Stability of Structure

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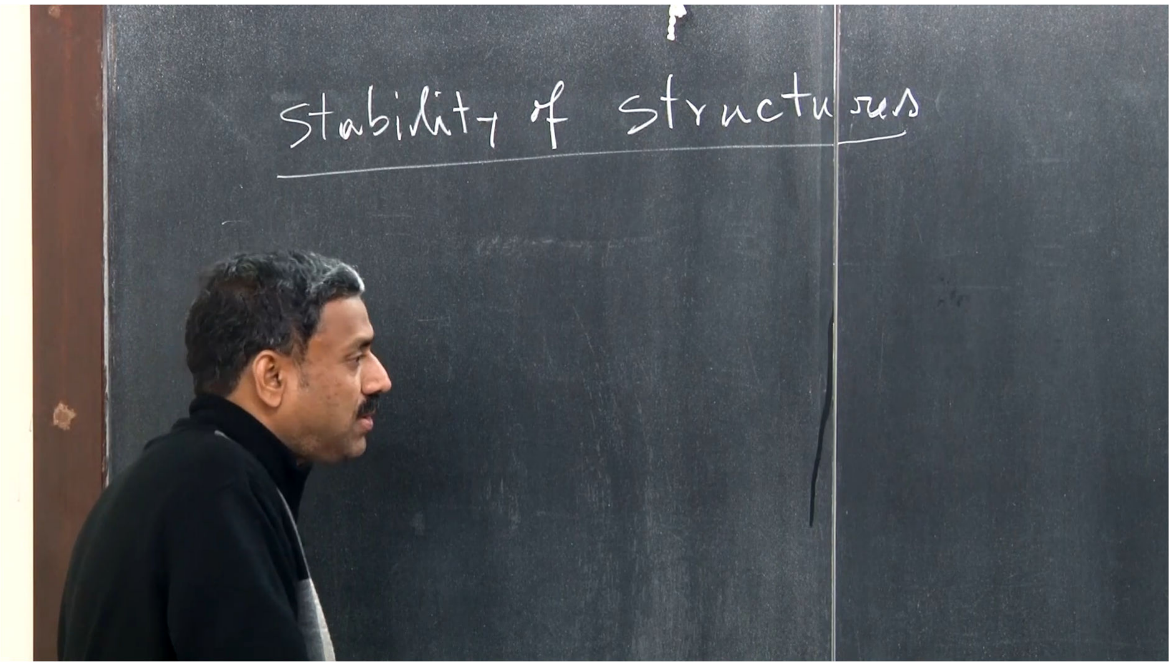
WEEK-01

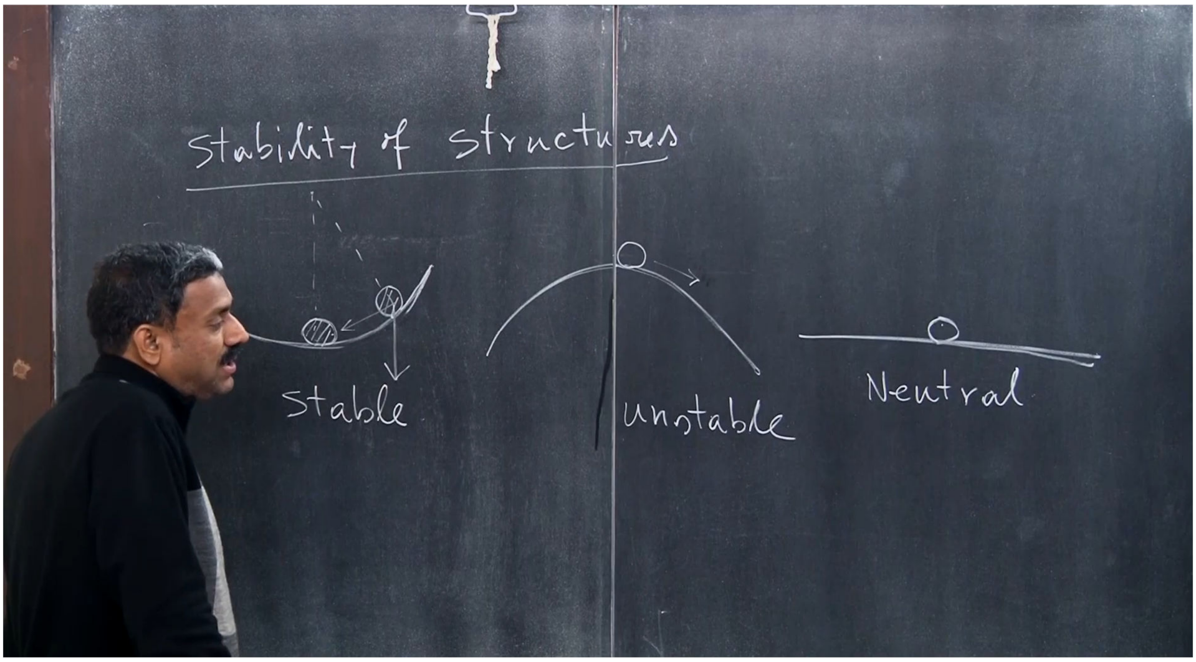
Lecture 2: Energy Approach for Structures with Two Degrees of Freedom

Okay good afternoon, everybody. So, we were discussing stability of structure. So, this is a second class. So, we have introduced the concept of structural stability ok and we have discussed that how stability is important ok. Why analysis of stability is important for both for design perspective as well as to avoid kind of instability in structures right which is one form of failure right. So, we also briefly recapitulate the contribution that were made by prominent scientists in this area, right.



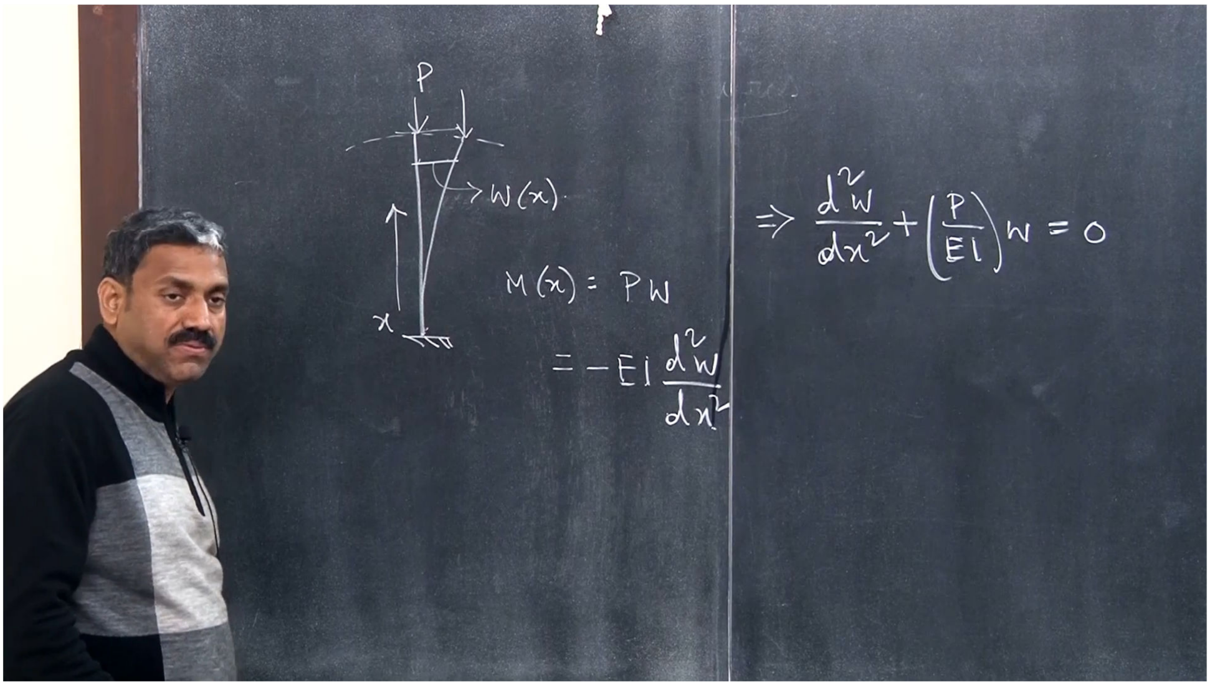
Starting from, the Euler buckling load in 17th century to modern concept of dynamic stability namely the aerodynamic flutter and then follower force instability of a simple beam or column rather under follower forces ok. And then we have started with the concept of stability we have started with the simple system that we have learned in our school level physics. that how we define stability there is we were considering 3 situations, where we were considering a ball that is resting on a concave bowl, right and then you can see that this ball is under equilibrium. Because, its weight is registered by the reaction exerted by this ball surface right and then, if you perturb this ball here then, because of its component of its weight it will come back to its original configuration and that is why it is stable configuration right. Consider the situation where this ball is kept on a convex surface and then if you perturb it little bit then it will diverge from its configuration and it will never be back to its original configuration.

So, this is unstable right. And there is a third possibility where this ball is kept on a horizontal surface. So, wherever you place the ball, it will be there.



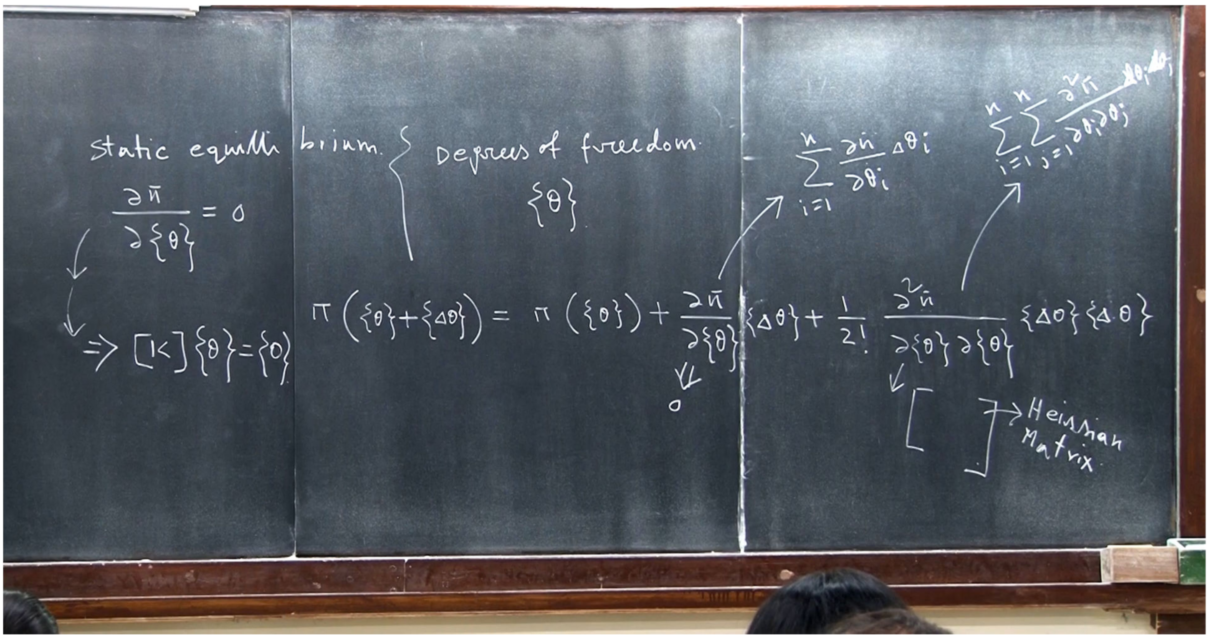
So, even if you perturb it, okay, it will remain on that surface, right. So, this is called neutral equilibrium. So, the concept of stability was explained using these three simple examples. qualitative example rather in our school level physics. So, we are trying to generalize this concept for our more advanced studies or treatment that we are going to give to structural stability okay. So, the first lesson in a stability of structure is that you have to consider equilibrium but in the perturbed configuration right. And knowingly or unknowingly you might know these things. if you can briefly recall that when you were taught the stability of column rather the buckling of column, then you have written down the beam bending equation in the perturbed configuration. Please recall that we know the beam bending equation right. EI $\frac{d^4w}{dx^4}$ is equal to minus q_x right or if you want to write in terms of bending moment then EI $\frac{d^2w}{dx^2}$ is equal to minus m_x right. So, m is the bending moment right. So, how this bending moment is coming there? So, we have considered a column right. And then we have written down the equation and perturbed. So, it was subjected to some force P compressive load axial compression. So, you must be recalling that this was perturbed and then there was some deflection $W(x)$ right and that is what it is going to cause bending moment $m(x)$ right. We are considering the same is x this coordinate right. So, if this is w then p into w right and their m is nothing but minus EI $\frac{d^2w}{dx^2}$ right. So, there we got the column buckling and that is where you solve this difference. So, now you are getting an equation which is $\frac{d^2w}{dx^2} + \frac{P}{EI}W = 0$ and that is what you solve to get the Euler critical rule right. So, why trying to emphasize because I want to relate with your undergraduate mechanics that you have learnt.

In stability you have to write the equilibrium in deformed configuration ok. So, that so you have to perturb it ok and then you have to see the equilibrium. For column it was simple because it was a one-dimensional system.



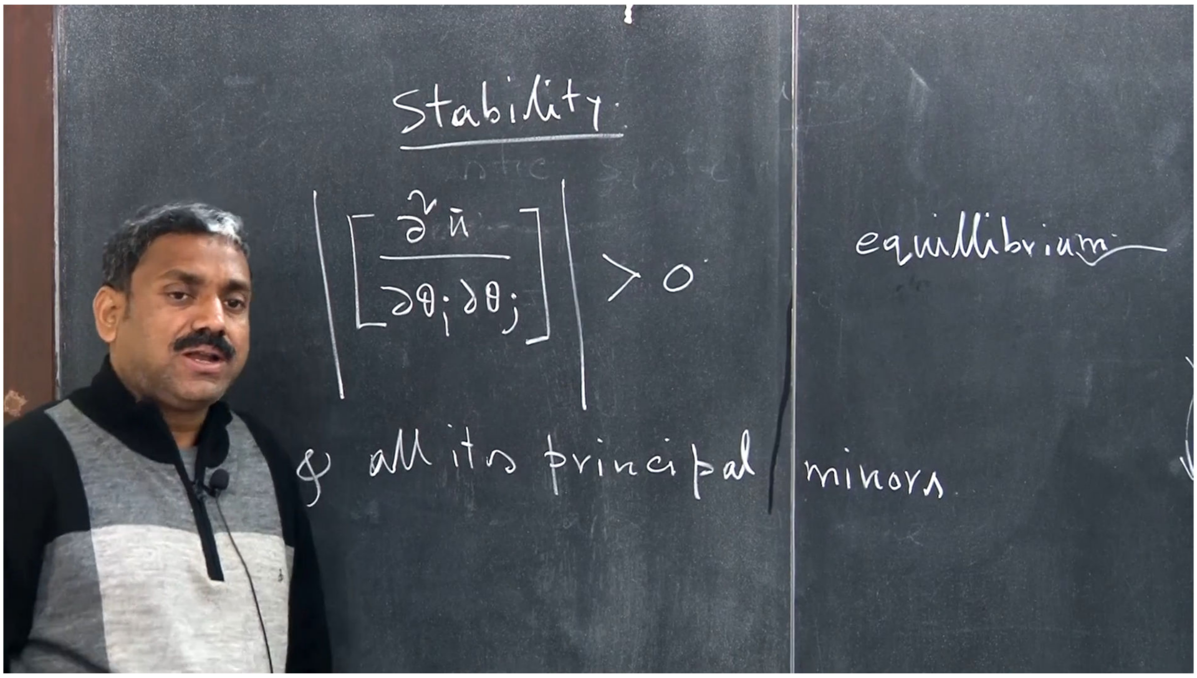
but if you want to extend it for the 3D continuum. You have to write down the equilibrium equation in terms of in deform configuration then the equilibrium equations will be little different. Here I will briefly touch upon those issues, you can recall that in your solid mechanics course right, if it is infinitely small deformation. then it was everything is expressing sort of Cauchy stress tensor right and that is where you have written that $\sigma_{ij,j} = 0$ that was equilibrium equation right in terms of Cauchy stress tensor. But then you are introduced with the other alternative stress tensor like first Piola-Kirchhoff stress tensor, second Piola-Kirchhoff stress tensor. if you can recall that in first and second Piola-Kirchhoff stress tensor we have distinguished the deformed and non-deformed configuration. So, if you consider second Piola-Kirchhoff stress tensor and first Piola-Kirchhoff stress tensor, then you will be able to write down the equilibrium equation in the deformed configuration in a continuous fashion. With all these things we will come later, okay. And you will see that we also briefly, explain that finite deformation, when the deformation is not infinite small, okay. then you have to have correct representation of the or you have to establish equivalency of the deformation measures ok. In order to correctly reproduce the strain energy functional or potential energy functional ok those things we will discuss briefly. Now what we have considered an example small example to illustrate this equation right. So, you can recall that We are considering elastic system. So, in this example we are considering elastic system. First let us discuss the elastic stability. This course primarily deals with the stability of elastic structure, elastic stability ok. We will hardly discuss the inelastic stability and others ok.

Nevertheless, those are advanced statement, and you can find in other courses. So, in elastic system we have considered that any elastic system will have some potential energy. we have developed right this is nothing but potential energy. and if you recall potential energy for an elastic system can be expressed in terms of strain energy minus work done. So, E is nothing but elastic strain energy, and how to calculate that, how to estimate that and W is the work done right, work done. So, then if you can recall in equilibrium when we consider the static equilibrium then we have obtained the if we consider the elastic system which have some degrees of freedom ok. So, elastic system degrees of freedom. we have defined as you see in terms of some vector θ right. So, $\frac{\partial \pi}{\partial \theta}$ must be equal to 0. So, you get the sets of equation for the equilibrium. So, that is the zeroth order information or first order information that we learn. So, this is for the for establishing the equilibrium. you will see that what will be equilibrium equation that is what we have used in statics even in dynamics to solve the system. Now for stability you have to consider higher order information. how? Please recall that we have to perturbing the system right. So, from the previous class. you can recall that we have done a Taylor series expansion. we have considered the expansion of potential energy functional π right. So, here it was θ and then we have put some perturbation to θ. So, $\theta + \Delta\theta$ right. So, then if you can recall $\pi(\theta + \Delta\theta) = \pi(\theta) + \frac{\partial \pi}{\partial \theta}\Delta\theta + \frac{1}{2!}\frac{\partial^2 \pi}{\partial \theta^2}(\Delta\theta)^2 + \cdots$ here or if you want to put it in that way. So, here these are when you can also express is a summation of $\pi = \sum_{i=1}^{n}\frac{\partial \pi}{\partial \theta_i}\Delta\theta_i$ right this one and this one you can also write there will be 2 summations over i and j. I am considering n degrees of freedom so then $\frac{\partial^2 \pi}{\partial \theta_i \partial \theta_j}\Delta\theta_i\Delta\theta_j$ right or instead of dθ. we can write $\Delta\theta$ okay instead of dθ here you write $\Delta\theta$ right. So, here you write $\Delta\theta_i\Delta\theta_j$ ok. So, you can see that this is what this will give you. So, here this you are going to get equilibrium equation right. kθ is equal to 0 or kθ is equal to if there is some external load then, there will be some load vector. Whereas this higher order term this here this higher order term will be another matrix of potential energy okay.



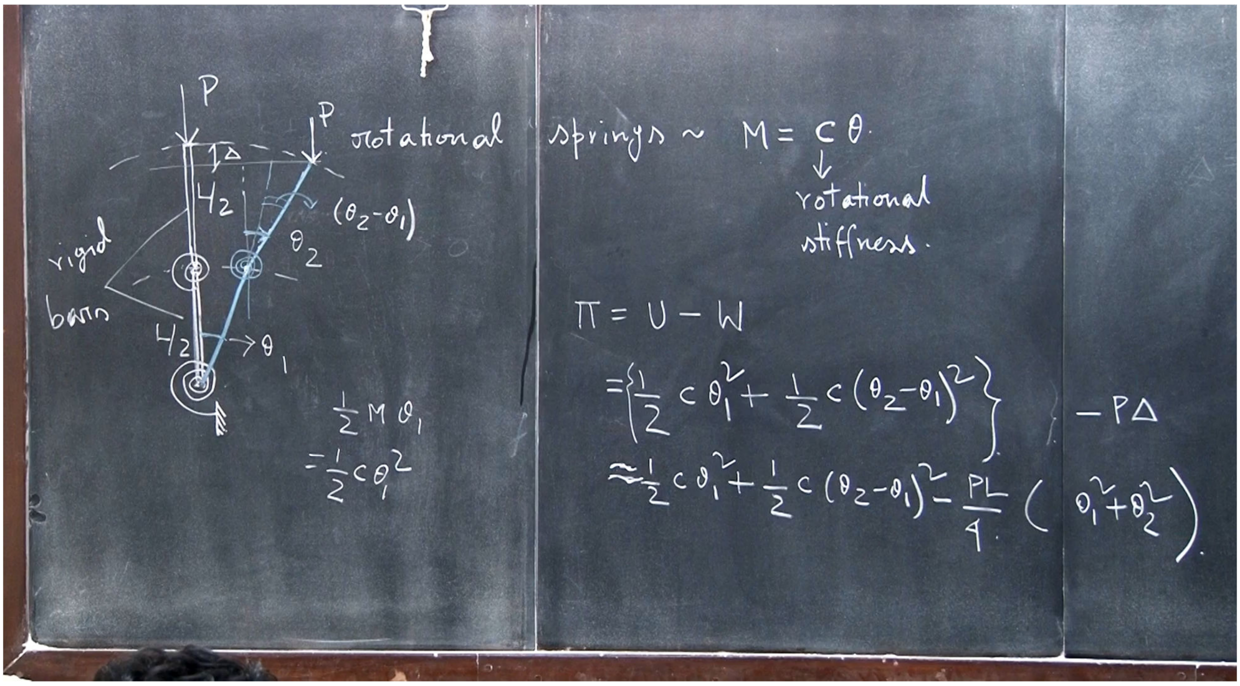
So, this is second derivative of the potential energy term right. So, this is Hessian matrix right. Hessian matrix of the potential energy and what we have seen that. When we are perturbing a system. so, from θ, we are going to $\Delta\theta$. So, then, this is being 0, right? So, it does not really matter whether $\Delta\theta$ is positive or negative, okay? Any perturbation, it can be positive side, it does not really matter, right? Since this is 0, this will, vanish. But then, so that means you have attained equilibrium, right? So, now because of that perturbation, if the alternate configuration is having higher strain energy, then the equilibrium that we are attaining will be stable right that means if this is greater than 0. because you see these are appearing in square right $\Delta\theta_i\Delta\theta_j$. so, basically this will always be what positive right. so, this term needs to be positive right.

Okay so how will ensure that? Well, if you make the determinant of the Hessian matrix positive and all its principal minors are positive, then that means this term is always positive, that means from θ to $\theta + \Delta\theta$. if you go to an alternate configuration due to this perturbation that will lead to a higher strain energy, okay, higher potential energy. That means our system will not attain that configuration. It will try to remain in its configuration. That means that equilibrium is stable. So, you see how stability. So, whether equilibrium is ensured by the gradient of the potential energy. when you want to ascertain the nature of equilibrium that means whether it is stable equilibrium or unstable equilibrium or neutral equilibrium.



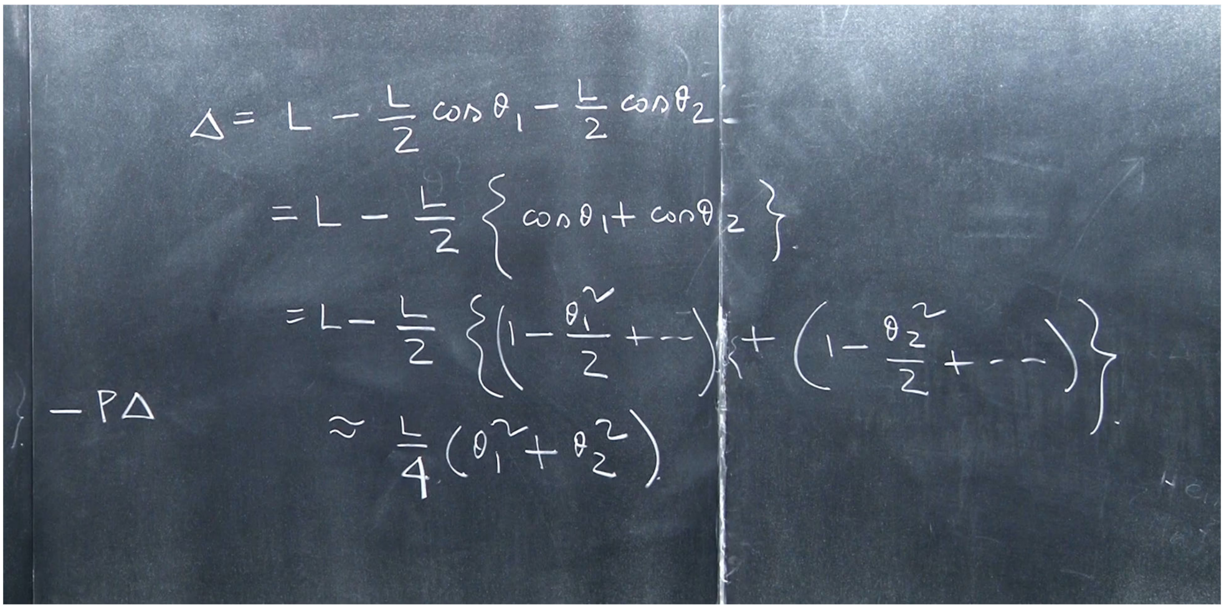
We have to explore on higher order derivative of the potential energy and that is where it is second order Hessian matrix ok. Now please note that you also have learned this Hessian concept in of those of you who did optimization right. because you must have done this steepest gradient method right search numerical search technique to attain the maxima or minima right. So, which direction your search will be guided that also depends on the sign of the Hessian matrix right. So, now you understand from here that, how we define stability of equilibrium ok. We have defined equilibrium by the gradient of a potential. this defines equilibrium that is what we have learnt over all our mechanics course minimization of the potential energy and which is the equilibrium right. Now for the stability of equilibrium which is a higher order information stability. You have to consider higher order derivative $\frac{\partial^2 \pi}{\partial \theta_i \partial \theta_j}$ okay. this determinant of this must be greater than 0. And all its principal minor. what is principal minor, because you just take diagonal term and then the sub determinant must be greater than 0 right. So, if it ensures then it is stable equilibrium. if it is negative that is unstable equilibrium. If determinant and principal minus are 0 then it is neutral equilibrium, clear. So, first we started see stability not necessarily that we have to follow a energy approach to analyze stability, but this is most intuitive. And a first class of stability started with this energy concept because this is most intuitive to us okay and please note that it is easy to deal with the potential energy functional. Energy approach we always love why? you can recall that red book energy approach in structural mechanics. that is a very celebrated book actually everybody all these great mechanics they read that book okay. Why because the energy approach is a very powerful approach right. Okay so we are going to illustrate this using a simple system that we have discussed in the in previous class. we just recapitulate whatever we have done in the previous class. So, now if you can recall that we were considering a 2 degree of freedom system, which is made of this rotational spring and this is a rigid bar that is also connected with another well middle ok. This is a rigid bar right., So, these are all rigid bar, and these are rotational spring. That means they will exact moment on application of rotation and it is subjected to some axial load P. And we were asked to study the stability of this system. So, the concept will be illustrated with respect to that simple toy problem okay. Now why we have considered this? Well, this is not a real structure but nevertheless it is a simple structure which has concentrated elasticity.

Because this is concentrated here right and I am assuming this is of length l/2, this is of length l/2 and all these two spring, rotational spring. Rotational springs are behaving moment is C into θ. θ is the relative rotation between the end, right. This is bending moment and C is the rotational stiffness, right. So, we have written down the potential energy. So, what if you can recall I am not going to write once again write down the equations. because I have done in previous class but you have the expression from the right. So, I am writing potential energy is nothing but strain energy minus work done. I am assuming that initially that are straight right, there is no imperfection. So, potential energy E was basically and I am perturbing it. Perturbing means when I am perturbing this. this is attaining a deformed configuration where this will be rotating. So, it will come here, and then with respect to this, this is going to come somewhere here. So, I am defining your this is angle is θ_1 and this angle from here it is I am defining this angle is θ_2. And this P is coming here right. So, that is the way the strings are in action. So, please note that I can use a different colour chalk. So, here was the spring was there and then here, the spring is there, and this rotation is happening with respect to this vertical axis, right? θ_1. So, please note that here strain energy is nothing but the half M into θ_1 right, and m is nothing but C into θ_1. So, half θ_1 square right, so half C θ_1^2 that was the energy in the first spring. what about second spring? Please note that in the second spring, it is the relative rotation that is important right. So, I have to expand it like this, so this angle.



So, this is rotation θ_1 with from the vertical axis, this is from vertical is θ_2. So, relative rotation here is nothing but $\theta_2 - \theta_1$, ok. So, then the next expression is half c $(\theta_2 - \theta_1)^2$, right. So, this is the expression for the strain energy, elastic strain energy and please note that this is concentrated elasticity, ok. And then what is work done? work done is done by this load P. load P, why? Because load P is coming to a distance, this distance vertically downward δ and this δ is what? δ

is nothing. So, I am writing P into the δ and this δ that is nothing but vertical deflection of point of application of load P will be what? It is nothing but what? L, initially these are all length L by 2, L by 2, this is length L by 2, this is also length L by 2, right? Initially there was this L minus L by 2 cosines of θ_1 minus L by 2 cosines of θ_2 right. Because projection of this one this is L by 2 cos θ_1 L by 2 cos θ_2. so, if sum of these two and so this much is basically the deformation. What we have used doesn't require any further simplification. Because it is appearing in terms of a polynomial, we also wanted to use some expansion of L by 2 oks. of cosine θ in Trigonometric function in terms of polynomial ok. And there is advantage to that, that I will explain you later. So, this is L if you can recall, this we have simplified and that was L minus L by 2 (cos θ_1 + cos θ_2) and that was L minus L by 2 $(1-\frac{\theta_1^2}{2!}+\cdots)$ means 2) higher terms I am neglecting ok .But you may consider it later we will consider this $\frac{\theta_2^2}{2!}$. So, this was what was this after simplification this was $\frac{L}{2}(\theta_1^2+\theta_2^2)$ right. L by 4, right? Okay. So, I am once again simplifying it. so, $\frac{1}{2}c\theta_1^2+\frac{1}{2}c(\theta_2-\theta_1)^2-\frac{PL}{4}(\theta_1^2+\theta_2^2)$ right.



Everything is in terms of a polynomial clear. So, we express the potential energy functional, why so because it is a function of function right in terms of polynomial. Now, we will do we will first find out the equilibrium. So, we will use the zeroth order information equilibrium. how will you ensure equilibrium? Well, $\frac{\partial\pi}{\partial\theta_1}=0$ and $\frac{\partial\pi}{\partial\theta_2}=0$. This if you differentiate this π with respect to θ_1

θ_2. So, you will get the simplified equation equations I am writing. This will give you first one equation will be $2C\theta_1 - C\theta_2 - \frac{PL}{2}\theta_1 = 0$ and the next one $C\theta_2 - C\theta_1 - \frac{PL}{2}\theta_2 = 0$. So, understand that. So, here you see that this is giving we are obtaining to what? To simultaneous equation but these are these are homogeneous system of equation right. So, if you want to write in matrix form if you combine them. then what you will get

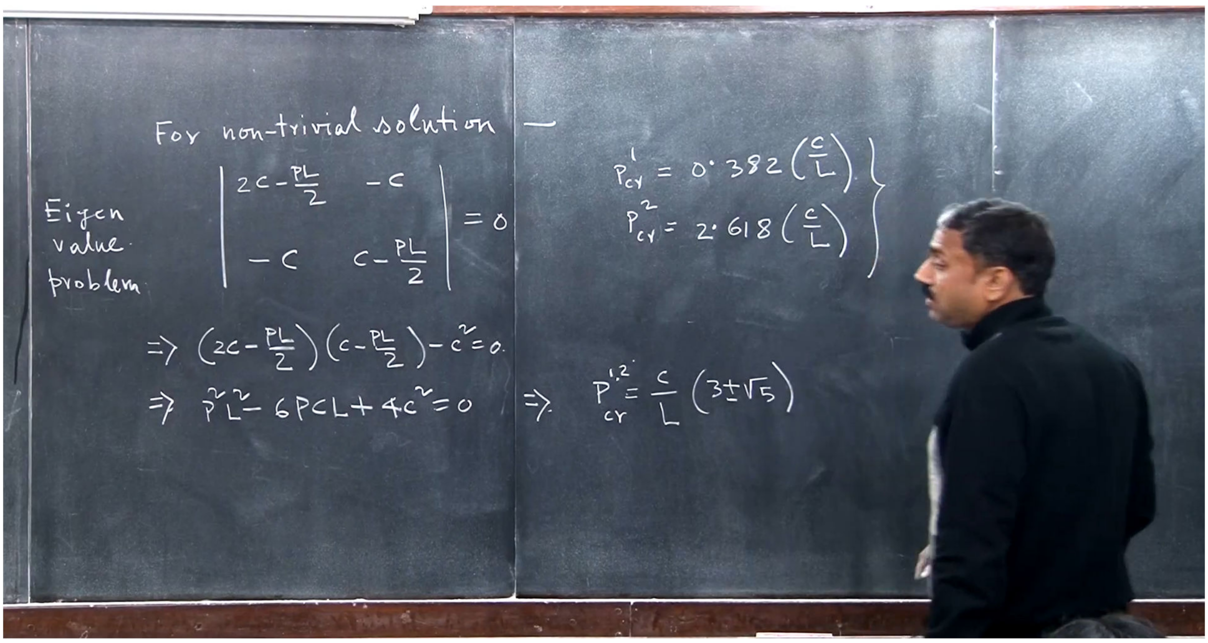
$$\begin{bmatrix} 2C - \frac{PL}{2} & -C \\ -C & C - \frac{PL}{2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the equilibrium itself is giving rise to a what to a homogeneous system of equations right, because these are homogeneous systems of equations right. Now you tell me if it is homogeneous system of equation then what? there can be a trivial solution where $\theta_1 = 0$ $\theta_2 = 0$ that means whatever be the be the system. if it is trivial that is a trivial configuration right. If $\theta_1 = 0$ $\theta_2 = 0$ then there are no other things. but vertical configuration there is no perturbation right. but if equilibrium has to be maintained in a perturb configuration right then you have to solve this.

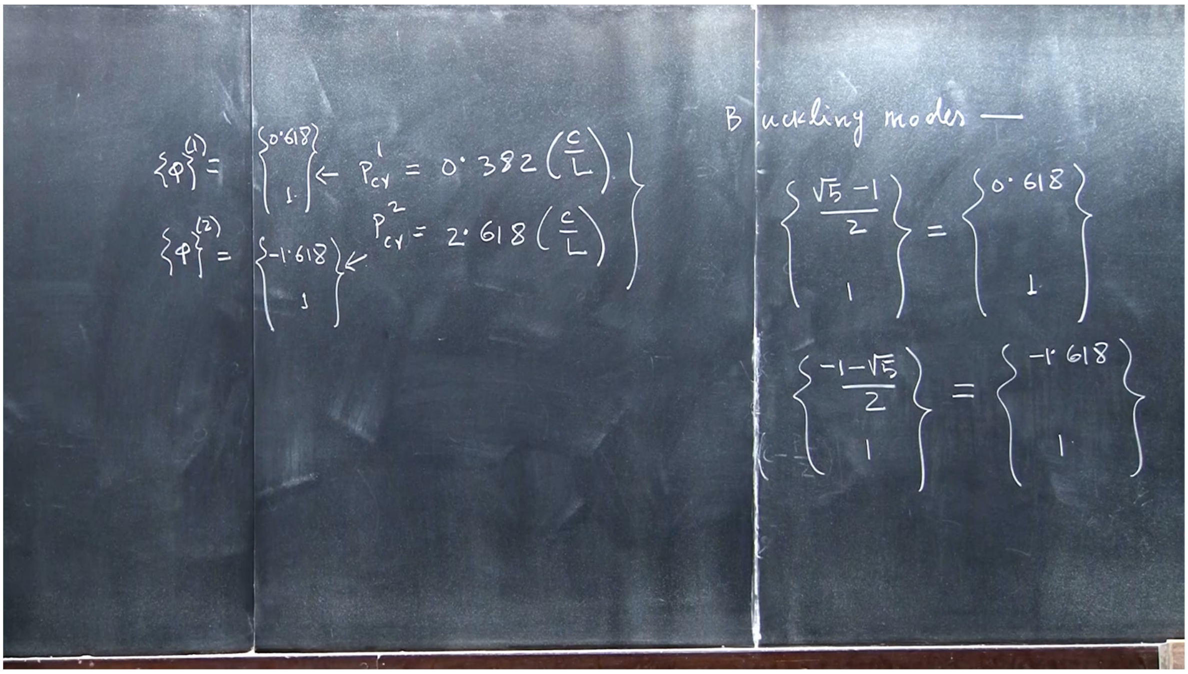
So, how will you get a non-trivial solution. For non-trivial solution what happens the system of equation determinant must be equal to 0 right. So,

$$\det \begin{bmatrix} 2C - \frac{PL}{2} & -C \\ -C & C - \frac{PL}{2} \end{bmatrix} = 0$$

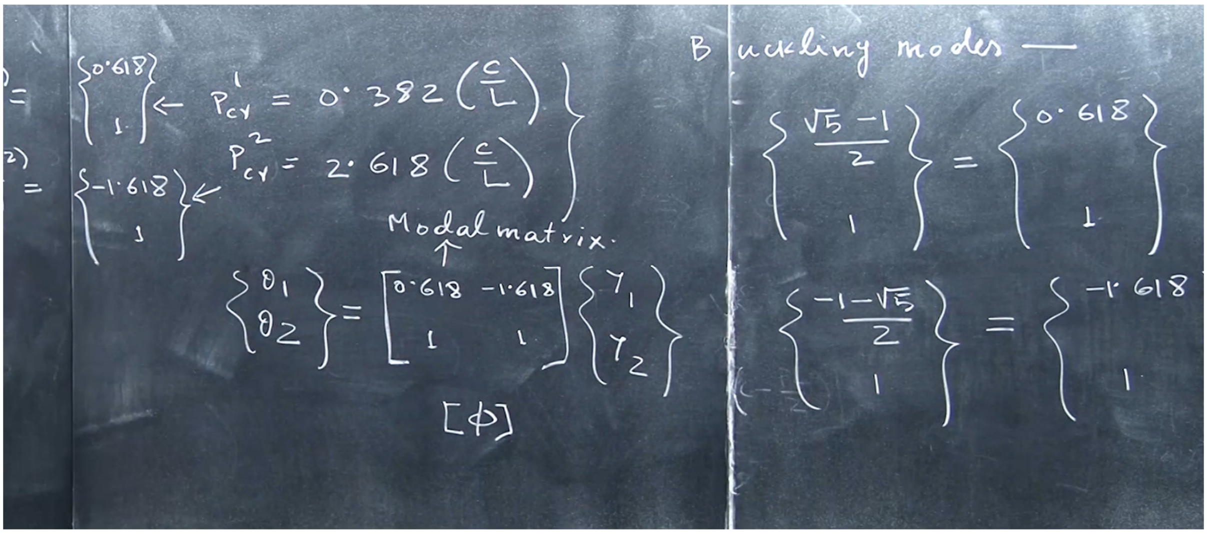
So, please note that here it led to an eigen value problem. So, this is what this is an eigen value problem right. when I am writing the determinant is vanishing that means it is a this leads to eigen value problem. So, what we see that stability analysis is for the given system and the same thing you can recall now from your mechanics column buckling formulation. See here it is an algebraic equation that is what its matrix form. but there it was in differential equation form EI $\frac{d^2w}{dx^2} + Pw = 0$. There it was also homogeneous equation right. But that was differential equation. that is also eigenvalue problem. But that was the solution was in terms of trigonometric function cosine and sine. So, that was we used to call it what trigonometric eigenvalue problem. whereas this is algebraic eigenvalue problem. You have seen similar kind of algebraic eigenvalue problem where when you study structural dynamics. In structural dynamics you free vibration problem you solve right that is nothing but a that giving rise eigenvalue problem. Eigenvalues give the natural frequency and eigenvectors give the mode safe for vibration. Here the eigenvalues will give you the values of the P, for which it will have a alternate configuration, whether it is stable, unstable or neutral. that we have to switch that means critical load you will get by solving this and the eigenvectors will give you the buckling mode. We are not we should not call it buckling but we can call it buckling, because it is also a column with distributed with concentrated elasticity right. A column will have a distributed elasticity but it is concentrated elasticity. So, this we will also from by solving this eigenvalue problem. we will get the critical load and the respectively Eigen buckling mode. Okay let us solve this, so you find the characteristic polynomial $(2C - \frac{PL}{2})(C - \frac{PL}{2}) - C^2 = 0$ and if you simplify it then you will get it, if you simplify it then what you are going to get. is well I mean let me write it down you will get an equation in the form of $P^2L^2 - 6PCL + 4C^2 = 0$ and this is a quadratic equation. So, it will have 2 values. it will have 2 roots right and it is expected because it is a 2 degree of freedom system. So, by solving this you will get P is 2 value ok. it is $\frac{C}{L}(3 \pm \sqrt{5})$ or I will just denote is P_{cr1}, P_{cr2}. So, P_{cr1} and P_{cr2}.



P_{cr1}, if you solve it then you will see the value will be $0.382\frac{C}{L}$ and here it is $2.618\frac{C}{L}$. So, at these two load you can have an alternate equilibrium configuration but that is in perturbed configuration right. So, system can remain in equilibrium in a perturbed configuration right and these are that the value. So, this is critical load 1 this is critical load 2 clear. So, other than these two, we will also find out what? Eigen buckling mode, right. So, we will see that buckling mode. How will you find out buckling mode? You just, so you will see that stability problem, linearized stability problem always led to an Eigen buckling problem, ok. So, how to find out the buckling mode right, you can find out the eigenvectors. So, eigenvector I am directly writing ok, without finding it out. So, eigenvector will be $\sqrt{5}-1$ into 1 or if you simplify 0.618 1 and here it is $-1-\sqrt{5}$ or it is -1.618 1. So, all of you know how to find out eigenvector, right? So, you basically consider the same system and you consider θ_1 to be 1, then you find out respective θ_2.



So, eigenvectors mean it is only in relative sense, right? You can always normalize it, by for a, you know using its L2 norm basically to, so that its norm is 1. But here I did not normalize, I just keep the top 1 as 1. So, here you see. that first this one the eigenvector. I am going to write P1, I am going to write P2 as this one is $[0.618,1]^T$ and here it is $[-1.618,1]^T$ ok. Now is that, so what are these represent? First one represents you see that, so this was the straight configuration right, this one right ok. So, first is the alternate configuration, so agent buckling mode represent the relative deformation of the 2 degrees of freedom ok. During when they attend this critical load ok. So, at P_{cr1}, P_{cr1} first critical load which is $0.382\frac{C}{L}$. If you see that 0.618 and 1 right. So, θ_1, θ_2. So, θ_1 means this is it is rotating like this, this is rotating like this ok. So, it is going here maybe, it is going here and this other thing is going here ok. So, this one is unity and then this one is 0.618 you see that but both are at the similar both are in the same direction you see that. So, both rotations are positive right either positive or negative right. So, eigen modes gives you relative distribution of their deformation not absolute right. This is mode say mode 1, mode 2. buckling mode 1 and then what about buckling mode 2? Buckling mode 2 here you see that the first one the θ_1 is negative right. So, what will happen here? See negative means if I put it like this, then the other one will come as it will be it will be something like this. this one, it is a minus 1.618 and then from here it is coming here, so then this is, this is 1, this unit, so this is clockwise and this is anticlockwise. So, if this one is moving this direction, the other one is moving this direction. So, the buckling mode 2, there you see that they are having opposite sign, right. So, do you find the similarity with the structural dynamics? There the mode shape represents the eigen modes, right? Free vibration, right? Here they are buckling mode, clear? Okay. Now here you must understand one thing that what we have learned that eigenvector can serve as a legitimate basis.

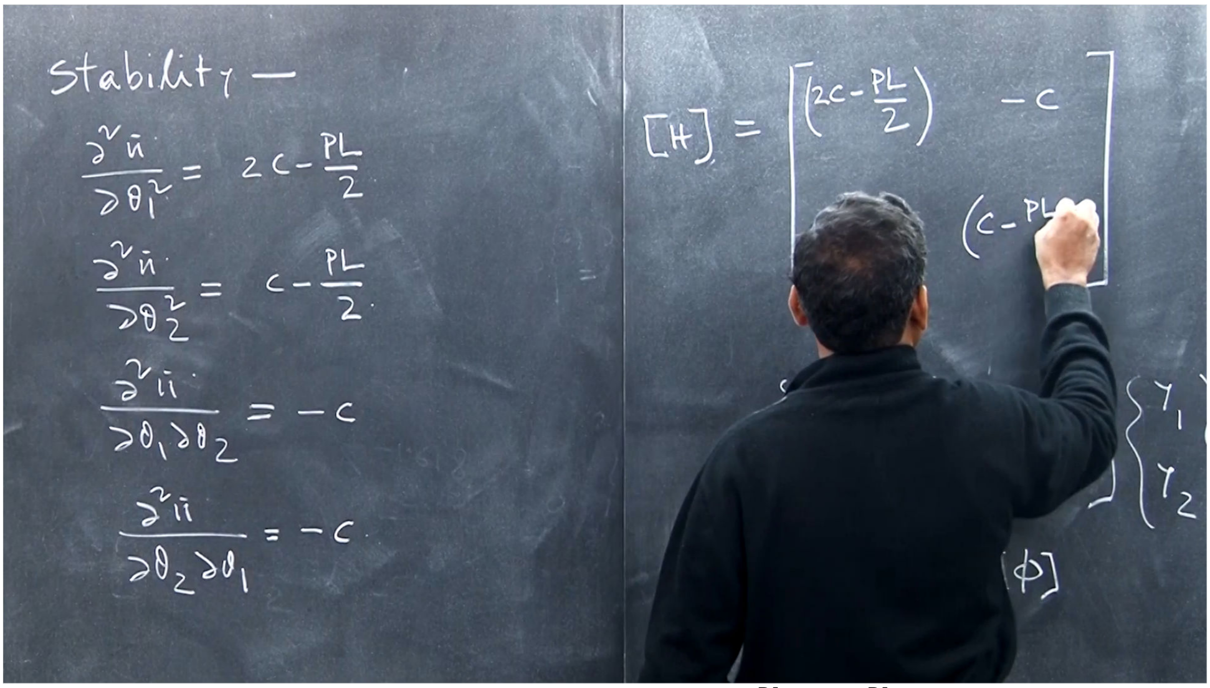


So, we can have a basis transformation, right? We have considered that θ_1, those are the value right. we can have and θ_1 θ_2 are coupled. we can consider that another basis y_1 y_2 right or θ_1 bar θ_2 by putting these eigenvectors where this is 0.618 this is 1 and this is minus 1.618 1. So, this is, I am calling it this matrix containing the eigenvectors. So, this is called modal matrix, right? This is called modal matrix. The way we have used this eigenvector to get the uncoupled equation from the coupled, here also, you please note that transformation, okay? Later we will talk about it. So, θ_1, θ_2 can be expressed. and this is kind of linear transformation, right? I mean some kind of similarity transformation. You can use this coordinate transformation, okay? Transformation of basis, okay? We will come there later but please recall that. Why because mode shapes give us a set of legitimate bases that can be used for coordinate transformation clear. Please note that we will come there later and I will tell you something interesting in stability context ok.

But here now. We have considered the first derivative. So, we have established our alternate equilibrium configuration, right. Now, we will see the nature of stability, ok. So, the stability, stability. So, for stability, what we will have to do? Second order derivative, right. So, we have to find out $\frac{\partial^2\pi}{\partial\theta_1^2}$. We have to find out $\frac{\partial^2\pi}{\partial\theta_2^2}$. We have to find out $\frac{\partial^2\pi}{\partial\theta_1\,\partial\theta_2}$. we have to find out $\frac{\partial^2\pi}{\partial\theta_2\,\partial\theta_1}$. So, here we can clearly see, so $\frac{\partial^2\pi}{\partial\theta_1^2}$ is nothing but $2C-\frac{PL}{2}$. If you can recall, we have found out the first derivative, right? We have found out the first derivative, so we just differentiate it once again. So, $\frac{\partial^2\pi}{\partial\theta_1^2}$ is nothing but $2C-\frac{PL}{2}$. Then $\frac{\partial^2\pi}{\partial\theta_2^2}$ is nothing but $C-\frac{PL}{2}$. another thing did you see that the critical load P_{cr} is a function of what? the alternate configuration, in alternate configuration the whatever load is a function of what C and N. That means it is the length geometric parameter as well as the stiffness, right. Why it is so? So, similar to buckling load, $\frac{\pi^2 EI}{L^2}$ EI is stiffness, L is geometric parameter, right. So, then $\frac{\partial^2\pi}{\partial\theta_1\,\partial\theta_2}$ is nothing but $-C$ ok. So, please write the Hessian matrix of the potential energy functional. So, Hessian matrix of the potential energy functional being

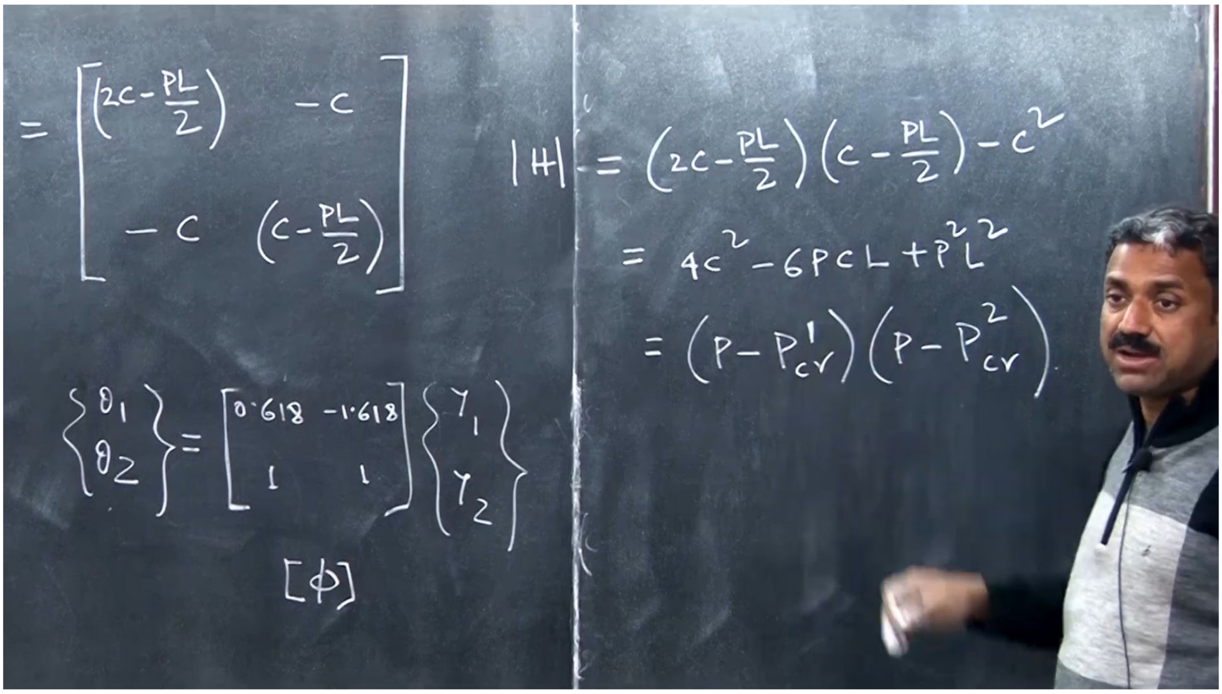
$$\begin{bmatrix} 2C-\frac{PL}{2} & -C \\ -C & C-\frac{PL}{2} \end{bmatrix}$$

Now, we have to see that well, I mean so we have to find out the determinant right. So, determinant need to be positive for the stability. Let us find out determinant.



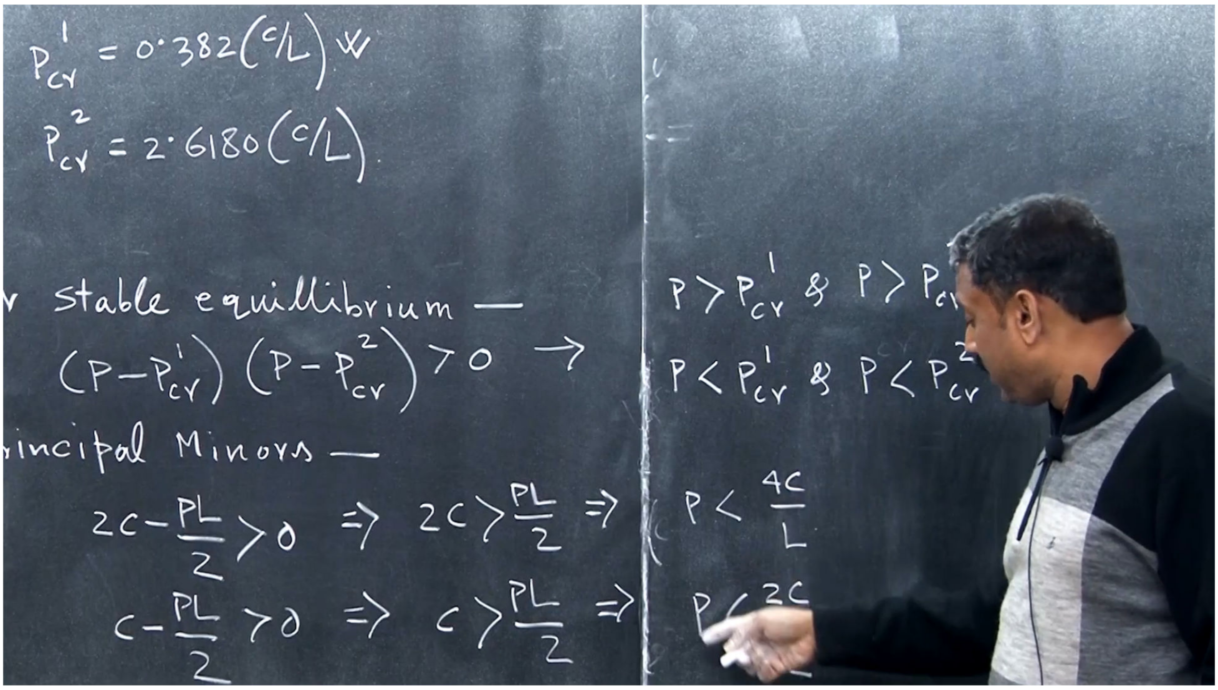
So, determinant if you find out what you will get? $(2C - \frac{PL}{2})(C - \frac{PL}{2}) - C^2$. You will see that this expression is same as the expression for the whatever you are getting for the quadratic equation, $-6PCL + P^2L^2$, ok. That means, may I write it, P, P minus P_{cr1} and may I factorize it like this, P_{cr2}, can I do it? this is not square please note that this refers to the fact one these one two superscripts refer to the first and second critical load. may I do it like that clear because it is the same polynomial same second quadratic polynomial, that so you can factor a like that, so that is the Hessian now other than Hessian.

What else? Hessian needs to be positive for stability, right? Keep it. So, this must be greater than 0 for stability, ok. For stable equilibrium, right? For stable equilibrium $(P - P_{cr1})(P - P_{cr2})$ must be greater than 0. That is the first condition, right? Clear? What about other condition? Principal minor. What is the principal minor? Principal minor is nothing but the diagonal term, right? So, this diagonal of this diagonal term. So principal minor now. What is the principle of this one, right? $2C - \frac{PL}{2}$ must be greater than 0, right? What does it imply? This implies that well you know $2C > \frac{PL}{2}$ or P must be less than $\frac{4C}{L}$, right? And the other one is WHAT? Other one is $C - \frac{PL}{2}$, right? Must be greater than 0. So here it is $C > \frac{PL}{2}$ or P less than $\frac{2C}{L}$. Understand that? Here you must recall the value of P_{cr1} and P_{cr2}. You must keep it with you.



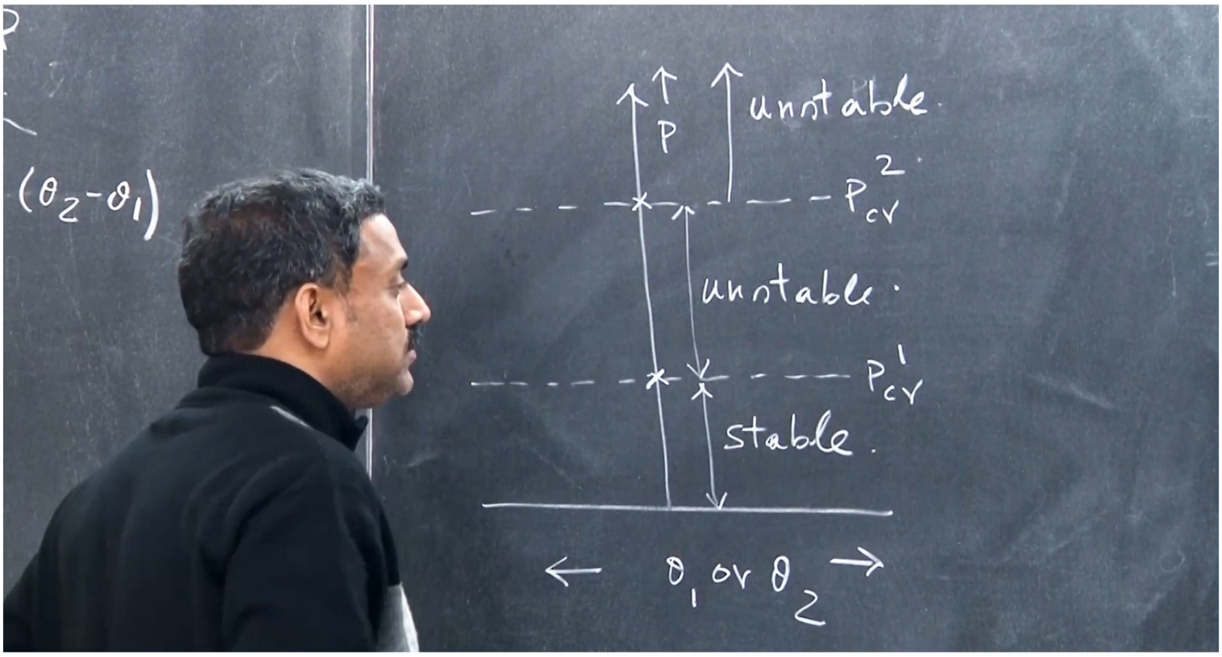
So, what was its value? 1 was what? What was its value? Yes, so it was I think 0.382, right? $0.382\frac{C}{L}$, C upon L and then it was $2.6180\frac{C}{L}$, right? Okay, now see. So, all these three conditions need to be satisfied for stability, right? Now you tell me, how can it be greater than 0? If this greater than this one, this also greater than this one, right? So, P must be greater than P, I mean P_{cr1} and P must be greater than P_{cr2}, right? This is one. Or I mean p must be greater than p_{cr1} and p must be also greater than p_{cr2}. But then if it is greater than p_{cr1}, it is automatically greater than p_{cr1}. You, see? Or there can be another thing, p less than p_{cr1} and p less than p_{cr2}. Right? There is another, because both are, if both are negative, then negative, negative, it will be positive, multiply, right? So, which one is more, but then if C, P must be less than $\frac{2C}{L}$. So which condition will satisfy? See, if P_{cr} is this. then, this must be less than $\frac{2C}{L}$, right? If it is 2C, less than $\frac{2C}{L}$, it is automatically less than $\frac{4C}{L}$. So, it must be less than $\frac{2C}{L}$ and also it needs to be less than this value. For it is getting satisfied right because if it is less than P_{cr1} it is automatically less than this you see clear. So, then it is only stable otherwise it is unstable right. So, I will just draw a diagram here now that I am removing this. Removing this so I will just drag draw this, this is θ_1 or θ_2 both either you can go this from this ok.

So, this I am drawing P, so there is 2 P_c this is one P_{cr1} P_{cr2} right, P_{cr1} and here it is P_{cr2} right. So, if it is P less than P_{cr1}, if P less than P_{cr1} this only this one then only it is stable. If it is here, it is unstable and here also it is unstable right. Do you understand? All of you understand because then only if it is less than P_{cr1}, this is satisfied, this is automatically satisfied and this is also satisfied.



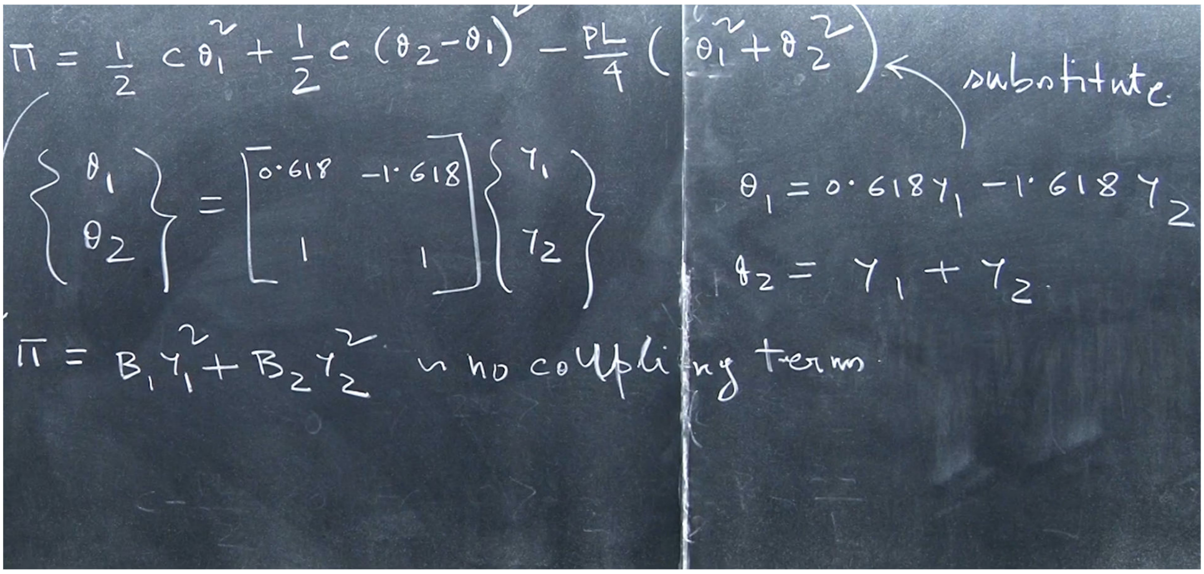
If P is less than P_{cr1}, this is negative, it is also negative, negatively positive. And then, if P is less than P_{cr1}, then only it is stable. Other all it is unstable. And when P is equal to P_{cr1}, what is happening? Double derivative, this is being 0. 0 means what? neutral that means at critical load it will be all neutral equilibrium right, it is neutral. And you also recall that when it is buckled basically just when it is just on the verge of buckling then it is what kind of equilibrium it is? Neutral. Because you can perturb it and you can keep it at any configuration you want, right? That is what we can see. So, what we can see that, this system will remain stable but then it will bifurcate into a state and at the first critical load. Why bifurcate? Because from here, here it is θ_1, θ_2, 0. Here θ_1, θ_2, 0. But then it is bifurcating, because if you want to attain an equilibrium other than θ_1, θ_2, 0, right? Then what will happen? Then it will, so what is happening? then if the load P is less than P_{cr1}, then whatever configuration you put it, you perturb it, right. First of all, then it will be in stable equilibrium configuration, right. It is stable, right. Equilibrium is only possible at P is equal to P_{cr1}, right.

But then this equilibrium configuration is neutral, right. And what is happening? Then when you are perturbing it, then either it is going to this side or it is going to this side, you understand? So, it is basically breaking the symmetry, okay. I will come there later when you incorporate because symmetry breaking is a more generalized concept that is used in physics. Your physics friend and others they use this. So, the system is bifurcating. This point, this point is what? This is a bifurcation. system is bifurcating right into an alternate state look from here-to-here θ_1 θ_2 0 but then system is bifurcating. of course, for θ_1 θ_2 has to maintain those kinds of modes of relative ratio between the two but then system is by far cutting into a state where it is losing its symmetry.



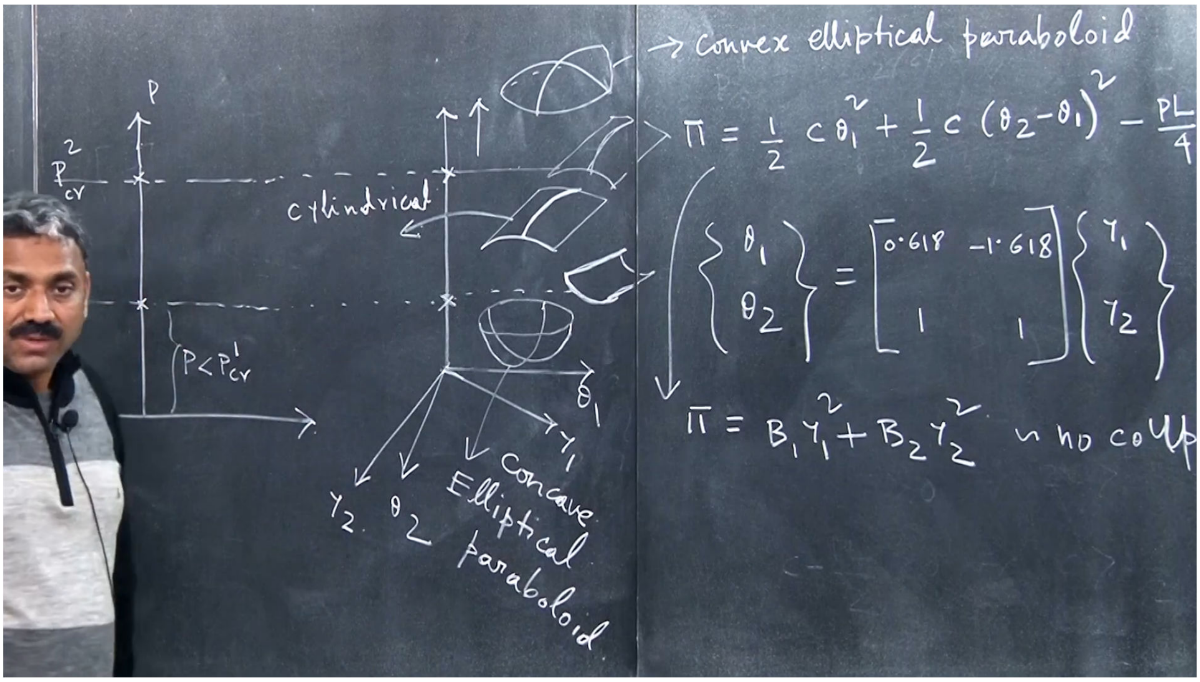
Why it is losing its symmetry because look when θ_1 θ_2 0 this is fine right it is cement but then when it is losing its stability, you see when it is in alternate configuration θ_1 θ_2 right then the system will remain there, so it will not come it goes slightly beyond that then, it will diverge towards this, it will not come you see so it will not distinguish between. Earlier it was not distinguishing whether it is why you used to call that breaking of symmetry because earlier it was symmetric keep it here keep it here if it is as long as P less than this one it will come it will come to this position right. The position is symmetric any of the configurations it can remain and it will come back to its because it is a stable right equilibrium right. But as soon as it is exceeded then the system is bifurcating into one direction you see consider the buckling what is up so same thing is buckling right so the column is column is there. So, if the load P is less than P critical if it is the axial load is less than critical load then. You just perturb it and leave it. Then whether you deflect it positive, negative it will come to original configuration. So, it is basically with respect to any perturbation it will remain in its symmetry, right? But as soon as its critical load is exceeded then what is happening? Then the system is bifurcating.

Bifurcating why? Because as soon as you are exceeding this, see when it is at critical then it is in neutral equilibrium. But slightly above this then it is unstable then it will diverge to in one direction that means it is no more coming back to original configuration will diverge in one direction. either if you are perturbing a positive direction, it will go on to be positive it is going to be negative reactions right. So, it distinguishes. so, that means the symmetry is broken. This phenomenon is called symmetry breaking. And why this is called bifurcation. Why it is called bifurcation? Because a stable configuration, is bifurcating in alternate configuration. This one is bifurcating. Clear? Okay. Now, so you understand the stability, all these things. Now, I will do one simple stuff. I will do one stuff that you can all recall that we have this eigenvector and then we have this modal matrix. I have written the modal matrix as I have written either you can write in terms of this minus 5 minus 1 2 minus 1 minus $\sqrt{5}$ 2 or you can write it in terms of these values when you are normalizing 0.6181 minus 1.6181 right. So, potential energy π was what? potential energy π was if you recall the expression $\frac{1}{2}c\theta_1^2 + \frac{1}{2}c(\theta_2 - \theta_1)^2 - \frac{PL}{4}(\theta_1^2 + \theta_2^2)$ right that was the expression right. what I am saying that if this is ϕ you can convert θ_1 θ_2 to be whatever these values are 0.6181, -1.6181 and here you write y_1 y_2. or you can write θ_1 so from here and then. So, then θ_1 become 0.618 y_1 - 1.618 y_2 and θ_2, it is a linear transformation, right? $y_1 + y_2$, then you substitute back and then you substitute here in the potential energy expression, substitute. Then if you substitute, what you are going to get π as? you will see you will get I am not writing the coefficients that you do yourself b_1 y_1^2 + b_2 y_2^2. So here look, here what is happening? this is square, this is square, this is square, this is θ_2^2 + θ_1^2, but there was one θ_1, θ_2 term, cross term, that means they are coupled.



In potential energy function, there are coupling term between this θ_1 and θ_2.but if we do this similarity transformation using the mode shape basis, using the eigenvector as the legitimate basis. then your potential energy function becomes function of this independent coordinate y_1 y_2 and there is no coupling term no coupling term you see that no coupling term. That is what you also have done. Why did you do that? In dynamics course, when you did eigenvector and you made the transformation and that is your stiffness matrix was become uncoupled, right. You pre-multiply and post-multiply with the eigenvector modal matrix, right.

Here, because that was generalized eigenvalue problem, here it was just eigenvalue problem and normal eigenvalue problem. So, what is happening? B_1, B_2 are always positive. Well, may not be positive. There will be, for this system B_1, B_2 will be always constant, right? It is a linear system. So, what we are trying to tell you is that, you see, this quadratic form of this potential energy function is very, very important. In deciding the nature of stability. If all the terms are square or quartic, then they are always positive. So square, square, positive. Now only, so then you do not require to find out anything else. Just see the potential energy function. If B_1 and B_2 are positive, then it is stable. clear. So, by using this kind of transformation of coordinate you can simplify the detection of the stability. where this b_1 and b_2 you see where b_1 is b_1 is positive or b_2 is positive both are positive it is stable right, because there is no coupling term θ_1 θ_2 you see and it will have a more important consequences later, I will come there later. But here I will show you something. What is happening when I am drawing this configuration right, this one is P, you see here. these are the important point right. So, here when p less than p_{cr} this regime stable regime p less than p_{cr1} this regime the potential energy functional will so here it is. I will now define the potential energy functional and, this is θ_1, this is θ_2, θ_1, θ_2 ok. This is y_1 and this is y_2. So, respective to this I will just respective to this I am going to put like this ok. So, here the potential energy function, here if you plot Here, you will see the potential energy functional will look like, so it is a look like an ellipsoidal paraboloid ok. So, it is a like this kind of surface elliptical paraboloid ok, here it is elliptical. So, potential energy function is always concave. Here in one direction, it is ellipse, another in direction it is parabolic. So, it is a solid object. Do you see that? So, it looks like a bowl ok. why it is when P less than P_{cr} that means any ball you put it so it is a concave surface of the potential energy and if it is a concave surface of the potential energy then it is it implies stable configuration right. Here what is happening when P is equal to P_{cr}? P is equal to P_{cr} what is happening is that it is being it will be becoming this kind of cylindrical, one thing is cylindrical. Cylindrical means what? In one coordinate it is concave but another coordinate it is neutral. Then when you are reaching here then it is being like this. but this configuration is also being like this. So, it is cylindrical but the surface is also convex ok and between this and this. it will be something like this unstable, so it will be something like this here something like this here and this will also be, so it is not This is also convex, this is also convex ,you see that and then when you come to this here both side will be, it will it is ellipsoid paraboloid but here what is happening is that here it will be it is convex elliptical paraboloid. Convex elliptical paraboloid here it is concave elliptical paraboloid ok. Here it is being cylindrical ellipsoid or cylindrical paraboloid whatever ok. Do you understand why this form is coming? If the potential energy functional being convex. then you see that its hessian is being positive all the time, okay. That is why it, because of the change of slope is also positive, you see. So, this convex, this is the potential energy function, here it is complete unstable.



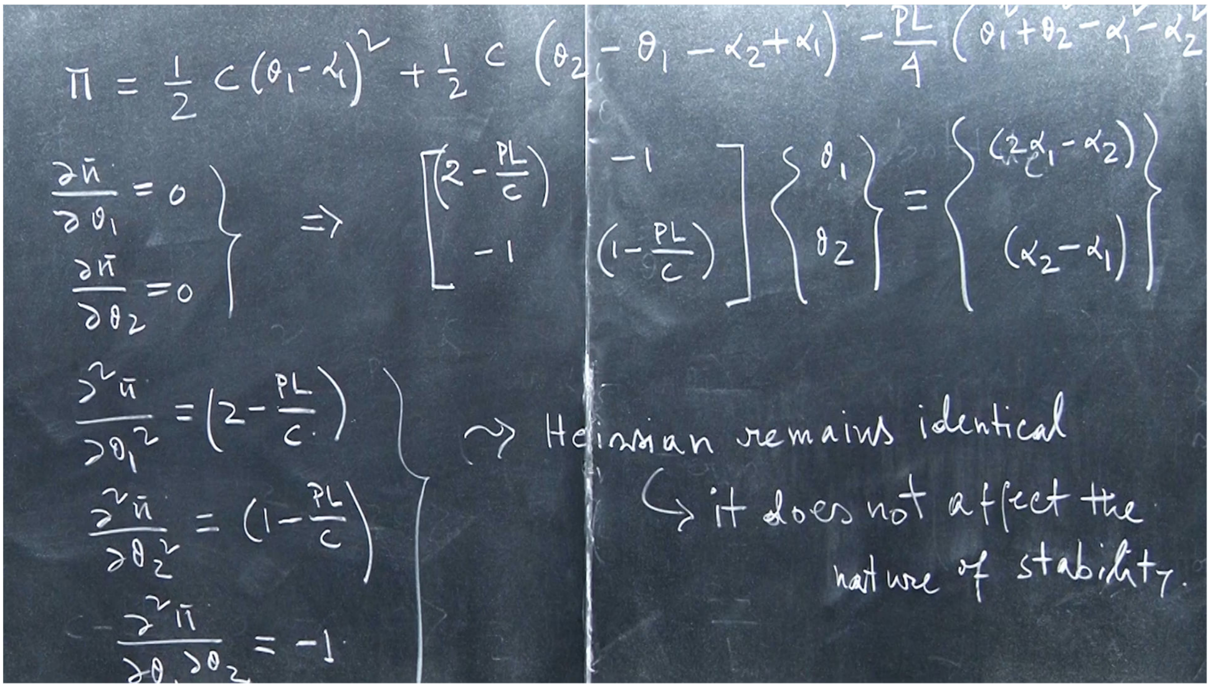
So, this is convex, this is completely concave, in between one, along one direction it being cylindrical. Cylindrical means what? If it is flat cylinder then it will remain in what? It will remain what? Neutral equilibrium that is what is happening P_{cr1} and P_{cr2} because P_{cr1} and P_{cr2} your Hessian is being 0, your Hessian is vanishing. How can Hessian, if Hessian is vanishing that means the equilibrium is neutral. That means here you see that at this point and this point, in this point here this surface needs to be one surface need to be cylindrical. Then only it will be neutral equilibrium. Clear? So, these and these, see here it will be like this and here it will be something See, if this one is along, this cylindrical axis aligned along y_2, then this cylindrical axis will be aligning along θ_1, okay. So, you see how here I am drawing it. Do you understand? See, at P is equal to P_{cr1}, what is happening? This, this, this, this. So, along other, along other it is neutral equilibrium, okay. Similarly, the other here it is aligned along θ_2, okay. I am not very particular about θ_1, θ_2, that you can decide who is who, but and in between, that means between this and this, this will not be perfectly cylinder. Cylinder will have a concave surface. Okay. So, in between these, in between these what will have? It will have these and these and but then it will have a concave surface. Clear? All of you understood what I am trying to. So, it is the shape of the potential energy functional that matters a lot in determining the stability.

Okay. Here it is fully concave, here it is cylindrical so one along one axis it is basically in neutral equilibrium because it is a hessian being 0 and things ok. Now what I am trying to tell you is, if it is an equation of parabola. whatever right because it is decoupled from here you can clearly you can see that what kind of equation it is ok. You can from your coordinate geometry you can recall the generalized equation of these conical sections right. And then special equations right, $y^2 = 4x$ for parabola. if it is in the its axis is along some other x = 0, y = 0. Otherwise generalized parabola we will all have this coupling term right, is not it? So, it is $x^2 + y^2 + 2gx + 2fy + 2hxy + c = 0$ that is the generalized conic section. But if it is uncoupled the way by coordinate transformation we did then for ellipse it is $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$. If it is aligned major axis minor axis is along the main axis right. So, this is uncoupled form which is much easier to deal with and understand that what kind of shape is this.

So, when I am talking about this elliptical paraboloid if you simplify it, you will get this kind of form ok. So, you understand the evolution of the potential energy surface and how it is affecting stability clear ok. So, now what I will do at least for this problem I will put the effect of imperfection. So, imperfection means, we will consider geometric imperfection that means earlier we assume this to be a perfect system but here what we will assume that initially this bar had some kind of little crooked kind of things, you see that. So, it was already having some imperfection of α_1, it was already having some imperfection in the terms of, in terms of α_2. α_1, α_2 are initial imperfections that means initial rotation. So instead of being perfectly straight, it is, they are already having little inclination. So, these are geometric imperfections. Imperfections α_1 and α_2 are the initial inclination of the rigid bars. If it is so, then how do we approach the problem? Potential energy function, how it will look like? Why? Strain energy minus work done. Strain energy will be $\frac{1}{2}c\ \theta_1$ is fine but earlier it was having α_1 imperfection. So, the relative rotation is $(\theta_1 - \alpha_1)^2$ + $\frac{1}{2}c$ here. θ_2 is also having imperfections. So, $(\theta_2 - \alpha_2) - (\theta_1 - \alpha_1)^2$ and then the work done is $\frac{PL}{4}$. Here you can understand that here it will be cosine of α_1 - cosine of θ_1 cosine of α_2 - cosine of θ_2. you understand why it is so right, because of the initial α_1 inclination it will have already $l\cos\alpha_1$ right and then it is further being $l\cos\theta_1$. So, $l\cos\theta_1$ $l\cos\alpha_1$ - $l\cos\theta_1$ will be the actual deformation that is what it is coming here right. So, if you know you do repeat the same exercise and expand this cosine function in terms of their series expansion and do the whole exercise. and if you do the whole exercise I am writing down the final expression okay. So, please have this potential energy function right so then I will just write down. you will see that what you will get ϕ is $\frac{1}{2}C(\theta_1-\alpha_1)^2+\frac{1}{2}C[(\theta_2-\theta_1)-(\alpha_2-\alpha_1)]^2-\frac{PL}{4}(\theta_1^2+\theta_2^2-\alpha_1^2-\alpha_2^2)$ right, and then you will see that see things will not of course imperfection will have some influence but I will just write down the equilibrium equation. So, $\frac{\partial\pi}{\partial\theta_1}=0$ and $\frac{\partial\pi}{\partial\theta_2}=0$. this will give you the system of equation you will keep. you will see that you will get this kind of things

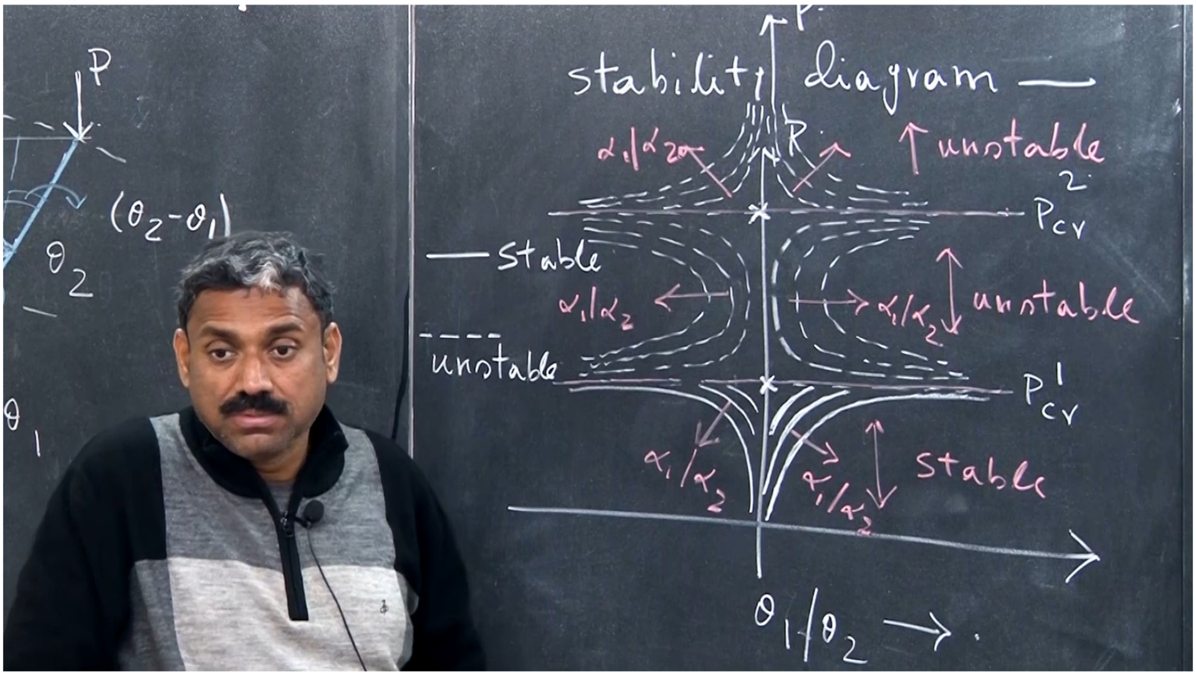
$$\begin{bmatrix} 2-\frac{PL}{C} & -1 \\ -1 & 1-\frac{PL}{C} \end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}=\begin{bmatrix}2\alpha_1-\alpha_2 \\ \alpha_2-\alpha_1\end{bmatrix}$$

So, you see unlike the eigenvalue problem, incorporation of imperfections is changing the problem.it is no more resulting into an eigenvalue problem. you know finding out the static equilibrium configuration right. because it is a non-homogeneous system of equations. So, the way you solve for the static equilibrium right. you can find out θ_1 θ_2 for given α_1 α_2 it is no more an eigenvalue problem. So, incorporation of imperfections you are basically breaking the type of the problem and then you will see the for the Hessian, $\frac{\partial^2\pi}{\partial\theta_1^2}$ is $2-\frac{PL}{C}\frac{\partial^2\pi}{\partial\theta_2^2}$, you will see that this equal nature of the equilibrium will not change, and you will see that I am explaining to you, $\frac{\partial^2\pi}{\partial\theta_1\,\partial\theta_2}$ is -1 whatever.



So, here Hessian remains identical Hessian remain identical. So, you see that here so hessian remain identical remains identical. So, it will imperfection is not affecting the stability but it is not always so please note that. just because it is a simple toy problem, so it will not for this problem, so it does not affect stability and the nature of stability. because these are constant terms, so second derivative So, here I am going to draw the same diagram right with imperfections. So, stability diagram I am going to draw the stability diagram. Stability diagram I am going to draw. Here I am drawing θ_1 or θ_2, I am going to put ϕ and then here, it is here 1, is here 2,

these are for perfect structure when the bifurcation is occurring, that is characterized by the value problem mathematically but here what is happening you will Please note that, these are the bifurcation point, here you see this, this is only stable, this is unstable. what is happening? See these are the imperfection, these are increasing α_1, α_2 value, α_1, α_2, increasing α_1, α_2, increasing, okay. Increasing α_1, α_2. This is for increasing α_1, α_2. What is happening is that basically see, so there is no bifurcation because all of this in imperfect system, these are called equilibrium path because the first derivative when we are making it 0, we are getting the equilibrium path. Equilibrium paths are asymptotically approaching the bifurcation load. See that there is no bifurcation because a single path is not bifurcating into two. So, incorporation of imperfections in this is eliminating the bifurcation. But these are called equilibrium path. Why? Because this path basically P, this is nothing but P, right? is the relationship which satisfy the equilibrium so P in terms of θ_1 θ_2 you can obtain from here, because that that is where the equilibrium is satisfied that is what I am calling them as equilibrium path you said but only these equilibrium paths are stable all other equilibrium path are unstable so stable equilibrium path are given by solid line whether these are unstable equilibrium path, unstable. And then all stable and unstable equilibrium path are asymptotically approaching the perfect system. That is what is the generalized characteristics of any system.



So, an imperfect system, the equilibrium path will asymptotically approach to the perfect system. because what is asymptote right you have found out asymptote pedal format in differential calculus right. So, this is the stability diagram here for these 2 degrees of freedom system elastic system. So, you must recall that what we have discussed, we have solved a simple example and by using energy approach we have seen how to find out stability. what is the nature of stability, how we can write down the functional expression for the potential energy approach. how we can do a coordinate transformation to make the coupled terms uncoupled, and that will facilitate finding out stability and how to find out whether it is stable equilibrium configuration or unstable equilibrium configuration and then the influence of uncertainty. Influence of imperfections. Clear? Thank you very much for today’s class.