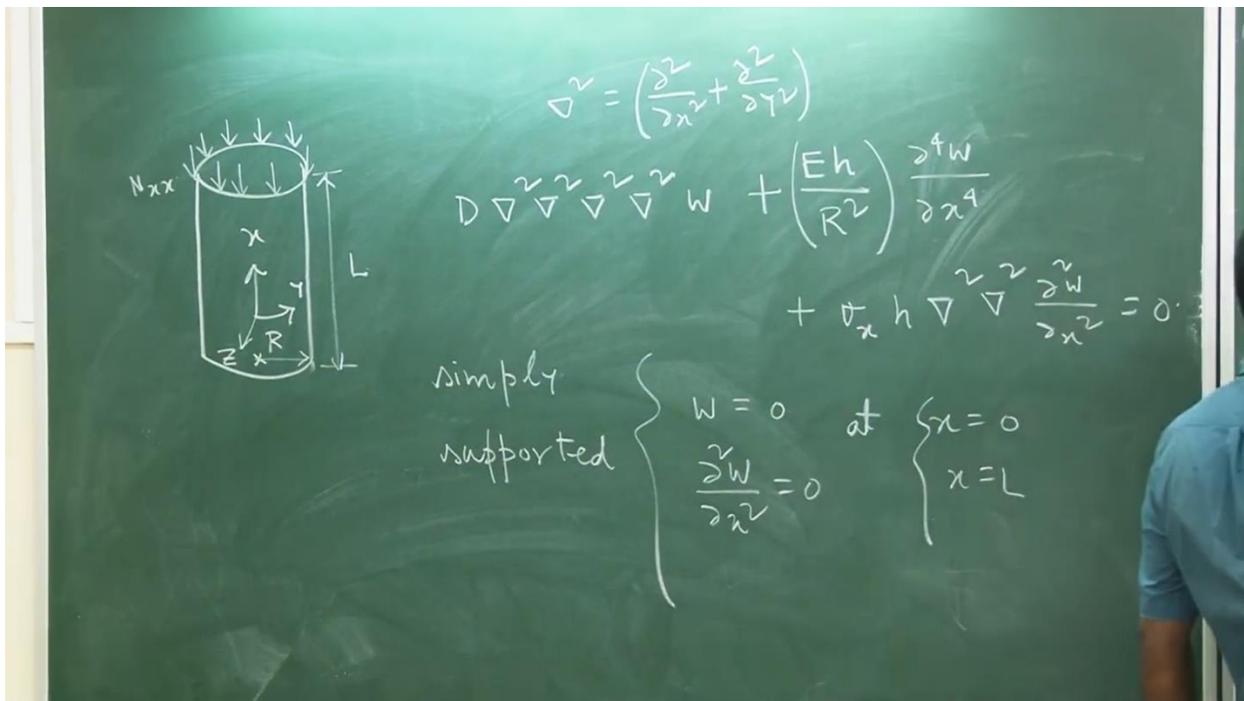
$$D\nabla^2\nabla^2\nabla^2\nabla^2w(x, y) - \nabla^2\nabla^2(N_{xx}^0w_{,xx} + 2N_{xy}^0w_{,xy} + N_{yy}^0w_{,yy}) + \left(\frac{Eh}{r^2}\right)w_{,xxxx} = 0.$$

So, this is Donnell's equation, okay? This is Donnell's equation. You can see in the book; that is why you will see this written as ∇^8 and this written as ∇^4 .



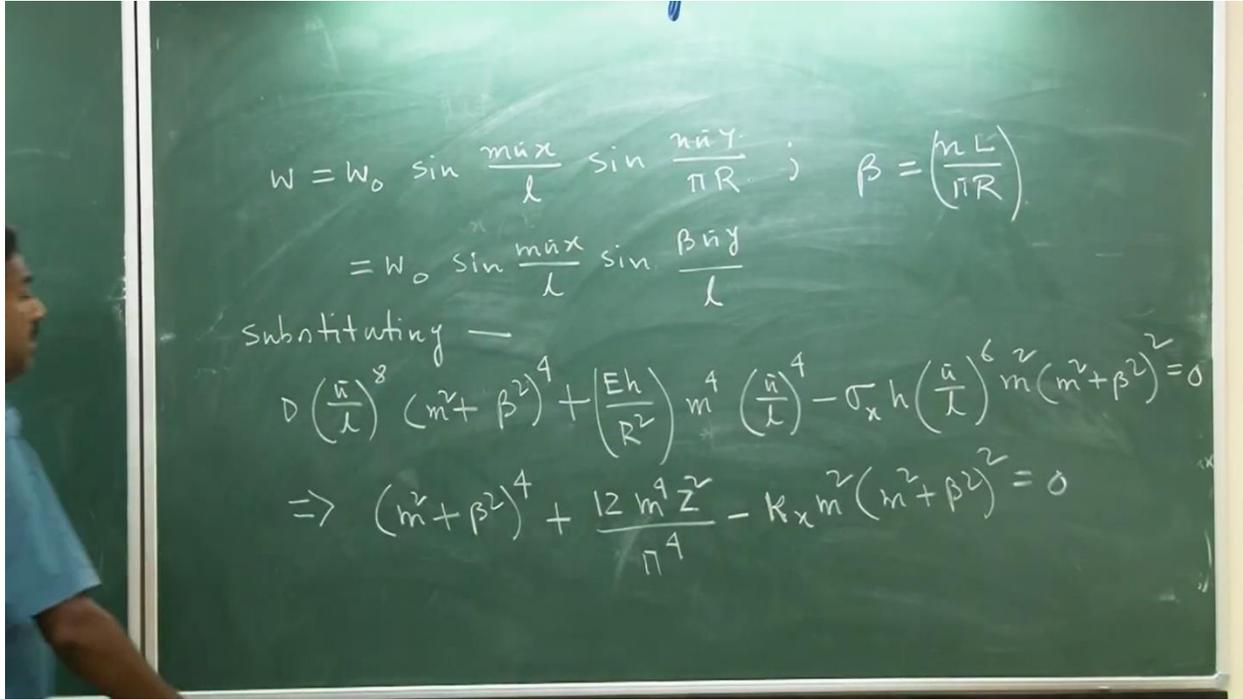
So, that is one thing you should please look into. The equations can be linearized and solved. How can it be linearized? We can consider that N_{xx0} ; these are constants, and then we can directly solve it, right? But one very important thing here is that we have assumed that these are zero, correct? What is deriving? That means the initial state is such that it remains constant. Derivatives are zero. What does it mean? That means there is no initial curvature when $w_{,xx}$ and $w_{,yy}$ are 0; there is no initial curvature. That means there are no initial imperfections. So, that is a very important assumption for the derivation of the Reynolds equation. There is no initial curvature. Only then is it valid, isn't it? So, initially, it was derived from the Donnell equation in 1934, but that is a bit different. So, he derived the linearized equation; even the derivation of the linearized equation itself was complicated. But then this derivation is taken from the book by Bazant; you know, he rederived it in 1984, Bazant. So, this is much simpler and people can follow; this is a little complicated to follow, but nevertheless, it can be fine. But there is a very important point that you must remember: for this to be true, Donnell's equation states that the initial curvature must be 0; that means there should not be any imperfections in it. This is very important. It can be linearized and solved, but otherwise not. In general, this equation is nonlinear, isn't it? In general, this equation is non-linear. As you can see, if N_{xx0} and N_{yy0} are initial, then as we go to the post-buckling regime, all of this will change. But this equation can be used, okay? Do that. So, these are the governing equations for cylindrical shells, and what I tried to emphasize to you is that there

is a generalized version of Donnell's equation, which is called the Donnell–Mushtari–Vlasov equation. That is true for any shell geometry, but if you want the generalized derivation, it is much more complicated. For any geometry, cylindrical geometry is much simpler, including cylindrical, spherical, and even that matter. The cone is also okay. But if you want to generalize it for any curvilinear system, then you have to use the theory of differential geometry. So that is there to understand the Mushtari-Vlasov equation, okay? But we will now work with this Donnell equation, okay, to solve simple problems. So, the first equation, once again, whatever we have solved using the axisymmetric assumption for the cylindrical field, is the same thing we are going to solve now, okay.



So, we are going to solve the cylindrical problem you know, under axial force. So, here is this, and then on top of that, this is L and this is R , so we want to solve it. So, for that, once again, we are going to write Donnell's equation. So $D \nabla^2 \nabla^2 \nabla^2 \nabla^2 w + \left(\frac{Eh}{R^2} \right) \frac{\partial^4 w}{\partial x^4} + \sigma_x h \nabla^2 \nabla^2 \frac{\partial^2 w}{\partial x^2} = 0$. So here, the boundary conditions are simply supported; you take the simply supported boundary condition, $w = 0$, right? I assume it is simply supported, so based on the bending moment and other factors, you know this is coming. And along y , that means along the circumference, so we are taking x here. And along the circumference here, y , and then out of plane z . So, $\partial^2 w / \partial y^2$ is a new 0, right? And it does okay. So, here let us assume that w is equal to $w_0 \sin(m\pi x/L) \sin(n\pi y/r)$,

okay. That is equal; that is the way. Of course, this is when we assume that this satisfies the boundary conditions; please note that. And then here what we see is that we adopt $\beta = nL/\pi R$. So, $nL = \pi R$ And then we can write $w_0 \sin(m\pi x/L) \sin(\beta\pi y)$.



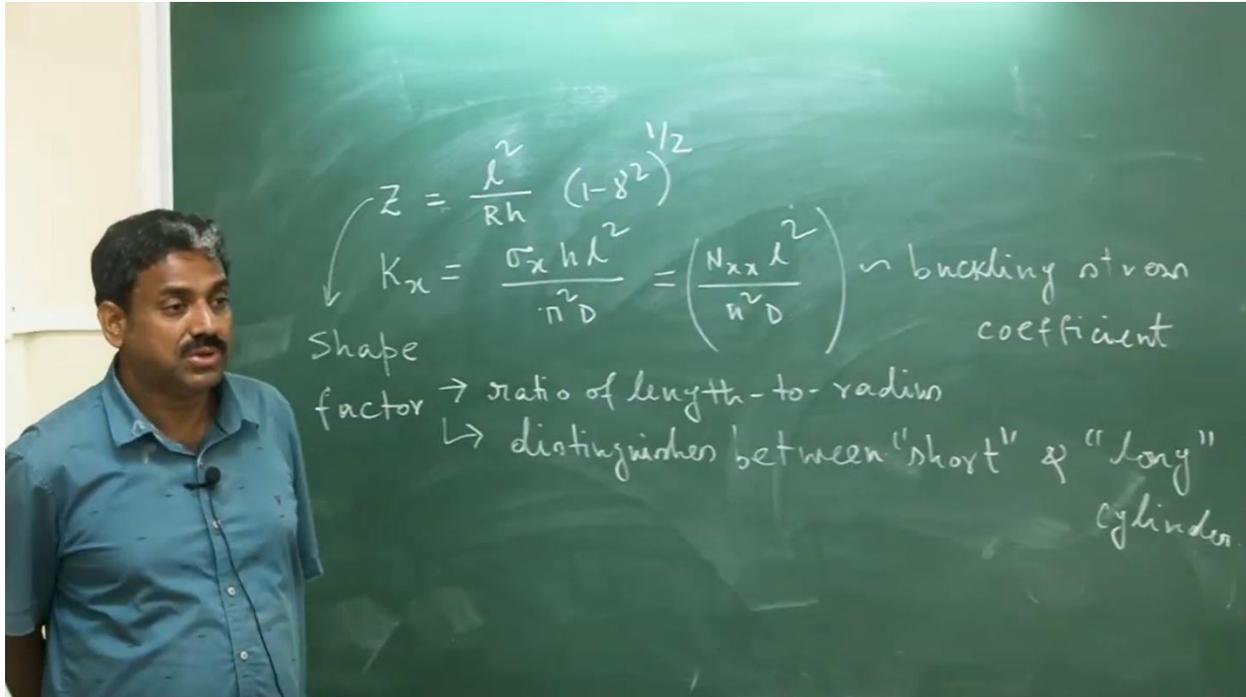
So, β is basically half the wave number, right? That is how you can see that m is the half wave number, $\pi x/L$; similarly, β is the half wave number here, okay? So then, substituting the equation, you can clearly see that all the derivatives become like the square of $n\pi/L$. This will be

$$D \left(\frac{\pi}{L}\right)^8 (m^2 + \beta^2)^4 + \left(\frac{Eh}{r^2}\right) \left(\frac{\pi}{L}\right)^4 m^4 - \sigma_x h \left(\frac{\pi}{L}\right)^6 m^2 (m^2 + \beta^2)^2 = 0.$$

So, if this assumption holds, it becomes an algebraic equation derived from the differential. As you understand, do you know what Nabla means? Nabla operator, what is ∇^2 ? $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. So, this is basically the difference; it just takes another y . So, that is where all the same x and the same π/L term will come from. So, (π/L) , since it is 8 orders, I am taking out $(\pi/L)^8$, right? And then from there, you can clearly see that with some simplification we can do, it will result in.

$$(m^2 + \beta^2)^4 + \frac{12m^4 z^2}{\pi^4} - k_x m^2 (m^2 + \beta^2)^2 = 0.$$

And there are additional parameters we define in this equation; we are simplifying it.



There are two parameters we are defining: one is z , and the other is κ_x . I will explain this string z and κ_x . So, z is called the batdrof parameter $L^2/(Rh)(1 - \nu^2)^{1/2}$, and then κ_x is nothing but $\sigma_x h L^2 / (\pi^2 D)$. And σ_x into h is nothing but N_{xx} , right? This is nothing but $\left(\frac{N_{xx} L^2}{\pi^2 D} \right)$; D is the flexural rigidity, right? And here is the Poisson's ratio ν . This z is a factor; Z is called the shape factor. And κ_x measures the ratio of the length to the race; this length is a , and κ_x is called the buckling stress coefficient. And this is the buckling stress coefficient, which is basically non-dimensionalizing the forces. The shape factor, z factor, is a measure of the ratio of the length to the use and the ratio of the length to the radius; this distinguishes between short and long cylinders. So, that means it essentially, you know, although a little generalized, the difference between short and long; short and long mean, you know, like short column and long column, right? The slenderness ratio, similarly, is a parameter known as the z parameter, which I will explain later. You know it is the ratio of length to radius, and it distinguishes between short and long objects. If the long cylinder behaves like a column, it is uncertain whether the short one behaves like a shell. Then in the next class, we will continue with this, okay? Thank you very much for today's class.