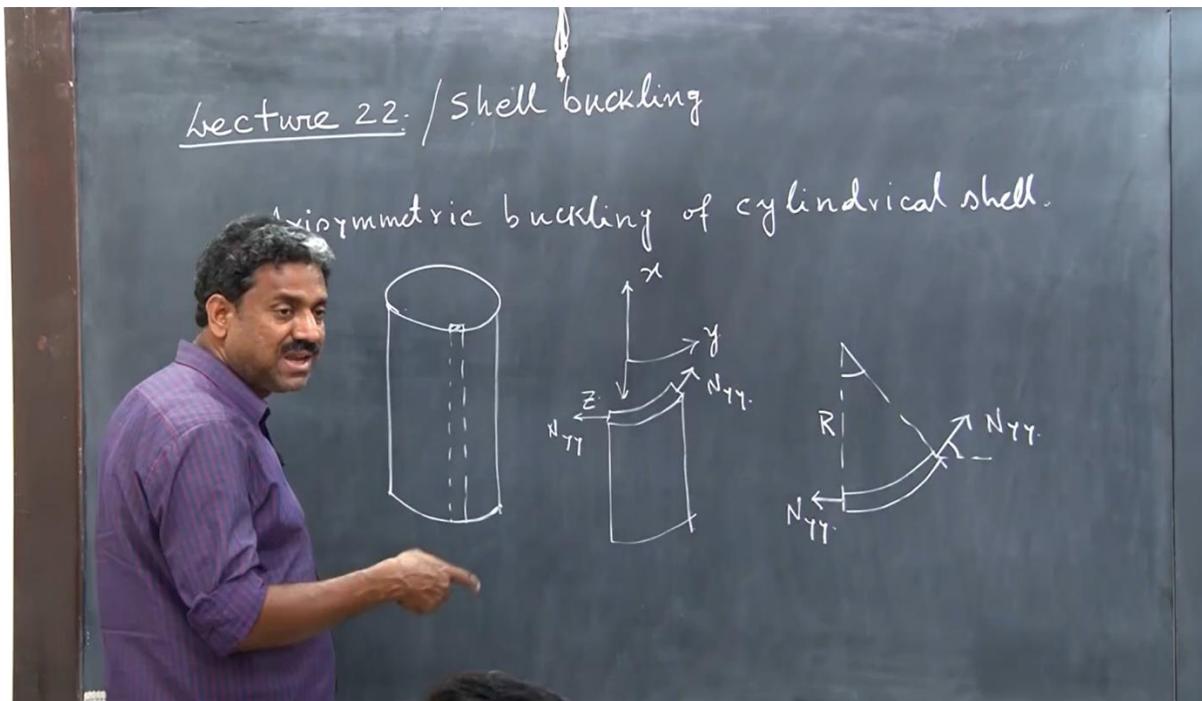


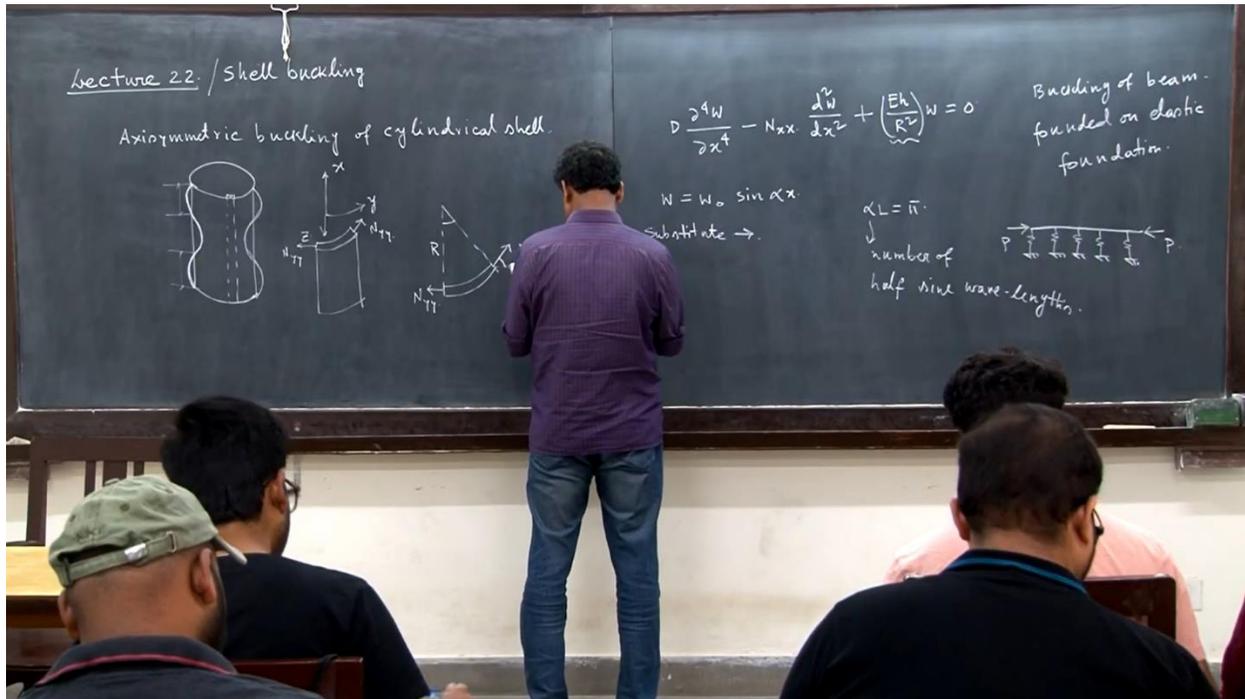
**Stability of structure**  
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**Department of Civil Engineering**  
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**WEEK-11**  
**Lecture 22: Stability Analysis of Shell**

So, we started our discussion on shell buckling and emphasized the importance of that, and we have started a discussion considering the accessibility of buckling in cylindrical shells, right? That is what we started. So essentially, we didn't consider any bending, you know. So, if you see the cylindrical shell, we essentially considered the bending for a strip; you know we have taken a small strip, right? And that is one-way bending. We did not consider any bending in that circumferential direction, right? and this adjunction that in the essentially the bend you know buckling of cylindrical shell as you see my take a buckling of cylindrical shell can be described reasonably using that the bending of a single strip you know. That is a reasonable assumption for this case.



And then, if you can recall, we have derived the governing equation. See, because of this curvature, there is, and our choice of coordinate system was something like this. So, the longitudinal direction was this  $x$ , the radial direction was this  $y$ , and  $Z$  was outward, right? Along the thickness right,

and then as you see, it will have this in-plane stresses right. So  $N_{yy}$ ,  $N_{yy}$  right,  $N_{yy}$ , and then because it is, you know. This angle and this angle will be the radius  $r$ . So, we are considering the component of  $N_{yy}$  along the radial direction, right? And basically, we will add to the effective external; I mean, we can consider that to be the transverse load, which is contributing to the effective external pressure, okay, analogous to, okay, and then we have written.



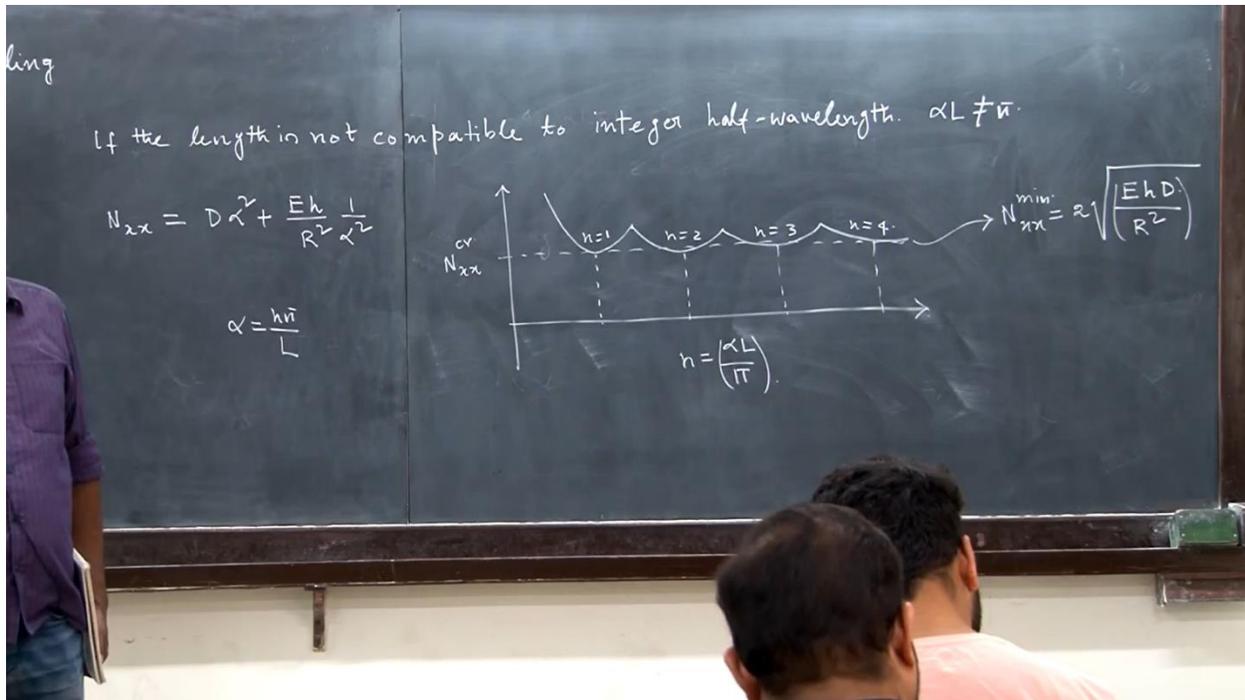
So, essentially, when we consider it to be, you know, the bending of a strip, then we can, and of course we have taken into account the component of the in-plane forces, okay. Then we have seen that we can essentially describe the bending using the bending in the  $x$  direction or longitudinal direction, okay, and That is what we derived in the previous class: the governing equation was

$$\frac{\partial^4 w}{\partial x^4} - N_{xx} \frac{d^2 w}{dx^2} + \left(\frac{Eh}{R^2}\right)w = 0.$$

So, if you see, this is very similar to what this equation is analogous to: the buckling of a beam founded on an elastic foundation, right? Or resting on an elastic foundation, right? Because for a beam now, of course, its flexural stiffness was  $EI$ , I mean  $D$ ; then this one, you know,  $Kw$ , this is the modulus of this subgrade reaction  $K$ , right? Because if you consider the buckling of a beam, you see that this beam, which is supported on an elastic foundation. And then  $P$  in both ends, then

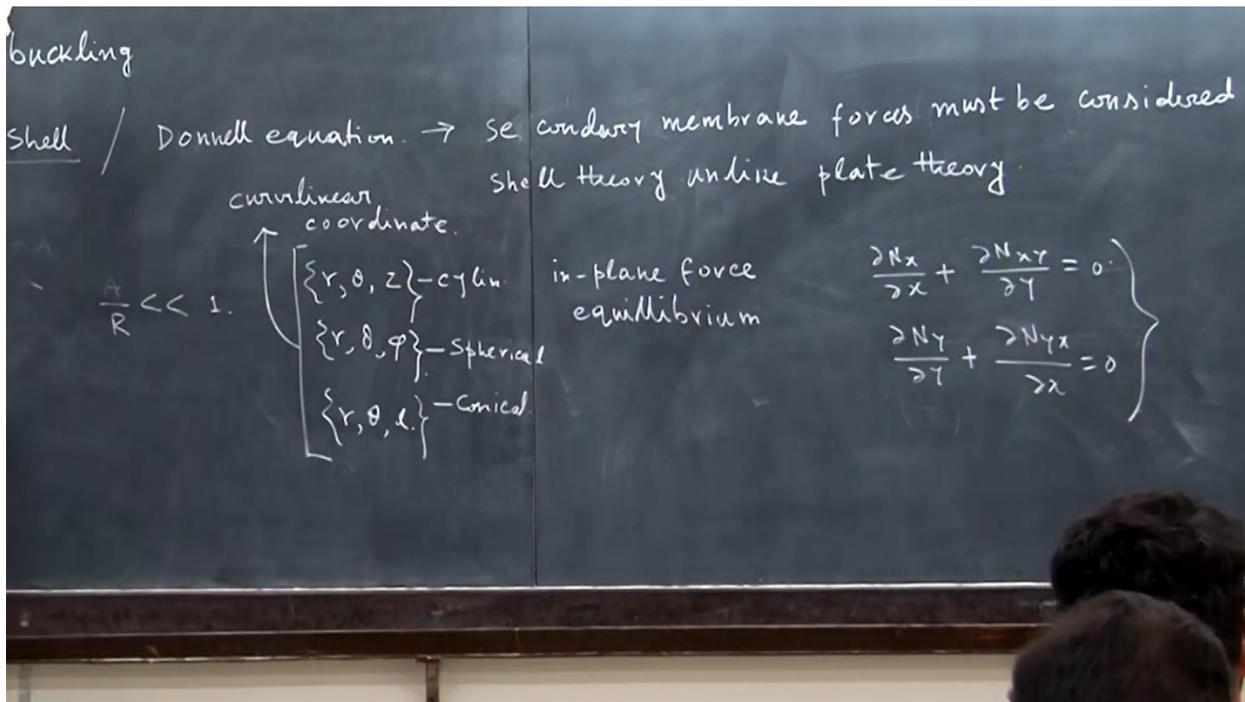
you are applying this for, as you see that the beam-column equation is what  $EI \frac{d^4 w}{dx^4}$ , right? Fine. Then whatever the axial force  $P \frac{d^2 w}{dx^2}$  is, plus the additional reaction forces that are going to come, is  $K$  into  $W$ ; essentially,  $K$  into  $W$  is coming. So,  $EH$  by  $R^2$  is essentially contributing, which will be analogous to the subgrade reaction, right? So now we can solve this. The solution we can obtain requires that you know the limitations of this governing equation and this kind of model, specifically the buckling behavior of a cylindrical shell. Cylindrical shell means it has only one curvature in its radial direction, while in other directions, it has an infinite radius of curvature. Okay, that is what I was saying. If it buckles in an axisymmetric manner, then you know. So, like this, you see that. So, essentially, the buckling buckles are having this pattern appearing along the longitudinal direction, right, in radial, and essentially this simplification we are adapting to describe this thing. So, we are essentially taking, bending is so dominant along longitudinal directions that we can essentially take a single strip over there. And then we can write down the governing equation for this, and then, of course, because of this curvature, the contribution of  $N_{yy}$  will be there. Along the radial direction, so that's what we have taken into account while deriving this equation, right? So, then we can assume the solution to be of the form we had earlier:  $\sin(\alpha x)$ , and here we are defining  $\alpha l$  to be  $\pi$ . So, it is basically a half wavelength,  $\alpha l$ , equal to  $\pi$ , because  $\alpha$  is the number of half sine wavelengths. Half sine wavelength means we are assuming this set to be sinusoidal, so how many alphas are there? This one, this one, that one you see, that is the alpha percentage. So then, with this kind of assumption, you substitute it into the equation, and substituting what we are going to get, If we write, you know this, you are going to get  $A_s$ .  $H$  is the thickness of the shell; shell thickness  $H$  can also be expressed in terms of  $T$ , but we are using  $H$  just to follow the convention in the book. Non-trivial solutions  $\sin(\alpha x)$  cannot be 0, so this one must be 0, this part is right. If you want to make it 0, then from there we are solving, you know. Solving for  $N_{xx}$  is a critical value. So that is the critical value that we are going to get right, and we see that  $\alpha L$  is equal to  $\pi$  because the half wavelength must be accommodated. So, basically,  $\alpha$  is nothing but  $\pi$  divided by  $L$ , right? So, you substitute for the value here. This, you will get  $d\alpha^2 + \frac{Eh}{r\alpha^2}$ , right? Now, the way for plate bending, what we did was try to minimize  $N_{xx}$  with respect to the number of half wavelengths, right  $\alpha$ , that is what. So, whatever we are doing is following whatever we have done in the plate equation, right? So, for the minimization of  $N_{xx}$ , you know,

let us put  $\frac{dN_{xx}}{d\alpha} = 0$ . So, essentially what, we get here is if you differentiate. for this  $\alpha$  is this and then substituting this  $\alpha$ , So, for that  $\alpha$ , we want to see how many half wavelengths there are and for which it can be minimum. So, in  $N_{xx}$ , you know the minimum or critical value will be. When we simplify it, you will get 2. So, that is the expression you are going to get. See, it is not always that. So, you get a very simple expression for the  $N_x$ : what will be the critical value of this  $N_x$ ? Because this was. The shell buckling of the shell under axial compression is what we have assumed. So, this was essentially the value of that right, buckling of a cylindrical shell under axial load. So, we have obtained a very nice expression. Of course, this expression although it will never reach ok, because this is very idealistic and I will explain to you what really happened there ok, it is not so simple shell buckling ok. For simplification, we have assumed that  $\alpha\pi$  is equal to that, which means  $\alpha$  is an integer, but it is not necessarily. So, it is not the case that an integer number of wavelengths will be accommodated within this length all the time. This expression is very simple, so you may think that well, and if we take a cylinder and taste it, we will get it, but it is not. So, if the length is not compatible, what we have seen is that in plate buckling, this is analogous to a plate that. If the half wavelengths are compatible with the length, that means the half wavelengths appear an integer number of times, right? Then it attains a minimum value; otherwise, on both sides, it increases, and that is what gives it a tongue-like structure, which you call Arnold's tongue, right? A similar kind of thing will also happen here, okay. If this is not compatible, then the length is not compatible. Integer half wavelength means  $\alpha L$  is not equal to  $\pi$ ; then it will increase on both sides, okay. So, we get an  $N_{xx} = d\alpha^2 + \frac{Eh}{r^2} \frac{1}{\alpha^2}$ . Well, that's good. So, this is  $\alpha$  depending on  $N\pi/L$ , okay. So, this is  $N = 1, N = 2, N = 3, N = 4$ . And you see that the minimum load is, what is the minimum? You have obtained here; this is nothing but  $2\sqrt{\frac{EhD}{R^2}}$ , okay. So, when you know this  $\alpha$  is an integer, okay? That means the length of the cylinder is good enough to accommodate an integer number of half wavelengths. It will attain that; it is irrespective of the number of wavelengths; it can be 1, 2, or 3, right? So, then if the minimum value is attained, the minimum value is given by this; otherwise, it will, the load is increasing okay, and that expression you have seen is nothing but  $d\alpha^2 + \frac{Eh}{r^2} \frac{1}{\alpha^2}$ . So, if  $n$  is not an integer then it will increase, right? And this is basically where that occurs, and for that, it gives an increasing number of wavelengths at which the modal transition occurs.



So, mode conversion: for the first mode, it will go to the second mode, which means that from a single wavelength, it will immediately jam into two wavelengths and then three wavelengths, something like that. So, this is very similar to the behavior that we have observed in plate buckling, right? Clear. So, this is a very simplified representation of shell buckling, and this is, I mean, this is based on the assumption that all cylindrical shells will be subjected to an axisymmetric mode of buckling, okay. And we can essentially represent the bending behavior using a single strip and incorporating the effect of curvature and circumferential direction in the well-derived equations. But then we get a minimum value; however, this does not really occur. The equation is very similar to, I mean, it is analogous to the equation, which goes about the buckling of a beam on an elastic foundation, and we obtain a very nice closed-form expression. A cylindrical shell hardly bends along the axis; I mean initially it might bend in an axisymmetric mode, right? But as soon as buckling is triggered, you know it is not only a single axisymmetric mode, but an anti-symmetric mode will also come into play. And it is essential that if you conduct an experiment, it is a combinational symmetric and axisymmetric mode, and in that case, it is not possible to have any assumption we made, okay. And also, what happens other than this, is that along with this bending in the longitudinal direction, this is also accompanied by bending in the transverse radial direction, right? And unlike plate, there will be, you know, even if we consider the first-order theory, there will be coupling between bending and membrane, turn, ok. That is what we are going to do now,

and we will follow the quasi-shallow shell theory in which we will assume that the buckling mode consists of wavelengths, each of which can be represented as a shallow shell. that means it is so we will derive this quasi-shallow shell theory you know. And buckling of that, and then we will derive the, from this we will obtain the Donnell equation, okay. This is a Donnell equation for shell buckling, but please note that even here, whatever we are doing is not what is observed in shell buckling. Shell buckling is very complicated; this is just one step of, in fact, the buckling up shell. If you see it, you know they started a long way. There have been efforts, you know, continuous efforts by many scientists, okay.

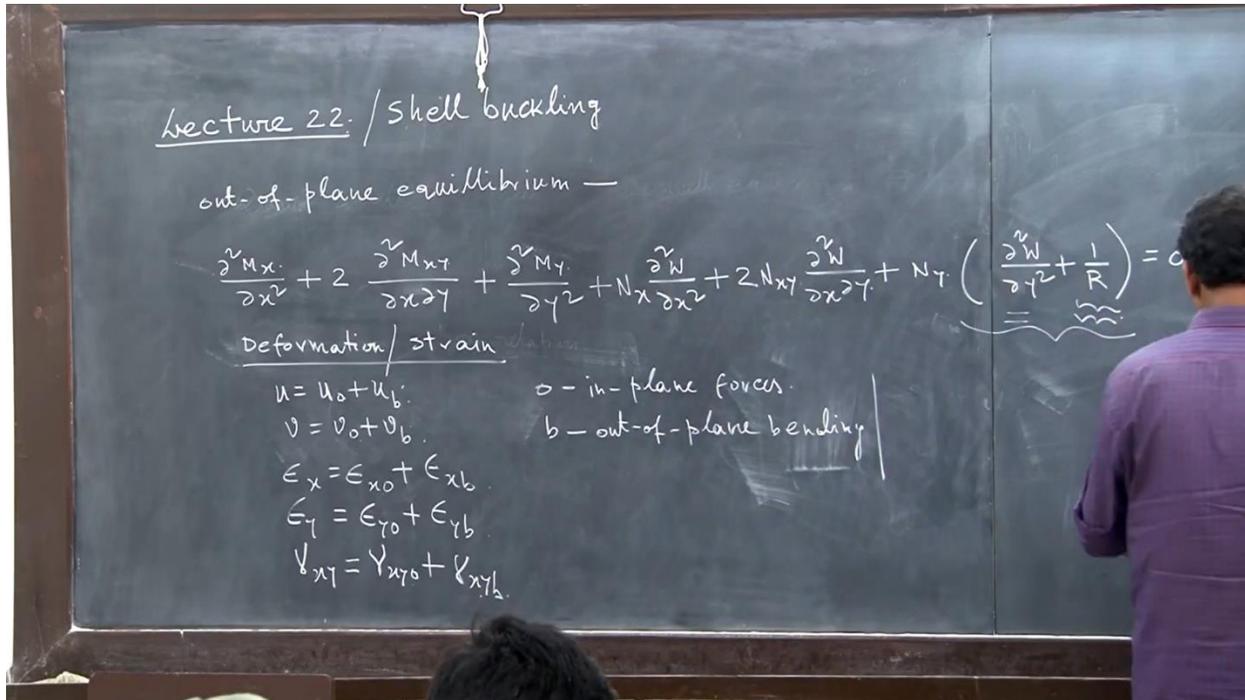


So, if you read the book by Bruce and Almorth, they themselves also contributed. Caladine, Bushnell, you know, and then, of course, von Karman, okay. And then there are a number of scientists, okay. And then there was Donnell, including Donnell, okay. So, initially with this, whatever we are doing here, we are using a simple equation to try to describe it, but it was not matching with what we observed experimentally. Nevertheless, let us see that. How this evolution occurs is actually okay in the theory. So, now we will see how Donnell's equations are derived, okay. So, quasi-Shallow theory means that you know. You have a shell, right? And then it has a radius of curvature. These are right, and then, of course, it will buckle, right? So, it will have this buckling wavelength, right, and it will have this amplitude, right? So, seeing this amplitude, you may assume that it is  $A$ . So,  $A/R$  is very, very negligibly smaller than 1. Then you know this is

called, like in each buckle, each half wavelength, you know, for the buckle describing the buckle is essentially behaving like a shallow shell, okay. So, that is what this is; one of the assumptions while deriving the Reynolds equation, okay, and then. Unlike the plate theory, in which we observe the coupling of bending and in-plane forces under moderately large deformation, right? We have seen that in the linearized theory, we assume that the axial forces, you know, the in-plane forces, are essentially constant throughout the plate, right? That is what the assumption was, and essentially, there is no coupling between, you know, in-plane forces and the out-of-plane bending, right. There was, unless you consider the moderate deformation, which was described with the von Kármán-Föppl equations, right. Unlike a plate, because of the curvature in a shell, the in-plane forces and out-of-plane bending are essentially coupled. So, from the very beginning, we have to consider the in-plane forces and the equilibrium of the in-plane forces as well. So, that is a difference between plate theory and shell theory for buckling. The bending in-plane forces and bending coupled bending. Coupling does not occur under small deformation or whatever in the pre-buckling regime, right? Or if you want to find out the critical load or whatever for the plate, the coupling occurs only in the post-critical regime. Unlike that, in shell, the buckling is set in from the very beginning, okay? So, that is what we are going to write. So, in plane equilibrium, So, Donnell, give this equation; here you can see that the secondary membrane forces. What is a secondary membrane, why is this secondary, why not primary, and what are primary membrane forces? By now, you understand that "secondary" means, see, initially, the plate is subjected to some axial force, right? And we assume that to be constant, and we find our critical value for that, right? So, we refer to that as primary forces in the pre-critical or even on the verge of buckling, right? Isn't it, but as soon as it buckles and then moderate deformation takes place, right, then there is a redistribution of those forces, right? And because of that, there are secondary forces that occur, right? Isn't it? And that is what we call, but here in the Donnell equation in the plate, in the shell, secondary member forces have to be considered along with the primary forces. You understand that, right? And why that issue? Because of this, okay? It must be considered; I am writing it, must be. Considered in a linear shell theory and unlike plate theory. There is a big difference, okay? So, in planning equilibrium of the inflated forces. You may wonder why we are not considering  $R$ ,  $\theta$ , and others, because that is more evident in plate bending theory. In plate bending theory, we have considered the cylindrical coordinate system  $\{R, \theta, z\}$  for the cylinder, and for the spherical coordinate system,  $\{R, \phi, \theta\}$  Right, this was for cylindrical shells, and this is for spherical, right?

Or dome, and there was for, you know, conical. What was the quadrilateral system?  $\{R, \theta, L\}$  and then slant height, right? And of course, all these things can be combined using a generalized curvilinear quadrilateral, right? Curvilinear coordinate system, right? You have learned this one in the bending theory of plates, the concept of the generalized curvilinear coordinate system. Covariant base vector, contravariant base vector, right, Gauss-Codazzi relationship, right, anticlastic curvature, synclastic curvature—all these things you have learned, right? Good, the first fundamental form of surface, the second fundamental form of surface, all these things you have learned, right? Good. Here we are not going to that much, okay? We are essentially restricting ourselves, and we are considering— even you will see that there are three very simple problems for which we derive the linearized equation for SHELL buckling. Once again for the cylindrical shell, we essentially adopt  $x$  along the longitudinal axis,  $y$  along the radial axis, and  $z$  along the thickness axis. So, we avoid all these things. Whatever, nevertheless, all these things are extremely important because please note that everything can be combined into one generalized coordinate system. It can be specialized to all of these, which is useful when you derive finite element theory, because although we are deriving all the theory, etc. You know in research nowadays, all the investigations are essentially except for very few who still want to do analytical solutions and things using perturbation asymptotic expansion, and things people do finite element. In formulating finite elements, if you know these things, it will be easier; you can easily follow the paper. Most of the literature is written using Gervais linear coding conventions. You have to know both the indicial notation and be conversant with the indicial notations, summation, conversion, and, along with that curve, you know the differential geometry. Why are you taught about the curvilinear coordinate system, the theory of surfaces, and others? A little bit of differential geometry you should know, okay? All of you have learned this metric tensor, right? See, all these discussions are also very much essential to your friends, you know, in physics—these people who do large theoretical physics, high-energy physics, and study the theory of relativity and others, right? There, the first lesson they learn in, you know, differential geometry, okay, and then the difference is that. Our metric tensor is essentially a spatial coordinate, right? Our space and time are separated, but their time is also, so they define unit tensors, you know, unit metric tensors based on that. So, you can see the complexities involved, okay. So, mathematics, you know, theory-wise, mathematics-wise, you know, both are in the same line, okay. This we have already derived. So, you know this; I am just reiterating one thing. So, this equation must be satisfied by the secondary

forces, right? This is the equilibrium equation for in-plane and out-of-plane equilibrium. Out of plain this, I am not deriving; you are going to understand that we have already derived for the plate, right? So, you can understand which term is contributing what.



Now we are considering only cylindrical objects;

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{d^2 w}{dx^2} + 2 N_{xy} \frac{d^2 w}{dx dy} + N_y \left( \frac{d^2 w}{dx^2} + \frac{1}{R^2} \right) = 0.$$

Please note that for spherical ones, things will be more complicated. So, this is the third equation. Of course, then, you understand where this  $1/r$  is coming from, right? This one.  $1/r$  is coming from the curvature terms, one second from the bending, right? And this is the radius of curvature. So,  $1/r$  is the radius of curvature, right? So, along with whatever curvature is due to deformation, the existing curvature in the geometry is also contributing to that. Please try to understand this term, okay? Both are curvature terms, right? Okay. I am writing down all the individual components of the theory, and then I will finally write the equation, and that is essentially what the equation is. Donnell's equation. So, it's not force deformation; actually, this is just the deformation component, the deformation and strain component, okay. So, you see that  $U = U_0 +$

$U_b$  stands for whatever is in the plane, okay, and  $B$  stands for the contribution from the bending, okay. 0 was for the in-plane forces, and  $B$  is for the out-of-plane bending. Basically, bending is always out of plane, right? So, since we see buckling, in the pre-buckling regime, basically, there is no bending, right? So, all were in plain forces; that is what I mean, and that is basically the, you know, ground state, right? So, that is what I am defining: 0,0,0,0,0 ok. And then once it buckles, of course, that will be accompanied by bending; that is what the component of bending is coming from, understand that. Then I will write the moment-curvature relation. You know this thing that we have all derived.

Moment-curvature relation.

$$M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \gamma \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \gamma \frac{\partial^2 w}{\partial x^2} \right)$$

$$M_{xy} = D(1-\gamma) \frac{\partial^2 w}{\partial x \partial y}$$

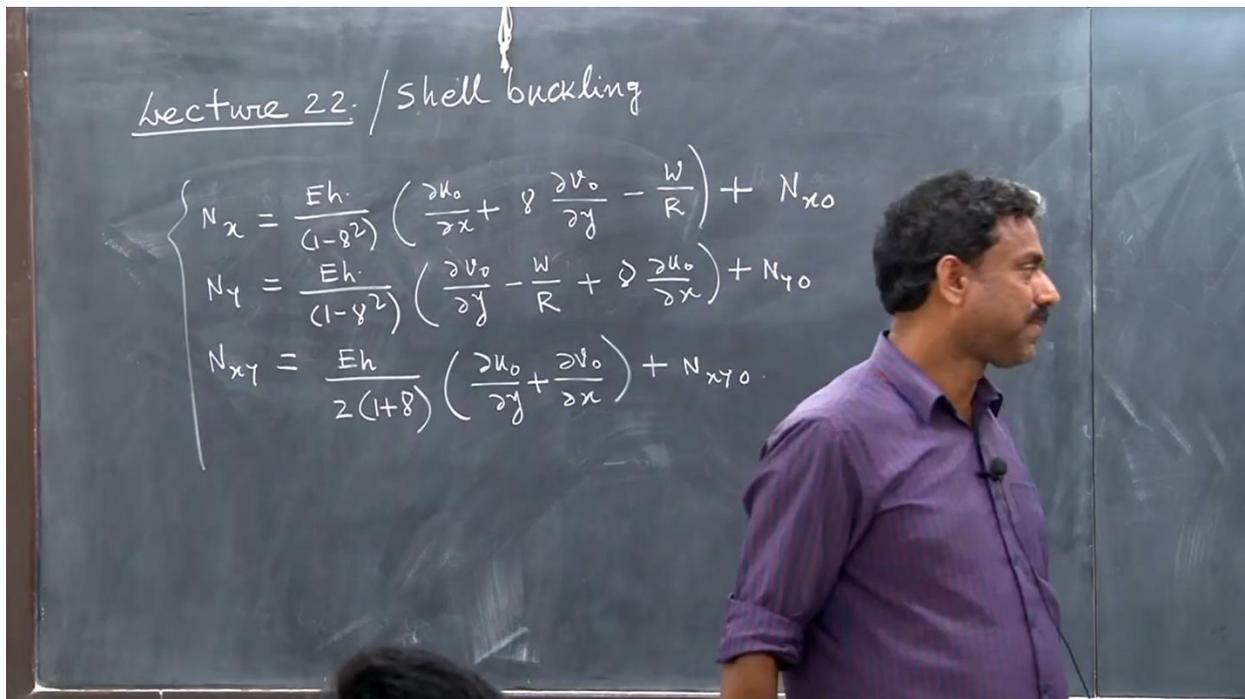
Strain-displacement relation.

$$\epsilon_{y0} = \frac{\partial v_0}{\partial y} - \frac{w}{R}$$

$$\gamma_{xy0} = \left( \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right)$$

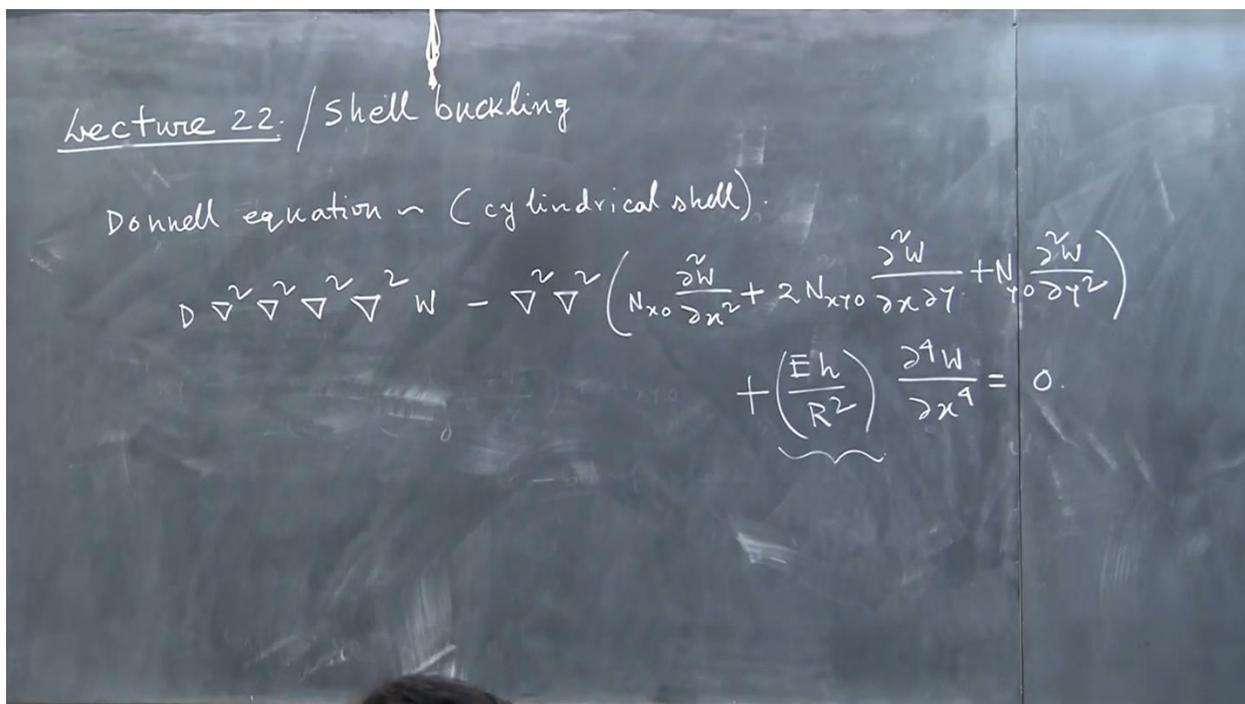
Depending on the, you know, sign convention, there will be a negative sign, right? And cross, you know, these are the bending moments expressed in terms of curvature. And then the torsion moment expresses the cross curvature. Then I will just write the strain component, you know, strain displacement relationship. So, as you see, will be  $\epsilon_{y0} = \frac{\partial v_0}{\partial y} - w/r$ . And then subtract  $w/r$  from where  $w/r$  is coming, okay? See, it is radially expanding, right? So,  $2\pi r$ , and then the way this term is coming:  $2\pi r + w - 2\pi r$  divided by  $r$ ; that is what  $w/r$ .  $w$  is the out-of-plane deflection, right? And  $\frac{\partial v_0}{\partial y}$ , you have learned from the small deformation theory, right?  $\epsilon$  fine. So, this is 0, okay. So, I am removing. So, essentially what we are going to do is I will now write down

the expressions for  $M_x$ ,  $M_y$ , and  $M_{xy}$ , okay? So, you have obtained the expression for the bending moment, you have obtained the strain and expressed the strain in terms of the displacement component, and we have the equilibrium equation, right? So, the things that are remaining are  $N_x$ ,  $N_y$ , and  $N_{xy}$ , okay, in terms of in-plane forces, right? The strain, you know, we will express in terms of in-plane. We will use the stress-strain relationship to express this in terms of stresses, and then stresses in terms of strain, and strain in terms of the displacement gradient.

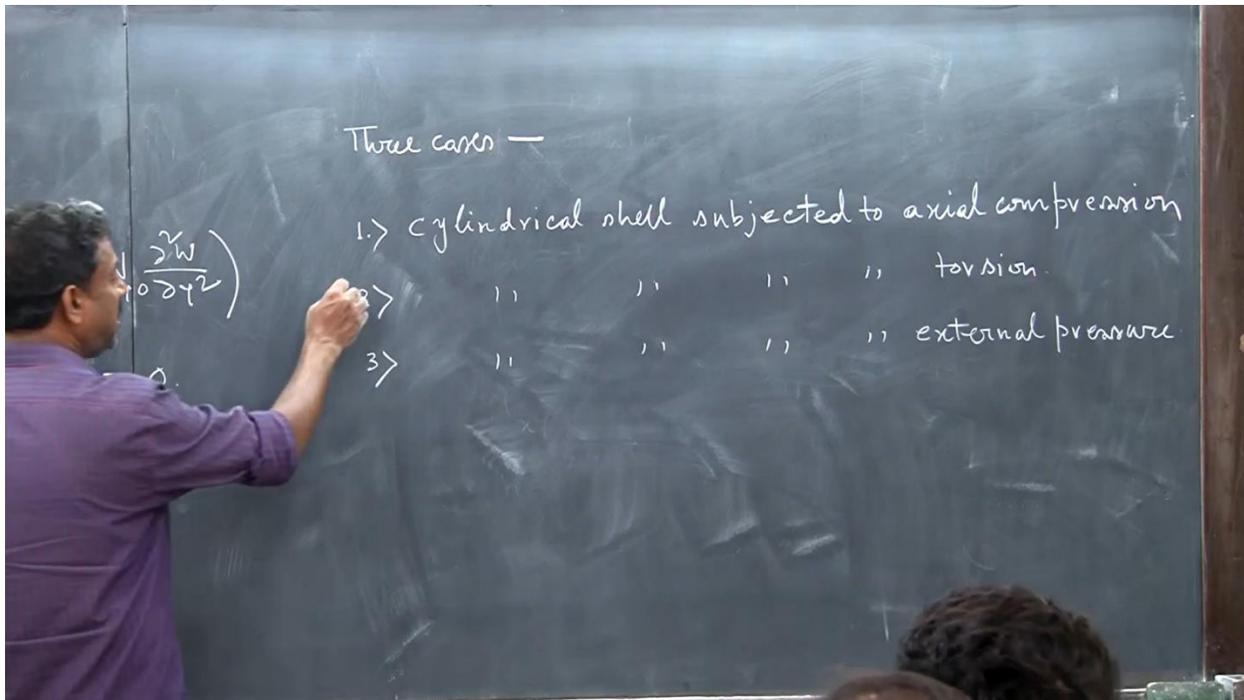


And then ultimately, we will substitute equilibrium into this equation. Now, middle surface stresses are okay, other than whatever is externally applied. Primary  $N_{x0}$  and  $N_{y0}$ ; there is also a secondary one, which is coming due to bending. That will also be expressed; that is what I am just writing, okay.  $N_x = \frac{Eh}{(1-\gamma^2)} \left( \frac{\partial u_0}{\partial x} + \gamma \frac{\partial v_0}{\partial y} - w/r \right) + N_{x0}$  This is the primary one, and this is the secondary one, okay. Please note that  $W$  will come only due to bending, right? And now I understand why, from the very beginning, this coupling will be there; understand that. I will try to explain using Donnell's equation itself. Okay, wait please. So, now that all components we have are okay, all of this will substitute into the equilibrium equation. And then we will just reorient the term, okay, and we are doing it essentially for the cylindrical shell for simplification. There can be very generalized equations, okay? That we should have also learned, but then I wanted to avoid

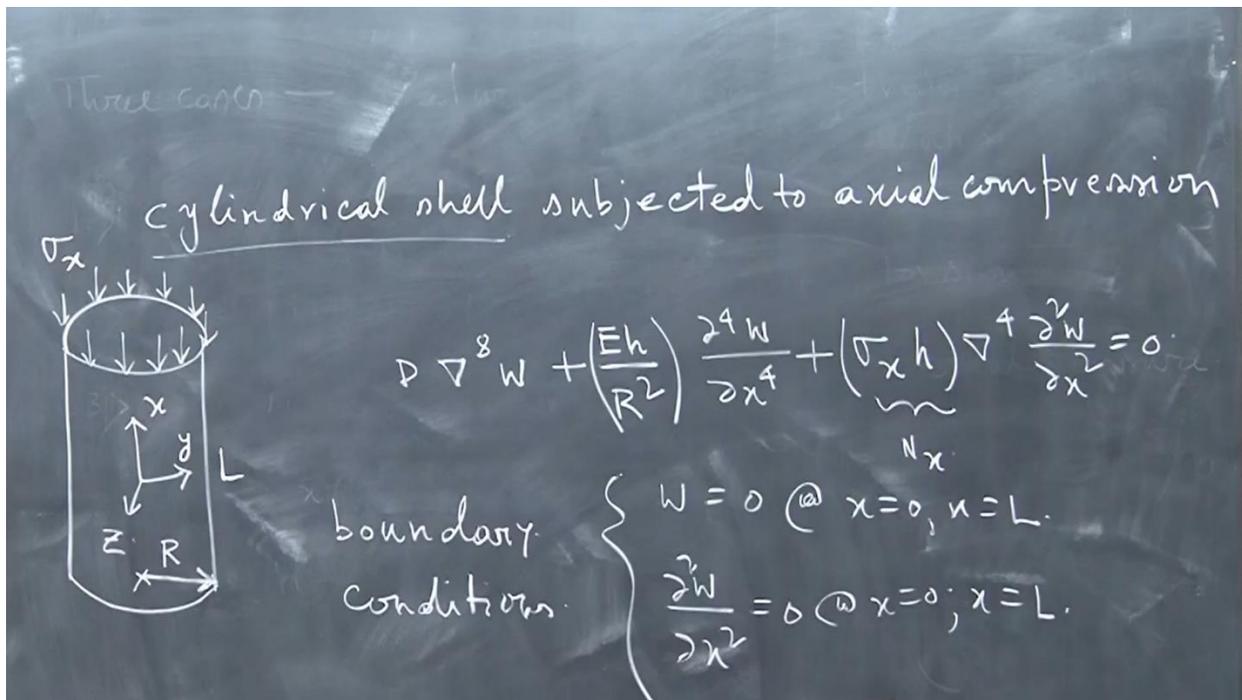
the mathematical complexity. Okay, I want to make the discussion simple, but essentially, we will try to focus on the physics of the problem. Okay. What really happens, okay, with the demonstration on the cylindrical shell is that the cylindrical shell, you know. If we understand that, it can be extrapolated to other shells; essentially, whatever is happening due to additional curvature, etc., there might be a qualitative difference. Quantitative difference, but not qualitative, okay, things. So, then substitute the ultimate equation you will get; this is Donnell's equation. Donnell equation, this is an 8th order equation, ok. Once again, the way for the plate, you have derived this, okay.  $\nabla^2 \nabla^2 w(x, y) = \frac{q(x)}{D}$ , right?



Similar equation; the only thing is that  $D \nabla^2 \nabla^2 \nabla^2 \nabla^2 W - \nabla^2 \nabla^2 \left( N_{x0} \frac{\partial^2 w}{\partial x^2} + 2N_{xy0} \frac{\partial^2 w}{\partial x \partial y} + N_{y0} \frac{\partial^2 w}{\partial y^2} \right) + \left( \frac{Eh}{R^2} \right) \frac{\partial^4 w}{\partial x^4} = 0$ , so essentially 8, 8 to the power of ok. You see, this is the equation ultimately; you see curvature in the cylindrical shell, which is coming in this term, right? So, this is the Donnell equation for, of course, a cylindrical shell.

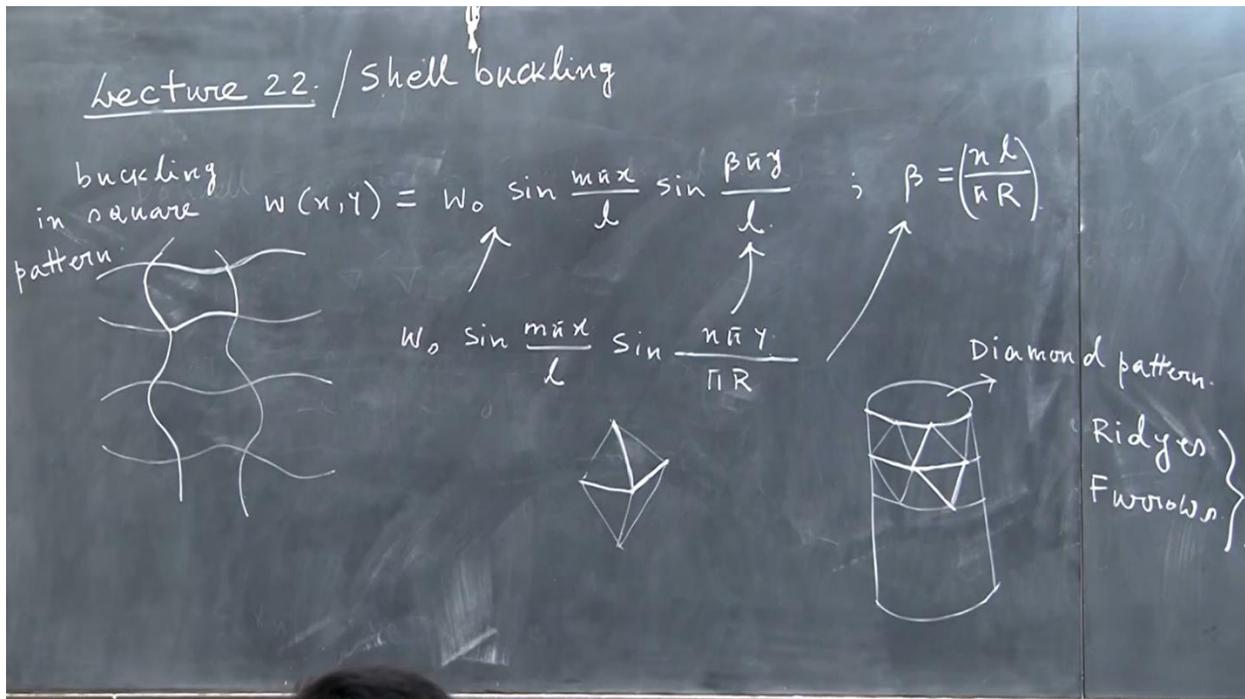


So, we will use this equation to study the three cases we can study, okay. Although we will not have time to study three cases, only three simple ones, we will only study the first one, you know, the cylindrical shell subjected to axial compression. A right cylindrical shell subjected to torsion can result in torsional buckling. A cylindrical shell subjected to external pressure. External pressure, and then when it is subjected to external pressure, you know there will be. Hoop stresses develop right circumferential hoop stresses, and because of that, it will also buckle right. So, these three simple analytical solutions can be obtained. Although all this analytical solution essentially the first one is meaningless. Because what happens, what is predicted by this equation for the second and two, is reasonably good to a certain extent because those are not imperfection sensitive. Once again, imperfection sensitivity will come into the picture when it is subjected to not all the shell buckling problems; not all the loading modes are imperfection sensitive. Please note that a cylindrical shell subjected to axial compression is notoriously imperfection-sensitive. So that is what it also mobilizes: huge, you know, post-critical non-linearity and even pre-buckling non-linearity, and that is what it basically collapses—whatever the little, you know, theory that we are trying to erect, okay.



But for the other, it might be a little closer. Okay, fine. So, the first one I am going to show is what resistance you get and then how the experimental result or actual behavior of the shell does not even comply with that, okay. We will consider a cylindrical shell subjected to axial compression. One second, please note that the cylindrical shell subjected to axial compression has been studied using the first example. Axisymmetric, assuming that there must be an axisymmetric buckling mode, and essentially the bending can be represented by studying the bending of a strip. And essentially, we will have a one-way bending, and we established this analogy with the beam on an elastic foundation, and we have seen. So, this is another way you see that increasingly we are imposing more and more complexity, okay. So, this is when we are going to show how the equations will look. If you are considering axial compression of a cylindrical shell. So,  $\sigma_x h$  is basically assuming that this is  $\sigma_x$  and  $h$  is the thickness, so that is nothing but  $N_x$ , okay. So, the coordinate axes you know are  $x$ ,  $y$ , and  $z$ . We assume the solution, and we assume the boundary conditions are correct. Assume simply supported; thus,  $w = 0$  at  $x = 0$  and  $x = L$ , and then we also assume that. Because if you assume this is simply supported out of plane for restraint, then see, because it is axisymmetric, you know, and then, you know. If we assume  $w = 0$  in the  $y$  direction, the bending expression indicates that the bending moment needs to be 0, right? So, from there, if we simplify, then the other derivatives will drop; only this derivative will remain, so that

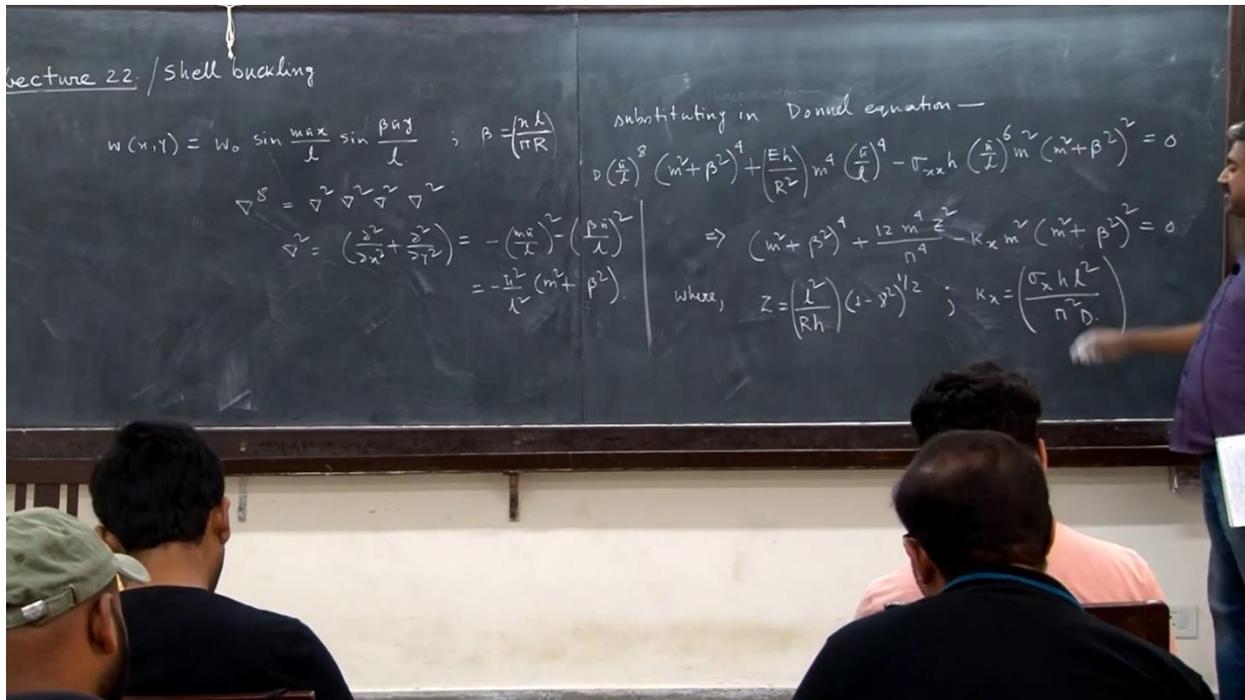
must be 0, ok. So, this is essentially the boundary conditions. Now we can assume a solution. In terms of  $x$  and  $y$ , we assume the solution by satisfying the boundary condition.



So, when you satisfy the boundary condition, we will have  $w_0 \sin \left( \frac{m\pi x}{l} \right) \sin \left( \frac{\beta\pi y}{l} \right)$ , where  $\beta = \left( \frac{nl}{\pi R} \right)$ . So, essentially what we try to do is we first assume, I mean this one is coming from somewhere, you know, this is coming; we assume this is  $w_0 \sin \left( \frac{m\pi x}{l} \right)$ , right? And then we assume  $\sin L$  is basically  $\pi$  times  $r$ , right? Two  $\pi r$ , rather  $\pi$  times  $r$ , and then it is  $n\pi y$ ,  $n\pi y$  by  $\pi r$ ;  $\pi r$  is the length, right? So, we assume this to be  $\beta$ , and then we substitute something like this, okay. Essentially consisting of half a side wavelength. The difference between the first one and here is that here we consider both ways of bending, bending along  $x$  as well as bending along  $y$ , and for both types of bending, we assume that we can approximate using a sinusoidal function. So, essentially, what are we assuming? That would be that we are restricting a priority so that it will buckle in a square pattern. Why? Because you see that this will. So, you know, along the  $x$  direction, it is all a sinusoidal kind of pattern. No, along the  $r$  direction also, it is this kind of pattern. So, you see how this sinusoidal pattern, you know, this curve is coming. Okay. So, we verify that it is buckling with this kind of assumption: buckling in a square pattern. So, basically, there is a problem, you know, when we say that while it is buckling in the square pattern, using a

sinusoidal approximation is a huge simplification of the problem. If you do the experiment and see what happens, you know it does not buckle in the square pattern at all. Now, if you take a cylindrical, unfortunately, we do not have a setup to do this experiment. This is a very interesting experiment, and it is a very interesting physics experiment compared to the one you are doing now. When it buckles, you will see that it develops a pattern that looks like a diamond. So, it does not buckle under a square pattern; it buckles under a diamond pattern. How does this diamond pattern look? The diamond pattern is something like this, right? So, essentially what happened is that if there is this kind of cylindrical shell. And what will happen is that a portion of this will come out, and then, So, it consists of two ridges and two furrows. See what are ridges and what are furrows? See, this is coming upward, right? This is, it is coming out. The next way is coming inward. So, that is what this kind of diamond pattern is, okay? This is a diamond pattern. And then, you see the diamond pattern, the inclined pattern, and what happened is that one will come out and one will go in. It will be something like this, okay? So, it consists of a number of ridges and a number of furrows, okay? It consists of ridges and furrows. The white buckle on the square pattern can be explained; I will explain later what happened. When it tries to buckle, essentially the buckling needs to be that the cylinder experiences the membrane stretching, because as you are pushing me, I am going to stretch it like this, right? So, the membrane stretching is happening, and how membrane stretching causes out-of-plane deflection; because of out-of-plane deflection, there is an in-plane strain, Von Karman strain, right? So, that tries to happen at first. That's what load increases. But then, suddenly, it goes to a configuration with a diamond pattern. And all this diamond pattern, if you see, if you sum this up, you know, this, you know, whatever is coming up, you know these ridges; you will see that the length of these ridges will essentially be the same as the circumference of the cylinder. So, what does it mean? That means, essentially, it is an inextensible mapping of the surface. And when there is inextensible mapping, that means what you understand is that essentially there is no, you know, maybe initially there may be a little membrane stretch, but later stretches essentially show that there is no contribution of strain energy from membrane stretching. The entire contribution of strain energy is consumed by bending. So, it is a pure bending, you know, in a select that is corroborated by the inextensible mapping of the surface, and this kind of pattern—you have to be careful to do this experiment. It cannot be; what do you have to do? You have to search on YouTube or in a book. You know you will see this picture. So, you have to take this cylinder and put a mandrel inside it so, that it is radial, so the

cylinder basically rests on a mandrel. That means it does not go inward, okay? You see that? In order to develop this pattern, it will be mounted on a mandrel, radially supported by a mandrel, and then you gradually apply the load using a. Of course, the displacement controller strain control experiment is okay, and gradually this pattern will develop. If you do not do it, you will not see this beautiful pattern in the end, and it is propagating over the length. In fact, we wrote a paper and then. We were all ignorant, you know, my students and I were. So, I was wrongly associating some pattern we saw in the finite element simulation with the Yoshimura pattern. This pattern is called the Yoshimura pattern, named after a Japanese scientist, Yoshimura. I like that this is a huge nonlinear phenomenon, okay? I will explain it later, but, So, then the reviewer corrected us by stating that we were wrong. Of course, we studied, and I suggest that you read more; the more you read, the more you learn. That's how little you know, okay? That was told by one of my professors at IIT Bombay, you see. Anyway, whatever we obtained from here, I will show you, and then I will introduce you to parameters that are extremely important. But then please note that it does not corroborate that. In fact, later Donnell himself tried to solve this equation using this, assuming this diamond pattern instead of a sinusoidal pattern. But he also failed. You know why he failed? He failed because he did not consider non-linearity, neither the pre-buckling non-linearity nor the post-critical non-linearity. Unless and until you consider those nonlinearities, you know, this modal interaction will never occur, and the interactions of these imperfections with the erosion of the membrane strain will not be captured. And if you do not capture the non-linearity, you will not be able to capture the actual behavior that is observed experimentally. So, there is a huge buzz that has flustered you for more than several decades, and ultimately it is von Karman. Who solves it using asymptotic techniques and gives a meaningful solution to this problem? Now it is all well understood. So, if you take and substitute into the equation, then what you will get. So, you understand why these things are coming, because it is all these nabla, I can just show it to you, okay. So, if you assume like this, then it is something like  $\nabla^2$ ,  $\nabla^2$ ,  $\nabla^2$ ,  $\nabla^2$ , right? So, then you know you differentiate all the, so this is nothing but one  $\nabla^2$ ; if you operate one  $\nabla^2$ , then you will see that it is nothing but the Laplacian operator, right? So,  $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ . So, if you differentiate them on this, then you will essentially get  $-\left(\frac{m\pi}{l}\right)^2 - \left(\frac{\beta y}{l}\right)^2$ , and you will get  $-\frac{\pi^2}{l^2}(m^2 + \beta^2)$ , okay.



So, 1 is this one, so it is yours, so there are 4 times this operation going on, and essentially it will be raised to the fourth power, understand that, right? Okay, so then from there, if you simplify, you will see that this will obtain an equation; I will simplify it with some variable

$(m^2 + \beta^2)^4 + \frac{12M^4 Z^2}{\pi^4} - K_x m^2 (m^2 + \beta^2) = 0$ .  $k_x$  is equal to  $\left(\frac{\sigma_x h l^2}{\pi^2 d}\right)$ ;  $k_x$  is essentially, you see, a stress coefficient, right? And  $Z$  is an important parameter;  $Z$  is a safe parameter, as you can see it is essentially a safe parameter for the shell. Why? Because it is a function, of course it is one of the functions of Poisson's ratio. Because it is a two-way bending, their coupling will be there. So, essentially, it involves the length, radius of curvature, and thickness, right? So, it is a function of, what is  $L/R$ ? It is a slenderness ratio, isn't it?  $L/H$  is the length. So, essentially, it is a geometric parameter that will control the buckling. Understand the critical role, right? So, that is why it is a shape parameter. And this parameter is called the Batdorf parameter. Bat, B-A-T, Bat Dorf, D-O-R-F. This is called the Batdorf parameter. Name the scientists who have investigated this. Now, once again, So,  $Z$  factor, you know,  $Z$  factor essentially helps in distinguishing between short and long shells and also thick and thin shells.

## Lecture 22. / Shell buckling

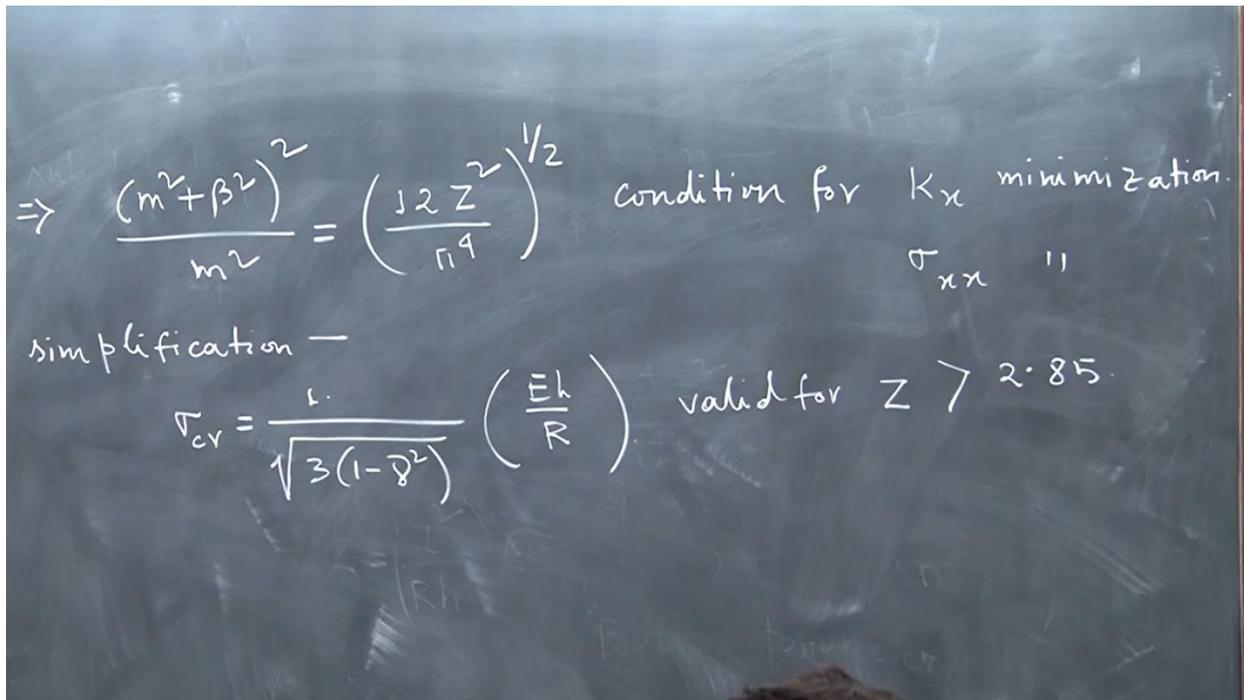
$Z$ -factor essentially helps in distinguishing between short/long shell and thick/thin shell.

$$\frac{\partial K_x}{\partial \left( \frac{(m^2 + \beta^2)^2}{m^2} \right)} = 1 + \frac{12 Z^2}{\pi^4} \left\{ - \frac{1}{\frac{m^4}{(m^2 + \beta^2)^4}} \right\} = 0.$$

See why, because  $L/R$ , you will distinguish between long and short shells. So, what happens if the shell is too long? What happens, you know? Its shell action will not be there; it will essentially buckle like a column under axial force, right? So, it will differentiate between the buckling of short and long shells, and then when we take  $L/H$ , where  $H$  is the thickness, it determines whether it is a thin shell or a thick shell; you understand that differentiation. Anyway, that is why it is a Batdorf parameter; it is an important parameter. Now, what we will do is clearly see that it is a function, unlike the previous one we considered. One way of bending it was only a single wavelength, you know. Single wave number right  $m$  and  $\beta$  earlier was only a single wave number, but now it is a two-way bending, so two-way bending, so wave number and this  $m$  and  $\beta$ . You know when you are taking the square root, you know that it is the combined wave number. So, essentially, this is a wave number parameter; right now, the way you have seen it is with changing wave numbers. There are more conversions that may occur, right? So, we have to minimize this with respect to the combined wavenumber. So, we can take this to be a combined wavenumber parameter and differentiate this to minimize the  $K_x$ .  $K_x$  is essentially related to the critical stress, right? For that reason, it contains what? Axial force parameters  $\sigma_x$  and  $\sigma_{xcr}$ , the critical values of which we can

find out. So, if we do that,  $\frac{\partial K_x}{\partial \left(\frac{(m^2 + \beta^2)^2}{m^2}\right)}$ . And then you will see from there that you will get a

condition for minimization. So, the condition is that if you simplify this  $\frac{(m^2 + \beta^2)^2}{m^2}$ , it is  $\left(\frac{12Z^2}{\pi^4}\right)^{\frac{1}{2}}$ .



This is the condition for  $K_x$  minimization, and when you are minimizing  $K_x$ , it also means  $\sigma_{xx}$  minimization, okay. So, this is the condition: what we see is that there must be a relationship established between the effective wave number and the wave numbers associated with the board shape of buckling, as well as the shape parameter of the shell, right? So, from here on, some simplification, simplifying on some simplification, we will get

$$\sigma_{CR} = \frac{1}{\sqrt{3(1-\nu^2)}} \cdot \left(\frac{Eh}{R}\right).$$

See, I mean this is a little, you know, comparable to our previous solution, right, when we assume that it is the one-way bending and then axisymmetric, right? So, that  $EH$  by  $R$  term was also there, but what was not there is basically this coupling term, because including Poisson's ratio, this term was not there. So, this is valid for  $y$ , valid for  $Z$ , and valid for the batdorf parameter greater than 2.85 because the batdorf parameter has to be positive. Okay, I will show you why. Batdorf, this is important. So, then I will stop here today; next, we will see.