Stability of structure

Prof: Sudib Kumar Mishra

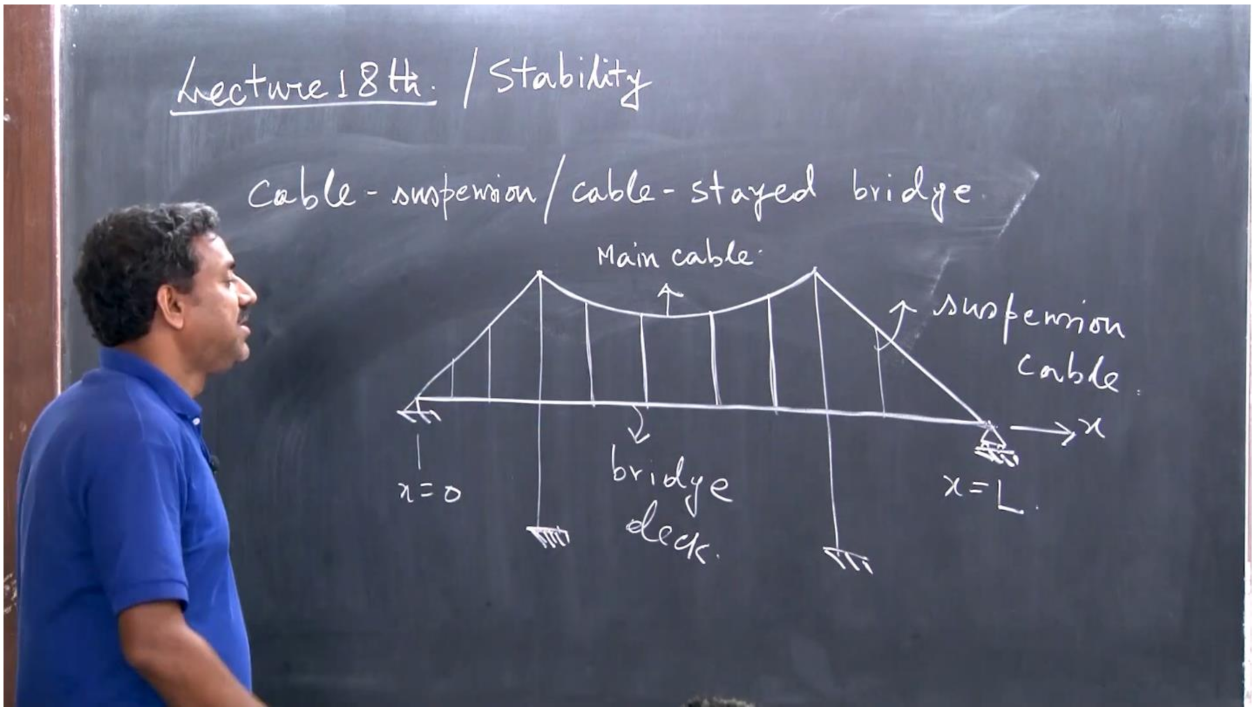
Department of Civil Engineering

IIT KANPUR

WEEK-10

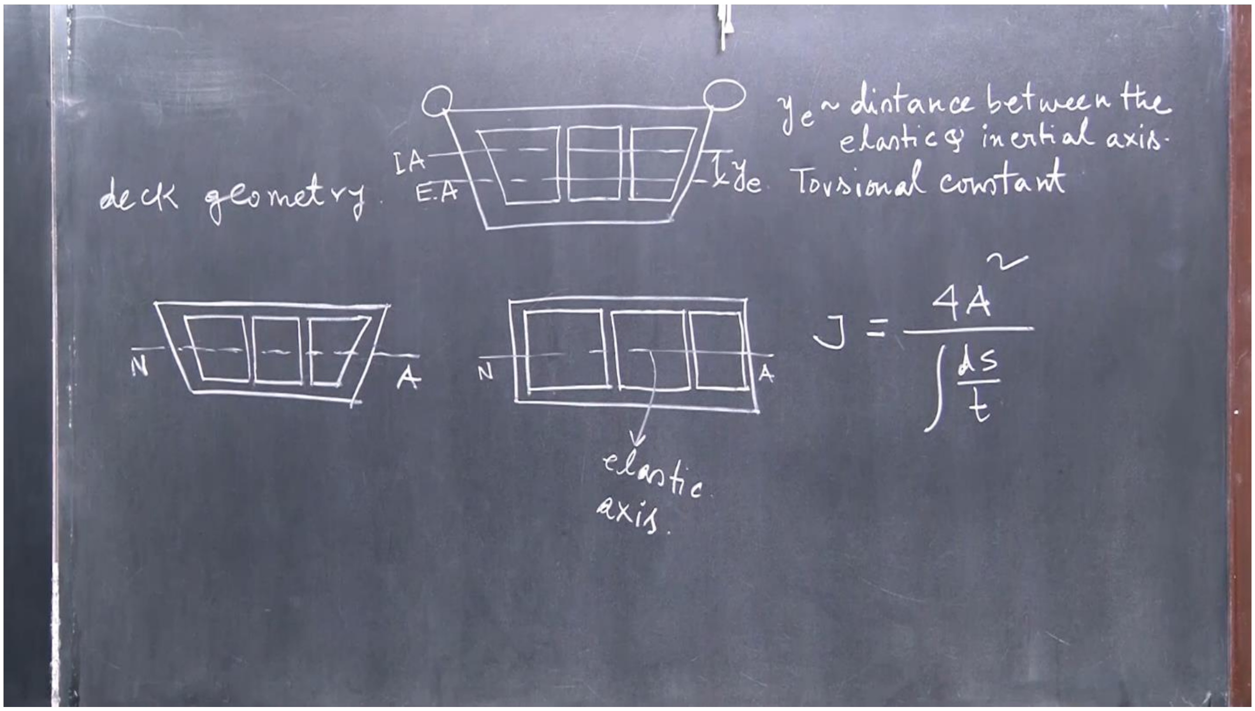
Lecture 19: Aerodynamic Instability Analysis of Tacoma Bridge

Okay, welcome to the 19th lecture on the stability of structures. So, if you see, until now we have covered several topics under dynamic instability. The first one we started with was why dynamic instability is important and why we need to consider dynamic instability. We started with follower force, and we have seen that the static method of analyzing stability does not hold in the case of follower force because there is a non-conservative component. Then we have to study dynamic stability to solve this problem. Then we started with the parametric resonance, where the column is subjected to a pulsating load, actually a pulsating load. We have done an energy analysis, and we have established the condition of instability. We have also seen that this leads to a mathematical problem, a differential equation, and the Matthew function is the solution for that; we have plotted the Matthew function to identify the stable and unstable boundaries and the transition from stability to instability. Then we started an example of gyroscopic stability, which is more common in mechanical engineering systems. The rotating shaft, which is subjected to axial force and axial compression, has been seen to rotate with some angular velocity when subjected to axial compression. Then we have derived the equation of motion, and we have seen that it leads to a simultaneous equation, you know, a differential equation, and then the respective matrix is not symmetric. So, it's a nonadjoint system. Okay. We have not solved it, but we have shown why the system is leading to an asymmetric system of equations. So, by solving this system of equations, you will get the condition for stability. So, those examples we have also solved. So, right now we are going to discuss another example of dynamic instability, which is very relevant to our hearts, you know, as civil engineers, because we are all aware of the infamous collapse of the 1940 Tacoma Bridge in the state of Washington in the United States, right? In this analysis, I'm not suggesting that this dynamic instability is fundamentally different from the others, but rather that we will approach it using a different formalism. Okay. So we are all aware of this 1940 Tacoma bridge, right? So, this happened in the state of Washington, right? It was an infectious example because this basically triggered a number of research activities on aerodynamic instability in bridges. And a little Aeroelasticity is involved here, so we'll try to analyze that this happens because of the aerodynamic instability. The collapse is happening due to aerodynamic instability. We'll discuss it. We'll keep the treatment simple yet comprehensive. Okay. So, all of us are aware of this bridge collapse. We have learned about it from our school-level physics.



Those of you who have studied the physics book every holiday know that on the cover page there was a snapshot of the collapse of this bridge. The bridge was basically vibrating vigorously, and it was flexural vibration accompanied by torsional vibration. Okay, so intense torsional flexural vibration. If you go to YouTube, there are videos of the collapse of the Tacoma Bridge. Please go see that video. Okay. Then you will realize that the collapse happened. So, the vibration started building up, and it went beyond that, and then it collapsed. Actually, all these cables started failing, and the deck was basically, you know, DE segmented, and it fell. But unfortunately, there were no casualties because it was not in use. It was just, you know, about to be commissioned. Okay. There was only a dog that you know died because of this. Okay. And there was a car. So, this fellow went for a walk. "But it was a significant incident. Before that, our knowledge base was not really adequate to consider aerodynamics in the design against aerodynamic instability. Afterwards, however, a great deal of research was taken up, and now the subject is much better understood. "

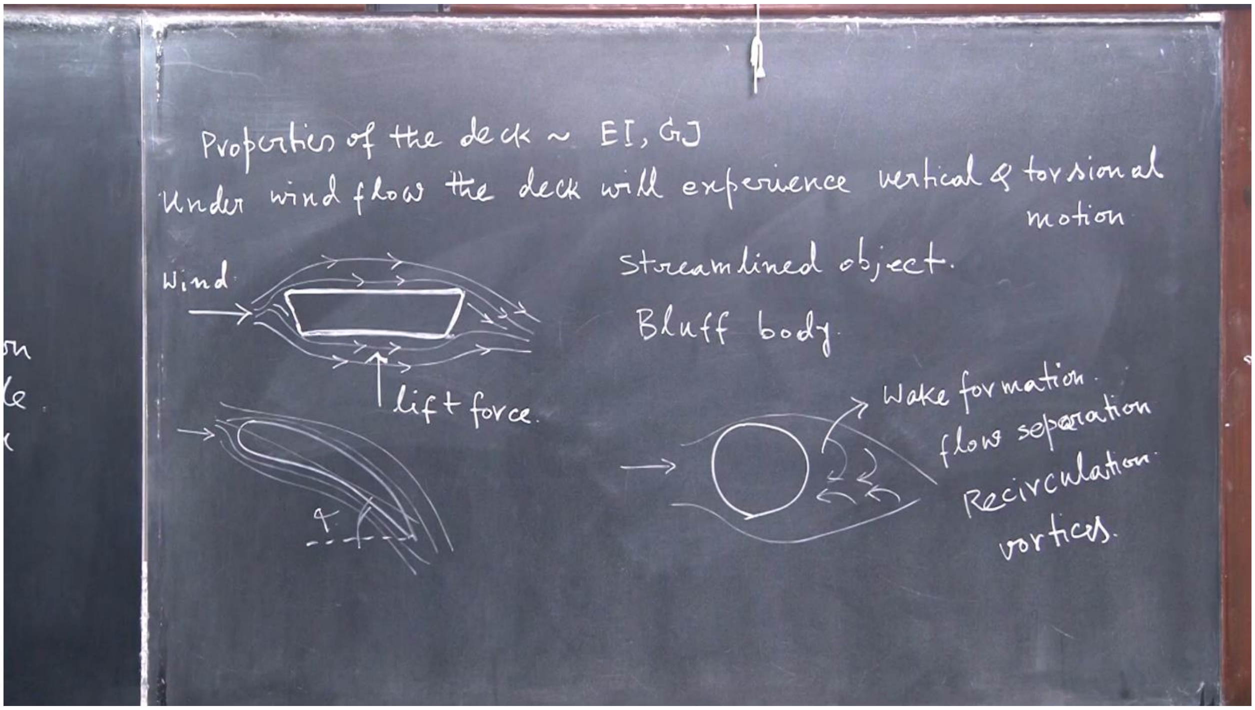
So, these analyses are in line with the Aeroelastic instability in streamline body. The analysis is similar to that of flutter-type aerodynamic instability in aircraft wings, as well as several other examples. So, if we consider a very simple model for the bridge, you know the Takoma Bridge was a cable-suspension bridge, right? Or it can be a cable-stayed bridge, okay?



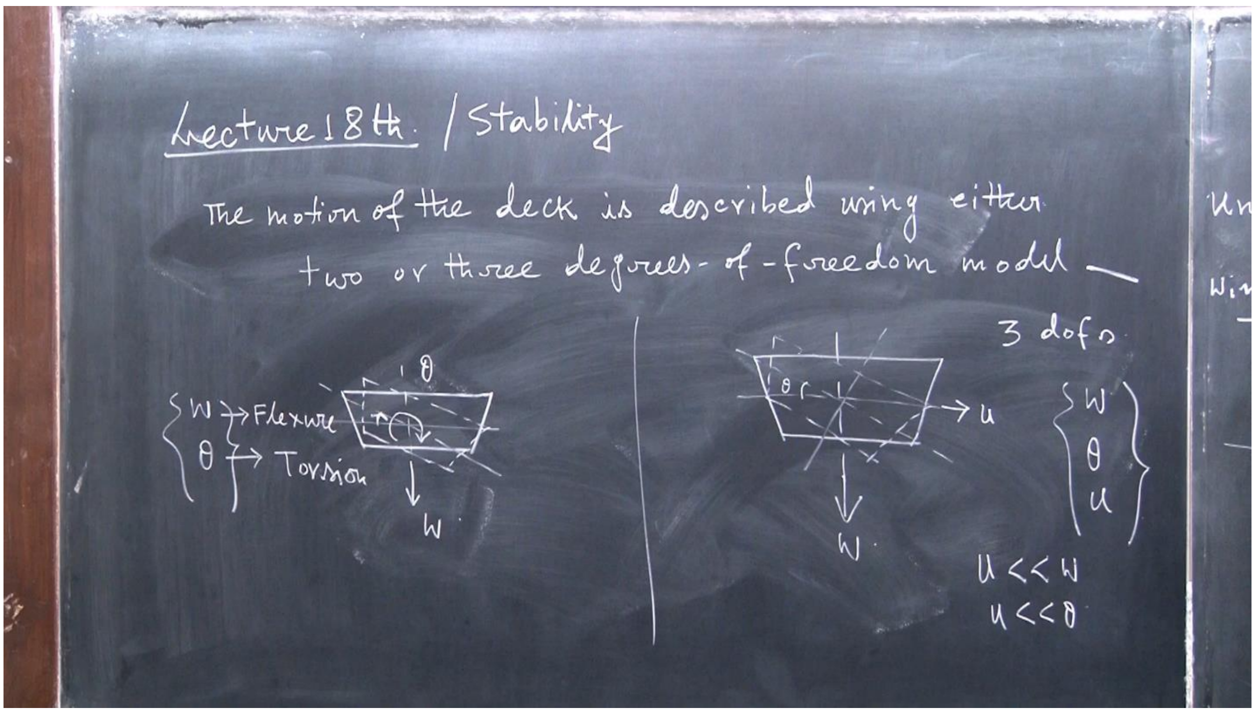
So, I'm drawing; it was like a simple line diagram. So, it was like this, and then there were two towers, you know, pylons, and then this was a supporting cable. And then it was all supported by this cable, and from this, suspenders were there. Huh? And all this, so this is the main cable. Okay. So, from the main cable, it was all suspended. Okay. These cables, which are holding this and this, are the bridge deck. This is the bridge deck. Huh? Bridge deck. And then these are the suspension cables. And this is the main cable. Now, of course, the bridge deck can be of complicated geometry. Most of the time, it is something like this: this kind of deck can have all this bridge deck. You know it is something like this; it can have several cellular structures in between a multicellular bridge deck. So, what happens with this kind of trapezoidal or rectangular shape, as you know? These are all thin wall sections, but they are not in proportion; however, you see these are the multicellular sections. Okay, so that is typically bridge deck geometry. So, you can always find the torsional constant. Now you can find out what the torsional constant is. See, these are the closed sections, right? For the closed section, you can recall

$$J = \frac{4A^2}{\int \frac{ds}{t}}$$

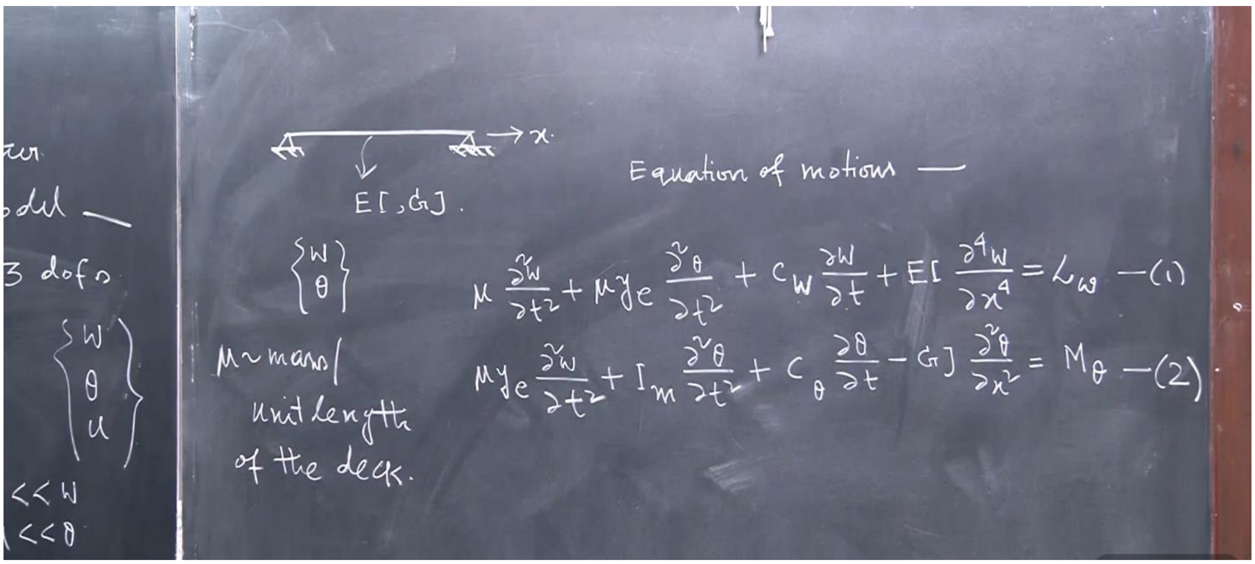
So, in v, when the warping is very little here, although this is a thin wall section, warping will be very little. Because it is a constant for warping, you do not need to consider it because it is quite stiff against warping. Okay. So, you can only consider the shear transfer constant when you know, uh, the torsional constant. And you can also find out the flexural stiffness; it's not a problem because you can find out what the neutral axis is for this, okay? An inertial axis, if you want, okay. You can have a neutral axis; you can have a CG. For this, the neutral axis will pass through the CG, right? But if it has a concentrated mass, similarly, this is the neutral axis, or we can call this axis the elastic axis, right? But if you know the neutral axis and the inertial axis will be the same, but if there is concentrated mass somewhere, you put concentrated mass; sometimes those are required, you know, for control purposes. Yeah, then it will have a different axis of inertia and a different axis of elastic axis. So, I'm considering one such example, you know, a lumped mass and things, so you will have two different axes; one is the inertial axis, and the other one is the elastic axis. This is the elastic axis. And this is the inertial axis. So, there will be some vertical distance y. Okay. So, this is the distance between the elastic and inertial axes. Okay. So, when the structure fails, what kind of vibrations can be expected? So, these are the longitudinal axes. Please note that these are longitudinal axes, right? The longitudinal axis can be denoted as x. So, this x is equal to 0, and this is x = L (longitudinal axis). But then there can be a transverse axis, right? So, the transverse axis is in this direction, away from the board, right? Now what will happen is that you can see which direction is more susceptible to wind, which has less lateral stiffness. Of course, when it is trying to bend laterally, right? So, we are going to consider the geometry of the deck; you understand, right, and all the parameters, right? So, you can find out all the properties of the bridge deck, right? The properties of the bridge deck $\sim [EI, GJ]$. Okay, all of these things you see when you consider the bridge deck geometry. So, the deck geometry is really slender compared to the overall structure—it's a very thin object, like a bridge deck. Now, when it is subjected to lateral wind, the wind flows across the deck in this direction. The geometry of the deck, although I'm sketching it in a simplified way here, could actually be smoother or shaped differently in practice. So, what happens is that when the wind comes, it generates streamlines around the deck. You know what a streamline is, right? Streamlines represent the path that a fluid particle follows in steady flow. So, as the flow passes the deck, the streamlines bend and adjust around its shape.



This behavior is different from what you see with a flat surface like glass—here, because the deck is more streamlined, the flow tends to follow its contour. Please try to understand the aerodynamic streamlined objects. There are two kinds of streamlined objects: one is a bluff body. A streamlined body is one where, when wind flows over it, the streamlines remain attached to the surface and do not separate from the body. The flow stays in contact with the body as it moves around it. For example, you might recall from your undergraduate fluid mechanics course the shape of an airfoil. An airfoil is designed so that the air streamlines follow its surface smoothly with minimal separation. So, when this flow is happening, the streamlines remain attached to the body. You can see that the streamlines do not separate; instead, they stay in contact with the surface of the body, follow its shape, and then continue downstream. But if the bluff body is there, for example, you have a circular cylinder, and then the flow is going on, what will happen? A wake region forms behind the cylinder. In that wake, you often observe what is known as the von Kármán vortex street—alternating vortices that are shed from either side of the cylinder. So, as the flow separates, a wake forms behind the body, and when the Reynolds number increases, the flow becomes more unstable in that wake region. What happens then? Multiple vortices are formed and shed periodically, creating that alternating vortex pattern we call the von Kármán vortex street. Yes, so in this case, there is flow separation. With separation, you get recirculation regions in the wake, and within these regions, recirculating vortices are formed. Bodies that cause this kind of behavior are called bluff bodies. So, what is the difference between a streamlined body and a bluff body? A streamlined body allows the streamlines to remain in contact with its surface. The flow stays attached, so there are no major flow separation and no large wake formation behind it. On the other hand, a bluff body causes the flow to separate early. This leads to the formation of a large wake and the shedding of vortices—such as the von Kármán vortex street—that are observed behind circular cylinders and similar shapes. Most bridge decks are designed as streamlined bodies. In a streamlined body, lift and drag forces are the main forces acting on it—you've studied lift and drag, right? For a streamlined body, these forces dominate. In contrast, a bluff (or block) body experiences flow separation, leading to the formation of vortices and other complex flow patterns behind it. Most bridge decks are designed to be streamlined bodies. In a streamlined body, the dominant forces are lift and drag—you've all learned about lift and drag, right? In contrast, for a bluff (or block) body, there is significant flow separation, which creates a large pressure difference and leads to the formation of vortices and other complex flow patterns. This upstream side difference is equal to nothing but the drag force. Right? The drag force will be important. The lift will be little. On the other end here in the streamline body, there will be significant lift. And if you recall, for this kind of body, when you have a certain chord, the lift coefficient CL was derived as something like CL = cosα, right? This depends on the inclination of the body. Anyway, for now, let us appreciate the fact that, for this kind of bridge deck geometry, it is a streamlined body. In such a body, the lift component dominates over the drag, while the drag remains minimal. So, essentially, the forcing that is going to come on this body will be predominantly lift force. And under the lift force, since the flow is a dynamic fluid flow, the lift force itself will fluctuate; it will vary with time and can induce vibrations. So, this will cause vibrations in the vertical plane—lateral as well as vertical vibrations. Not only that, but it can also induce torsional motion. By torsion, I mean that the deck will experience a twisting motion. So, you have flexure in the vertical plane, and this will be accompanied by torsion. This describes the overall motion of the bridge deck. The motion of the bridge deck under wind flow will cause the bridge deck to experience vertical and torsional motion. The vertical vibration occurs due to the fluctuating lift force, while torsion arises because the flow exerts forces that also fluctuate, causing twisting. This results in coupled torsional–flexural vibrations. You might wonder why there isn't much lateral vibration—there can be a small amount of lateral motion, but it is usually minor. In the plane of the wind lateral vibration, right? There can be lateral vibration because of whatever little drag force is coming and the fluctuating drag force. But that is because the wind velocity, of course, wind velocity if you consider wind time, these are all random loading. Right? There will be a mean component and a fluctuating component. Although we are not considering the fluctuating component, because of the small drag, there may be some lateral deflection, but we usually don't consider it. The amplitude of lateral vibration is much smaller compared to that of the vertical vibration, which is the dominant motion.



The bridge deck experiences both vertical deflection and torsion. We idealize the motion of the bridge deck using a two-degree-of-freedom system: one is the vertical vibration or deflection, denoted by W, and the other is the torsional rotation of the deck, denoted by θ. By neglecting the lateral deflection to be negligible, we consider a two-dimensional idealization. The motion of the deck is described using either a two- or three-degree-of-freedom model. In a two-degree-of-freedom model, we consider the deck's vertical motion, W, and its torsional rotation, θ. Thus, W represents the vertical deflection, and θ represents the angular rotation of the deck. Right, W represents the flexural (vertical) motion, and θ represents the torsional motion. That's the two-degree-of-freedom model. You could also extend this to a three-degree-of-freedom model if needed. If we consider a more detailed model, we can use a three-degree-of-freedom system, which is a bit more complicated. In this case, the deck can have W for vertical deflection, θ for torsion, and an additional degree of freedom, U. So, the three-degree-of-freedom model can include W, θ, and U. However, U is usually much smaller than both W and θ. So unnecessarily, it can either have a two-degree-of-freedom model or a three-degree-of-freedom model. You can see this is a three-degree-of-freedom model, right? The thing is that the only problem with the three degrees of freedom model is that it will involve many more unknown coefficients, something called flutter derivatives.

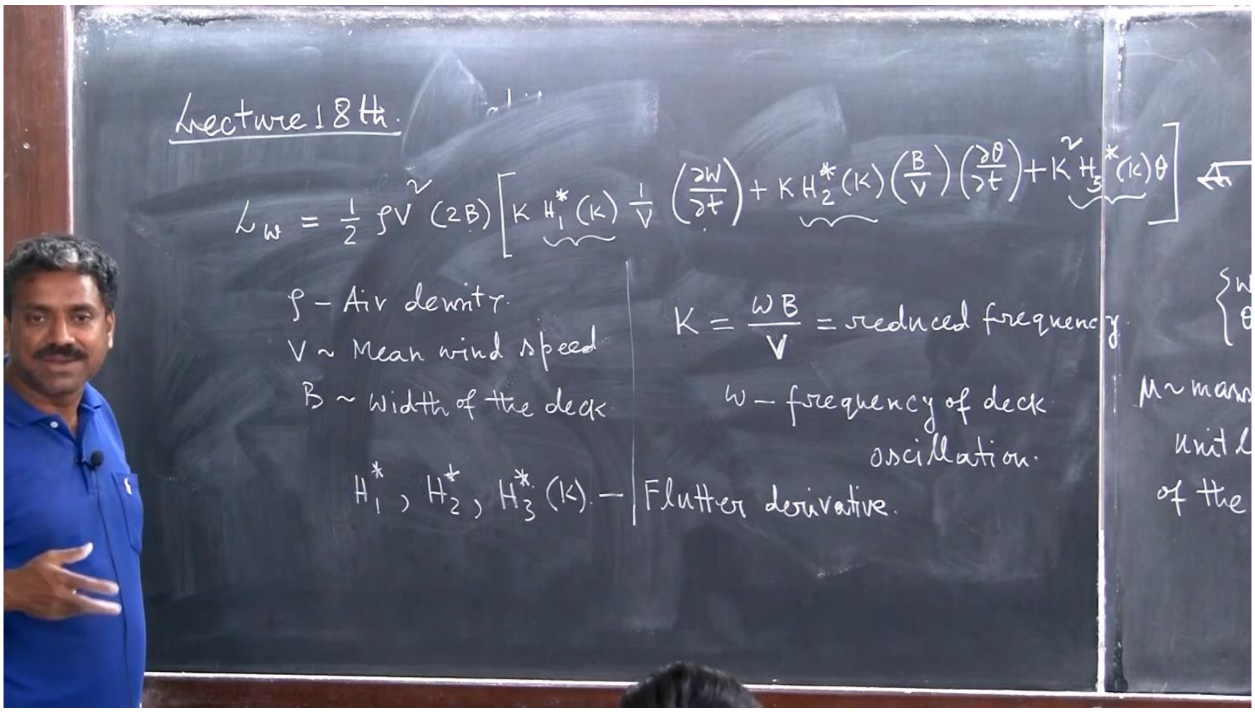


Later, I will come there. Okay, two degrees of freedom model is what we are going to analyze because this is simple. We can write down the equation of motion using the Lagrangian approach. You all know how to derive the equation of motion, right? I won't go through the full derivation here; I'll just write it directly. Of course, you are all familiar with the vibrations of continuous systems. Right now, I'm simplifying the system by condensing it into something like a beam model. In this representation, all contributions—such as the stiffness of the suspenders and the stiffness of the bridge deck—are effectively combined. The suspenders also contribute to the overall flexural stiffness of the deck. All of this is condensed into an equivalent stiffness term EI, GJ. Here, x is the coordinate along the deck, and the degrees of freedom are W for vertical deflection and θ for torsional rotation. So, two degrees of freedom. So now I'm going to write down the equation of motion. As you know, W and θ will be coupled—they influence each other.

Alright, let's directly write the equation of motion. The one is you write [Music]. Please see, let's try to understand this. It is nothing but the inertia term from mu, which is the mass per unit length of the deck. So, it's a lateral, you know, vertical vibration, you know,

$$\mu\frac{d^2w}{dt^2}+\mu y\frac{d^2\theta}{dt^2}$$

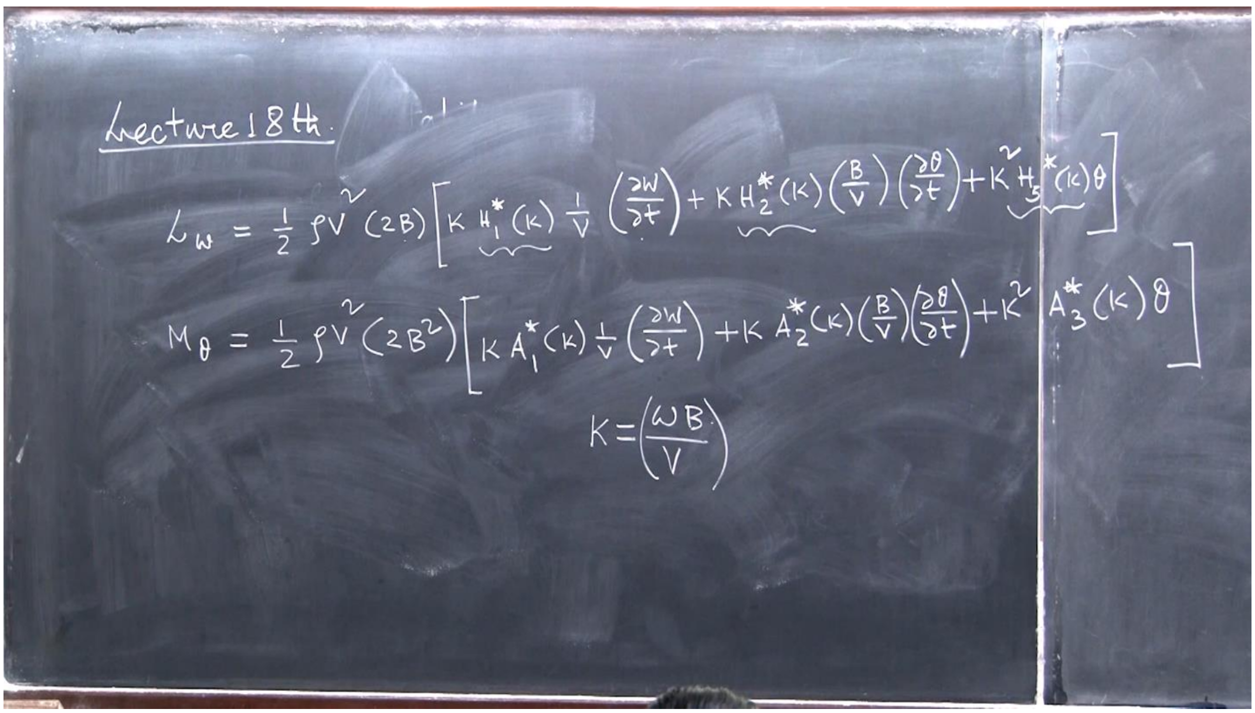
Why? Because the coupling between the flexure and torsion.



What is y? y is the difference between the elastic axis and the inertia axis. Right? What is the difference between the inertial axis and the elastic axis? For many cases, if it is symmetric, you know, and if there is no concentration, etc., then you will be zero. Okay, you do not need to consider Y then, right? Then CW CWD DWDT it is for damping, right? Yes. The damping term ($C_w\dot{w}$), together with the elastic restoring force from bending stiffness ($EI\frac{\partial^4 w}{\partial x^4}$), must balance the applied aerodynamic lift force L_w. This equation expresses the force equilibrium of the system. "Okay, we will write it, and then the equation of motion for the rotation is μ. Once again, for dimensional matching, EI will appear here since this term corresponds to the rotational motion." You see that $I_M\ddot{\theta}$ represents the mass moment of inertia term. The term $\mu y\ddot{w}y$ shows the coupling between torsional motion and vertical deflection. In addition, we have the damping terms and the flexural stiffness contributions. Finally, M_θ is the aerodynamic moment generated by the lift force. How do we find the L_W and M_θ? The aeroelasticians have derived the expressions for the lift and the corresponding moment. Okay, whatever we are writing here now, this is a little involved. I will write it, and then I will explain to you what this is. Okay, this is the way it is expressed: this L_W is

$$L_W = \frac{1}{2}\rho V^2(2B)\left[KH_1^*(K)\frac{1}{V}\frac{\partial w}{\partial t} + KH_2^*(k)\left(\frac{B}{V}\right)\left(\frac{\partial \theta}{\partial t}\right) + k^2H_3^*(k)\theta\right]$$

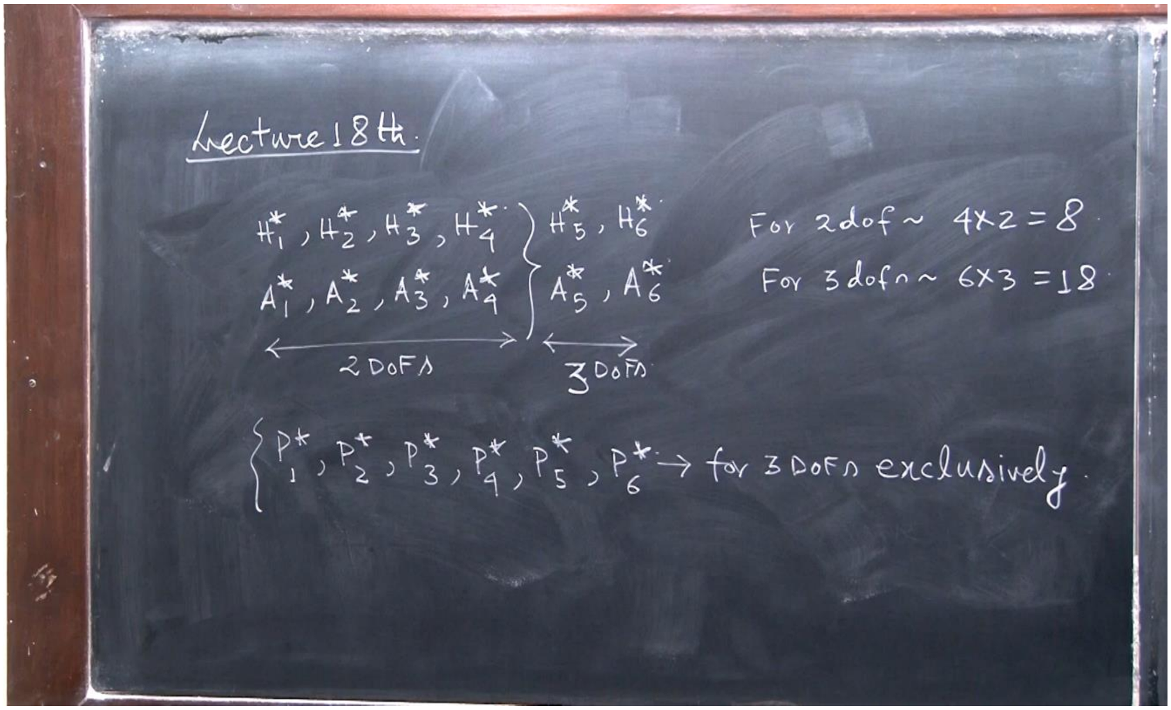
Row is nothing but the density of the air. This is the velocity of the wind. Once again, I am considering only that I am assuming as if the wind is flowing at a constant velocity. We are not considering fluctuations, but this is a simple simplified analysis. B is the wind velocity, row is the density of the air, B is the width of the deck, and K is a parameter. K is called reduced frequency. Okay. Reduced frequency, which is nothing but $\frac{\omega B}{V}$, is called reduced frequency. This is the important elastic parameter. "Ω is the frequency of oscillation of the deck, and B represents the deck width. The reduced frequency, K, is an important parameter defined using the wind speed and deck dimensions. The functions H_1^*, H_2^*, and H_3^* are aerodynamic transfer functions that depend on this reduced frequency. " Do you see that? Here, θ represents the rotation of the deck. You can observe that the lift force depends not only on the vertical velocity of the deck but also on its rotational velocity and the rotation itself. Do you see that? The lift force depends on the response of the deck itself. In other words, the loading on the structure depends on the structure's own motion, which is why this is called a parametric system. In parametric excitation, we have seen that during parametric resonance, the axial force acting on a structure, such as a column, becomes a parameter of the system. This is because the force itself influences the system's parameters directly in the equation of motion. You can also see that the geometric stiffness matrix depends on the axial force, whether it is in compression or tension. A compressive axial force tends to reduce the stiffness, while a tensile force can increase it. So, the system becomes parametric—that's why we call it a parametric system. You understand? In parametric resonance, it is called 'parametric' because the system's parameters, rather than just external forces, are varying. The condition for parametric resonance is also different from that of ordinary resonance. The exciting frequency needs to be two times the frequency of the lateral vibration, not the conventional resonance that you have learned in dynamics. So, all the other parameters are clear, right? The loading itself becomes a function of the deck geometry. Of course, the air density also plays a role, since it determines the aerodynamic pressure acting on the deck. So, the load must be a function of the air—specifically, the air density. It also depends on the deck width and the response of the structure, including both its velocity and displacement. The important point is why the reduced frequency becomes so significant. Here, three parameters have been identified: H_1^*, H_2^*, and H_3^*. All of these are functions of the reduced frequency, and they are referred to as the flutter derivatives. So, what are flutter derivatives, and why do we use the star (*) to identify them as such? These are important parameters that characterize the aerodynamic forces and moments on the structure as functions of the reduced frequency. Flutter derivatives, as functions of the reduced frequency, are essentially parameters that relate the aerodynamic lift forces acting on the structure to the structure's response. Do you understand that? And flutter derivatives have a very interesting property. Now, to obtain the flutter derivative, you have to conduct experiments in a wind tunnel. Okay, you can also study this using CFD (computational fluid dynamics) or fluid-structure interaction simulations, but the most reliable approach is to build a physical model and conduct routine tests. In our national internal facility, we perform such testing to determine the flutter derivatives. For conventional geometries, like this type of deck, we already have a good understanding of how these flutter derivatives depend on the reduced frequency, which I will demonstrate. And not that all the flutter derivatives will be equally effective. Some flutter derivatives will be more dominant over the others. Okay. And sometimes you can neglect some of the flutter derivatives. Understand? So, by introducing flutter derivatives, you can basically bypass complex fully structured interaction that you would otherwise need to do. Do you understand? Using internal experiments, you can extract experimental data and perform system identification to determine these parameters. This is essentially an inverse or identification problem—a concept you may encounter later in structural health monitoring. So, in the winter experiment, we have the model, and we always instrument the model. We subjected it to different wind velocities. "Since we have the structural responses available and we also know the wind velocity, we can use the governing equation. With the measured responses and forces, we can determine all of these parameters." "So, we can determine H_1^* and H_2^*. Since these flutter derivatives depend on the deck geometry as well as the reduced frequency of oscillation, we can plot their variation with the reduced frequency." Okay, now there's an important point here. I have presented two models: one is a two-degree-of-freedom model, and the other is a three-degree-of-freedom model. Right? These are all reduced-order models. A bridge, in reality, has thousands or even millions of degrees of freedom, but we can condense the system down to just two or three for analysis. That's what we do in wind tunnel testing. Even with a two-degree-of-freedom model, you still need H_1^*, H_2^*, H_3^*, and there is also an H_4^* that is not included here. There are four flutter derivatives for the lift force (L_W) and another four for the moment, so 4 + 4 equals 8—meaning you need to identify eight flutter derivatives to fully express the lift force and moment. Similarly, if you consider a three-degree-of-freedom model, that means you have deflection, rotation, torsion, and also lateral translation. Then, for a more detailed model, you won't just have four flutter derivatives — you will have six: H_1^*, H_2^*, H_3^*,..., H_6^*. Similarly, for the moment, you will have six derivatives: A_1^*, A_2^*, A_3^*,..., A_6^*. In addition, there are six other derivatives that describe the coupled effects between lift and moment.



So, you will have 6 times three; how much is that? 18. As you can see, the dimensionality of the problem increases exponentially—from 8 to 18—which is why we often neglect the lateral degree of freedom. It's usually not significant, so we focus only on torsion, rotation, and vertical deflectionYou understand that it is necessary to reduce the number of flutter derivatives required to express the forcing as a function of the response of the structure, and I have already described how to determine these flutter derivatives through wind tunnel testing that involves solving some identification problems. Identification is nothing but solving an optimization problem, essentially by measuring the response through adequate instrumentation and also subjecting the model to specific wind velocity. That's exactly what we do at the National Wind Tunnel Facility. From there, you can develop different algorithms—you can tweak the mathematics a bit, use a stochastic algorithm, or a deterministic one. This is where your thesis work comes in. Right? Anyway, so you understand what a flutter derivative is, right? And how the number of flutter derivatives increases with the incorporation of additional degrees of freedom, right? Similarly, we can express that K in this expression is very important for the reduced frequency $\left(\frac{\omega B}{V}\right)$. Okay. Now I write M_θ. M_θ can also similarly be expressed as

$$M_\theta = \frac{1}{2}\rho V^2(2B)^2\left[KA_1^*(K)\frac{1}{V}\left(\frac{\partial w}{\partial t}\right) + KA_2^*(k)\left(\frac{B}{V}\right)\left(\frac{\partial \theta}{\partial t}\right) + k^2 A_3^*(k)\theta\right]$$

Lift force is denoted H* and moment is denoted A*. Once again, please note that K is nothing but $\frac{\omega B}{V}$. V is the wind velocity, and V is the liquid. Then omega is the frequency of oscillation. Do you see that here? It's a function of the lateral deflection velocity, the angular velocity, and the rotation. Right? Once again, the Flutter derivatives A_1^*, A_2^*, and A_3^*. Right? Achieving derivatives is also determined in a similar way. Please note that in a two-degree-of-freedom system, here at least we can consider 3 + 3 = 6 total derivatives. Okay. But actually, there are 18, and then if we consider another degree of freedom, that will be 18. Huh. Anyway, once again there will be a reduction in this. Okay. So, do you understand that? Now you can substitute L_W and M_θ here, right? You can substitute it here. So, two equations, right? Now, once you obtain all these things, then you may remove all of this. You noted it down, right? Then what will happen? We can solve this equation for a two-degree-of-freedom system. What is the best way to solve this equation of motion, huh? So right now, the flutter derivatives are H_1^*, H_2^*, H_3^*, and sometimes you also put H_4^*. Then you also have A_1^*, A_2^*, A_3^*, A_4^*. These eight are used for a two degree of freedom model. Okay. Now, if you want to add these, they are for two degrees of freedom. Then, if you require more, then what will happen? This is H_5^*, H_6^*, and then A_5^*, A_6^*. These are another two degrees of freedom, and these are the three degrees of freedom. So, this plus this and in addition you will have for three degrees of freedom P_1^*, P_2^*, P_3^*, P_4^*, P_5^*, P_6^*. Because these are exclusively for three degrees of freedom. Do you understand that? For two degrees of freedom, these are 4 * 2 = 8. And for a three-degree-of-freedom elastic model, there are 6 * 3 = 18. Okay.



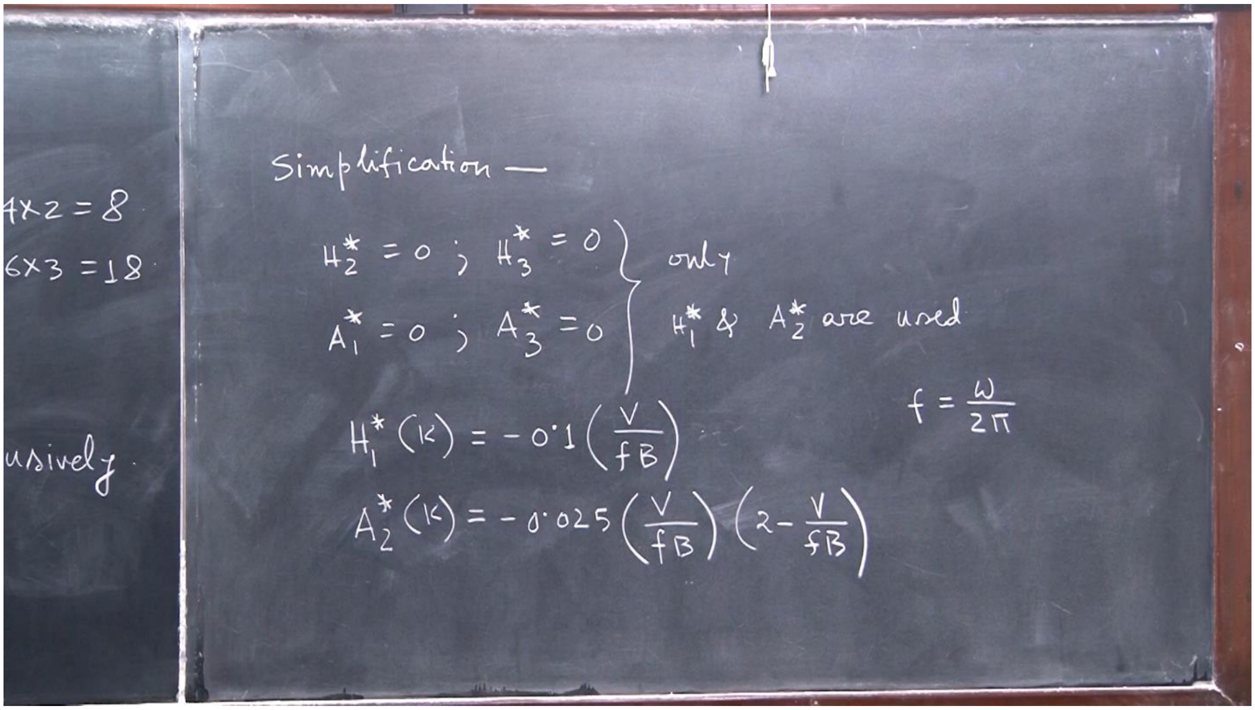
Now we've evaluated them from testing the aerolastic model. Now we'll solve the equation. So, the best way to solve this is that you know there will be little simplification. Okay, the simplification is here. The simplification for our case, specifically for the Tacoma Bridge geometry, is as follows: H_2^* and H_3^* are both zero. Similarly, A_1^* and A_3^* are also zero. Therefore, the only non-zero terms are H_1^* and A_2^*, which are the only ones used in this analysis; all others are zero. Okay, that's clear. Now, there is one further simplification that has already been made by others. H_1^* is a function of k and can be expressed as:

$$H_1^* = -0.1\left(\frac{V}{FB}\right)$$

Similarly, A_2^* as a function of k is:

$$A_2^* = -0.025\left(\frac{V}{FB}\right)\left\{2 - \left(\frac{V}{FB}\right)\right\}$$

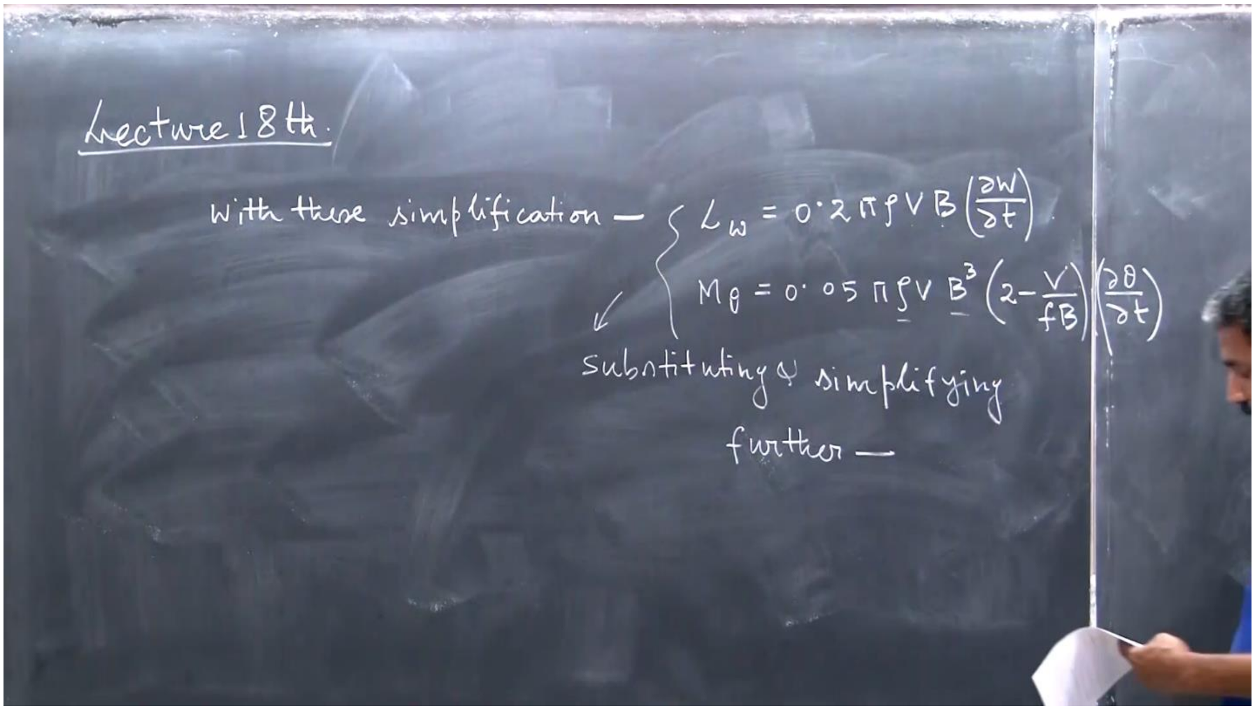
It is important to note something interesting: in these expressions, $F_B = \frac{\omega}{2\pi}$, which is simply the frequency in Hz. These results come from aerodynamic model testing carried out on this type of bridge deck geometry in wind-tunnel facilities.



From these tests, researchers plotted H2, H3, A1, and A3 as functions of v/fb. And they have seen that these kinds are kind of neglected, and these kinds of functional expressions they have found out. Okay. But these expressions are not universal. Please note that for different bridge geometries, these results will vary. However, for the flat plate bridge deck—and specifically for the Tacoma Narrows Bridge—these are the values obtained from the wind-tunnel testing. An interesting point to note here is that $H_1^*(k)$ does not change its sign—it remains consistent. However, in the case of $A_2^*(k)$, for certain values, it can switch from positive to negative. This sign change is quite significant and worth paying attention to, as we will illustrate later. But A_2^* for a certain value of wind, this will change its behavior, which is very interesting and that we will come to later. Okay. Now, the equations will be further simplified because all other flutter derivatives are zero. So, I will rewrite the equations now. So, what will I do? With this simplification, L_W (the lift becomes) = $0.2\pi\rho B\left(\frac{\partial w}{\partial t}\right)$. You see that lift becomes a function only of the vertical velocity. It is now expressed as a function of the wind speed, the deck width, and the air density. You can see that the form becomes much simpler since we have substituted the functional expressions of the flutter derivatives. Moreover, for the moment equation, the contribution becomes zero. The expression for the aerodynamic moment is:

$$M_\theta = 0.05\pi\rho V B^3 \left\{2 - \left(\frac{V}{F_B}\right)\right\}\left(\frac{\partial\theta}{\partial t}\right)$$

Here, f is the oscillation frequency in Hz, while ω is the oscillation frequency in radians per second, with the relation: $\omega = 2\pi f$.



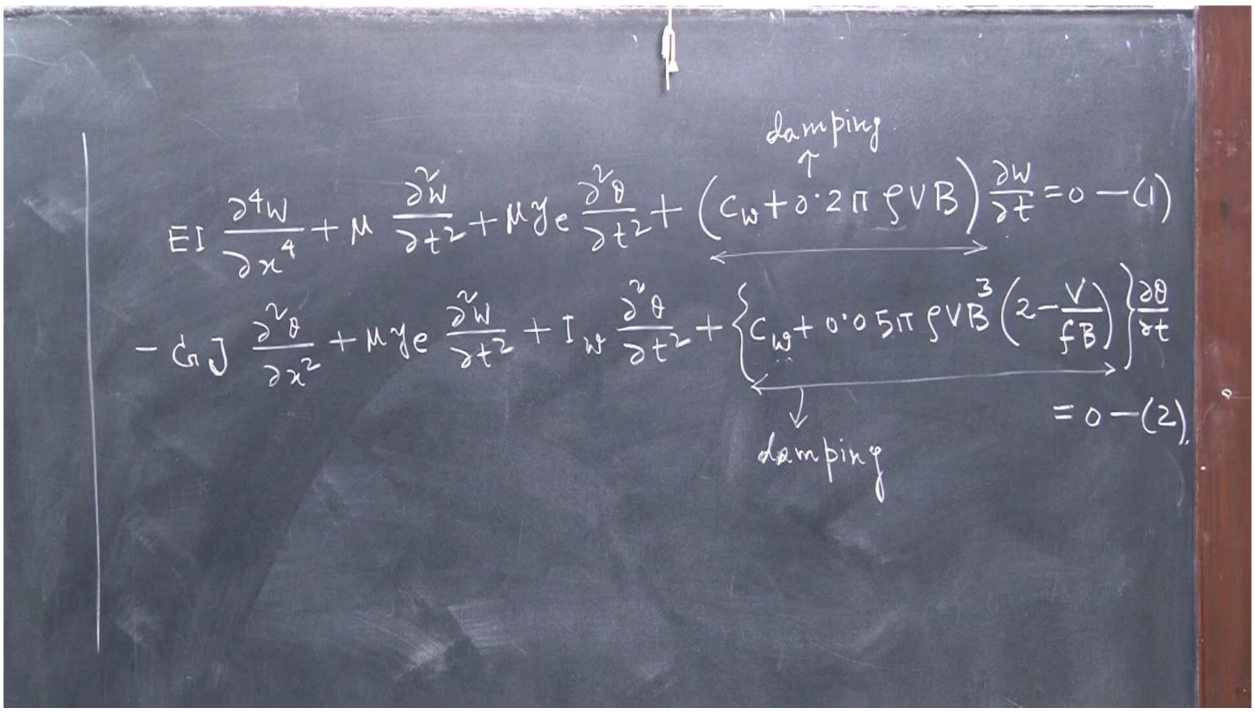
The angular velocity is simply $\frac{\partial\theta}{\partial t}$. You can also see that the expression depends on air density ρ, deck width B, wind speed V, and angular velocity. The key difference to note is that L_W always retains a definite value, whereas M_θ can switch from positive to negative. In other words, the aerodynamic moment may change its direction and nature depending on the wind speed. The consequences of this behavior will become clear in the subsequent discussion. Okay. Now I will substitute and simplify. Simplifying further, what we are going to get is basically

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} + \mu y_e\frac{\partial^2\theta}{\partial t^2} + (c_w + 0.2\pi\rho V B)\frac{\partial w}{\partial t} = 0$$

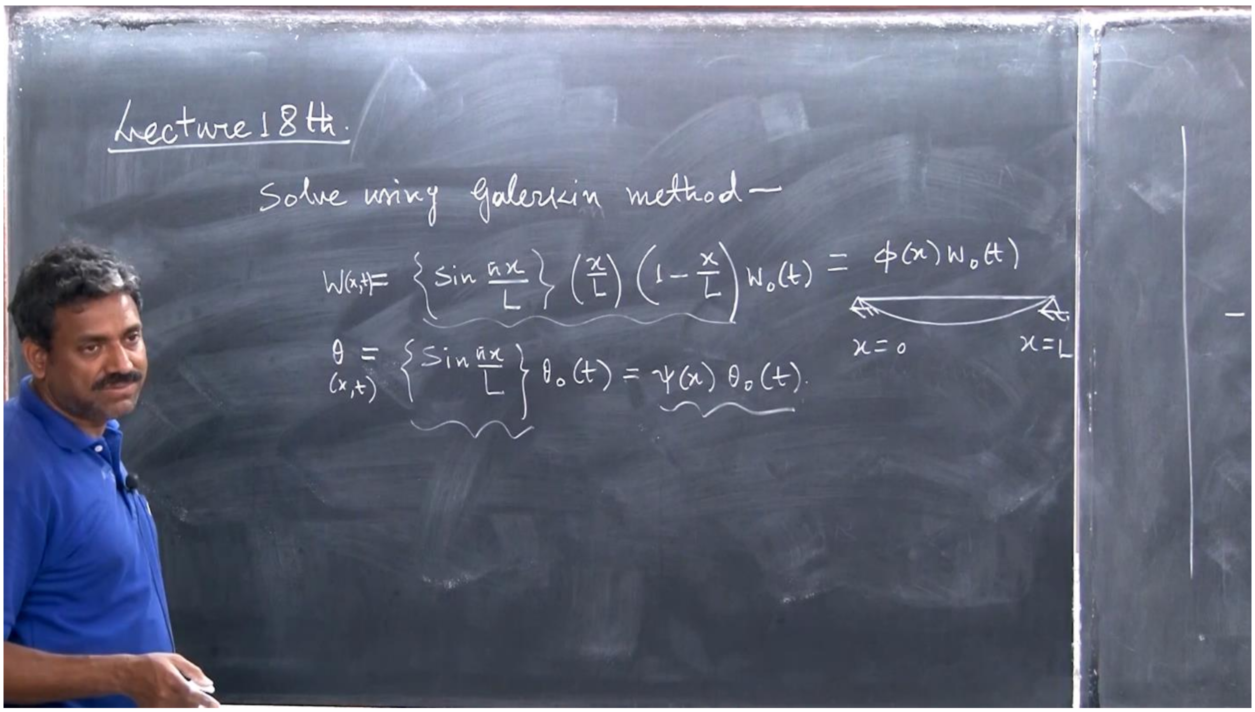
Do you know what j is? Please tell me what j is. j is nothing but the St. Venant torsional constant, right? And for a closed section, you know how to evaluate it—there is no warping contribution, right? That's because it is a closed section.

$$-GJ\frac{\partial^2\theta}{\partial x^2} + \mu y_e\frac{\partial^2 w}{\partial t^2} + I_w\frac{\partial^2\theta}{\partial t^2} + \left[C_w + 0.05\pi\rho V B^3\left\{2 - \left(\frac{v}{F_B}\right)\right\}\right]\frac{\partial\theta}{\partial t} = 0$$

; this is equation (2). You see that this is a damping term, right? Effectively, I'm basically collecting the term or grouping the term to find out the effective damping. You see, this is the effective damping because it is associated with the respective velocities—vertical velocity, flexural velocity, and torsional velocity.

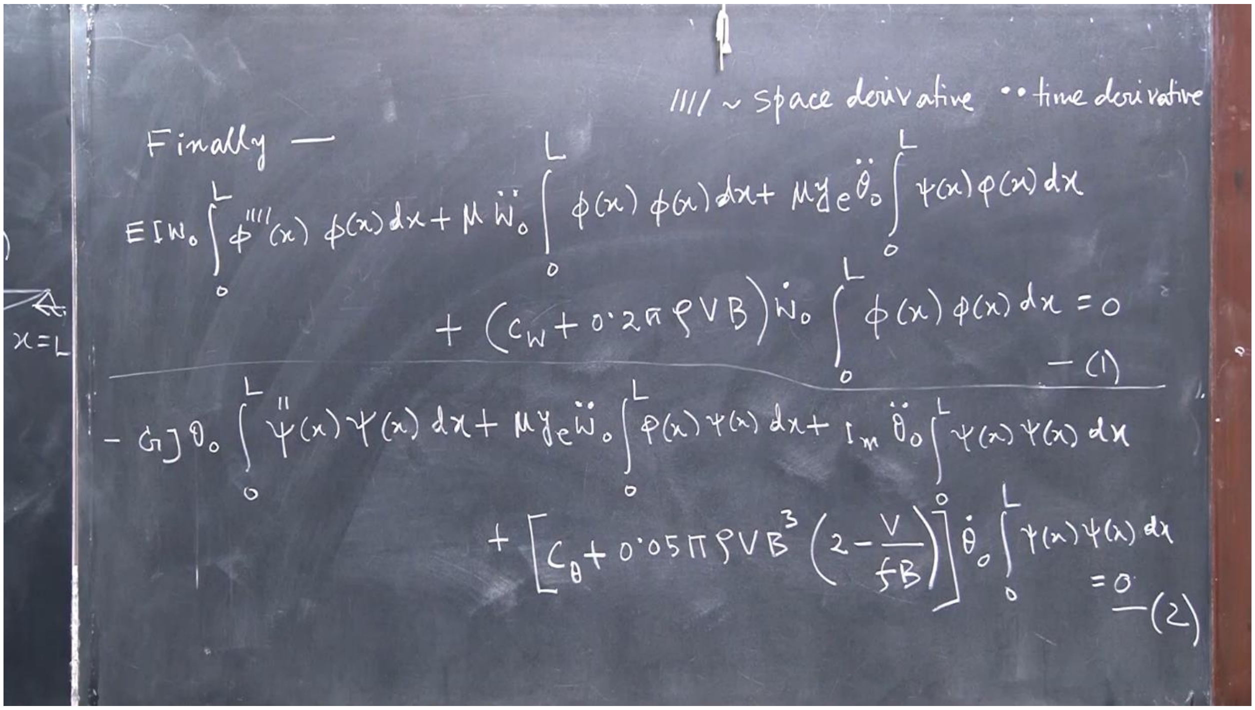


The interesting part is that for the flexural velocity, this damping is always positive, so there is no issue. Damping—and positive damping—means it will always help in dissipating energy, as in viscous damping. Here, viscous damping is represented by C_W, but there is also a contribution from the forcing. So, it has two parts: the lift and the force due to wind or airflow, which effectively modify the damping in the system. With viscous damping, an additional term appears. This term is associated with the system, and there is no problem—it actually increases or enhances the damping, at least for vertical or flexural vibrations. "But what is the situation regarding torsional vibration? You see, C_W—this is the discussed part—but for certain values of wind speed, it has the potential to become negative." Negative damping means what? Instead of dissipating energy, it will import energy to the system, and if it keeps on importing energy to the system, it will lead to instability and an unbounded response. "So here, do you see the potential for instability? This is contributed to by the torsional vibration, not the flexural vibration, although both of them are coupled." Thus, the wind excitation effectively modifies the system's effective damping. This is a very important observation—please note it. These two terms are referred to as aerodynamic damping, whereas C_W represents viscous damping. The aerodynamic damping terms, caused by wind excitation, dissipate energy and are non-conservative forces. Now, we need to solve this differential equation. Of course, it is a system of two coupled differential equations.



The best way to solve it is to assume that the beam is simply supported at both ends. You can then use the Galerkin method to solve it. While using the Galerkin method, we must assume forms for W and θ that satisfy the boundary conditions. Let us assume the beam is simply supported. For a simply supported beam of length L, we can assume the mode shape to be $\sin\left(\frac{\pi x}{L}\right)$. This effectively reduces the system to a simply supported beam. So, we can assume the deflection W_0 as $\sin\left(\frac{\pi x}{L}\right)$ and θ similarly as $\sin\left(\frac{\pi x}{L}\right)$. However, if we do it exactly like this, a small problem arises. Therefore, we need to include some additional terms. Additional things mean looking at the x=0 deflection, which needs to be zero at x=l and w. So, we may let x be l to be another function and 1 - x to be another function over here. Yes, at x=0 it will be zero. At x=12, this will also be zero. So let us assume that this is okay. So, W depends on both x and t. Correspondingly, we can write

W_0 as a function of time and θ_0 as a function of time. So, this is basically the spatial term, and this is the time-dependent term—similar to the method of separation of variables. We can denote the spatial function as f(x) and the time-dependent function as $W_0(t)$, or simply s(t). So, space and time are separated the way we do in the separation of variables. Okay.



So then at least the differential equation in terms of space will be eliminated. We will make it algebraic. From the differential operator, we will make the operator algebraic. We will get the time one, and then we can solve. Okay. By step-by-step integration, you do not need to solve the time because these are homogeneous equations. You can see this will lead to a value problem. Okay. Huh? So that we understand the idea, we substitute this into the equation. This is how the Galerkin technique works: after substitution, we multiply the equation by the weight function. These are the trial functions that you need to substitute into the equation, and then you multiply them by the test function. For the first equation, use f(x) as the weight function, and for the second, use Ψ(x) as the weight function. In the Bubnov-Galerkin method, the trial and test functions are the same. All of you must follow this procedure. If you have done finite element analysis or the Galerkin method in other classes, this will be familiar, so I am not going into all the details here. We have to choose this function, which I have explained by satisfying the geometric boundary condition, right? We substitute the trial function into the equation, multiply it by the weight function, and then integrate over the domain from 0 to L. Let me write down the first step so you can see how it works. I will write the final step, and you can do the details yourself. Finally, we will get an expression like

$$EIw_0 \int_0^L \phi'''' (x)\phi(x)dx + \mu\ddot{w}_0 \int_0^L \phi (x)\phi(x)dx + \mu y_e\ddot{\phi}_0 \int_0^L \Psi (x)\phi(x)dx$$

$$+ (c_w + 0.2\pi\rho VB)\dot{w}_0 \int_0^L \phi (x)\phi(x)dx = 0 - (1)$$

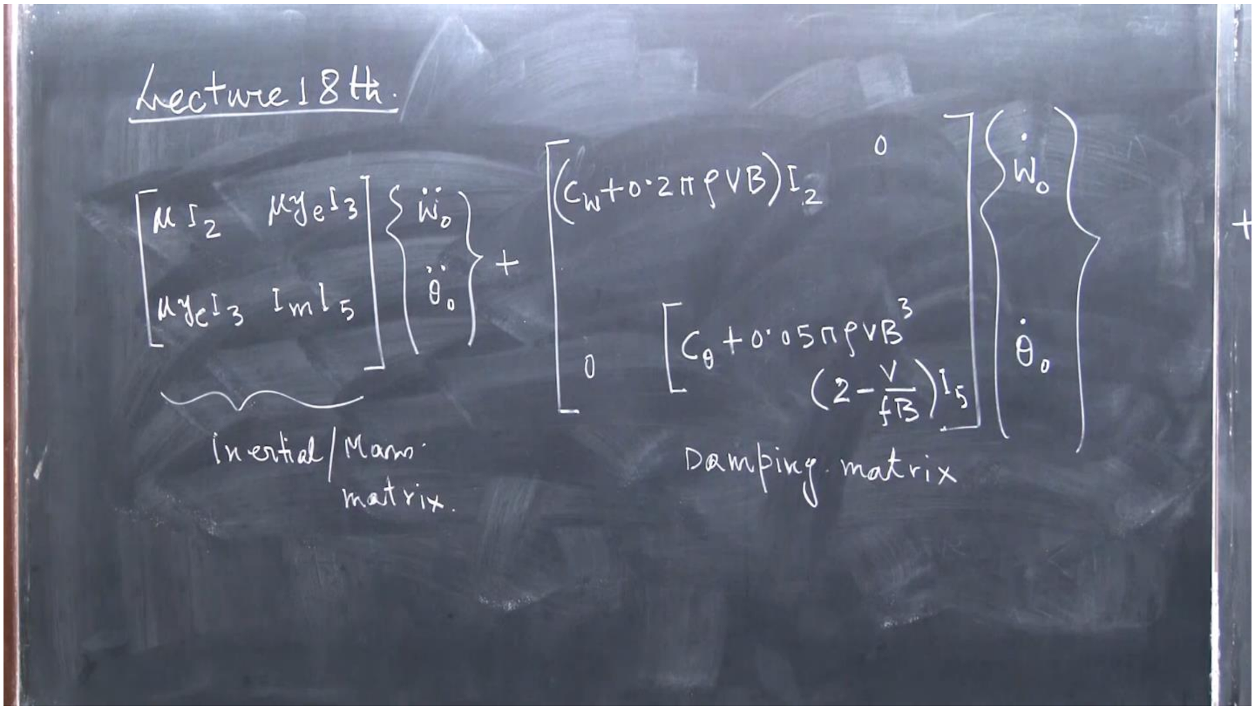
This is how it will look, with all the weight functions included. Okay. And Ψ is coming because the approximation theta C is there. Clear? This is one equation, the first equation. And for the second one, you will get

$$-GJ\theta_0 \int_0^L \Psi'' (x)\Psi(x)dx + \mu y_e\ddot{w}_0 \int_0^L \phi (x)\Psi(x)dx + I_m\ddot{\theta}_0 \int_0^L \Psi (x)\Psi(x)dx + \left[c_\theta + 0.05\pi\rho VB^3 \left(2 - \frac{v}{f_b}\right)\right] \dot{\theta}_0 \int_0^L \Psi (x)\Psi(x)dx = 0$$

Please note there is slightly, you know, please do not mingle these two equations. Okay. Please note that when I'm writing this class (") these are space derivatives, okay? We conventionally use them, and then when I'm writing this, you know, (¨) dot, this is referring to the time derivative. The number of dots implies the order of the derivative, okay? This is the fourth order space derivative, and this one is a time derivative, right?

$$GJ\theta_0 \int_0^L \Psi'' (x)\Psi(x)dx + \mu y_e\ddot{w}_0 \int_0^L \phi (x)\Psi(x)dx + I_m\ddot{\theta}_0 \int_0^L \Psi (x)\Psi(x)dx + \left[c_\theta + 0.05\pi\rho VB^3 \left(2 - \frac{v}{f_b}\right)\right] \dot{\theta}_0 \int_0^L \Psi (x)\Psi(x)dx = 0.$$

Please see the equation how it looks. Huh? For the first equation, the trial function and test function are φ. For the second equation, the trial and test functions are Ψ. There are some cross terms involving the product of Ψ and φ, which arise as a consequence of the coupling between torsional motion and vertical deflection. This coupling occurs only in the terms that contain y, where y represents the distance between the elastic axis and the inertial axis. All these coupling terms that contain y involve the product of two different functions, Ψ(x) and φ(x).

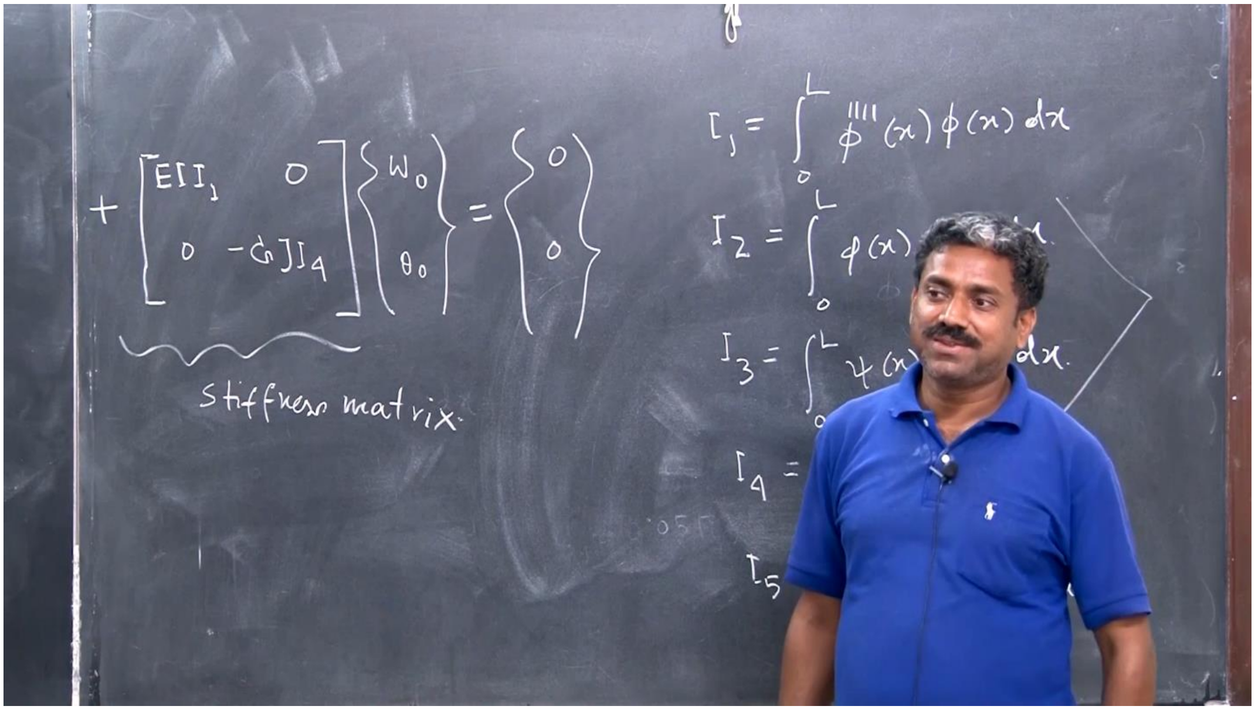


One of them acts as the trial function, while the other serves as the test function. When we perform the integration from 0 to L, the terms involve Ψ(x) and φ(x), both of which contain sinusoidal components such as $\sin\left(\frac{\phi(x)}{L}\right)$. In addition, the first function also includes polynomial terms like $\left(\frac{x}{L}\right)$ and $\left(1-\frac{x}{L}\right)$. So, if you integrate finally, then what you will get, I'm writing in matrix form. So, this matrix form, okay, if you integrate, then you will get, huh, I'm writing in matrix form, then you will get it like

$$\begin{bmatrix} \mu I_2 & \mu y_e I_3 \\ \mu y_e I_3 & I_m I_5 \end{bmatrix} \begin{bmatrix} \ddot{W}_0 \\ \ddot{\theta}_0 \end{bmatrix} + \begin{bmatrix} (c_w + 0.2\pi\rho V I_2) & 0 \\ 0 & \left[c_\theta + 0.05\pi\rho V B^3 \left(2 - \frac{V}{F_B}\right) I_5\right] \end{bmatrix} \begin{Bmatrix} \dot{W}_0 \\ \dot{\theta}_0 \end{Bmatrix}$$

$$+ \begin{bmatrix} EII_1 & 0 \\ 0 & -GJI_4 \end{bmatrix} \begin{Bmatrix} W_0 \\ \theta_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Then this integral I'm writing I_1 is nothing but the integral from 0 to L $\phi(x)\phi(x)dx$. I_2, I_3, those are nothing but the integral $I_2 = \int_0^L \phi\,(x)\phi(x)dx$.



I_3 is nothing but $\int_0^L \Psi\,(x)\phi(x)dx$, $I_4 = \int_0^L \Psi''(x)\Psi(x)dx$, and $I_5 = \int_0^L \Psi(x)\Psi(x)dx$. Please note that the coupling terms are I_2 and I_3. You see that these two are coupling terms; the rest of the things and all the others include only different derivatives, but these are the same function. The same function only I_3 and I_4 are different. Okay, so that's what you see here: I_2 and I_5 are there, and then there is I_1, I_4, and then here I_2, I_3, I_3, I_2, I_3. So, this is a system of homogeneous equations, right? Please write it down first. These represent the stiffness terms—in other words, the stiffness matrix. The next set corresponds to the damping matrix, which depends on velocity. Finally, the remaining terms represent the inertia, or more precisely, the mass matrix. So, one term corresponds to stiffness, another to damping, and the last to inertia (mass). So, I hope you've understood it up to this point. It's a homogeneous system of equations. Initially, it wasn't homogeneous because the lift and moment terms were present. However, we made it homogeneous by assimilating those terms into the damping expression since they are functions of velocity. So, they ultimately contribute to the damping, which is why they are referred to as aerodynamic damping. This aerodynamic damping is associated with the translational deflection — that is, the vertical velocity — while the other terms are related to the torsional velocity. Once again, because of the presence of the $\left(2 - \frac{V}{f_b}\right)$ term, this damping can change from positive to negative, leading to instability. In the next class, we'll see how we can derive stability information from this and determine the critical velocity—that is, the critical wind speed at which instability is triggered. This type of instability is known as aeroelastic flutter, which was the phenomenon responsible for the collapse of the Tacoma Narrows Bridge. Okay, thank you very much.